A Cantor set in the plane that is not $\sigma$-monotone

by

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Abstract. A metric space $(X, d)$ is monotone if there is a linear order $<$ on $X$ and a constant $c$ such that $d(x, y) \leq cd(x, z)$ for all $x < y < z$ in $X$, and $\sigma$-monotone if it is a countable union of monotone subspaces. A planar set homeomorphic to the Cantor set that is not $\sigma$-monotone is constructed and investigated. It follows that there is a metric on a Cantor set that is not $\sigma$-monotone. This answers a question raised by the second author.

1. Introduction. The following notions were introduced in [4]:

Definition 1.1. A metric space $(X, d)$ is called

- monotone if there is a linear order $<$ on $X$ and a constant $c$ such that $d(x, y) \leq cd(x, z)$ for all $x < y < z$ in $X$,
- $\sigma$-monotone if it is a countable union of monotone subspaces.

Topological properties of monotone and $\sigma$-monotone spaces are investigated in [1]. We quote some results: A subspace of a monotone metric space is monotone. A metric space with a dense monotone subspace is monotone. Every monotone space topologically embeds into a linearly orderable metrizable topological space, but does not have to be linearly orderable. Every separable monotone space topologically embeds into the line. The topological dimension of a $\sigma$-monotone space is at most 1. Every ultrametric space is monotone. Every topologically discrete metric space is $\sigma$-monotone, but not necessarily monotone.

Fractal properties of monotone and $\sigma$-monotone sets in Euclidean spaces, namely porosity, Hausdorff measures and Hausdorff dimensions and rectifiability, and functions with a $\sigma$-monotone graph are investigated in [2].

An application appears in [3], where $\sigma$-monotone sets serve as a tool for a characterization of Borel sets in a Euclidean space $\mathbb{R}^n$ that map onto a...
cube $[0,1]^m$ ($m \leq n$) by a quasi-Lipschitz mapping, i.e. a mapping that is $\beta$-Hölder for each $\beta < 1$.

The very first application of $\sigma$-monotone spaces appears in [4]: Let $\dim$ and $\dim_H$ denote, respectively, the topological and Hausdorff dimensions. It is shown that every analytic $\sigma$-monotone metric space $X$ contains a Lipschitz preimage of every self-similar set $S$ satisfying the strong separation condition with $\dim_H S < \dim_H X$. A number of results are derived from this theorem. E.g., any analytic metric space $X$ contains a universal measure zero set $E \subseteq X$ such that $\dim_H E \geq \dim X$; any analytic $\sigma$-monotone metric space $X$ contains a universal measure zero set $E \subseteq X$ such that $\dim_H E \geq \dim X$; and any analytic set $X \subseteq \mathbb{R}^n$ contains a universal measure zero set $E \subseteq X$ such that $\dim_H E = \dim_H X$.

As explained in [4], the following question raised in [5] is of particular interest:

**Question 1.2 ([5]).** Is every compatible metric on the Cantor set $\sigma$-monotone?

The goal of the present paper is to provide a negative answer to this question by constructing a set $X$ in the plane that is homeomorphic to the Cantor set but is not, as a metric subspace of the Euclidean plane, $\sigma$-monotone.

In Section 2 we state and prove a combinatorial lemma that is essential for the construction of the set $X$, which is performed in Section 3. In Section 4 we calculate the linear Hausdorff measure of $X$ and its Hausdorff dimension. In Section 5 we prove that every $\sigma$-monotone subset of $X$ is contained in a $\sigma$-compact set that is meager in $X$ and has linear Hausdorff measure zero. In particular, $X$ is not $\sigma$-monotone.

Throughout the paper we use the following notation and terminology. $\mathbb{N}$ denotes the set of all positive integers, *excluding zero*. The cardinality of a set $A$ is denoted $|A|$. A metric on a metrizable space is *compatible* if it induces the topology of $X$. If $(X, \rho)$ is a metric space and $x \in X$, the symbol $B_\rho(x, r)$ (or just $B(x, r)$) denotes the closed ball centered at $x$ with radius $r$. If $X, Y$ are metric spaces, a mapping $f : X \rightarrow Y$ is termed *bi-Lipschitz* if it is bijective and both $f$ and its inverse are Lipschitz mappings. Of course, a bi-Lipschitz mapping is a homeomorphism. The metric spaces $X, Y$ are *bi-Lipschitz equivalent* if there is a bi-Lipschitz mapping $f : X \rightarrow Y$.

We will need the following facts established in [1].

