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Wirtinger presentations for higher dimensional manifold knots obtained from diagrams

by

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Abstract. A Wirtinger presentation of a knot group is obtained from a diagram of the knot. T. Yajima showed that for a 2-knot or a closed oriented surface embedded in the Euclidean 4-space, a Wirtinger presentation of the knot group is obtained from a diagram in an analogous way. J. S. Carter and M. Saito generalized the method to non-orientable surfaces in 4-space by cutting non-orientable sheets of their diagrams by some arcs. We give a modification to their method so that one does not need to find and describe such arcs on the diagram. This method is easily generalized to higher dimensional manifold knots, which may not be locally flat.

1. Introduction. A Wirtinger presentation of a knot group is obtained from a diagram of the knot (cf. [2, 4, 20, 21]). For a 2-knot or a closed oriented surface M in $\mathbb{R}^4$, a Wirtinger presentation of the knot group $\pi_1(\mathbb{R}^4 - M)$ is obtained from a (broken surface) diagram in an analogous way. Consider a diagram of M in $\mathbb{R}^3$, and label the sheets of the diagram by $x_1, \ldots, x_s$, where s is the number of sheets. Each x_i is regarded as a meridian element of the knot group $\pi_1(\mathbb{R}^4 - M)$ and the knot group is generated by $x_1, \ldots, x_s$. Each double curve (connected component of the double point set in $\mathbb{R}^3$) induces a relator of the form $x_i x_j x_i^{-1} x_l^{-1}$ or $x_i x_j^{-1} x_i^{-1} x_l$, where x_i is the label of the upper sheet and x_j, x_l are the lower sheets around the double curve. The exponents are the signs of the intersections of the oriented sheets x_i, x_j, x_l and a small loop around the double curve (cf. [3, 25]). For a non-orientable surface in $\mathbb{R}^4$, this method does not apply directly. If a sheet of a broken surface diagram contains an orientation-reversing loop, then one cannot assign a meridian element to the sheet. This happens even

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in diagrams with simple configurations as in Figure 1 (the diagrams present a standard projective plane and a standard Klein bottle in $\mathbb{R}^4$). The first diagram has a single sheet which is a Möbius band. The second has two sheets; one is a disk and the other is a punctured Klein bottle. To avoid such a bad situation, J. S. Carter and M. Saito divided the non-orientable sheets into orientable pieces by some arcs satisfying a certain condition (see [3]). One has to find such arcs and describe them on the diagram. This process is a little bothersome when the configuration of the diagram is not simple. In this paper we give an alternative method to divide the non-orientable sheets, which is quite elementary. In fact, the diagram itself gives the information on the division, and the argument is valid in higher dimensional cases. Moreover our argument does not require a surface in $\mathbb{R}^4$, or higher dimensional manifold knot, to be locally flat. We work in the PL (or smooth) category.

