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Topological compactifications

by

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Abstract. We study those compactifications of a space such that every autohomeomorphism of the space can be continuously extended over the compactification. These are called H-compactifications. Van Douwen proved that there are exactly three H-compactifications of the real line. We prove that there exist only two H-compactifications of Euclidean spaces of higher dimension. Next we show that there are 26 H-compactifications of a countable sum of real lines and 11 H-compactifications of a countable sum of Euclidean spaces of higher dimension. All H-compactifications of discrete and countable locally compact spaces are described.

1. Introduction. A compactification γX of a space X is said to be an *H-compactification* if each autohomeomorphism of X can be continuously extended to a mapping of γX into γX. This notion was studied by several authors: in [Smi94] it is called 'equivariant extension', in [dGM60] it is called 'G-compactification' (where G is a subgroup of the group of all autohomeomorphisms of X), and in [vD79] the name 'topological compactification' is used. The last name is probably the most suitable since it expresses the fact that the compactification is defined only with respect to the topology of the base space and does not depend on the concrete representation of the space. However, we prefer the shorter term introduced above.

Informally, a compactification is an H-compactification if and only if it can be defined only in terms of topological properties of the given space. Clearly, the Alexandroff one-point compactification of a non-compact locally compact space X (denoted by αX) and the Čech–Stone compactification βX of a Tychonoff space X are always H-compactifications.

It is known that the Sorgenfrey line as well as the spaces of rational and irrational numbers admit only one H-compactification and that there are exactly three H-compactifications of the real line (see [vD79]). In the same paper it is noted that there are at least eleven H-compactifications of

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a countable sum of real lines. In this work we study Euclidean spaces of higher dimension and their countable sums.

For a continuous mapping $f\colon X \to Y$ of Tychonoff spaces we denote by $\beta f\colon \beta X \to \beta Y$ the only continuous extension of f. For brevity we write X^* for the Čech–Stone remainder $\beta X \setminus X$. We denote by $\mathcal{H}(X)$ the group of all autohomeomorphisms of a space X. For $x \in X$ the set $\{h(x)\colon h \in \mathcal{H}(X)\}$ is called the *orbit* of the point x.

For all other unexplained notions we refer to [Eng89] and [Cha76]. All spaces in this paper are supposed to be Tychonoff.

2. Lattice structure. In this section, we give a characterization of an H-compactification by bounded continuous functions which are extendable over the compactification. Consequently, we observe that the H-compactifications of a locally compact space form a lattice. The notion of a homogeneous mapping will be helpful.

DEFINITION 1. We say that a continuous mapping $f\colon X \to Y$ is *homogeneous* if for every $h \in \mathcal{H}(X)$ there exists $g \in \mathcal{H}(Y)$ such that $fh = gf$.

Let us emphasize that this notion depends essentially on the codomain of f. Clearly a compactification γX is an H-compactification if and only if the inclusion mapping $X \to \gamma X$ is homogeneous. It is easy to verify that the identity as well as constant mappings are always homogeneous and that the composition of two homogeneous mappings is again homogeneous. Neither products nor sums of homogeneous mappings are in general homogeneous. Two simple lemmas are stated below.

LEMMA 2. *A diagonal mapping $\Delta f_i\colon X \to \prod Y_i$ is homogeneous provided that every mapping $f_i\colon X \to Y_i$ is homogeneous.*

LEMMA 3. *Let $f\colon X \to Y$ be a homogeneous mapping. Then its corestrictions $f\colon X \to f(X)$ and $f\colon X \to \overline{f(X)}$ are also homogeneous.*

Recall that the set of all compactifications of a space X (up to equivalence) with natural order is a complete upper semilattice. If X is locally compact we even get a complete lattice. For more details see [Cha76, p. 16].

PROPOSITION 4. *The set of all H-compactifications of a space X is a complete subsemilattice of the semilattice of all compactifications of X.*

Proof. For a set of H-compactifications $\gamma_i X$ we denote by $\gamma_i\colon X \to \gamma_i X$ the inclusion mappings. The least upper bound of these compactifications is given by $\overline{\gamma(X)}$ where $\gamma = \Delta\gamma_i$. This is an H-compactification because γ and its corestriction $\gamma\colon X \to \overline{\gamma(X)}$ are homogeneous by Lemmas 2 and 3. ∎

PROPOSITION 5. *Let γX be a compactification of a space X and let $\sim$ be an equivalence relation for which $\gamma X = \beta X/{\sim}$. Then the following* *conditions are equivalent:*

(i) *γX is an H-compactification.*

(ii) *For every $h \in \mathcal{H}(X)$ and $x, y \in \beta X$ with $x \sim y$ we have $\beta h(x) \sim \beta h(y)$.*

(iii) *Every homeomorphism $h \in \mathcal{H}(X)$ is uniformly continuous with respect to the unique uniformity on γX.*

(iv) *For every $f \in \mathcal{C}^*(X)$ which is continuously extendable over γX and for every $h \in \mathcal{H}(X)$ the function fh can be continuously extended over γX.*

Proof. (i)⇒(ii). Take any $h \in \mathcal{H}(X)$ and $x, y \in \beta X$. Denote by $\varphi\colon \beta X \to \gamma X$ the only extension of the identity on X. Note that $x \sim y$ iff $\varphi(x) = \varphi(y)$. By (i) there is a $g \in \mathcal{H}(\gamma X)$ extending h, from which we get $\varphi\beta h = g\varphi$ since both sides are the same on a dense subset of βX. Thus if $\varphi(x) = \varphi(y)$ we get $\varphi\beta h(x) = g\varphi(x) = g\varphi(y) = \varphi\beta h(y)$ and hence $\beta h(x) \sim \beta h(y)$.

(ii)⇒(iii). For any $h \in \mathcal{H}(X)$ define $g \in \mathcal{H}(\beta X/{\sim})$ by the condition $g([x]_\sim) = [\beta h(x)]_\sim$. The mapping g is uniformly continuous and extends h. Thus h is also uniformly continuous.

(iii)⇒(iv) Pick $h \in \mathcal{H}(X)$ and $f \in \mathcal{C}^*(X)$ continuously extendable to a function $\bar{f}\colon \gamma X \to \mathbb{R}$. Since h is uniformly continuous by (iii) and $\bar{f}$ is also uniformly continuous, the composition fh is uniformly continuous. Thus by Theorem 8.3.10 from [Eng89, p. 447] there is a continuous extension of fh over γX.

(iv)⇒(i). Let $h \in \mathcal{H}(X)$. We want to show that h is continuously extendable to a mapping $\gamma X \to \gamma X$. By a special case of Theorem 3.2.1 from [Eng89, p. 136], a continuous mapping $h\colon X \to \gamma X$ can be continuously extended over γX if and only if for every pair E, F of disjoint closed subsets of γX the preimages $h^{-1}(E)$ and $h^{-1}(F)$ have disjoint closures in Y.

To verify this condition let E and F be as above. There exists $f \in \mathcal{C}(\gamma X)$ such that $E \subseteq f^{-1}(0)$ and $F \subseteq f^{-1}(1)$. By (iv) there exists $e \in \mathcal{C}(\gamma X)$ such that $fh = e\restriction_X$. Finally,

$$\overline{h^{-1}(E)} \cap \overline{h^{-1}(F)} \subseteq \overline{(fh)^{-1}(0)} \cap \overline{(fh)^{-1}(1)} \subseteq e^{-1}(0) \cap e^{-1}(1) = \emptyset. \ \blacksquare$$

Using the previous theorem we get the following result.