**Lemma 1.3.** A metric space that is bi-Lipschitz equivalent to a monotone space is monotone.

**Lemma 1.4.** If $X$ is $\sigma$-monotone, then it is a countable union of closed monotone subspaces.
2. Polygons

**Definition 2.1.** Consider the cyclic group $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$ for $n \in \mathbb{N}$. Define a metric on $\mathbb{Z}_n$ by

$$\rho_n(i, j) = \min(|i - j|, n - |i - j|).$$

The metric spaces $(\mathbb{Z}_n, \rho_n)$ are thought of as abstract regular polygons. They serve as building blocks for the construction of $X$. The following combinatorial lemma is crucial.

**Lemma 2.2.** Let $A \subseteq \mathbb{Z}_n$, $|A| \geq 3$. For each linear order $<$ on $A$ there are $i, j, k \in A$ such that $i < j < k$ and

$$\frac{\rho_n(i, j)}{\rho_n(i, k)} \geq \frac{1}{2} \frac{|A| - 1}{n - (|A| - 1)}.$$

**Proof.** Denote $|A| = m$. If $n = m = 3$, then $1 = \rho_n(i, j)$ for any distinct $i, j \in A$. If $n > m = 3$, then $1 \leq \rho_n(i, j) \leq n/2$ for any distinct $i, j \in A$. In either case, the inequality trivially holds for any triple $i, j, k$ of distinct elements of $\mathbb{Z}_n$. We shall thus assume that $|A| > 3$. Let $A = \{a_z : z \in \mathbb{Z}_m\}$ be the unique increasing enumeration of $A$ (with respect to the natural order).

Throughout the proof, addition is modulo $m$. Denote by $\ell$ the integer part of $m/2$.

Order $\mathbb{Z}_m$ by $z < z'$ iff $a_z < a_{z'}$. Let $N = \{z \in \mathbb{Z}_m : z < z + \ell\}$. If $N = \emptyset$, then $z > z + \ell$ for all $z$ and thus

$$0 > \ell > 2\ell > 3\ell > \cdots > m\ell = 0.$$  

Therefore $N \neq \emptyset$. If $N = \mathbb{Z}_m$, then $z < z + \ell$ for all $z$ and thus

$$0 < \ell < 2\ell < 3\ell < \cdots < m\ell = 0.$$  

Therefore $N \neq \mathbb{Z}_m$. So there are $z_1 \in N$ and $z_2 \notin N$. Let $z$ be the last term in the sequence $z_1, z_1 + 1, \ldots, z_2 - 1$ that satisfies $z \in N$. Clearly $z + 1 \notin N$. Consider two cases:

- $z + \ell \triangleleft z + 1$: Since $z \in N$, we have $z \triangleleft z + \ell \triangleleft z + 1$. Put $i = a_z$, $j = a_{z+\ell}$ and $k = a_{z+1}$. Clearly $i < j < k$. There are at most $n - m$ nodes strictly between $a_z$ and $a_{z+1}$. Hence $\rho_n(i, k) \leq n - (m - 1)$. There are at least $\ell - 1$ nodes strictly between $a_z$ and $a_{z+\ell}$ both clockwise and counterclockwise. Hence $\rho_n(i, j) \geq \ell$. Thus

$$\frac{\rho_n(i, j)}{\rho_n(i, k)} \geq \frac{\ell}{n - (m - 1)} \geq \frac{1}{2} \frac{|A| - 1}{n - (|A| - 1)}.$$  

- $z + \ell \triangleright z + 1$: Obviously $|A| > 3$ yields $z + \ell \neq z + 1$. Thus $z + \ell \triangleright z + 1$. Since $z + 1 \notin N$, we have $z + 1 + \ell \triangleleft z + 1 \triangleleft z + \ell$. Put $i = a_{z+1+\ell}$, $j = a_{z+1}$ and $k = a_{z+\ell}$. Clearly $i < j < k$. The inequality (1) follows by the same reasoning as above. ■
Remark 2.3. Since the set $A \subseteq (\mathbb{Z}_n, \rho_n)$ in the lemma is finite, it is trivially monotone. The lemma says that a constant witnessing its monotonicity cannot be less than $\frac{1}{2} \frac{|A| - 1}{n - (|A| - 1)}$. In particular, a constant witnessing monotonicity of $(\mathbb{Z}_n, \rho_n)$ cannot be less than $(n - 1)/2$.

Remark 2.4. The particular case $A = \mathbb{Z}_n$ can be phrased this easy way: Whatever linear order $\prec$ a regular polygon is equipped with, there are neighboring vertices $x, z$ and a vertex $y$ opposite to them such that $x \prec y \prec z$.

