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A Cantor set in the plane that is not σ-monotone

by

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Abstract. A metric space (X,d) is *monotone* if there is a linear order $<$ on X and a constant c such that $d(x,y) \le cd(x,z)$ for all $x<y<z$ in X, and *σ-monotone* if it is a countable union of monotone subspaces. A planar set homeomorphic to the Cantor set that is not σ-monotone is constructed and investigated. It follows that there is a metric on a Cantor set that is not σ-monotone. This answers a question raised by the second author.

1. Introduction.

The following notions were introduced in [4]:

Definition 1.1. A metric space (X,d) is called

- *monotone* if there is a linear order $<$ on X and a constant c such that $d(x,y) \le cd(x,z)$ for all $x<y<z$ in X,
- *σ-monotone* if it is a countable union of monotone subspaces.

Topological properties of monotone and σ-monotone spaces are investigated in [1]. We quote some results: A subspace of a monotone metric space is monotone. A metric space with a dense monotone subspace is monotone. Every monotone space topologically embeds into a linearly orderable metrizable topological space, but does not have to be linearly orderable. Every separable monotone space topologically embeds into the line. The topological dimension of a σ-monotone space is at most 1. Every ultrametric space is monotone. Every topologically discrete metric space is σ-monotone, but not necessarily monotone.

Fractal properties of monotone and σ-monotone sets in Euclidean spaces, namely porosity, Hausdorff measures and Hausdorff dimensions and rectifiability, and functions with a σ-monotone graph are investigated in [2].

An application appears in [3], where σ-monotone sets serve as a tool for a characterization of Borel sets in a Euclidean space $\mathbb{R}^n$ that map onto a

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cube $[0,1]^m$ ($m \leq n$) by a quasi-Lipschitz mapping, i.e. a mapping that is β-Hölder for each $\beta < 1$.

The very first application of σ-monotone spaces appears in [4]: Let dim and $\dim_{\mathsf{H}}$ denote, respectively, the topological and Hausdorff dimensions. It is shown that every analytic σ-monotone metric space X contains a Lipschitz preimage of every self-similar set S satisfying the strong separation condition with $\dim_{\mathsf{H}} S < \dim_{\mathsf{H}} X$. A number of results are derived from this theorem. E.g., any analytic metric space X contains a universal measure zero set $E \subseteq X$ such that $\dim_{\mathsf{H}} E \geq \dim X$; any analytic σ-monotone metric space X contains a universal measure zero set $E \subseteq X$ such that $\dim_{\mathsf{H}} E \geq \dim X$; and any analytic set $X \subseteq \mathbb{R}^n$ contains a universal measure zero set $E \subseteq X$ such that $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$.

As explained in [4], the following question raised in [5] is of particular interest:

Question 1.2 ([5]). Is every compatible metric on the Cantor set σ-monotone?

The goal of the present paper is to provide a negative answer to this question by constructing a set X in the plane that is homeomorphic to the Cantor set but is not, as a metric subspace of the Euclidean plane, σ-monotone.

In Section 2 we state and prove a combinatorial lemma that is essential for the construction of the set X, which is performed in Section 3. In Section 4 we calculate the linear Hausdorff measure of X and its Hausdorff dimension. In Section 5 we prove that every σ-monotone subset of X is contained in a σ-compact set that is meager in X and has linear Hausdorff measure zero. In particular, X is not σ-monotone.

Throughout the paper we use the following notation and terminology. $\mathbb{N}$ denotes the set of all positive integers, *excluding zero*. The cardinality of a set A is denoted $|A|$. A metric on a metrizable space is *compatible* if it induces the topology of X. If (X,ρ) is a metric space and $x \in X$, the symbol $B_\rho(x,r)$ (or just $B(x,r)$) denotes the closed ball centered at x with radius r. If X, Y are metric spaces, a mapping $f : X \to Y$ is termed *bi-Lipschitz* if it is bijective and both f and its inverse are Lipschitz mappings. Of course, a bi-Lipschitz mapping is a homeomorphism. The metric spaces X, Y are *bi-Lipschitz equivalent* if there is a bi-Lipschitz mapping $f : X \to Y$.

We will need the following facts established in [1].

Lemma 1.3. *A metric space that is bi-Lipschitz equivalent to a monotone space is monotone.*

Lemma 1.4. *If X is σ-monotone, then it is a countable union of closed monotone subspaces.*

2. Polygons

DEFINITION 2.1. Consider the cyclic group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ for $n \in \mathbb{N}$. Define a metric on $\mathbb{Z}_n$ by

$$\rho_n(i,j) = \min(|i-j|, n-|i-j|).$$

The metric spaces $(\mathbb{Z}_n, \rho_n)$ are thought of as abstract regular polygons. They serve as building blocks for the construction of X. The following combinatorial lemma is crucial.