PROPOSITION 6. *The set of all H-compactifications of a locally compact space X is a complete sublattice of the lattice of all compactifications of X.*

Proof. For a given family of H-compactifications $\gamma_i X$ denote by C_i the set of all continuous functions from $\mathcal{C}^*(X)$ that are extendable over $\gamma_i X$. Let $C = \bigcap C_i$. The greatest lower bound of the compactifications $\gamma_i X$ is given by $\overline{\gamma(X)}$ where $\gamma\colon X \to \mathbb{R}^C$ is defined by $f(x)_e = e(x)$. We are going to verify condition (iv) of Proposition 5. Let $h \in \mathcal{H}(X)$ and let $f \in \mathcal{C}^*(X)$ be continuously extendable over $\overline{\gamma(X)}$. Then $f \in C$. Consequently, $fh \in C_i$

for every i because $\gamma_i X$ is an H-compactification. Hence $fh \in \bigcap C_i = C$ is continuously extendable over $\overline{\gamma(X)}$. ∎

3. Countable, discrete and Euclidean spaces. What does the set of all H-compactifications look like? This question seems to be hard to answer in some cases. For example, for a rigid space (i.e. admitting no non-trivial homeomorphisms) every compactification is an H-compactification and there can be a huge number of them (e.g. when a dense rigid subset of the real line is taken there exist at least continuum many). However, we can describe the lattices of all H-compactifications of discrete spaces, countable locally compact spaces, Euclidean spaces and countable sums of Euclidean spaces. This is possible since these spaces possess a rich structure of homeomorphisms.

3.1. Discrete spaces. The two lemmas below will be used in proving Theorem 9.

LEMMA 7. *Let X be a space and γX and δX be two compactifications such that δX is an H-compactification of X and there exists a continuous mapping $f\colon \delta X \to \gamma X$ extending the identity on X. If f is a homogeneous mapping then γX is an H-compactification of X.*

Proof. The inclusion $\gamma\colon X \to \gamma X$ is a homogeneous mapping, being the composition of two homogeneous mappings, f and the inclusion $X \to \delta X$. ∎

Note that the opposite implication in the previous lemma is not true in general. Just consider the space $X = \beta\omega \oplus \omega$ and a continuous mapping $f\colon \beta X \to \alpha X$ which extends the identity on X. Note that $\beta X = \beta\omega \oplus \beta\omega$ is an H-compactification of X and f is not homogeneous.

LEMMA 8. *Let $\sim$ be an equivalence relation on a space X. Suppose moreover that equivalence classes are either singletons or unions of orbits. Then the quotient mapping $q\colon X \to X/{\sim}$ is homogeneous.*

Proof. Fix $h \in \mathcal{H}(X)$ and define $g\colon X/{\sim} \to X/{\sim}$ by $g([x]) = [h(x)]$, where $[x]$ denotes the equivalence class of x. This mapping is well-defined because equivalence classes are either singletons or unions of orbits. It is a bijection satisfying $qh = gq$. For an open set $G \subseteq X/{\sim}$ we can see that $H = h^{-1}q^{-1}(G)$ is open. If $x \in H$ then $[x] \subseteq H$. Hence $q(H)$ is open. The equality $q(H) = g^{-1}(G)$ implies that g is continuous. Similarly g^{-1} is continuous and so g is the required homeomorphism. ∎

THEOREM 9. *The only H-compactifications of a discrete space D of cardinality $\kappa \geq \omega$ are of the form $\beta D/F_\lambda$ where $\omega \leq \lambda \leq \kappa^+$ and*

$$F_\lambda = \beta D \setminus \bigcup\{\overline{A}\colon A \subseteq D,\ |A| < \lambda\}.$$

Proof. Recall that the system $\{\overline{A}\colon A \subseteq D\}$ is a clopen base of βD. Hence every set F_λ is closed and $\beta D/F_\lambda$ is a compactification of D. This is an H-compactification by Lemma 7 because the quotient mapping $q_\lambda\colon \beta D \to \beta D/F_\lambda$ is homogeneous by Lemma 8. Note that $F_\omega = \beta D \setminus D$ and $F_{\kappa^+} = \emptyset$ and hence $\beta D/F_\omega = \alpha D$ and $\beta D/F_{\kappa^+} = \beta D$.

Let $f\colon \beta D \to \gamma D$ be a continuous mapping onto some compactification γD extending the identity on D. Suppose that γD is not equivalent to any $\beta D/F_\lambda$. This means that for every $\omega \le \lambda \le \kappa^+$, either $|f(F_\lambda)| \ge 2$, or $f{\restriction}_{\beta D\setminus F_\lambda}$ is not one-to-one. Let λ be the least cardinal such that $f{\restriction}_{\beta D\setminus F_\lambda}$ is not one-to-one. Clearly $\omega < \lambda \le \kappa^+$. By minimality of λ we derive that $\lambda = \mu^+$ is a successor cardinal and $|f(F_\mu)| \ge 2$.

The proof is completed by using the lemma below. ∎

Lemma 10. *Let D be a discrete space of cardinality $\kappa \ge \omega$. Suppose $f\colon \beta D \to \gamma D$ is the only continuous extension of the identity on D onto some compactification γD. If there exists $\omega \le \mu \le \kappa$ for which $|f(F_\mu)| \ge 2$ and $f{\restriction}_{\beta D\setminus F_{\mu^+}}$ is not one-to-one then γD is not an H-compactification of D.*

Proof. There exist two points $u, v \in F_\mu$ such that $f(u) \ne f(v)$ and two distinct points $x, y \in \beta D \setminus F_{\mu^+}$ such that $f(x) = f(y)$. Let U and V be open neighbourhoods of u and v respectively satisfying $f(U) \cap f(V) = \emptyset$. We can find $A, B \subseteq D$ such that $u \in \overline{A} \subseteq U$ and $v \in \overline{B} \subseteq V$. Note that $|A|, |B| \ge \mu$. From the fact that $x, y \notin F_{\mu^+}$ we derive the existence of two disjoint subsets $M, N \subseteq D$ of cardinalities less than μ^+ such that $x \in \overline{M}$ and $y \in \overline{N}$. We can also require $|D \setminus (M \cup N)| = \kappa$.

It is easy to see that there exists a $b \in \mathcal{H}(D)$ such that $b(M) \subseteq A$ and $b(N) \subseteq B$. Let $h \in \mathcal{H}(\beta D)$ be the only continuous extension of b. Note that $h(x) \in \overline{A}$ and $h(y) \in \overline{B}$. Suppose for contradiction that γD is an H-compactification and find $g \in \mathcal{H}(\gamma D)$ extending b. Two continuous mappings fh and gf are equal on D and because D is dense in βD we derive $fh = gf$. On the other hand we have $fh(x) \in f(U)$ and $fh(y) \in f(V)$ and $gf(x) = gf(y)$. But $f(U)$ and $f(V)$ are disjoint, hence $fh(x) \ne fh(y)$, and this contradicts the equality $fh = gf$. ∎

Corollary 11. *The only H-compactifications of ω are $\alpha\omega$ and $\beta\omega$.*

3.2. Countable metrizable spaces. As mentioned in the paper of de Groot and McDowell [dGM60], 'it is of interest to ask for conditions under which every autohomeomorphism of a given space can be extended to a suitable metric compactification'. In this section we study this problem in the class of countable metrizable spaces. Let us recall the frequently used fact that the Alexandroff one-point compactification of a non-compact separable locally compact metrizable space is always metrizable.

Proposition 12. *A non-compact countable metrizable space admits a metrizable H-compactification if and only if it is locally compact. In this case the only metrizable H-compactification is the one-point compactification.*

Proof. Suppose that X is a countable metrizable space and γX an arbitrary metrizable H-compactification. Denote by Y the set of all isolated points of X.

We claim that Y is dense in X. Suppose it is not. Hence $X \setminus \overline{Y}$ is an open non-empty set no points of which are isolated. We can find a non-empty clopen set $Q \subseteq X \setminus Y$. Since Q is a non-empty countable metrizable space without isolated points, it is homeomorphic to the rational numbers $\mathbb{Q}$ (see [vE86, p. 17]). By a result of van Douwen [vD79] there exists exactly one H-compactification of $\mathbb{Q}$, namely $\beta\mathbb{Q}$. The inclusion $Q \to X$ is homogeneous, which implies that the closure of Q in γX is homeomorphic to βQ. This contradicts the assumption that γX is metrizable. So the claim is proved.