3. Construction of the set. In this section we define a rather regular compact set in the plane that

- is homeomorphic to the Cantor set,
- has positive and finite Hausdorff length,
- is not $\sigma$-monotone.

The set is constructed so that every nonempty open subset contains for each $n$ a bi-Lipschitz copy of the polygon $(\mathbb{Z}_n, \rho_n)$ described in the previous section. It is a continuous image of a cartesian product of the groups $\mathbb{Z}_n$, provided with a suitable metric.

Definition 3.1. Let

$$Z = \prod_{n=1}^{\infty} \mathbb{Z}_n = \{ x \in (\mathbb{N} \cup \{0\})^\mathbb{N} : x(n) < n \text{ for all } n \in \mathbb{N} \}.$$

Provide $Z$ with the product topology. Recall Definition 2.1 and define a metric $\rho$ on $Z$ by

$$\rho(x, y) = \sup_{n \in \mathbb{N}} \frac{\rho_n(x(n), y(n))}{n!}.$$  

Since $Z$ is a countable product of finite topological groups, it is obviously a zero-dimensional compact topological group. It has no isolated points. Thus it is homeomorphic to the Cantor ternary set.

The verification that $\rho$ is a metric compatible with the topology of $Z$ is straightforward. Thus $(Z, \rho)$ is a metric space homeomorphic to the Cantor set.

Since

$$\frac{\rho_n(x(n), y(n))}{n!} \geq \frac{1}{n!} \geq \frac{m}{m!} > \frac{\rho_m(x(m), y(m))}{m!}$$

whenever $n < m$ and $x(n) \neq y(n)$, $\rho$ is equivalently described by the following formula that we shall often use: if $x \neq y$, then

$$\rho(x, y) = \frac{\rho_{\kappa(x,y)}(x(\kappa(x, y)), y(\kappa(x, y)))}{\kappa(x, y)!},$$
where

\[(3) \quad \kappa(x, y) = \min\{n : x(n) \neq y(n)\}.\]

Throughout the paper, \(Z^* = \bigcup_{m=1}^{\infty} \prod_{n=1}^{m} \mathbb{Z}_n\) denotes the set of initial segments of elements of \(Z\). For \(E \subseteq Z\) set

\[E^* = \{p \in Z^* : p \subseteq x \text{ for some } x \in E\}.\]

For \(p \in Z^*\), \([p] = \{x \in Z : p \subseteq x\}\) denotes the cylinder consisting of all elements of \(Z\) that extend \(p\). Note that cylinders are clopen sets that form a base for the topology of \(Z\). Recall that \(|p|\) denotes the cardinality = length of \(p\). If \(p \in Z^*\) and \(k \geq 0\) is an integer, then \(p \dashv k\) denotes the usual concatenation.

The usual product (= Haar) measure on \(Z\) is denoted \(\mu_Z\). Note that \(\mu_Z([p]) = 1/|p|!\) for all \(p \in Z^*\).

We now define a planar set \(X\) that is bi-Lipschitz equivalent to the metric space \(Z\). It is the set announced at the beginning of the section.

**Definition 3.2.** Define a mapping \(\phi : Z \to \mathbb{R}^2\) thus:

\[
\phi(x) = \left(\sum_{n=1}^{\infty} \frac{\cos 2\pi \frac{x(n)}{n}}{2(n-1)!}, \sum_{n=1}^{\infty} \frac{\sin 2\pi \frac{x(n)}{n}}{2(n-1)!}\right) = \sum_{n=1}^{\infty} \frac{\exp(2\pi i \frac{x(n)}{n})}{2(n-1)!}.\]

Let \(X = \phi(Z) \subseteq \mathbb{R}^2\) and provide it with the Euclidean metric.

**Proposition 3.3.** \(\phi : (Z, \rho) \to X\) is bi-Lipschitz.

Fig. 1. The set \(X = \phi(Z)\). The two triangular shapes are centered at the endpoints of a segment. The six square shapes are centered at the vertices of the triangles. The twenty four pentagonal shapes are centered at the vertices of the squares, et cetera ad infinitum. One of the vertices of the pentagons is shown magnified in the circle.
**Proof.** Let \(x, y \in \mathbb{Z}, x \neq y\). Throughout the proof we use the following notation: \(\kappa = \kappa(x, y), j = \rho_\kappa(x(\kappa), y(\kappa))\),

\[
S_n = \exp\left(\frac{2\pi i x(n)}{n}\right) - \exp\left(\frac{2\pi i y(n)}{n}\right).
\]

Note that \(\kappa \geq 2, j \geq 1 \) and \(\rho(x, y) = j/\kappa!\).