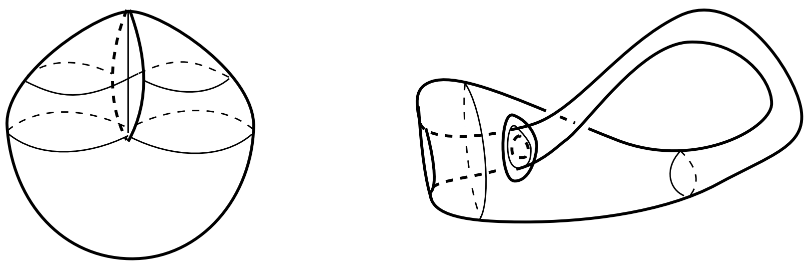

Fig. 1

A method to obtain a Wirtinger presentation for PL and locally flatly (or smoothly) embedded surfaces in $\mathbb{R}^4$ using diagrams was given by T. Yajima [25] (based on an idea of [16]) and a clear exposition (without a proof) is found in [3] including the non-orientable case. Existence of a Wirtinger presentation for such surfaces in $\mathbb{R}^4$ had been known before using diagrams. For example, Fox's method in [5] gives such a presentation. A similar method was used in [17]. The latter method works quite well if a surface in $\mathbb{R}^4$ is given in the motion picture form and the configuration of the surface satisfies a certain condition as on p. 134 of [5]. It is proved in [9, 12] that any (PL locally flat or smooth) surface in $\mathbb{R}^4$ is deformable into such a form. Existence of a Wirtinger presentation for higher dimensional orientable smooth manifold knots was stated by J. Simon in [23]. His idea is to use Fox's method inductively (the details for the 2-knot case are in [24], for example). Yajima [27] proved algebraically that the Kervaire conditions [14, 15] imply existence of a Wirtinger presentation, and hence a higher dimensional orientable (smooth or PL locally flat) manifold knot group has a Wirtinger presentation if the group has a trivial second homology (there exist a lot of manifold knot groups with non-trivial second homology [1, 7, 18, 19, 23]).

For related topics on higher dimensional manifold knot groups, we refer to [6, 7, 8, 10, 11, 14, 15, 23, 26, 27].

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2. Regular sheets. Let M be a closed (not necessarily connected) n-manifold embedded piecewise linearly (or smoothly) in $\mathbb{R}^{n+2}$, and let $p : \mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ be the projection. We suppose that M is in general position with respect to the projection p. The singularity set Δ that is the closure of the multiple point set $\{y \in \mathbb{R}^{n+1} \mid |p^{-1}(y) \cap M| > 1\}$ in $\mathbb{R}^{n+1}$ (or its preimage in M) is naturally regarded as an $(n-1)$-dimensional stratified complex. It is well known that the $(n-1)$-dimensional strata consist of transverse double points; we call them the *double point strata* and denote by Δ^1. (This is seen by a general position argument which does not require M to be locally flat, cf. [22].) Lower dimensional strata are in general complicated. For $n = 2$, the 0-dimensional strata consist of triple points and branch points if M is locally flat (cf. [3, 13]); if M is not locally flat, then there may be some cone points over classical knot diagrams in small 2-spheres in $\mathbb{R}^3$. For our purpose, classification of lower dimensional strata is not required at all since they do not contribute to the knot group.

The singularity set Δ divides $p(M)$ (or M in the preimage) into some pieces. Each piece (connected component of $p(M)-\Delta$) is an open n-manifold embedded in $\mathbb{R}^{n+1}$ consisting of regular points of $p(M)$, which we call an *open regular sheet*. Let $N(\Delta)$ be a regular neighborhood of Δ in $\mathbb{R}^{n+1}$. We call a component of $\mathrm{cl}(p(M) - N(\Delta))$ a *regular sheet*, where cl means closure. A regular sheet is a deformation retract of an open regular sheet. For example, the first (broken surface) diagram in Figure 1 has a single regular sheet which is a 2-disk. The second has two sheets; one is a 2-disk and the other is a punctured annulus.

LEMMA 1. (*Open*) *regular sheets are* 2-*sided* (*or co-orientable*).

Proof. If M is orientable, each (open) regular sheet is orientable and hence it is co-orientable in $\mathbb{R}^{n+1}$. If M is non-orientable and if there is a non-orientable regular sheet Σ, take an orientation-reversing loop, say c, in the interior of the sheet Σ. We may assume that it is a simple loop (if $\dim(M) > 3$, it is obvious; if $\dim(M) = 2$, modify the loop if necessary). Push the loop c out of Σ in $\mathbb{R}^{n+1}$ along the normal direction, obtaining a loop c'. Since c is an orientation-reversing loop of Σ, the regular neighborhood $N(c; \mathbb{R}^{n+1})$ of c in $\mathbb{R}^{n+1}$ is a twisted I-bundle over $N(c; \Sigma)$ and hence we may assume that $|c' \cap N(c; \Sigma)| = 1$. On the other hand, $|c' \cap p(M)| = 0$

mod 2, since any $\mathbb{Z}_2$-intersection number of cycles vanishes in $\mathbb{R}^{n+1}$. Hence c' has another intersection with $p(M)$ off $N(c;\Sigma)$. This contradicts the fact that c is a loop on Σ. ∎