For any $x, y \in \gamma X \setminus X$ we can fix sequences $x_n, y_m \in Y$ converging to x and y respectively because γX is metrizable and isolated points of X are dense in γX . We can assume moreover that all the points x_n, y_m are distinct. We can define a homeomorphism $h \in \mathcal{H}(X)$ by

$$h(z) = \begin{cases} x_n & \text{if } z = y_n \text{ and } n \text{ is odd,} \\ y_n & \text{if } z = x_n \text{ and } n \text{ is odd,} \\ z & \text{otherwise,} \end{cases}$$

because the sets $\{x_n : n \in \omega\}$ and $\{y_m : m \in \omega\}$ are clopen and discrete. There is an extension $g \in \mathcal{H}(\gamma X)$ of h for which necessarily $x = h(x) = y$. Thus $\gamma X \setminus X$ contains at most one point. Therefore X is locally compact.

Since the one-point compactification of a locally compact separable metrizable space is again metrizable, we are done. ∎

In what follows, it is of importance that every countable locally compact space is metrizable. To see this, recall that network weight is the same as weight for compact spaces [Eng89, p. 127] and thus any countable locally compact space is first countable. Moreover by the Urysohn metrization theorem [Eng89, p. 260] such a space is metrizable because it is second countable and regular.

For any ordinal δ denote by X^δ the δth Cantor–Bendixson derivative of X. By $\operatorname{rank} X$ we denote the least ordinal δ for which X^δ is empty. This is reasonable only for scattered spaces. For any space X define $\operatorname{top} X = X^\delta$ if $\operatorname{rank} X = \delta + 1$ and $\operatorname{top} X = \emptyset$ otherwise. Notice that for any scattered compact space its Cantor–Bendixson rank cannot be a limit ordinal.

It is helpful to define another ordinal rank, $\operatorname{rank}_c X$, associated to every scattered space X. It is the least ordinal δ for which X^δ is compact. Note that $\operatorname{rank}_c X \leq \operatorname{rank} X$.

Below we give a series of statements useful for the proof of Theorem 16.

LEMMA 13 (Mazurkiewicz–Sierpiński [MS20]). *Let K and L be countable and compact spaces. Then K is homeomorphic to L if and only if* $\operatorname{rank} K = \operatorname{rank} L$ *and* $\operatorname{top} K \cong \operatorname{top} L$.

COROLLARY 14. *The orbits of a countable locally compact space X are precisely the sets $X^\delta \setminus X^{\delta+1}$ where $\delta < \operatorname{rank} X$. Moreover for any $x, y \in X^\delta \setminus X^{\delta+1}$ there is a homeomorphism $h \in \mathcal{H}(X)$ such that $h(x) = y$ and $h\restriction_{X^{\delta+1}}$ is the identity.*

Proof. Clearly the sets $X^\delta \setminus X^{\delta+1}$ are preserved by all homeomorphisms since the Cantor–Bendixson derivative is a topological notion. On the other hand suppose we have two distinct points $x, y \in X^\delta \setminus X^{\delta+1}$. There exist disjoint compact clopen neighbourhoods P and Q of x and y respectively for which $P \cap X^\delta = \{x\}$ and $Q \cap X^\delta = \{y\}$. The sets P and Q are homeomorphic by Lemma 13 and since they are clopen we can extend this homeomorphism by the identity to a homeomorphism of the whole space. ∎

PROPOSITION 15. *Let X and Y be two countable locally compact spaces. Then X is homeomorphic to Y if and only if* $\operatorname{rank} X = \operatorname{rank} Y$, $\operatorname{rank}_c X = \operatorname{rank}_c Y$ *and* $\operatorname{top} X \cong \operatorname{top} Y$.

Proof. The direct implication is clear. If one of the two spaces is compact, so is the other, since $\operatorname{rank}_c X = \operatorname{rank}_c Y$, and by Lemma 13 we get the desired result. Thus suppose they are both non-compact and consider their one-point compactifications $\alpha X = X \cup \{\infty_x\}$ and $\alpha Y = Y \cup \{\infty_y\}$. There are three possibilities:

- $\operatorname{rank} X = \operatorname{rank}_c X$. This implies that $\operatorname{rank} \alpha X = \operatorname{rank} X + 1 = \operatorname{rank} Y + 1 = \operatorname{rank} \alpha Y$ and $\operatorname{top} \alpha X = \{\infty_x\} \cong \{\infty_y\} = \operatorname{top} \alpha Y$.
- $\operatorname{rank} X = \operatorname{rank}_c X + 1$. In this case $\operatorname{rank} \alpha X = \operatorname{rank} X = \operatorname{rank} Y = \operatorname{rank} \alpha Y$ and $\operatorname{top} \alpha X = \operatorname{top} X \cup \{\infty_x\} \cong \operatorname{top} Y \cup \{\infty_y\} = \operatorname{top} \alpha Y$.
- $\operatorname{rank} X > \operatorname{rank}_c X + 1$. Then $\operatorname{rank} \alpha X = \operatorname{rank} X = \operatorname{rank} Y = \operatorname{rank} \alpha Y$ and $\operatorname{top} \alpha X = \operatorname{top} X \cong \operatorname{top} Y = \operatorname{top} \alpha Y$.

In all cases the assumptions of Lemma 13 are satisfied, hence there exists a homeomorphism $h\colon \alpha X \to \alpha Y$. It is not always the case that $h(\infty_x) = \infty_y$ so we need to find a homeomorphism $g \in \mathcal{H}(\alpha Y)$ for which $g(h(\infty_x)) = \infty_y$. However its existence is a consequence of Corollary 14 for $\delta = \operatorname{rank}_c X$ since $\infty_x \in (\alpha X)^\delta \setminus (\alpha X)^{\delta+1}$ and thus $h(\infty_x), \infty_y \in (\alpha Y)^\delta \setminus (\alpha Y)^{\delta+1}$.
Finally $gh\restriction_X$ is the required homeomorphism. ∎

THEOREM 16. *Let X be a countable locally compact space. Then the only H-compactifications of X are of the form $\beta X / F_\delta$ where*

$$F_\delta = \bigcap \{\overline{X^\epsilon} \colon \epsilon < \delta\} \cap X^*$$

for $1 \leq \delta \leq \operatorname{rank}_c X + 1$. *Thus the lattice of all H-compactifications of* X *is a chain isomorphic to the ordinal* $\operatorname{rank}_c X + 1$ *when* $\operatorname{rank}_c X$ *is finite, and to* $\operatorname{rank}_c X + 2$ *when* $\operatorname{rank}_c X$ *is infinite.*

Proof. Note that the Čech–Stone compactification of X is equivalent to $\beta X/F_\delta$ for $\delta = \operatorname{rank}_c X + 1$. Suppose γX is an H-compactification distinct from βX and denote by $f\colon \beta X \to \gamma X$ the only extension of the identity on X. Let δ be the least non-zero ordinal for which there exist distinct points $x \in \beta X \setminus \overline{X^\delta}$ and $y \in \beta X$ such that $f(x) = f(y)$. Note that both x and y are elements of the remainder and clearly $\delta \leq \operatorname{rank}_c X$ since $\overline{X^{\operatorname{rank}_c X}} \subseteq X$. Our aim is to prove that γX is equivalent to $\beta X/F_\delta$.

Pick any point $z \in F_\delta$ distinct from x and y. We wish to show that every neighbourhood U of z contains a point which is mapped to $f(y)$ by f. Take a clopen neighbourhood O of x such that $O \cap X^\delta = \emptyset$ and $y, z \notin O$ and a clopen neighbourhood P of z with $P \subset U$, $y \in P$ and $P \cap \emptyset$. We distinguish two possibilities in order to define a sequence $Z = \{z_n \colon n \in \omega\}$.