Since \(S_n = 0\) for all \(n < \kappa\) and \(|S_n| \leq 2\) for all \(n\),

\[
\frac{|S_\kappa|}{2(\kappa - 1)!} - \sum_{n=\kappa}^{\infty} \frac{1}{n!} \leq |\phi(x) - \phi(y)| \leq \frac{|S_\kappa|}{2(\kappa - 1)!} + \sum_{n=\kappa}^{\infty} \frac{1}{n!}.
\]

The vectors \(\exp\left(\frac{2\pi i x(\kappa)}{\kappa}\right)\) and \(\exp\left(\frac{2\pi i y(\kappa)}{\kappa}\right)\) span an angle of \(2\pi j/\kappa\). Therefore

\[
|S_\kappa| = 2 \sin \frac{2\pi j/\kappa}{2} = 2 \sin \frac{\pi j}{\kappa}.
\]

Use elementary estimates \((2/\pi)t \leq \sin t \leq t\) that hold for all \(t \in [0, \pi/2]\) to get

\[
\frac{4j}{\kappa} \leq |S_\kappa| \leq \frac{2\pi j}{\kappa}.
\]

One more inequality we need:

\[
\sum_{n=\kappa}^{\infty} \frac{1}{n!} \leq \frac{1}{\kappa!} \sum_{i=0}^{\infty} \frac{1}{(\kappa + 1)^i} \leq \frac{1}{\kappa!} \left(1 + \frac{1}{\kappa}\right).
\]

Combining these inequalities with (4) and recalling that \(\kappa \geq 2\) and \(\rho(x, y) = j/\kappa!\) yields on one hand

\[
|\phi(x) - \phi(y)| \leq \frac{1}{(\kappa - 1)!} \frac{\pi j}{\kappa} + \frac{3/2}{\kappa!} \leq \frac{j}{\kappa!} \left(\pi + \frac{3/2}{j}\right) \leq \rho(x, y)(\pi + 3/2)
\]

and on the other hand

\[
|\phi(x) - \phi(y)| \geq \frac{1}{(\kappa - 1)!} \frac{2j}{\kappa} - \frac{3/2}{\kappa!} \geq \frac{j}{\kappa!} \left(2 - \frac{3/2}{j}\right) \geq \frac{1}{2} \rho(x, y),
\]

which shows that \(\phi\) is bi-Lipschitz.

Most properties of \(\mathbb{Z}\) and its subsets we are after, e.g. monotonicity, Hausdorff measure zero, Hausdorff dimension, meagerness, are bi-Lipschitz invariant. Thus, as regards these properties, \(\mathbb{Z}\) and \(X\) are indistinguishable. In the next section we show that the mapping \(\phi\) doubles Hausdorff measure.

**4. Hausdorff measure of the set.** We now calculate Hausdorff measures and dimensions of \(\mathbb{Z}\) and \(X\). The \(s\)-dimensional Hausdorff measure and related approximation pre-measures are denoted \(\mathcal{H}^s\) and \(\mathcal{H}_s^\delta\), respectively, and \(\dim_H\) denotes Hausdorff dimension. A refinement of the above proposition is needed:
Lemma 4.1. For each $\varepsilon > 0$ there is $\delta > 0$ such that if $\rho(x, y) < \delta$, then $|\phi(x) - \phi(y)| \geq (2 - \varepsilon)\rho(x, y)$.

Proof. Fix $\varepsilon > 0$ small enough to satisfy

$$\pi(1 - \varepsilon) - (1 + \varepsilon) \geq 2. \tag{8}$$

Choose $m \in \mathbb{N}$ subject to $\varepsilon > 1/m$ and

$$\sin t \geq (1 - \varepsilon)t \quad \text{whenever} \quad 0 \leq t \leq \frac{2\pi}{\varepsilon m} \tag{9}$$

and put $\delta = 1/m!$.