3. How to get a presentation. Let M, p and Δ be as before. By a *diagram* of M we mean the image $p(M)$ equipped with over-under information at each transverse double point. The knot group $\pi_1(\mathbb{R}^{n+2}-M)$ has the following "Wirtinger type" presentation: Let $\Sigma_1,\ldots,\Sigma_s$ be the regular sheets of the diagram of M, where s is the number of regular sheets. By Lemma 1, they are co-orientable in $\mathbb{R}^{n+1}$. Fix a co-orientation of each regular sheet. Generators of the group presentation are $x_1,\ldots,x_s$, which are represented by meridian loops of the regular sheets with a base point $*$ in $\mathbb{R}^{n+2}$ (explained later). Relators of the group presentation correspond to the double point strata and are of the form $x_i x_j^{\varepsilon_1} x_k^{\varepsilon_2} x_l^{\varepsilon_3}$ and $x_i x_k^{\varepsilon_2}$ where $\varepsilon_1,\varepsilon_2,\varepsilon_3\in\{1,-1\}$. Precisely speaking, for each double point stratum, consider a small loop in $\mathbb{R}^{n+1}$ intersecting the four regular sheets around the stratum, say $\Sigma_i,\Sigma_j,\Sigma_k,\Sigma_l$. By changing the starting point and the orientation of the loop if necessary, we may assume that $\Sigma_i,\Sigma_j,\Sigma_k,\Sigma_l$ appear in this order along the loop, Σ_i and Σ_k are above Σ_j and Σ_l, and the loop intersects Σ_i in the direction of the co-orientation of Σ_i. Then we have two relators $x_i x_k^{\varepsilon_2}$ and $x_i x_j^{\varepsilon_1} x_k^{\varepsilon_2} x_l^{\varepsilon_3}$ (or equivalently $x_i x_k^{\varepsilon_2}$ and $x_i x_j^{\varepsilon_1} x_i^{-1} x_l^{\varepsilon_3}$), where $\varepsilon_1,\varepsilon_2,\varepsilon_3$ are $+1$ (or -1, resp.) if the loop intersects $\Sigma_j,\Sigma_k,\Sigma_l$ in the direction of (or in the opposite direction to, resp.) their respective co-orientations.

THEOREM 2. *The group presentation described above is a presentation of the knot group $\pi_1(\mathbb{R}^{n+2}-M)$. In particular, any PL* (*or smooth*) *manifold knot group has a Wirtinger type presentation.*

REMARK. Suppose that there is a relator $x_i x_j^{\varepsilon_1} x_i^{-1} x_l^{\varepsilon_3}$ with $\varepsilon_3=\varepsilon_1$ in the above presentation. If one prefers that $\varepsilon_3=-\varepsilon_1$, then one has to introduce a new generator, say x_l', and a relator $x_l' x_l$ to the presentation, so that the relator $x_i x_j^{\varepsilon_1} x_i^{-1} x_l^{\varepsilon_3}$ may be replaced with $x_i x_j^{\varepsilon_1} x_i^{-1} x_l'^{-\varepsilon_1}$.

For a given Wirtinger type presentation, it is easy to construct a PL locally flat (or smooth) surface in $\mathbb{R}^4$ whose knot group has that presentation (see 14.2.1 of [11]). Thus we have the following.

COROLLARY 3. *For any PL* (*or smooth*) *manifold knot M in $\mathbb{R}^{n+2}$, there exists a PL locally flat* (*or smooth*) *surface F in $\mathbb{R}^4$ such that $\pi_1(\mathbb{R}^{n+2}-M)\cong\pi_1(\mathbb{R}^4-F)$.*

4. Proof of Theorem 2. First we explain the elements $x_1,\ldots,x_s$ of $\pi_1(\mathbb{R}^{n+2}-M)$. Let $\Sigma=\mathrm{cl}(p(M)-N(\Delta))$, which is the union of regular sheets. We denote by M_+ the part of M homeomorphic to Σ via p, and by M_- the closure of the complementary part in M:

$$M_+ = M \cap p^{-1}(\Sigma) = M \cap p^{-1}(\mathrm{cl}(\mathbb{R}^{n+1} - N(\Delta))),$$
$$M_- = M \cap p^{-1}(N(\Delta)).$$