- If $\delta = \epsilon + 1$ define $\{z_n \colon n \in \omega\}$ to be a sequence contained in $P \cap (X^\epsilon \setminus X^\delta)$ whose limit is $\infty \in \alpha X$. This is possible since $P \cap X^\epsilon$ is a closed and non-compact subset of X and $X^\epsilon \setminus X^\delta$ is dense in X^ϵ.
- When δ is a supremum of ordinals $\{\delta_n \colon n \in \omega\}$ less than δ one can find a sequence $\{z_n \colon n \in \omega\}$ with limit $\infty \in \alpha X$ and $z_n \in P \cap X^{\delta_n} \setminus X^{\delta_n + 1}$.

At this moment let us mention that

$$\emptyset \neq \overline{Z} \setminus Z \subseteq \bigcap_{\epsilon < \delta} \overline{X^\epsilon} \setminus X^\delta. \tag{1}$$

Since Z and X^δ are two disjoint closed subsets of X there exists a clopen set $N \subseteq X$ containing the sequence Z and disjoint from X^δ. Denote by M the clopen set $O \cap X$. It follows that $\operatorname{rank} M = \operatorname{rank}_c M = \delta = \operatorname{rank}_c N = \operatorname{rank} N$. Moreover if δ is an isolated ordinal we get $\operatorname{top} M \cong \omega \cong \operatorname{top} N$, as otherwise $\operatorname{top} M = \emptyset = \operatorname{top} N$. Thus by Proposition 15 the sets M and N are homeomorphic. Since they are clopen and disjoint we can define a homeomorphism $h \in \mathcal{H}(X)$ such that $h(M) = N$, $h(N) = M$ and h is the identity on the complement of $M \cup N$. Since $x \in \overline{M}$ we get $\beta h(x) \in U$ and $\beta h(y) = y$. There exists an extension $g \in \mathcal{H}(\gamma X)$ of h, because γX is an H-compactification. We have $gf = f\beta h$ and since $f(x) = f(y)$ we get $f(\beta h(x)) = f(\beta h(y)) = f(y)$. Consequently, $\beta h(x)$ is a point in U which is mapped by f to $f(y)$. As U was an arbitrary neighbourhood of z, by continuity of f we get $f(z) = f(y)$.

We have just proved that $\gamma X \leq \beta X/F_\delta$. If we assume this inequality is strict we get a contradiction with the minimality of δ.

In order to prove that all the above mentioned H-compactifications are mutually distinct we have to verify that $F_\epsilon \neq F_\delta$ for $1 \leq \epsilon < \delta \leq \operatorname{rank}_c X + 1$.

It suffices to check that $\overline{X^\delta} \setminus X \subsetneq \bigcap_{\epsilon<\delta} \overline{X^\epsilon} \setminus X$ for $1 \le \delta \le \operatorname{rank}_c X$, which is a consequence of (1). ■

Example 17. The space $\omega \times (\omega+1)^n$ admits exactly $n+2$ H-compactifications.

Remark 18. When dealing with countable non-locally compact spaces the situation seems to be more complicated even in the range of metrizable spaces.

3.3. Euclidean spaces. Van Douwen proved in [vD79] that there exist only three H-compactifications of the real line, namely $\alpha\mathbb{R}$, $[-\infty,+\infty]$ and $\beta\mathbb{R}$. Continuing this research, we show that there are only two H-compactifications of Euclidean spaces of higher dimension.

The following definitions are given in order to state Lemma 22, Theorem 24 and Theorem 29 succinctly. These statements are formulated for a general space X instead of the Euclidean space $\mathbb{R}^n$, mainly because the proofs then seem to be more transparent.

Definition 19. If $\mathcal{U}$ is a collection of subsets of a metric space X then the *mesh* of $\mathcal{U}$ is

$$\operatorname{mesh} \mathcal{U} = \sup\{\operatorname{diam} U \colon U \in \mathcal{U}\} \in [0,+\infty].$$

Definition 20. We say that an open subset U of a space X has *property* $(*)$ if whenever $E \subseteq U$ is closed in X and $V \subseteq U$ is open and non-empty, there is a homeomorphism $h \in \mathcal{H}(X)$ such that $h(E) \subseteq V$ and h is the identity on the complement of U.

Definition 21. We say that a space X has *property* $(**)$ if there is a number $N \in \omega$ such that for every $\epsilon > 0$ there is an open cover $\mathcal{U}$ which can be expressed as a union of N discrete subcollections with mesh less than ϵ, where each $U \in \mathcal{U}$ has property $(*)$.

We note that property $(**)$ is automatically connected with the number N.

Lemma 22. *Let X be a separable locally compact metric space with property $(**)$. Let $M = 2N$. Then:*

- (i) *For every closed set F contained in an open set H there exist a closed discrete set $C \subseteq F$ and closed sets $F_0,\dots,F_{M-1}$ such that $F = \bigcup F_i$ and for every $i < M$ and each open neighbourhood G of C there is an $h \in \mathcal{H}(X)$ such that $h(F_i) \subseteq G$ and h is the identity on the complement of H.*
- (ii) *For every pair of disjoint closed sets $F, F' \subseteq X$ there exist disjoint closed discrete sets $C, C' \subseteq X$ and closed sets $F_0,\dots,F_{M-1}, F'_0,\dots, F'_{M-1}$ such that $F = \bigcup F_i$, $F' = \bigcup F'_j$ and for any $i,j < M$ and*

neighbourhoods G and G' of C and C' respectively there exists a homeomorphism $h \in \mathcal{H}(X)$ such that $h(F_i) \subseteq G$ and $h(F'_j) \subseteq G'$.

Proof. (i) Fix a closed set F and an open set H containing F. Our first claim is that there exists an open cover $\mathcal{V}$ of F in H which can be expressed as a union of M discrete collections of sets with property $(*)$.

Let us find a chain of compact sets $\{K_n : n \in \omega\}$ where $K_n \subset \operatorname{int} K_{n+1}$ and $K_0 = \emptyset$. This can be done since X is locally compact and of countable weight. Denote by d_n the distance of K_n from $K_{n+2} \setminus \operatorname{int} K_{n+1}$ and by ϵ_n an arbitrary positive number less than $\frac{1}{3}\min\{d_0, \dots, d_n\}$ and $\operatorname{dist}(F \cap K_n, X \setminus H)$. There exists an open cover $\mathcal{U}^n = \mathcal{U}^n_0 \cup \dots \cup \mathcal{U}^n_{M-1}$ of X of mesh less than ϵ_n, consisting of sets with property $(*)$ where every collection $\mathcal{U}^n_i$ is discrete. Define now

$$\mathcal{V}_i = \begin{cases} \bigcup_{n \text{ odd}} \{U \in \mathcal{U}^{n+1}_i : U \cap F \cap (K_{n+1} \setminus \operatorname{int} K_n) \neq \emptyset\} & \text{for } i < N, \\ \bigcup_{n \text{ even}} \{U \in \mathcal{U}^{n+1}_{i-N} : U \cap F \cap (K_{n+1} \setminus \operatorname{int} K_n) \neq \emptyset\} & \text{for } N \le i < M. \end{cases}$$

The collections $\mathcal{V}_i$ are discrete and consist of sets with property $(*)$. Then $\mathcal{V} = \bigcup_{i<M} \mathcal{V}_i$ is the required cover of F contained in H. Thus the claim is proved.

Since metrizable spaces are paracompact according to the Stone theorem [Eng89, p. 300] we can find a closed indexed refinement $\{E(V) : V \in \mathcal{V}\}$ whose union is F, because by Remark 5.1.7 from [Eng89, p. 301] every open cover of a regular paracompact space has a closed indexed refinement. Observe that the set $F_i = \bigcup\{E(V) : V \in \mathcal{V}_i\}$ is closed, because it is the union of a discrete collection of closed sets. Suppose that C is a selecting set from $\{E(V) : V \in \mathcal{V}\}$. It is a closed discrete set.

Suppose that G is an open neighbourhood of C and fix $i < M$. For every $V \in \mathcal{V}_i$ there exists $h_V \in \mathcal{H}(X)$ which sends the set $E(V)$ into $G \cap V$ and which is the identity on the complement of V. This is because V has property $(*)$.