LEMMA 2.2. *Let $A \subseteq \mathbb{Z}_n$, $|A| \geq 3$. For each linear order $\prec$ on A there are $i, j, k \in A$ such that $i \prec j \prec k$ and*

$$\frac{\rho_n(i,j)}{\rho_n(i,k)} \geq \frac{1}{2} \frac{|A|-1}{n-(|A|-1)}.$$

Proof. Denote $|A| = m$. If $n = m = 3$, then $1 = \rho_n(i,j)$ for any distinct $i, j \in A$. If $n > m = 3$, then $1 \leq \rho_n(i,j) \leq n/2$ for any distinct $i, j \in A$. In either case, the inequality trivially holds for any triple i, j, k of distinct elements of $\mathbb{Z}_n$. We shall thus assume that $|A| > 3$. Let $A = \{a_z : z \in \mathbb{Z}_m\}$ be the unique increasing enumeration of A (with respect to the natural order).

Throughout the proof, addition is modulo m. Denote by ℓ the integer part of $m/2$.

Order $\mathbb{Z}_m$ by $z \lhd z'$ iff $a_z \prec a_{z'}$. Let $N = \{z \in \mathbb{Z}_m : z \lhd z + \ell\}$. If $N = \emptyset$, then $z \rhd z + \ell$ for all z and thus

$$0 \rhd \ell \rhd 2\ell \rhd 3\ell \rhd \cdots \rhd m\ell = 0.$$

Therefore $N \neq \emptyset$. If $N = \mathbb{Z}_m$, then $z \lhd z + \ell$ for all z and thus

$$0 \lhd \ell \lhd 2\ell \lhd 3\ell \lhd \cdots \lhd m\ell = 0.$$

Therefore $N \neq \mathbb{Z}_m$. So there are $z_1 \in N$ and $z_2 \notin N$. Let z be the last term in the sequence $z_1, z_1 + 1, \ldots z_2 - 1$ that satisfies $z \in N$. Clearly $z + 1 \notin N$. Consider two cases:

• $z + \ell \lhd z + 1$: Since $z \in N$, we have $z \lhd z + \ell \lhd z + 1$. Put $i = a_z$, $j = a_{z+\ell}$ and $k = a_{z+1}$. Clearly $i \prec j \prec k$. There are at most $n - m$ nodes strictly between a_z and a_{z+1}. Hence $\rho_n(i,k) \leq n - (m-1)$. There are at least $\ell - 1$ nodes strictly between a_z and $a_{z+\ell}$ both clockwise and counterclockwise. Hence $\rho_n(i,j) \geq \ell$. Thus

$$\frac{\rho_n(i,j)}{\rho_n(i,k)} \geq \frac{\ell}{n-(m-1)} \geq \frac{1}{2} \frac{|A|-1}{n-(|A|-1)}. \tag{1}$$

• $z + \ell \unrhd z + 1$: Obviously $|A| > 3$ yields $z + \ell \neq z + 1$. Thus $z + \ell \rhd z + 1$. Since $z + 1 \notin N$, we have $z + 1 + \ell \lhd z + 1 \lhd z + \ell$. Put $i = a_{z+1+\ell}$, $j = a_{z+1}$ and $k = a_{z+\ell}$. Clearly $i \prec j \prec k$. The inequality (1) follows by the same reasoning as above. ■

REMARK 2.3. Since the set $A \subseteq (\mathbb{Z}_n, \rho_n)$ in the lemma is finite, it is trivially monotone. The lemma says that a constant witnessing its monotonicity cannot be less than $\frac{1}{2}\frac{|A|-1}{n-(|A|-1)}$. In particular, a constant witnessing monotonicity of $(\mathbb{Z}_n, \rho_n)$ cannot be less than $(n-1)/2$.

REMARK 2.4. The particular case $A = \mathbb{Z}_n$ can be phrased this easy way: Whatever linear order $\prec$ a regular polygon is equipped with, there are neighboring vertices x, z and a vertex y opposite to them such that $x \prec y \prec z$.

3. Construction of the set. In this section we define a rather regular compact set in the plane that

- is homeomorphic to the Cantor set,
- has positive and finite Hausdorff length,
- is not σ-monotone.

The set is constructed so that every nonempty open subset contains for each n a bi-Lipschitz copy of the polygon $(\mathbb{Z}_n, \rho_n)$ described in the previous section. It is a continuous image of a cartesian product of the groups $\mathbb{Z}_n$, provided with a suitable metric.