As is usual with knot diagrams, we assume that M_+ is contained in $\mathbb{R}^{n+1} \times [0,1) \subset \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$ and M_- is contained in $\mathbb{R}^{n+1} \times (-1,0]$. More precisely, we assume that

$$M_+ = (\partial\Sigma) \times [0,1/2] \cup \Sigma \times \{1/2\},$$
$$M_- \cap \mathbb{R}^{n+1} \times \{0\} = (\partial\Sigma) \times \{0\}.$$

Take a base point $*$ in $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}$ whose last coordinate is sufficiently large. For each $i \in \{1,\dots,s\}$, consider a small arrow a_i in $\mathbb{R}^{n+1}$ intersecting the interior of Σ_i transversely in the direction of the co-orientation of Σ_i. Put a copy of a_i in $\mathbb{R}^{n+1} \times \{0\}$, which we denote by the same symbol a_i. Let x_i be an element of $\pi_1(\mathbb{R}^{n+1} \times [0,\infty) - M_+, *)$ represented by a loop $b_i^0 \cdot a_i \cdot b_i^1$ in $\mathbb{R}^{n+1} \times [0,\infty) - M_+$ with base point $*$, where b_i^0 is a straight path from $*$ to the initial point of a_i and b_i^1 is a straight path from the terminal point of a_i to $*$.

Lemma 4. (1) *The element $x_i \in \pi_1(\mathbb{R}^{n+1} \times [0,\infty) - M_+, *)$ does not depend on the arrow a_i.*

(2) *The group $\pi_1(\mathbb{R}^{n+1} \times [0,\infty) - M_+, *)$ is a free group generated by $x_1,\dots,x_s$.*

Proof. (1) Let a_i' be another small arrow in $\mathbb{R}^{n+1}$ intersecting the interior of Σ_i in the direction of the co-orientation of Σ_i, and let x_i' be an element of $\pi_1(\mathbb{R}^{n+1} \times [0,\infty) - M_+, *)$ obtained by use of a_i'. Since Σ_i is connected, we can slide the arrow a_i' onto a_i along Σ_i; then x_i' is a conjugate of x_i by an element represented by a loop in $\mathbb{R}^{n+1} \times [0,\infty) - M_+$ which is disjoint from $\Sigma \times [0,1]$. Since the loop is null-homotopic in $\mathbb{R}^{n+1} \times [0,\infty) - M_+$, we have $x_i = x_i'$.

(2) From the above argument we see that if $s = 1$, then $\pi_1(\mathbb{R}^{n+1} \times [0,\infty) - M_+, *)$ is a free group generated by x_1. We prove the assertion by induction on s. If $s > 1$, consider a regular neighborhood $N(\Sigma_1)$ of Σ_1 in $\mathbb{R}^{n+1}$ and consider a cone V_1 in $\mathbb{R}^{n+1} \times [0,\infty)$ over the copy $(\partial N(\Sigma_1)) \times \{0\}$ of $\partial N(\Sigma)$ with the base point $*$ as the cone vertex. The cone V_1 divides $\mathbb{R}^{n+1} \times [0,\infty) - M_+$ into two pieces, say H_1 and H_2, such that $H_1 \cap H_2 = V_1$ and H_1 contains the component $M_1^{(1)}$ of M_+ corresponding to Σ_1. It is not difficult to see that H_1 is a deformation retract of $\mathbb{R}^{n+1} \times [0,\infty) - M_+^{(1)}$ and H_2 is a deformation retract of $\mathbb{R}^{n+1} \times [0,\infty) - (M_+ - M_+^{(1)})$. In particular, we have $\pi_1(H_1, *) \cong \pi_1(\mathbb{R}^{n+1} \times [0,\infty) - M_+^{(1)}, *)$

and $\pi_1(H_2, *) \cong \pi_1(\mathbb{R}^{n+1} \times [0, \infty) - (M_+ - M_+^{(1)}), *)$. By induction hypothesis, the former is a free group generated by x_1 and the latter is a free group generated by $x_2, \dots, x_s$. Applying the van Kampen theorem along V_1, we see that $\pi_1(\mathbb{R}^{n+1} \times [0, \infty) - M_+, *)$ is a free group generated by $x_1, \dots, x_s$. ∎