Since the collection $\mathcal{V}_i$ is discrete, we can construct $h \in \mathcal{H}(X)$ which is roughly speaking, the composition of all these h_V. Consequently, h sends the set F_i into G.

(ii) For two disjoint closed sets F and F' we can find open sets H and H' whose closures are disjoint and which contain F and F' respectively. By (i) there exist closed discrete sets C and C' and closed sets $F_0, \dots, F_{M-1}$, $F'_0, \dots, F'_{M-1}$ such that whenever $i, j < M$ and G and G' are neighbourhoods of C and C' respectively, there exist $h, h' \in \mathcal{H}(X)$ such that $h(F_i) \subseteq G$, $h'(F'_j) \subseteq G'$, $h\restriction_{X\setminus H} = \mathrm{id}$ and $h'\restriction_{X\setminus H'} = \mathrm{id}$. Then $h'' = h'h$ is the required homeomorphism. ∎

Definition 23. Let X be a topological space and $\mathcal{P}$ a collection of subspaces of X. Then X is called *n-homogeneous with respect to $\mathcal{P}$* if for any sets $C_i, D_i \in \mathcal{P}$ for $i < n$ satisfying $C_i \cong D_i$ and $C_i \cap C_j = \emptyset = D_i \cap D_j$ for $i \neq j$ there exists a homeomorphism $h \in \mathcal{H}(X)$ such that $h(C_i) = D_i$ for every $i < n$.

When $n = 1$ we just say *homogeneous with respect to $\mathcal{P}$*. Note that for example homogeneity with respect to points is equivalent to the classical notion of homogeneity.

Theorem 24. *Let X be a separable locally compact but non-compact metric space with property $(**)$ and 2-homogeneous with respect to closed discrete sets. Then αX and βX are the only H-compactifications of X.*

Proof. Suppose that γX is an H-compactification of X distinct from the one-point compactification. Take arbitrary disjoint closed sets F and F' in X. Our aim is to prove that their closures in γX are also disjoint, because then we infer that γX is equivalent to βX.

By Lemma 22 there exist two closed discrete sets C, C' and closed sets $F_0, \ldots, F_M, F'_0, \ldots, F'_M$ with properties mentioned above. Since $F = \bigcup F_i$ and $F' = \bigcup F'_j$ it is enough to prove that for any $i, j < M$ the closures of F_i and F'_j are disjoint in γX.

Note that there exist two countable infinite closed discrete sets D and D' of X whose closures in γX are disjoint, because $\gamma X \setminus X$ contains at least two points. Since X is 2-homogeneous with respect to closed discrete sets and since γX is an H-compactification, the closures of C and C' in γX are disjoint. Hence we can separate them by open sets G and G' in X whose closures in γX are disjoint.

By Lemma 22 we can find $h \in \mathcal{H}(X)$ with $h(F_i) \subseteq G$ and $h(F'_j) \subseteq G'$. Consequently, the closures of $h(F_i)$ and $h(F'_j)$ in γX are disjoint and since γX is an H-compactification, the closures of F_i and F'_j are also disjoint.

Thus we have proved that γX is equivalent to βX. ∎

We need to show that Euclidean spaces of dimension at least two satisfy the assumptions of Theorem 24. This is partially done in Proposition 26. The following notion will be useful in the proof.

Definition 25. A space X is called *strongly locally homogeneous* if for every $x \in X$ and each neighbourhood U of x there exists a neighbourhood V of x in U such that for every $y \in V$ we can find a homeomorphism of X which sends x to y and is the identity on the complement of V.

Note that if we have an open connected set U in a strongly locally homogeneous space X and two points $x, y \in U$, there is always a homeomorphism $h \in \mathcal{H}(X)$ for which $h(x) = y$ and which is the identity on the complement of U. This is because the set $\{g(x) \colon g \in \mathcal{H}(X), g\restriction_{X\setminus U} = \mathrm{id}\}$ is clopen in U and thus it has to be equal to U.

It is a well-known fact that Euclidean spaces are strongly locally homogeneous (see [vM01, p. 64]).

PROPOSITION 26. *Let $n \geq 2$. Then every bijection of closed discrete subsets of $\mathbb{R}^n$ can be extended to a homeomorphism of the whole space.*

Proof. Let C and D be closed discrete subsets of $\mathbb{R}^n$ of cardinality $m \leq \omega$, and $b\colon C \to D$ a bijection. We will use the structure of the normed linear space $(\mathbb{R}^n, \|\cdot\|)$. The words *span* and *conv* mean *linear hull* and *convex hull* respectively. Let $S \subset \mathbb{R}^n$ consist of those points which either lie on a line through two points from $C \cup D$ or whose distance from two distinct points from $C \cup D$ is the same. This set is a countable union of nowhere dense sets, so a set of the first category in $\mathbb{R}^n$. Hence by the Baire theorem $\mathbb{R}^n \setminus S \neq \emptyset$; without loss of generality we can assume that $0 \in \mathbb{R}^n \setminus S$.

Enumerate now the set C as $\{c_i \colon i < m\}$ and denote $r_i = \|c_i\|$, $\varphi_i = c_i/\|c_i\| \in \mathbb{S}^{n-1}$, $d_i = b(c_i)$ and $s_i = \|d_i\|$. By the first paragraph we know that $r_i \neq r_j$, $\varphi_i \neq \varphi_j$ and $s_i \neq s_j$ for $i < j < m$. Fix $i < m$ for a while. Denote by ϵ_i the positive distance of the set $\bigcup\{\mathrm{span}(0, c) \colon c \in C, \|c\| < r_i\} \cup C \setminus \{c_i\}$ from the segment $\mathrm{conv}(c_i, s_i\varphi_i)$. Denote by U_i the set of all points whose distance from $\mathrm{conv}(c_i, s_i\varphi_i)$ is less than $\frac{1}{3}\epsilon_i$. Since $\mathbb{R}^n$ is strongly locally homogeneous and the open set U_i is connected it follows that there exists $h_i \in \mathcal{H}(\mathbb{R}^n)$ such that $h_i(c_i) = s_i\varphi_i$ and h_i is the identity on the complement of U_i. Since the collection $\{U_i \colon i < m\}$ is discrete we can define $h \in \mathcal{H}(\mathbb{R}^n)$ by

$$h(x) = \begin{cases} h_i(x) & \text{if } x \in U_i, \\ x & \text{otherwise.} \end{cases}$$

We proceed similarly to find a homeomorphism h'. Let ϵ_i' be a positive distance of the point s_i from the set $\{s_j \colon j \neq i\}$ and denote by U_i' the set of all points whose distance from $s_i\mathbb{S}^{n-1}$ is less than $\frac{1}{3}\epsilon_i'$. We can find $h_i' \in \mathcal{H}(\mathbb{R}^n)$ such that $h_i'(s_i\varphi_i) = d_i$ and h_i' is the identity on the complement of U_i'. Since the collection $\{U_i' \colon i < m\}$ is discrete we can define $h' \in \mathcal{H}(\mathbb{R}^n)$ by

$$h'(x) = \begin{cases} h_i'(x) & \text{if } x \in U_i', \\ x & \text{otherwise.} \end{cases}$$

Now it is enough to define h'' as the composition $h'h$ to get the desired homeomorphism for which $h''(c_i) = h'h(c_i) = h'(s_i\varphi_i) = d_i = b(c_i)$. ∎

REMARK 27. The proof of Proposition 26 can be done more easily for Euclidean spaces of dimension at least three. In the plane the situation is more complicated. Note that this proposition obviously fails to be true for $n = 1$.