DEFINITION 3.1. Let

$$\mathsf{Z} = \prod_{n=1}^{\infty} \mathbb{Z}_n = \{x \in (\mathbb{N} \cup \{0\})^{\mathbb{N}} : x(n) < n \text{ for all } n \in \mathbb{N}\}.$$

Provide Z with the product topology. Recall Definition 2.1 and define a metric ρ on Z by

$$\rho(x,y) = \sup_{n \in \mathbb{N}} \frac{\rho_n(x(n), y(n))}{n!}.$$

Since Z is a countable product of finite topological groups, it is obviously a zero-dimensional compact topological group. It has no isolated points. Thus it is homeomorphic to the Cantor ternary set.

The verification that ρ is a metric compatible with the topology of Z is straightforward. Thus (Z, ρ) is a metric space homeomorphic to the Cantor set.

Since

$$\frac{\rho_n(x(n), y(n))}{n!} \geq \frac{1}{n!} \geq \frac{m}{m!} > \frac{\rho_m(x(m), y(m))}{m!} \tag{2}$$

whenever $n < m$ and $x(n) \neq y(n)$, ρ is equivalently described by the following formula that we shall often use: if $x \neq y$, then

$$\rho(x,y) = \frac{\rho_{\kappa(x,y)}\big(x(\kappa(x,y)), y(\kappa(x,y))\big)}{\kappa(x,y)!},$$

where

$$\kappa(x,y) = \min\{n : x(n) \neq y(n)\}. \tag{3}$$

Throughout the paper, $\mathsf{Z}^\bullet = \bigcup_{m=1}^{\infty} \prod_{n=1}^{m} \mathbb{Z}_n$ denotes the set of initial segments of elements of Z. For $E \subseteq \mathsf{Z}$ set

$$E^\bullet = \{p \in \mathsf{Z}^\bullet : p \subseteq x \text{ for some } x \in E\}.$$

For $p \in \mathsf{Z}^\bullet$, $[\![p]\!] = \{x \in \mathsf{Z} : p \subseteq x\}$ denotes the cylinder consisting of all elements of Z that extend p. Note that cylinders are clopen sets that form a base for the topology of Z. Recall that $|p|$ denotes the cardinality = length of p. If $p \in \mathsf{Z}^\bullet$ and $k \geq 0$ is an integer, then $p^\frown k$ denotes the usual concatenation.

The usual product (= Haar) measure on Z is denoted μ_{Z}. Note that $\mu_{\mathsf{Z}}([\![p]\!]) = 1/|p|!$ for all $p \in \mathsf{Z}^\bullet$.

We now define a planar set X that is bi-Lipschitz equivalent to the metric space Z. It is the set announced at the beginning of the section.

Definition 3.2. Define a mapping $\phi : \mathsf{Z} \to \mathbb{R}^2$ thus:

$$\phi(x) = \left(\sum_{n=1}^{\infty} \frac{\cos 2\pi \frac{x(n)}{n}}{2(n-1)!}, \sum_{n=1}^{\infty} \frac{\sin 2\pi \frac{x(n)}{n}}{2(n-1)!}\right) = \sum_{n=1}^{\infty} \frac{\exp\left(2\pi i \frac{x(n)}{n}\right)}{2(n-1)!}.$$

Let $\mathsf{X} = \phi(\mathsf{Z}) \subseteq \mathbb{R}^2$ and provide it with the Euclidean metric.

Proposition 3.3. *$\phi : (\mathsf{Z}, \rho) \to \mathsf{X}$ is bi-Lipschitz.*

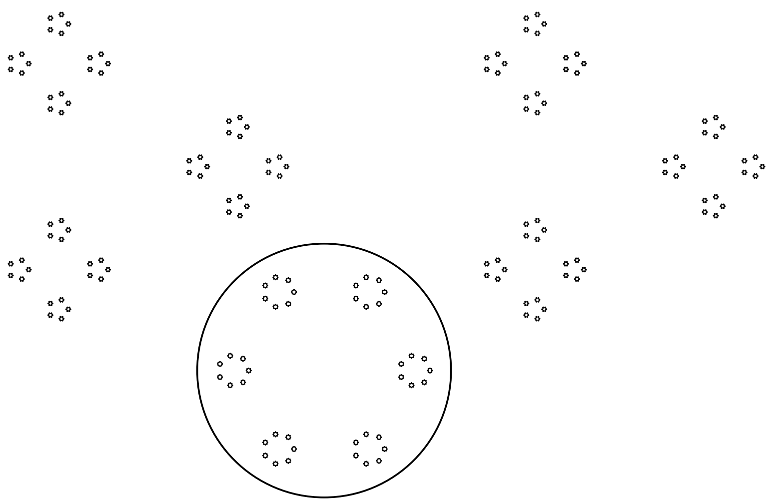

Fig. 1. *The set* $\mathsf{X} = \phi(\mathsf{Z})$. The two triangular shapes are centered at the endpoints of a segment. The six square shapes are centered at the vertices of the triangles. The twenty four pentagonal shapes are centered at the vertices of the squares, *et cetera ad infinitum.* One of the vertices of the pentagons is shown magnified in the circle.