The images of $x_1, \dots, x_s$ under the inclusion-induced homomorphism $\pi_1(\mathbb{R}^{n+1} \times [0, \infty) - M_+, *) \to \pi_1(\mathbb{R}^{n+2} - M, *)$ are denoted by the same symbols. They are the generators of the group presentation given in Theorem 2.

Proof of Theorem 2. Put $\Delta = \Delta^1 \cup \Delta^2$, where Δ^1 is the double point strata (the $(n-1)$-dimensional strata) and Δ^2 is the lower dimensional strata. We divide the regular neighborhood $N(\Delta)$ as follows: Let $N(\Delta^2)$ be a regular neighborhood of Δ^2 in $\mathbb{R}^{n+1}$, and put $W_2 = \mathrm{cl}(\mathbb{R}^{n+1} - N(\Delta^2))$. Let $N(\Delta^1)$ be a regular neighborhood of $\Delta^1 \cap W_2$ in W_2, and put $W_1 = \mathrm{cl}(\mathbb{R}^{n+1} - N(\Delta^2) - N(\Delta^1))$. We assume that $N(\Delta)$ is the union of $N(\Delta^2)$ and $N(\Delta^1)$.

Let A_1 and A_2 be arcs in a cylinder $D^2 \times [-1, 0]$ as in Figure 2. Notice that $N(\Delta^1)$ is a trivial D^2-bundle over $\Delta^1 \cap W_2$, since each component of Δ^1 (a double point stratum) has a trivialization determined from the four regular sheets around it. We identify $N(\Delta^1)$ with $(\Delta^1 \cap W_2) \times D^2$ and $N(\Delta^1) \times [-1, 0]$ with $(\Delta^1 \cap W_2) \times (D^2 \times [-1, 0])$. We may assume that M_- restricted to $N(\Delta^1) \times [-1, 0]$ is $(\Delta^1 \cap W_2) \times (A_1 \cup A_2) \subset (\Delta^1 \cap W_2) \times (D^2 \times [-1, 0])$. By a routine argument in knot theory using the van Kampen theorem inductively, we see that $\pi_1((\mathbb{R}^{n+1} \times [0, \infty) - M_+) \cup (N(\Delta^1) \times [-1, 0] - M_-), *)$ has a group presentation as in Theorem 2.

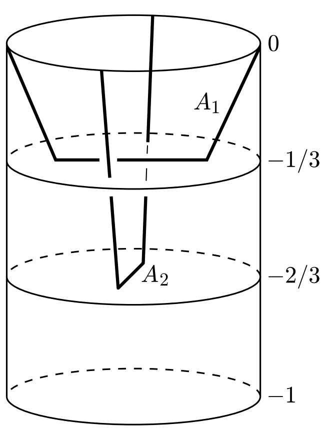

Fig. 2

Since $\varDelta^2$ has codimension more than two in $\mathbb{R}^{n+1}$, we have

$$\begin{aligned}
&\pi_1(\mathbb{R}^{n+2} - M)\\
&\cong \pi_1((\mathbb{R}^{n+1} \times [0,\infty) - M_+) \cup (\mathbb{R}^{n+1} \times (-\infty,0] - M_-) - \varDelta^2 \times (-\infty,0], *)\\
&\cong \pi_1((\mathbb{R}^{n+1} \times [0,\infty) - M_+) \cup (\mathbb{R}^{n+1} \times [-1,0] - M_-) - N(\varDelta^2) \times [-1,0], *)\\
&\cong \pi_1((\mathbb{R}^{n+1} \times [0,\infty) - M_+) \cup (N(\varDelta^1) \times [-1,0] - M_-), *).
\end{aligned}$$

Therefore the knot group $\pi_1(\mathbb{R}^{n+2} - M)$ has the required presentation. ■

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