Corollary 28. *There are exactly two H-compactifications of $\mathbb{R}^n$ for $n \geq 2$, namely $\alpha\mathbb{R}^n$ and $\beta\mathbb{R}^n$.*

Proof. It is enough to check that the assumptions of Theorem 24 are satisfied. The fact that $\mathbb{R}^n$ is 2-homogeneous with respect to closed discrete sets follows easily from Proposition 26. To verify the assumptions of Lemma 22 consider the maximum metric ρ on $\mathbb{R}^n$ and denote by $B_\rho(x,r)$ the open ball with centre x and diameter r. Put $N = 2^n$. For arbitrary $\epsilon > 0$ we can take the cover $\mathcal{U} = \bigcup\{\mathcal{U}_j : j \in 2^n\}$ of $\mathbb{R}^n$ where

$$\mathcal{U}_j = \{B_\rho(\epsilon i/3, \epsilon/4) \colon i \in \mathbb{Z}^n,\ i_k \equiv j_k \pmod 2\}.$$

All collections $\mathcal{U}_j$ are discrete and they consist of sets with property $(*)$, because every ball $B_\rho(x,r)$ in $\mathbb{R}^n$ has property $(*)$. Thus $\mathbb{R}^n$ has property $(**)$.

For completeness we should add that the two H-compactifications are distinct, which is however an immediate consequence of the fact that there exists a continuous bounded function with no limit at infinity. ∎

3.4. Products of ω and Euclidean space. In this section we are going to describe all the H-compactifications of the product $\omega \times \mathbb{R}^n$. It is shown that there are always finitely many of them, but the situation in the case $n = 1$ is different from the case $n \geq 2$. Some of these H-compactifications can be obtained via applying compactifications α, β and φ, where φ denotes the *Freudenthal compactification* which is the largest compactification with zero-dimensional remainder. For a locally compact space X it always exists and can be described as a quotient of βX where every component of $\beta X \setminus X$ shrins to a point. Some additional facts concerning the Freudenthal compactification can be found in [Dom03].

For a space X we define W_0 to be the quotient space $\omega \times \alpha X/(\omega \times \{\infty\})$. The space W_1 is obtained from $\omega \times \alpha X$ by adding one point ∞ where base neighbourhoods of ∞ are of the form

$$\{\infty\} \cup \bigcup_{n>m} \{n\} \times (\alpha X \setminus K_n)$$

where K_n are compact subsets of X and $m \in \omega$. The space W_2 arises by adjoining a point ∞ to X. Base neighbourhoods of ∞ are of the form

$$\{\infty\} \cup \bigcup_{n>m} \{n\} \times (X \setminus K_n)$$

where K_n are compact subsets of X and $m \in \omega$. Finally a space W_3 is given by $Y \cup \omega^*$ where Y is an open set in W_3 and base neighbourhoods of a point $x \in \omega^*$ are of the form

$$(\omega^* \cap \overline{M}^{\beta\omega}) \cup \bigcup_{n\in\omega} \{n\} \times (X \setminus K_n)$$

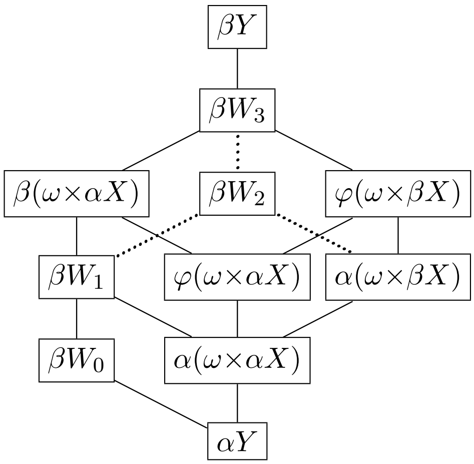
where $M \subseteq \omega$, $x \in \overline{M}^{\beta\omega}$ and K_n are compact subsets of X.

We use Hasse diagrams to describe a lattice or a partially ordered set. Here, two objects are connected with a line if the upper one is a minimal element strictly bigger than the lower one. For more information about Hasse diagrams we refer to the first chapter of [Bir67, p. 4].

The following result can be applied to $X = \mathbb{R}^n$ where $n \geq 2$ since the assumptions were verified in the proof of Corollary 28.

THEOREM 29. *Let X be a non-compact connected separable locally compact metric space with property $(**)$ and 2-homogeneous with respect to closed discrete sets. Then the space $Y = \omega \times X$ admits exactly eleven H-compactifications which form a lattice described below.*

(2)



NOTATION 30. We will use a consequence of Proposition 5 that all H-compactifications of the (locally compact) space Y are in a natural one-to-one correspondence with those closed equivalence relations $E \subseteq (\beta Y \setminus Y)^2$ such that $\beta h \times \beta h(E) = E$ for every $h \in \mathcal{H}(Y)$. Equivalences with this property are called *invariant*. Note that closed invariant equivalences of a locally compact space form a lattice antiisomorphic to the lattice of all H-compactifications.

Denote by $\pi\colon Y \to \omega$ the projection onto the first coordinate, by X_n the set $\{n\} \times X$, by C the closed set

$$\beta Y \setminus \bigcup\{\overline{K}^{\beta Y} : K \subseteq Y \text{ and } K \cap X_n \text{ is compact for every } n\}$$

and by Δ the diagonal of the square $(\beta Y \setminus Y)^2$.

Moreover, define three equivalences on Y^* as follows:

$$E_C = \Delta \cup \{(x, y) \in C \times C\},$$
$$E_* = \Delta \cup \{(x, y) \in Y^* \times Y^* \colon \beta\pi(x), \beta\pi(y) \in \omega^*\},$$
$$E_= = \Delta \cup \{(x, y) \in Y^* \times Y^* \colon \beta\pi(x) = \beta\pi(y)\}.$$

It can be verified that these equivalences are closed. Since X is connected (and thus any homeomorphism of Y sends a set X_n onto some X_m) we see that the mapping π is homogeneous. Consequently, all the three equivalences above are invariant.

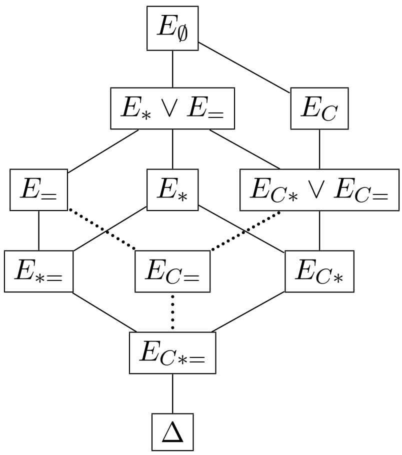
Proof of Theorem 29. It is interesting to mention that the four equivalences $E_C, E_*, E_=$ and Δ generate the lattice of all closed invariant equivalences on Y^*, as will follow. For any $S \subseteq \{C, *, =\}$ we denote by E_S the closed invariant equivalence $\bigwedge_{i\in S} E_i$. Our claim is that the only closed one-generated invariant equivalences are the E_S except $E_=$. So take a pair $(x, y) \in (\beta Y \setminus Y)^2$ and observe that if $x = y$ then this pair generates Δ, otherwise we have to distinguish the following possibilities. We give a complete proof only in the case of $E_{C*=}$ since the other cases are very similar.