Proof. Let $x, y \in \mathsf{Z}$, $x \neq y$. Throughout the proof we use the following notation: $\kappa = \kappa(x, y)$, $j = \rho_\kappa(x(\kappa), y(\kappa))$,

$$S_n = \exp\left(2\pi i \frac{x(n)}{n}\right) - \exp\left(2\pi i \frac{y(n)}{n}\right).$$

Note that $\kappa \geq 2$, $j \geq 1$ and $\rho(x, y) = j/\kappa!$.

Since $S_n = 0$ for all $n < \kappa$ and $|S_n| \leq 2$ for all n,

$$\frac{|S_\kappa|}{2(\kappa-1)!} - \sum_{n=\kappa}^{\infty} \frac{1}{n!} \leq |\phi(x) - \phi(y)| \leq \frac{|S_\kappa|}{2(\kappa-1)!} + \sum_{n=\kappa}^{\infty} \frac{1}{n!}. \tag{4}$$

The vectors $\exp\left(2\pi i \frac{x(\kappa)}{\kappa}\right)$ and $\exp\left(2\pi i \frac{y(\kappa)}{\kappa}\right)$ span an angle of $2\pi j/\kappa$. Therefore

$$|S_\kappa| = 2 \sin \frac{2\pi j/\kappa}{2} = 2 \sin \frac{\pi j}{\kappa}. \tag{5}$$

Use elementary estimates $(2/\pi)t \leq \sin t \leq t$ that hold for all $t \in [0, \pi/2]$ to get

$$\frac{4j}{\kappa} \leq |S_\kappa| \leq \frac{2\pi j}{\kappa}. \tag{6}$$

One more inequality we need:

$$\sum_{n=\kappa}^{\infty} \frac{1}{n!} \leq \frac{1}{\kappa!} \sum_{i=0}^{\infty} \frac{1}{(\kappa+1)^i} \leq \frac{1}{\kappa!}\left(1 + \frac{1}{\kappa}\right). \tag{7}$$

Combining these inequalities with (4) and recalling that $\kappa \geq 2$ and $\rho(x, y) = j/\kappa!$ yields on one hand

$$|\phi(x) - \phi(y)| \leq \frac{1}{(\kappa-1)!} \frac{\pi j}{\kappa} + \frac{3/2}{\kappa!} \leq \frac{j}{\kappa!}\left(\pi + \frac{3/2}{j}\right) \leq \rho(x, y)(\pi + 3/2)$$

and on the other hand

$$|\phi(x) - \phi(y)| \geq \frac{1}{(\kappa-1)!} \frac{2j}{\kappa} - \frac{3/2}{\kappa!} \geq \frac{j}{\kappa!}\left(2 - \frac{3/2}{j}\right) \geq \frac{1}{2}\rho(x, y),$$

which shows that ϕ is bi-Lipschitz. ∎

Most properties of Z and its subsets we are after, e.g. monotonicity, Hausdorff measure zero, Hausdorff dimension, meagerness, are bi-Lipschitz invariant. Thus, as regards these properties, Z and X are indistinguishable. In the next section we show that the mapping ϕ doubles Hausdorff measure.

4. Hausdorff measure of the set. We now calculate Hausdorff measures and dimensions of Z and X. The s-dimensional Hausdorff measure and related approximation pre-measures are denoted $\mathcal{H}^s$ and $\mathcal{H}^s_\delta$, respectively, and $\dim_{\mathsf{H}}$ denotes Hausdorff dimension. A refinement of the above proposition is needed:

LEMMA 4.1. *For each $\varepsilon > 0$ there is $\delta > 0$ such that if $\rho(x,y) < \delta$, then*

$$|\phi(x) - \phi(y)| \geq (2-\varepsilon)\rho(x,y).$$

Proof. Fix $\varepsilon > 0$ small enough to satisfy

$$\pi(1-\varepsilon) - (1+\varepsilon) \geq 2. \tag{8}$$

Choose $m \in \mathbb{N}$ subject to $\varepsilon > 1/m$ and

$$\sin t \geq (1-\varepsilon)t \quad \text{whenever} \quad 0 \leq t \leq \frac{2\pi}{\varepsilon m} \tag{9}$$

and put $\delta = 1/m!$.