We refer to the previous proof, including notation. Suppose $\rho(x, y) < \delta$. Then $\kappa \geq m$. Combine (7) with $\varepsilon > 1/\kappa$ and (4) to get

$$|\phi(x) - \phi(y)| \geq \frac{|S_\kappa|}{2(\kappa - 1)!} - \frac{1 + \varepsilon}{\kappa!}. \tag{10}$$

Distinguish two cases: If $j \geq 2/\varepsilon$, use (10) and (6):

$$|\phi(x) - \phi(y)| \geq \frac{2j}{\kappa!} - \frac{1 + \varepsilon}{\kappa!} = \frac{2j}{\kappa!} \left(1 - \frac{1 + \varepsilon}{2j}\right) \geq \frac{2j}{\kappa!} \left(1 - \frac{\varepsilon}{2}\right) \geq \rho(x, y)(2 - \varepsilon).$$

If $j < 2/\varepsilon$, use (10), (5), (9) with $t = \pi j/\kappa$ and (8):

$$|\phi(x) - \phi(y)| \geq \frac{2\pi j(1 - \varepsilon)}{2\kappa!} - \frac{1 + \varepsilon}{\kappa!} = \frac{j}{\kappa!} \left(\pi(1 - \varepsilon) - \frac{1 + \varepsilon}{j}\right) \geq \rho(x, y)(\pi(1 - \varepsilon) - (1 + \varepsilon)) \geq 2\rho(x, y). \blacklozenge$$

Proposition 4.2.

(i) $\mathcal{H}^1(E) = \frac{1}{2} \mu_Z(E)$ for any Borel set $E \subseteq Z$. In particular, $\mathcal{H}^1(Z) = 1/2$ and $\dim_H Z = 1$.

(ii) $\mathcal{H}^1(E) = 2\mathcal{H}^1(\phi^{-1}[E]) = \mu_Z(\phi^{-1}[E])$ for any Borel set $E \subseteq X$. In particular, $\mathcal{H}^1(X) = 1$ and $\dim_H X = 1$.

Proof. We prove (i) first. For $n \in \mathbb{N}$ denote by $[n/2]$ the integer part of $n/2$. Note that for all $p \in Z^*$,

$$\text{diam } [p] = \frac{[(|p| + 1)/2]}{(|p| + 1)!} \leq \frac{1}{2|p|!}. \tag{11}$$

Let $A \subseteq Z$ be a Borel set. We first estimate $\text{diam } A$ from below by $\mu_Z(A)$. Since the closure $\overline{A}$ of $A$ is compact, there are $x, y \in \overline{A}$ such that $\rho(x, y) = \text{diam } A$. Therefore there are unique $n \in \mathbb{N}$ and $j \leq [n/2]$ such that $\text{diam } A = j/n!$. Hence there is $p \in Z^*$ with $|p| = n - 1$ such that $A \subseteq [p]$ and $\rho_n(x(n), y(n)) \leq j$ for all $x, y \in A$. Consequently:
• If $j < \lfloor n/2 \rfloor$, then there is $i \in \mathbb{Z}_n$ such that $x(n) \in \{i, i + 1, \ldots, i + j\}$ for all $x \in A$. It follows that $A \subseteq \bigcup_{k=i}^{i+j} \lceil p^{-k} \rceil$. Therefore (11) yields
\[
\mu_{Z}(A) \leq \sum_{k=i}^{i+j} \mu_{Z}(\lceil p^{-k} \rceil) = (j + 1) \frac{1}{n!} = \frac{j + 1}{j} \frac{j}{n!} \leq 2 \text{ diam } A.
\]

• If $j = \lfloor n/2 \rfloor$, then $A \subseteq \lceil p \rceil$ and (11) yield
\[
\mu_{Z}(A) \leq \mu_{Z}(\lceil p \rceil) = \frac{1}{(n - 1)!} = \frac{n}{n/2} \frac{[n/2]}{n!} \leq 2 \frac{n}{n - 1} \text{ diam } A.
\]

In either case, $\text{diam } A \geq \frac{1}{2}(1 - 1/n) \mu_{Z}(A)$.

Let now $E$ be a Borel set. Let $m \in \mathbb{N}$ and $\delta < 1/m$! and suppose $\{A_{k}\}$ is a cover of $E$ by Borel sets of diameters at most $\delta$. Then $\text{diam } A_{k} \geq \frac{1}{2}(1 - 1/m) \mu_{Z}(A_{k})$ for all $k$. Thus
\[
\sum_{k} \text{diam } A_{k} \geq \frac{1}{2} \left(1 - \frac{1}{m}\right) \sum_{k} \mu_{Z}(A_{k}) \geq \frac{1}{2} \left(1 - \frac{1}{m}\right) \mu_{Z}(E).
\]

Therefore $\mathcal{H}_{\delta}^{1}(E) \geq \frac{1}{2}(1 - 1/m) \mu_{Z}(E)$. Let $m \to \infty$ to get $\mathcal{H}^{1}(E) \geq \frac{1}{2} \mu_{Z}(E)$.