- $x, y \in C$.
  - $\beta\pi(x), \beta\pi(y) \in \omega^*$.
    - $\beta\pi(x) = \beta\pi(y)$. To prove that (x, y) generates the equivalence $E_{C*=}$ take any $(x', y') \in E_{C*=} \setminus \Delta$ and a neighbourhood $G \times H$ of (x', y') in $\beta Y \times \beta Y$. We want to find $h \in \mathcal{H}(Y)$ for which $(\beta h(x), \beta h(y)) \in \overline{G} \times \overline{H}$. Since $x', y' \in C$ and $\beta\pi(x') = \beta\pi(y') \in \omega^*$ there exists an infinite set $K \in \beta\pi(x')$ such that for every $n \in K$ the closures of $G \cap X_n$ and $H \cap X_n$ in Y are non-compact. We can assume that $\omega \setminus K$ is infinite. Suppose that E and F are disjoint closed subsets of $L \times X$ for which $x \in \overline{E}$ and $y \in \overline{F}$ where L is an infinite subset of ω such that $\omega \setminus L$ is infinite. Let $b\colon \omega \to \omega$ be an arbitrary bijection for which $b(L) = K$.
      For fixed $n \in L$ apply Lemma 22 to the space X_n and closed sets $E \cap X_n$ and $F \cap X_n$ to get closed discrete sets $C^n, D^n \subseteq X_n$ and closed sets $E_i^n, F_j^n \subseteq X_n$ where $i < M$ with suitable properties. Since
      $$E = \bigcup_{i<M} \bigcup_{n\in\omega} E_i^n \quad \text{and} \quad F = \bigcup_{j<M} \bigcup_{n\in\omega} F_j^n,$$
      there exist indices $i, j < M$ such that $x \in \overline{\bigcup\{E_i^n : n \in \omega\}}$ and $y \in \overline{\bigcup\{F_j^n : n \in \omega\}}$. Since X is 2-homogeneous with respect to closed discrete sets we can find a homeomorphism $h_n\colon X_n \to X_{b(n)}$ such that $h_n(C^n) \subseteq G \cap X_{b(n)}$ and $h_n(D^n) \subseteq H \cap X_{b(n)}$. Lemma 22 yields $g_n \in \mathcal{H}(X_n)$ such that $g_n(E_i^n) \subseteq h_n^{-1}(G \cap X_{b(n)})$ and $g_n(F_j^n) \subseteq h_n^{-1}(H \cap X_{b(n)})$.
      For $n \in \omega \setminus L$ we let h_n be the natural homeomorphism $X_n \to X_{b(n)}$ and g_n the identity on X_n. Now $h = \bigcup\{h_n g_n : n \in \omega\} \in \mathcal{H}(Y)$ is the desired homeomorphism.
    - $\beta\pi(x) \neq \beta\pi(y)$. In this case the pair (x, y) generates E_{C*}.

    - $\beta\pi(x) \in \omega$ or $\beta\pi(y) \in \omega$.
        - $\beta\pi(x) = \beta\pi(y)$. Then (x, y) generates $E_{C=}$.
        - $\beta\pi(x) \neq \beta\pi(y)$. Then (x, y) generates E_C.
- $x \notin C$ or $y \notin C$.
    - $\beta\pi(x), \beta\pi(y) \in \omega^*$.
        - $\beta\pi(x) = \beta\pi(y)$. Then (x, y) generates $E_{*=}$.
        - $\beta\pi(x) \neq \beta\pi(y)$. Then (x, y) generates E_*.
    - $\beta\pi(x) \in \omega$ or $\beta\pi(y) \in \omega$.
        - $\beta\pi(x) = \beta\pi(y)$. This cannot happen.
        - $\beta\pi(x) \neq \beta\pi(y)$. Then (x, y) generates $E_\emptyset$.

We conclude that each closed invariant congruence can be obtained as the join of some family of one-generated equivalences E_S and Δ. Thus there exist at most $2^9 = 512$ of them. However there are obvious inclusions $\Delta \subseteq E_S \subseteq E_T$ for $T \subseteq S \subseteq \{C, *, =\}$, which reduces this number substantially.

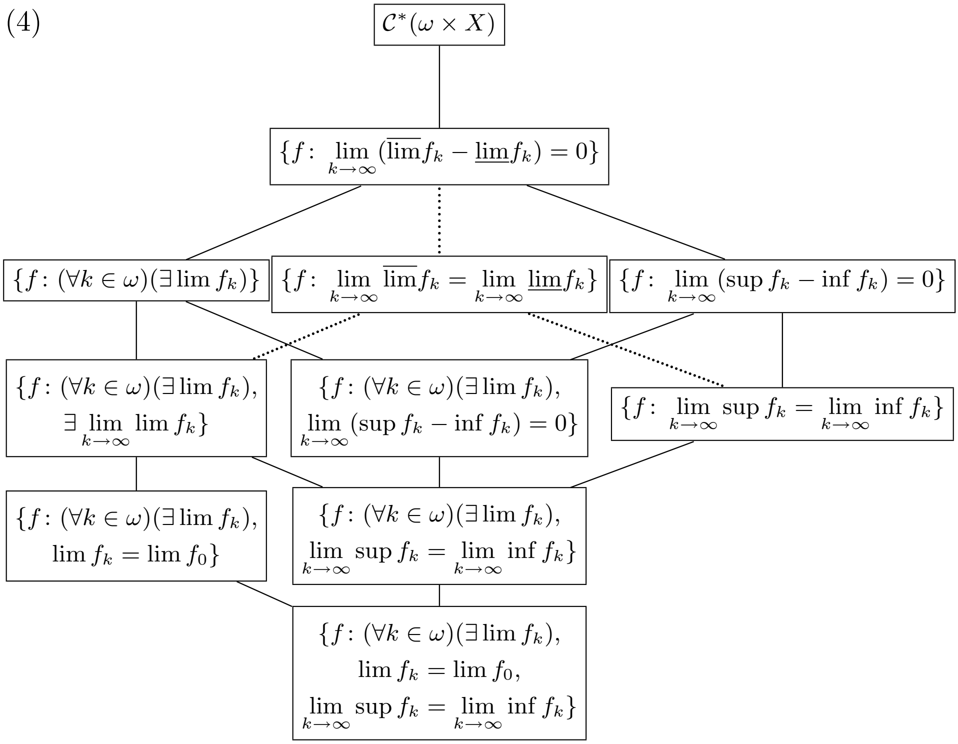
Observe that $E_* = E_{*=} \vee E_{C*}$, because if $(x, y) \in E_* \setminus \Delta$ we can find $x', y' \in C$ such that $\beta\pi(x) = \beta\pi(x')$ and $\beta\pi(y) = \beta\pi(y')$ and thus $(x, x') \in E_{*=}$, $(x', y') \in E_{C*}$ and $(y', y) \in E_{*=}$, hence $(x, y) \in E_{*=} \vee E_{C*}$. Similarly it can be shown that $E_= = E_{*=} \vee E_{C=}$ and $E_\emptyset = E_C \vee E_{*=}$. If we put together these equalities and the obvious inclusions we see that the lattice of all closed invariant equivalences is given by the Hasse diagram

(3) 

It remains to verify that this lattice corresponds to the lattice mentioned in the statement, which is routine. ∎

Remark 31. Below we give a characterization of all H-compactifications of the space $\omega \times X$ from Theorem 29 using rings of continuous functions. For simplicity we denote by f_k the restriction $f{\restriction}_{\{k\}\times X}$ whenever $f \in \mathcal{C}^*(\omega \times X)$. The symbol $\lim f_k$ means $\lim_{x\to\infty} f_k(x)$ where $X \cup \{\infty\}$ is the one-point compactification of X. The symbols $\overline{\lim}$ and $\underline{\lim}$ denote limes superior and limes inferior respectively.

(4)



Corollary 32. *The only H-compactifications of the space $Z = \omega \times \mathbb{S}^n$ for $n \geq 2$ are the Alexandroff one-point compactification αZ, the Freudenthal compactification φZ and the Čech–Stone compactification βZ.*

Proof. Since there is a homogeneous embedding of $\omega \times \mathbb{R}^n$ onto a dense subspace of $\omega \times \mathbb{S}^n$, every H-compactification of $\omega \times \mathbb{S}^n$ is at the same time an H-compactification of $\omega \times \mathbb{R}^n$. From Theorem 29 we derive that only the three cases given can occur. ∎

The situation is different for $n = 1$. Van Douwen noted in [vD79] that there exist at least eleven H-compactifications of $\omega \times \mathbb{R}$. In fact there are exactly 26 of them. However it would be a long-distance run to prove this completely, because there is a lot of routine work. Therefore we omit some details in the proof.