We refer to the previous proof, including notation. Suppose $\rho(x,y) < \delta$. Then $\kappa \geq m$. Combine (7) with $\varepsilon > 1/\kappa$ and (4) to get

$$|\phi(x) - \phi(y)| \geq \frac{|S_\kappa|}{2(\kappa-1)!} - \frac{1+\varepsilon}{\kappa!}. \tag{10}$$

Distinguish two cases: If $j \geq 2/\varepsilon$, use (10) and (6):

$$\begin{aligned}|\phi(x) - \phi(y)| &\geq \frac{2j}{\kappa!} - \frac{1+\varepsilon}{\kappa!} = \frac{2j}{\kappa!}\left(1 - \frac{1+\varepsilon}{2j}\right) \\ &\geq \frac{2j}{\kappa!}\left(1 - \varepsilon\frac{1+\varepsilon}{4}\right) \geq \frac{2j}{\kappa!}\left(1 - \frac{\varepsilon}{2}\right) \geq \rho(x,y)(2-\varepsilon).\end{aligned}$$

If $j < 2/\varepsilon$, use (10), (5), (9) with $t = \pi j/\kappa$ and (8):

$$\begin{aligned}|\phi(x) - \phi(y)| &\geq \frac{2\pi j(1-\varepsilon)}{2\kappa!} - \frac{1+\varepsilon}{\kappa!} = \frac{j}{\kappa!}\left(\pi(1-\varepsilon) - \frac{1+\varepsilon}{j}\right) \\ &\geq \rho(x,y)(\pi(1-\varepsilon) - (1+\varepsilon)) \geq 2\rho(x,y). \ \blacksquare\end{aligned}$$

PROPOSITION 4.2.

(i) $\mathcal{H}^1(E) = \frac{1}{2}\mu_{\mathsf{Z}}(E)$ *for any Borel set* $E \subseteq \mathsf{Z}$. *In particular,* $\mathcal{H}^1(\mathsf{Z}) = 1/2$ *and* $\dim_{\mathsf{H}} \mathsf{Z} = 1$.

(ii) $\mathcal{H}^1(E) = 2\mathcal{H}^1(\phi^{-1}[E]) = \mu_{\mathsf{Z}}(\phi^{-1}[E])$ *for any Borel set* $E \subseteq \mathsf{X}$. *In particular,* $\mathcal{H}^1(\mathsf{X}) = 1$ *and* $\dim_{\mathsf{H}} \mathsf{X} = 1$.

Proof. We prove (i) first. For $n \in \mathbb{N}$ denote by $[n/2]$ the integer part of $n/2$. Note that for all $p \in \mathsf{Z}^\bullet$,

$$\operatorname{diam} [\![p]\!] = \frac{[(|p|+1)/2]}{(|p|+1)!} \leq \frac{1}{2|p|!}. \tag{11}$$

Let $A \subseteq \mathsf{Z}$ be a Borel set. We first estimate $\operatorname{diam} A$ from below by $\mu_{\mathsf{Z}}(A)$. Since the closure $\overline{A}$ of A is compact, there are $x, y \in \overline{A}$ such that $\rho(x,y) = \operatorname{diam} A$. Therefore there are unique $n \in \mathbb{N}$ and $j \leq [n/2]$ such that $\operatorname{diam} A = j/n!$. Hence there is $p \in \mathsf{Z}^\bullet$ with $|p| = n-1$ such that $A \subseteq [\![p]\!]$ and $\rho_n(x(n), y(n)) \leq j$ for all $x, y \in A$. Consequently:

• If $j < [n/2]$, then there is $i \in \mathbb{Z}_n$ such that $x(n) \in \{i, i+1, \dots, i+j\}$ for all $x \in A$. It follows that $A \subseteq \bigcup_{k=i}^{i+j} [\![p^\frown k]\!]$. Therefore (11) yields

$$\mu_{\mathsf{Z}}(A) \leq \sum_{k=i}^{i+j} \mu_{\mathsf{Z}}([\![p^\frown k]\!]) = (j+1)\frac{1}{n!} = \frac{j+1}{j}\,\frac{j}{n!} \leq 2\operatorname{diam} A.$$

• If $j = [n/2]$, then $A \subseteq [\![p]\!]$ and (11) yield

$$\mu_{\mathsf{Z}}(A) \leq \mu_{\mathsf{Z}}([\![p]\!]) = \frac{1}{(n-1)!} = \frac{n}{[n/2]}\,\frac{[n/2]}{n!} \leq 2\frac{n}{n-1}\operatorname{diam} A.$$

In either case, $\operatorname{diam} A \geq \frac{1}{2}(1-1/n)\mu_{\mathsf{Z}}(A)$.