To prove the opposite inequality it suffices to show $\mathcal{H}^{1}(Z) \leq 1/2$. Let $m$ be as above and $\delta = 1/m!$. Cover $Z$ by the family $\mathcal{E} = \{\lceil p \rceil : p \in \mathbb{Z}^{*}, |p| = m\}$. Since $|\mathcal{E}| = m!$, (11) yields
\[
\mathcal{H}_{\delta}^{1}(Z) \leq \sum_{|p|=m} \text{diam } \lceil p \rceil \leq m! \cdot \frac{1}{2m!} = \frac{1}{2}.
\]

Let $m \to \infty$ to get $\mathcal{H}^{1}(Z) \leq 1/2$.

The family $\mathcal{E}$ can also be used to show that $\mathcal{H}^{1}(X) \leq 1$: Since $\text{diam } \phi(\lceil p \rceil) \leq \sum_{n=|p|}^{\infty} 1/n!$ for all $p \in \mathbb{Z}^{*}$, one may use (7) to show that given any $\varepsilon > 0$ there is $m$ large enough so that the family $\{\phi E : E \in \mathcal{E}\}$ is a cover of $X$ by sets of diameters below $(1 + \varepsilon)/m!$ and thus witnesses $\mathcal{H}_{\delta}^{1}(X) \leq 1 + \varepsilon$ with $\delta = (1 + \varepsilon)/m!$. Let $m \to \infty$ and $\varepsilon \to 0$ to get $\mathcal{H}^{1}(X) \leq 1$.

It remains to prove that $\mathcal{H}^{1}(E) \geq 2 \mathcal{H}^{1}(\phi^{-1}[E]) = \mu_{Z}(\phi^{-1}[E])$ for every set $E \subseteq X$. But that follows immediately from Lemma 4.1 and part (i). ■

5. The set is not $\sigma$-monotone. We now show that large subsets of $Z$ and $X$ are not $\sigma$-monotone. To that end we prepare a combinatorial lemma.

**Definition 5.1.** Let $E \subseteq Z$. For $p \in E^{*}$ denote
\[
\deg_{E}(p) = |\{k \in \mathbb{Z}_{|p|+1} : p^{-k} \in E^{*}\}|,
\]
the number of immediate successors of $p$ in $E^{*}$.

Say that a node $p \in E^{*}$ is bad if
\[
\forall \alpha < 1 \forall n \in \mathbb{N} \exists q \in E^{*} \ (p \subseteq q \& |q| \geq n \& \deg_{E}(q) > \alpha(|q| + 1)).
\]
Say that a node \( p \in E^\bullet \) is good if it is not bad, i.e.

\[
\exists \alpha < 1 \exists n \in \mathbb{N} \ \forall m \geq n \ (p \subseteq q \ \& \ |q| \geq m \ \Rightarrow \ \deg_E(q) \leq \alpha(|q| + 1)).
\]

**Lemma 5.2.** If \( E \subseteq \mathbb{Z} \) is monotone, then each node \( p \in E^\bullet \) is good.

**Proof.** Suppose \( < \) and \( c \) are the order and the constant witnessing the monotonicity of \( E \). Choose \( n \in \mathbb{N} \) such that

\[
\alpha := \frac{2c}{1 + 2c} + \frac{1}{n + 1} < 1.
\]

Aiming at a contradiction assume there is a bad node \( p \in E^\bullet \). Thus there is an extension \( q \supseteq p \) such that \(|q| \geq n \) and \( \deg_E(q) > \alpha(|q| + 1) \). Therefore

\[
\deg_E(q) - 1 > \left( \frac{2c}{1 + 2c} + \frac{1}{(|q| + 1)} \right) (|q| + 1) - 1 = \frac{2c(|q| + 1)}{1 + 2c}.
\]

Denote \( m = |q| + 1 \) and \( A = \{ a \in \mathbb{Z}_m : q \prec a \in E^\bullet \} \). Clearly \(|A| = \deg_E(q)\).

For each \( a \in A \) choose \( x_a \in E \) such that \( q \prec a \subseteq x_a \). Order \( A \) by \( a < b \) iff \( x_a < x_b \). Since \( < \) is a linear order, so is \( \prec \). Now apply Lemma 2.2. There are \( i,j,k \in A \) such that \( i < j < k \) and

\[
\frac{\rho_m(i,j)}{\rho_m(i,k)} \geq \frac{1}{2} \frac{|A| - 1}{m - (|A| - 1)} = \frac{1}{2} \frac{\deg_E(q) - 1}{m - (\deg_E(q) - 1)} \geq \frac{1}{2} \frac{2c}{m - \frac{2cm}{1 + 2c}} = c.
\]

Since \( \rho(x_a,x_b) = \rho_m(a,b)/m! \) for all \( a,b \in A \), the last estimate yields \( \rho(x_i,x_j) > c \rho(x_i,x_k) \) and since clearly \( x_i < x_j < x_k \), the points \( x_i,x_j,x_k \) in \( E \) break monotonicity of \( E \): a contradiction. 