Theorem 33. *There exist exactly* 26 *H-compactifications of the space $\omega \times \mathbb{R}$. They form a lattice described in the diagram* (6)*.*

Proof. We use the terminology of Notation 30 with X replaced by $\mathbb{R}$. Moreover, for any pair E, F of closed subsets of $\mathbb{R}$ define $o(E, F)$ to be the infimum of $k \in \omega$ such that there exist intervals (i.e. connected sets) $I_0, \ldots, I_k$ covering E and not intersecting F. Note that $o(E,F) = +\infty$ whenever E and F have non-empty intersection. By a simple consideration we derive that the numbers $o(E,F)$ and $o(F,E)$ are either both infinite or both finite, and their difference is at most one. For any $x, y \in \beta Y$ denote

$$\operatorname{over}(x,y) = \inf\{\sup_{n\in\omega} o(E \cap X_n, F \cap X_n) \colon x \in \overline{E},\ y \in \overline{F},\ E, F \subseteq Y\}$$

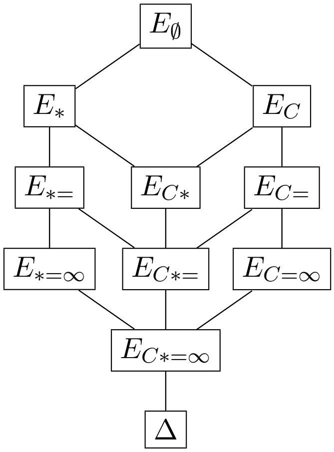
and define a closed invariant equivalence $E_\infty = \{(x,y) \in Y^* \times Y^* \colon \operatorname{over}(x,y) = \infty\}$. For any $S \subseteq \{C, *, =, \infty\}$ put $E_S = \bigwedge_{i\in S} E_i$. In this way we get at most sixteen equivalences but it is easily seen that $E_\infty \subseteq E_=$. Hence there are only twelve of them.

Our claim is that every one-generated closed invariant equivalence is of the form E_S for some $S \subseteq \{C, *, =, \infty\}$ or Δ. To verify this pick $(x,y) \in Y^* \times Y^*$ and denote by E the closed invariant equivalence generated by this couple. If $x = y$ we get $E = \Delta$. Otherwise we distinguish the possibilities below. An argument at each step is necessary, but it is omitted.

- $x, y \in C$.
  - $\beta\pi(x), \beta\pi(y) \in \omega^*$.
    - $\beta\pi(x) = \beta\pi(y)$.
      - $\operatorname{over}(x,y) = \infty$: $E = E_{C*=\infty}$.
      - $\operatorname{over}(x,y) < \infty$: $E = E_{C*=}$.
    - $\beta\pi(x) \neq \beta\pi(y)$: $E = E_{C*}$.
  - $\beta\pi(x) \in \omega$ or $\beta\pi(y) \in \omega$.
    - $\beta\pi(x) = \beta\pi(y)$.
      - $\operatorname{over}(x,y) = \infty$: $E = E_{C=\infty}$.
      - $\operatorname{over}(x,y) < \infty$: $E = E_{C=}$.
    - $\beta\pi(x) \neq \beta\pi(y)$: $E = E_C$.
- $x \notin C$ or $y \notin C$.
  - $\beta\pi(x), \beta\pi(y) \in \omega^*$.
    - $\beta\pi(x) = \beta\pi(y)$.
      - $\operatorname{over}(x,y) = \infty$: $E = E_{*=\infty}$.
      - $\operatorname{over}(x,y) < \infty$: $E = E_{*=}$.
    - $\beta\pi(x) \neq \beta\pi(y)$: $E = E_*$.
  - $\beta\pi(x) \in \omega$ or $\beta\pi(y) \in \omega$.
    - $\beta\pi(x) = \beta\pi(y)$: This cannot happen.
    - $\beta\pi(x) \neq \beta\pi(y)$: $E = E_\emptyset$.

Next, a partially ordered set of all one-generated closed invariant equivalences is given (this is not a lattice):

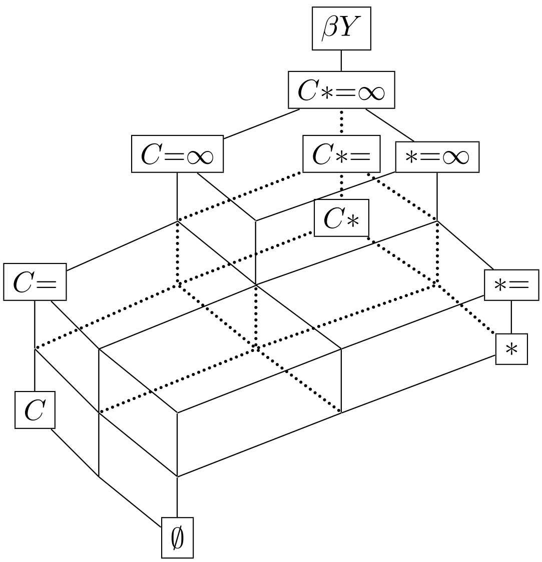
(5)



Remember that every closed invariant equivalence is a join of some subcollection of the 11 equivalences from the diagram (5). It can be counted by hand that there are 15 (unordered) pairs of equivalences in the diagram (5) that are not comparable, only 5 incomparable triples and clearly no such quadruples. Since $E_* = E_{C*} \vee E_{*=}$ and $E_\emptyset = E_C \vee E_* = E_C \vee E_{*=}$, there are in fact at most $12 = 15 - 3$ two-generated invariant closed congruences that are not one-generated and at most 3 three-generated ones that are not two-generated. Thus there are at most $11+12+3 = 26$ H-compactifications.

Finally, we get the Hasse diagram (6) of the lattice of all H-compactifications of $\omega \times \mathbb{R}$. For brevity we write for example '$C{=}$' instead of $\beta Y/E_{C=}$.

(6)



It remains to verify that these compactifications are pairwise non-equivalent. This is however a lot of routine work only. ∎

Remark 34. When dealing with the problem of finding all H-compactifications of $\omega\times\mathbb{R}$, one starts with compactifications obtained in a natural way in the form $\gamma(\omega\times\delta\mathbb{R})$ where γ and δ are substituted by α,β or φ. Even if we add $\alpha(\omega\times\mathbb{R})$ we obtain only 10 H-compactifications in this way.

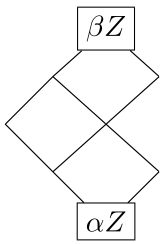
Corollary 35. *There exist exactly four H-compactifications of the space* $Z=\omega\times\mathbb{S}$. *These are (from the smallest to the biggest)* αZ, *the Freudenthal compactification* φZ, *the compactification* γZ *described below and* βZ.

The compactification γZ *can be described as* $\beta Z/{\sim}$ *where* $x\sim y$ *iff* $\sup\{o(E_n,F_n)\colon n\in\omega\}=+\infty$ *for every pair of closed sets* $E,F\subseteq Y$ *such that* $x\in\overline{E}$ *and* $y\in\overline{F}$. *In this situation* $A_n=A\cap\{n\}{\times}\mathbb{S}$ *for* $A\subseteq Z$, *and* $o(E_n,F_n)$ *is the infimum of* $k\in\omega$ *for which there exist connected sets* $I_0,\dots,I_k\subseteq Z_n$ *such that* $E_n\subseteq I_0\cup\dots\cup I_k\subseteq Z_n\setminus F_n$.

Proof. There is a homogeneous embedding $\omega\times\mathbb{R}\to\omega\times\mathbb{S}$ onto a dense subset and thus every H-compactification of $\omega\times\mathbb{S}$ is at the same time an H-compactification of $\omega\times\mathbb{R}$. The latter were classified in Theorem 33 from which it follows that only four of them occur here. ∎

Corollary 36. *The lattice of all H-compactifications of the space* $Z=\omega\times[-\infty,+\infty]$ *contains eight elements and is given by the Hasse diagram*

(7)



Proof. This is a consequence of Theorem 33 since there is a homogeneous embedding of $\omega\times\mathbb{R}$ onto a dense subset of Z. Exactly eight compactifications from the lattice (6) occur here. ∎

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