Let now E be a Borel set. Let $m \in \mathbb{N}$ and $\delta < 1/m!$ and suppose $\{A_k\}$ is a cover of E by Borel sets of diameters at most δ. Then $\operatorname{diam} A_k \geq \frac{1}{2}(1-1/m)\mu_{\mathsf{Z}}(A_k)$ for all k. Thus

$$\sum_k \operatorname{diam} A_k \geq \frac{1}{2}\Big(1-\frac{1}{m}\Big)\sum_k \mu_{\mathsf{Z}}(A_k) \geq \frac{1}{2}\Big(1-\frac{1}{m}\Big)\mu_{\mathsf{Z}}(E).$$

Therefore $\mathcal{H}^1_\delta(E) \geq \frac{1}{2}(1-1/m)\mu_{\mathsf{Z}}(E)$. Let $m \to \infty$ to get $\mathcal{H}^1(E) \geq \frac{1}{2}\mu_{\mathsf{Z}}(E)$.

To prove the opposite inequality it suffices to show $\mathcal{H}^1(\mathsf{Z}) \leq 1/2$. Let m be as above and $\delta = 1/m!$. Cover Z by the family $\mathcal{E} = \{[\![p]\!] : p \in \mathsf{Z}^\bullet,\ |p| = m\}$. Since $|\mathcal{E}| = m!$, (11) yields

$$\mathcal{H}^1_\delta(\mathsf{Z}) \leq \sum_{|p|=m} \operatorname{diam}[\![p]\!] \leq m! \cdot \frac{1}{2m!} = \frac{1}{2}.$$

Let $m \to \infty$ to get $\mathcal{H}^1(\mathsf{Z}) \leq 1/2$.

The family $\mathcal{E}$ can also be used to show that $\mathcal{H}^1(\mathsf{X}) \leq 1$: Since $\operatorname{diam}\phi([\![p]\!]) \leq \sum_{n=|p|}^{\infty} 1/n!$ for all $p \in \mathsf{Z}^\bullet$, one may use (7) to show that given any $\varepsilon > 0$ there is m large enough so that the family $\{\phi E : E \in \mathcal{E}\}$ is a cover of X by sets of diameters below $(1+\varepsilon)/m!$ and thus witnesses $\mathcal{H}^1_\delta(\mathsf{X}) \leq 1+\varepsilon$ with $\delta = (1+\varepsilon)/m!$. Let $m \to \infty$ and $\varepsilon \to 0$ to get $\mathcal{H}^1(\mathsf{X}) \leq 1$.

It remains to prove that $\mathcal{H}^1(E) \geq 2\mathcal{H}^1(\phi^{-1}[E]) = \mu_{\mathsf{Z}}(\phi^{-1}[E])$ for every set $E \subseteq \mathsf{X}$. But that follows immediately from Lemma 4.1 and part (i). ■

5. The set is not σ-monotone. We now show that large subsets of Z and X are not σ-monotone. To that end we prepare a combinatorial lemma.

DEFINITION 5.1. Let $E \subseteq \mathsf{Z}$. For $p \in E^\bullet$ denote

$$\deg_E(p) = |\{k \in \mathbb{Z}_{|p|+1} : p^\frown k \in E^\bullet\}|,$$

the number of immediate successors of p in $E^\bullet$.

Say that a node $p \in E^\bullet$ is *bad* if

$$\forall \alpha < 1\ \forall n \in \mathbb{N}\ \exists q \in E^\bullet\ \big(p \subseteq q\ \&\ |q| \geq n\ \&\ \deg_E(q) > \alpha(|q|+1)\big).$$

Say that a node $p \in E^\bullet$ is *good* if it is not bad, i.e.

$$\exists \alpha < 1\ \exists n \in \mathbb{N}\ \forall m \geq n\ \big(p \subseteq q\ \&\ |q| \geq m \Rightarrow \deg_E(q) \leq \alpha(|q|+1)\big). \tag{12}$$

LEMMA 5.2. *If $E \subseteq \mathsf{Z}$ is monotone, then each node $p \in E^\bullet$ is good.*

Proof. Suppose $<$ and c are the order and the constant witnessing the monotonicity of E. Choose $n \in \mathbb{N}$ such that

$$\alpha := \frac{2c}{1+2c} + \frac{1}{n+1} < 1.$$

Aiming at a contradiction assume there is a bad node $p \in E^\bullet$. Thus there is an extension $q \supseteq p$ such that $|q| \geq n$ and $\deg_E(q) > \alpha(|q|+1)$. Therefore

$$\deg_E(q) - 1 > \left(\frac{2c}{1+2c} + \frac{1}{(|q|+1)}\right)(|q|+1) - 1 = \frac{2c(|q|+1)}{1+2c}.$$