**Theorem 5.3.** If \( E \subseteq X \) is \( \sigma \)-monotone, then there is an \( F_\sigma \)-set \( F \supseteq E \) that is meager (in \( X \)) and such that \( \mathcal{H}^1(F) = 0 \). In particular, \( E \) is meager and \( \mathcal{H}^1(E) = 0 \).

**Proof.** According to Lemma 1.3 and Propositions 3.3 and 4.2 we may work in \( \mathbb{Z} \). Due to Lemma 1.4 \( E \) may be assumed to be \( F_\sigma \). Since meagerness and measure zero are countably additive properties, we may actually assume that \( E \) is closed and monotone. By the preceding lemma any node \( p \in E^\bullet \) satisfies condition (12). Consequently, there are \( \alpha < 1 \) and \( n \) such that

\[
\mu_Z(\{x \in E : p \subseteq x\}) \leq \prod_{m > n} \frac{\alpha m}{m} = \prod_{m > n} \alpha = 0.
\]

Since \( E^\bullet \) is countable and \( E \subseteq \bigcup_{p \in E^\bullet} \{x \in E : p \subseteq x\} \), it follows that

\[
\mu_Z(E) \leq \sum_{p \in E^\bullet} \mu_Z(\{x \in E : p \subseteq x\}) = 0.
\]

Apply Proposition 4.2 to get \( \mathcal{H}^1(E) = 0 \) and notice that if \( E \) were not meager, being closed it would contain a nonempty open set and thus would have positive measure. \( \blacksquare \)
Since $X$ is homeomorphic to the Cantor set, the theorem yields a negative answer to Question 1.2:

**Corollary 5.4.** There is a compatible metric on the Cantor set that is not $\sigma$-monotone.

The theorem says that $\sigma$-monotone subsets of $X$ are small with respect to measure and category. There are, however, monotone sets of large Hausdorff dimension. Perhaps it is not accidental: What if any compatible metric on the Cantor set admits a $\sigma$-monotone subset of full Hausdorff dimension? To date we do not have much to say about this phenomenon.

**Proposition 5.5.** There is a monotone closed set $F \subseteq X$ such that $\dim H F = 1$.

**Proof.** Work in $\mathbb{Z}$. Use the idea and notation of the proof of Proposition 4.2. For each $n \in \mathbb{N}$ set $F_n = \{0, 1, \ldots, \lfloor n/2 \rfloor\}$ and define

$$F = \prod_{n=1}^{\infty} F_n = \{x \in \mathbb{Z} : x(n) \leq \lfloor n/2 \rfloor\}.$$ 

Then $F$ is obviously compact. We first show that $\dim H F = 1$. Let $\mu_F$ be the uniformly distributed product measure on $F$. Clearly

$$\mu_F([p] \cap F) = \prod_{i=1}^{\lfloor p \rfloor} \frac{1}{\lfloor i/2 \rfloor + 1} \leq \prod_{i=1}^{\lfloor p \rfloor} \frac{2}{i} = \frac{2^{\lfloor p \rfloor}}{|p|!}, \quad p \in F^*.$$  

(13)

Proceed as in the proof of Proposition 4.2. Let $A \subseteq F$ be a Borel set and let $n, j \in \mathbb{N}$ with $j \leq \lfloor n/2 \rfloor$ be the unique numbers such that $\text{diam} A = j/n!$. There is $p \in F^*$ with $|p| = n-1$ such that $A \subseteq [p] \cap F$ and $\rho_n(x(n), y(n)) \leq j$ for all $x, y \in A$.

- If $j < \lfloor n/2 \rfloor$, then there is $i \in \mathbb{Z}_n$ such that

$$\mu_F(A) \leq \sum_{k=i}^{i+j} \mu_F(\lfloor p^k \rfloor \cap F) \leq (j + 1) \frac{2^n}{n!}.$$  

- If $j = \lfloor n/2 \rfloor$, then $A \subseteq [p]$. Hence

$$\mu_F(A) \leq \mu_F([p] \cap F) \leq \frac{2^{n-1}}{(n-1)!} = \frac{n}{2} \frac{2^n}{n!} \leq (\lfloor n/2 \rfloor + 1) \frac{2^n}{n!}.$$  

(13)