Denote $m = |q|+1$ and $A = \{a \in \mathbb{Z}_m : q^\frown a \in E^\bullet\}$. Clearly $|A| = \deg_E(q)$. For each $a \in A$ choose $x_a \in E$ such that $q^\frown a \subseteq x_a$. Order A by $a \prec b$ iff $x_a < x_b$. Since $<$ is a linear order, so is $\prec$. Now apply Lemma 2.2: There are $i, j, k \in A$ such that $i \prec j \prec k$ and

$$\frac{\rho_m(i,j)}{\rho_m(i,k)} \geq \frac{1}{2}\,\frac{|A|-1}{m-(|A|-1)} = \frac{1}{2}\,\frac{\deg_E(q)-1}{m-(\deg_E(q)-1)} > \frac{1}{2}\,\frac{\frac{2cm}{1+2c}}{m-\frac{2cm}{1+2c}} = c.$$

Since $\rho(x_a, x_b) = \rho_m(a,b)/m!$ for all $a, b \in A$, the last estimate yields $\rho(x_i, x_j) > c\rho(x_i, x_k)$ and since clearly $x_i < x_j < x_k$, the points x_i, x_j, x_k in E break monotonicity of E: a contradiction. ∎

THEOREM 5.3. *If $E \subseteq \mathsf{X}$ is σ-monotone, then there is an F_σ-set $F \supseteq E$ that is meager* (*in* X) *and such that $\mathcal{H}^1(F) = 0$. In particular, E is meager and $\mathcal{H}^1(E) = 0$.*

Proof. According to Lemma 1.3 and Propositions 3.3 and 4.2 we may work in Z. Due to Lemma 1.4, E may be assumed to be F_σ. Since meagerness and measure zero are countably additive properties, we may actually assume that E is closed and monotone. By the preceding lemma any node $p \in E^\bullet$ satisfies condition (12). Consequently, there are $\alpha < 1$ and n such that

$$\mu_{\mathsf{Z}}(\{x \in E : p \subseteq x\}) \leq \prod_{m>n} \frac{\alpha m}{m} = \prod_{m>n} \alpha = 0.$$

Since $E^\bullet$ is countable and $E \subseteq \bigcup_{p \in E^\bullet}\{x \in E : p \subseteq x\}$, it follows that

$$\mu_{\mathsf{Z}}(E) \leq \sum_{p \in E^\bullet} \mu_{\mathsf{Z}}(\{x \in E : p \subseteq x\}) = 0.$$

Apply Proposition 4.2 to get $\mathcal{H}^1(E) = 0$ and notice that if E were not meager, being closed it would contain a nonempty open set and thus would have positive measure. ∎

Since X is homeomorphic to the Cantor set, the theorem yields a negative answer to Question 1.2:

Corollary 5.4. *There is a compatible metric on the Cantor set that is not σ-monotone.*

The theorem says that σ-monotone subsets of X are small with respect to measure and category. There are, however, monotone sets of large Hausdorff dimension. Perhaps it is not accidental: What if any compatible metric on the Cantor set admits a σ-monotone subset of full Hausdorff dimension? To date we do not have much to say about this phenomenon.

Proposition 5.5. *There is a monotone closed set $F \subseteq \mathsf{X}$ such that $\dim_{\mathsf{H}} F = 1$.*

Proof. Work in Z. Use the idea and notation of the proof of Proposition 4.2. For each $n \in \mathbb{N}$ set $F_n = \{0, 1, \ldots, [n/2]\}$ and define

$$F = \prod_{n=1}^{\infty} F_n = \{x \in \mathsf{Z} : x(n) \le [n/2]\}.$$

Then F is obviously compact. We first show that $\dim_{\mathsf{H}} F = 1$. Let μ_F be the uniformly distributed product measure on F. Clearly

$$\mu_F([\![p]\!] \cap F) = \prod_{i=1}^{|p|} \frac{1}{[i/2]+1} \le \prod_{i=1}^{|p|} \frac{2}{i} = \frac{2^{|p|}}{|p|!}, \qquad p \in F^{\bullet}. \tag{13}$$

Proceed as in the proof of Proposition 4.2: Let $A \subseteq F$ be a Borel set and let $n, j \in \mathbb{N}$ with $j \le [n/2]$ be the unique numbers such that $\operatorname{diam} A = j/n!$. There is $p \in F^{\bullet}$ with $|p| = n-1$ such that $A \subseteq [\![p]\!] \cap F$ and $\rho_n(x(n), y(n)) \le j$ for all $x, y \in A$.

- If $j < [n/2]$, then there is $i \in \mathbb{Z}_n$ such that

$$\mu_F(A) \le \sum_{k=i}^{i+j} \mu_F([\![p^\frown k]\!] \cap F) \overset{(13)}{\le} (j+1)\frac{2^n}{n!}.$$

- If $j = [n/2]$, then $A \subseteq [\![p]\!]$. Hence

$$\mu_F(A) \le \mu_F([\![p]\!] \cap F) \overset{(13)}{\le} \frac{2^{n-1}}{(n-1)!} = \frac{n}{2}\frac{2^n}{n!} \le ([n/2]+1)\frac{2^n}{n!}.$$

In either case we have

$$\mu_F(A) \le \frac{2^n(j+1)}{n!}. \tag{14}$$

Fix $s < 1$ and let $m \in \mathbb{N}$ be large enough so that $n2^n \le (n!)^{1-s}$ for all $n \ge m$. Then

$$\frac{2^n(j+1)}{n!} \le \frac{n2^n}{n!} \le \frac{(n!)^{1-s}}{n!} \le \left(\frac{1}{n!}\right)^s \le \left(\frac{j}{n!}\right)^s$$

whenever $n \geq m$ and $1 \leq j \leq [n/2]$. So if $\operatorname{diam} A \leq 1/m!$, then (14) yields $\mu_F(A) \leq (\operatorname{diam} A)^s$. It follows that if F is covered by a family $\{A_k\}$ of sets with diameters not exceeding $1/m!$, then

$$\sum_k (\operatorname{diam} A_k)^s \geq \sum_k \mu_F(A_k) \geq \mu_F\Big(\bigcup_k A_k\Big) \geq \mu_F(F) = 1.$$

Therefore $\mathcal{H}^s(F) \geq \mathcal{H}^s_{1/m!}(F) \geq \mu_F(F) = 1$. Since $s < 1$ was arbitrary, we conclude that $\dim_{\mathsf{H}} F = 1$.

It remains to show that F is monotone. Let $<$ be the lexicographic order on F. Assume $x < y < z$. Three configurations are possible (recall (3)):

- $\kappa(x,z) = \kappa(y,z) < \kappa(x,y)$: Then (2) yields $\rho(x,y) \leq \rho(x,z)$.
- $\kappa(x,z) = \kappa(x,y) < \kappa(y,z)$: Then (2) yields $\rho(x,y) = \rho(x,z)$.
- $\kappa(x,y) = \kappa(x,z) = \kappa(y,z)$: Let κ denote this common value. Since $x < y < z$, we have $0 \leq x(\kappa) < y(\kappa) < z(\kappa) \leq [\kappa/2]$. Therefore
$$\rho_\kappa(x(\kappa), z(\kappa)) = z(\kappa) - x(\kappa) > y(\kappa) - x(\kappa) = \rho_\kappa(x(\kappa), y(\kappa))$$
and thus $\rho(x,z) \geq \rho(x,y)$, as required. ■

There are also non-σ-monotone subsets of X with small Hausdorff dimension:

Proposition 5.6. *There is a closed set $F \subseteq \mathsf{X}$ with $\dim_{\mathsf{H}} F = 0$ that is not σ-monotone.*

Proof. Work in Z. Choose a strictly increasing sequence $\langle n_k : k \in \mathbb{N}\rangle$ such that

$$n_1 \ldots n_k \leq (n_k!)^{1/k}$$

($n_k = 4^k$ will do), put $I = \{n_k : k \in \mathbb{N}\}$ and set

$$F = \{x \in \mathsf{Z} : x(n) = 0 \text{ if } n \notin I\}.$$

Fix $s > 0$ and let $k \in \mathbb{N}$ with $1/k < s$. Consider the family $\mathcal{E} = \{[\![p]\!] : p \in F^\bullet, |p| = n_k\}$. It is obviously a cover of F and $|\mathcal{E}| = n_1 \ldots n_k$. According to (11), $\operatorname{diam} E \leq 1/n_k!$ for each $E \in \mathcal{E}$. Therefore

$$\mathcal{H}^s_{1/n_k!}(F) \leq \sum_{E \in \mathcal{E}} (\operatorname{diam} E)^s \leq n_1 \ldots n_k \left(\frac{1}{n_k!}\right)^s \leq (n_k!)^{1/k} \left(\frac{1}{n_k!}\right)^{1/k} = 1.$$

Let $k \to \infty$ to get $\mathcal{H}^s(F) \leq 1$. Since this holds for all $s > 0$, we get $\dim_{\mathsf{H}} F = 0$.

It remains to show that F is not σ-monotone. Assume the contrary. Since F is compact, using Lemma 1.4 and the Baire category theorem there is a nonempty open (in F) set U that is monotone. There is $p \in F^\bullet$ such that $[\![p]\!] \cap F = \{x \in F : p \subseteq x\} \subseteq U$. Since $\deg_U(q) = n_k$ for any $k \in \mathbb{N}$ and any $q \in U^\bullet$ of length $n_k - 1$, the node p is bad (for U). Apply Lemma 5.2

to conclude that U is not monotone: a contradiction proving that F is not σ-monotone. ∎

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