In either case we have

$$\mu_F(A) \leq \frac{2^n (j + 1)}{n!}.$$  

(14)

Fix $s < 1$ and let $m \in \mathbb{N}$ be large enough so that $n2^n \leq (n!)^{1-s}$ for all $n \geq m$. Then

$$\frac{2^n (j + 1)}{n!} \leq \frac{n2^n}{n!} \leq \frac{(n!)^{1-s}}{n!} \leq \left( \frac{1}{n!} \right)^s \leq \left( \frac{j}{n!} \right)^s$$
whenever $n \geq m$ and $1 \leq j \leq \lfloor n/2 \rfloor$. So if diam $A \leq 1/m!$, then (14) yields $\mu_F(A) \leq (\text{diam } A)^s$. It follows that if $F$ is covered by a family $\{A_k\}$ of sets with diameters not exceeding $1/m!$, then

$$\sum_k (\text{diam } A_k)^s \geq \sum_k \mu_F(A_k) \geq \mu_F\left( \bigcup_k A_k \right) \geq \mu_F(F) = 1.$$ 

Therefore $\mathcal{H}^s(F) \geq \mathcal{H}^{s}_{1/m!}(F) \geq \mu_F(F) = 1$. Since $s < 1$ was arbitrary, we conclude that dim$_H F = 1$.

It remains to show that $F$ is monotone. Let $<\!$ be the lexicographic order on $F$. Assume $x < y < z$. Three configurations are possible (recall (3)):

- $\kappa(x, z) = \kappa(y, z) < \kappa(x, y)$: Then (2) yields $\rho(x, y) \leq \rho(x, z)$.
- $\kappa(x, z) = \kappa(x, y) < \kappa(y, z)$: Then (2) yields $\rho(x, y) = \rho(x, z)$.
- $\kappa(x, y) = \kappa(x, z) = \kappa(y, z)$: Let $\kappa$ denote this common value. Since $x < y < z$, we have $0 \leq x(\kappa) < y(\kappa) < z(\kappa) \leq [\kappa/2]$. Therefore

$$\rho_\kappa(x(\kappa), z(\kappa)) = z(\kappa) - x(\kappa) > y(\kappa) - x(\kappa) = \rho_\kappa(x(\kappa), y(\kappa))$$

and thus $\rho(x, z) \geq \rho(x, y)$, as required. $\blacksquare$

There are also non-$\sigma$-monotone subsets of $X$ with small Hausdorff dimension:

**Proposition 5.6.** There is a closed set $F \subseteq X$ with dim$_H F = 0$ that is not $\sigma$-monotone.

**Proof.** Work in $\mathbb{Z}$. Choose a strictly increasing sequence $\langle n_k : k \in \mathbb{N}\rangle$ such that

$$n_1 \ldots n_k \leq (n_k!)^{1/k}$$

($n_k = 4^k$ will do), put $I = \{n_k : k \in \mathbb{N}\}$ and set

$$F = \{x \in \mathbb{Z} : x(n) = 0 \text{ if } n \notin I\}.$$ 

Fix $s > 0$ and let $k \in \mathbb{N}$ with $1/k < s$. Consider the family $\mathcal{E} = \{[p] : p \in F^*, |p| = n_k\}$. It is obviously a cover of $F$ and $|\mathcal{E}| = n_1 \ldots n_k$. According to (11), diam $E \leq 1/n_k!$ for each $E \in \mathcal{E}$. Therefore

$$\mathcal{H}^{s}_{1/n_k!}(F) \leq \sum_{E \in \mathcal{E}} (\text{diam } E)^s \leq n_1 \ldots n_k \left( \frac{1}{n_k!} \right)^s \leq (n_k!)^{1/k} \left( \frac{1}{n_k!} \right)^{1/k} = 1.$$ 

Let $k \to \infty$ to get $\mathcal{H}^s(F) \leq 1$. Since this holds for all $s > 0$, we get dim$_H F = 0$.

It remains to show that $F$ is not $\sigma$-monotone. Assume the contrary. Since $F$ is compact, using Lemma 1.4 and the Baire category theorem there is a nonempty open (in $F$) set $U$ that is monotone. There is $p \in F^*$ such that $[p] \cap F = \{x \in F : p \subseteq x\} \subseteq U$. Since deg$_U(q) = n_k$ for any $k \in \mathbb{N}$ and any $q \in U^*$ of length $n_k - 1$, the node $p$ is bad (for $U$). Apply Lemma 5.2
to conclude that $U$ is not monotone: a contradiction proving that $F$ is not $\sigma$-monotone. ■

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