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Ample hierarchy

by

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Abstract. The ample hierarchy of geometries of stables theories is strict. We generalise the construction of the free pseudospace to higher dimensions and show that the n-dimensional free pseudospace is ω-stable n-ample yet not $(n+1)$-ample. In particular, the free pseudospace is not 3-ample. A thorough study of forking is conducted and an explicit description of canonical bases is given.

1. Introduction. Morley's renowned categoricity theorem [9] described any model of an uncountably categorical theory in terms of basic foundational bricks, so-called strongly minimal sets. A long-standing conjecture aimed to understand the geometry of a strongly minimal set in terms of three archetypal examples: a trivial set, a vector space over a division ring and an irreducible curve over an algebraically closed field. The conjecture was proven wrong [7] by obtaining in a clever fashion a non-trivial strongly minimal set which does not interpret a group. In particular, Hrushovski's new strongly minimal set does not interpret infinite fields, which follows from the fact that the resulting structure is CM-trivial. Recall that CM-triviality is a generalisation of 1-basedness and it prohibits a certain point-line-plane configuration which is present in Euclidean geometry. The simplest example of a CM-trivial theory that is not 1-based is the *free pseudoplane:* an infinite forest with infinite branching at every node. CM-trivial theories are rather rigid and in particular definable groups of finite Morley rank are nilpotent-by-finite [10].

Taking the pseudoplane as a guideline, a non-CM-trivial ω-stable theory which does not interpret an infinite field was constructed in a purely combinatorial way [2]. The structure so obtained is of infinite rank, and it remains still open whether the construction could be modified to produce one of finite Morley rank. In [11, 4] a whole hierarchy of new geometries

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(called n-ample) was exhibited, infinite fields being at the top of the classification. Evans suggested that his example could be used to show that the hierarchy is strict, though no proof was given.

The goal of this article is to generalise the aforementioned construction to higher dimensions in order to show that the N-dimensional pseudospace is N-ample yet not $(N+1)$-ample, thus showing that the ample hierarchy is proper. After a thorough study of the pseudospace, we are able to simplify the combinatorics behind the original construction. In particular, we characterise non-forking and give explicit descriptions of canonical bases of finitary types over certain substructures. Moreover, we show that the theory of the pseudospace has weak elimination of imaginaries.

Tent [12] obtained the same result earlier independently; however, we present a different construction and axiomatisation of the free pseudospace for higher dimensions. We are indebted to Tent for pointing out that the prime model of the 2-dimensional free pseudospace could be seen as a building. We would like to express our gratitude to Yoneda for a careful reading of a first version of this work. We thank the referee for helpful remarks.

2. Ample concepts. Throughout this article, we assume a certain knowledge of stability theory, in particular non-forking and canonical bases. We refer the reader to [13] for a gentle and careful explanation of these notions. Throughout this article, we work inside a sufficiently saturated model of a first-order theory T and all sets are small subsets of it.

We first state a fact, which we believe is common knowledge, that will be used repeatedly.

FACT 2.1. *Given a stable theory T and sets A, B, C and D, if* $\mathrm{acl}^{\mathrm{eq}}(B)\cap \mathrm{acl}^{\mathrm{eq}}(C) = \mathrm{acl}^{\mathrm{eq}}(A)$ *and* $D \mathop{\smile\!\!\!\!\!\!\;|}_A BC$, *then*

$$\mathrm{acl}^{\mathrm{eq}}(DB)\cap \mathrm{acl}^{\mathrm{eq}}(DC) = \mathrm{acl}^{\mathrm{eq}}(DA).$$

Proof. Pick e in $\mathrm{acl}^{\mathrm{eq}}(DB)\cap \mathrm{acl}^{\mathrm{eq}}(DC)$. The independence $D \mathop{\smile\!\!\!\!\!\!\;|}_A BC$ implies that $\mathrm{Cb}(De/BC)$ lies in $\mathrm{acl}^{\mathrm{eq}}(B)\cap \mathrm{acl}^{\mathrm{eq}}(C) = \mathrm{acl}^{\mathrm{eq}}(A)$, so e lies in $\mathrm{acl}^{\mathrm{eq}}(DA)$. ∎

Recall now the definition of CM-triviality and n-ampleness [11, 4].

DEFINITION 2.2. Let T be a stable theory. The theory T is 1-*based* if for every pair of algebraically closed (in T^{eq}) subsets $A \subset B$ and every real tuple c, the canonical base $\mathrm{Cb}(c/A)$ is algebraic over $\mathrm{Cb}(c/B)$. Equivalently, for every algebraically closed set A (in T^{eq}) and every real tuple c, the canonical base $\mathrm{Cb}(c/A)$ is algebraic over c.

The theory T is *CM-trivial* if for every pair of algebraically closed (in T^{eq}) subsets $A \subset B$ and every real tuple c, if $\mathrm{acl}^{\mathrm{eq}}(Ac)\cap B = A$, then $\mathrm{Cb}(c/A)$ is algebraic over $\mathrm{Cb}(c/B)$.

The theory T is called *n-ample* if there are $n+1$ real tuples satisfying the following conditions (possibly working over parameters):

(1) $\mathrm{acl}^{\mathrm{eq}}(a_0,\dots,a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_0,\dots,a_{i-1},a_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(a_0,\dots,a_{i-1})$ for every $0 \le i < n$,
(2) $a_{i+1} \mathop{\smile\!\!\!\!\!\!\;|}_{a_i} a_0,\dots,a_{i-1}$ for every $1 \le i < n$,
(3) $a_n \not\mathop{\smile\!\!\!\!\!\!\;|} a_0$.

By inductively choosing models $M_i \supset a_i$ such that

$$M_i \mathop{\smile\!\!\!\!\!\!\;|}_{a_i} M_0,\dots,M_{i-1},a_{i+1},\dots,a_n,$$

Fact 2.1 allows us to deduce the following, which was already remarked in [10, Corollary 2.5] in the case of CM-triviality.

REMARK 2.3. In the definition of n-ampleness, we can replace all tuples by models.

COROLLARY 2.4. *A stable theory T is n-ample if and only if T^{eq} is.*

Clearly, every 1-based theory is CM-trivial. Furthermore, a theory is 1-based if and only if it is not 1-ample; it is CM-trivial if and only if it is not 2-ample [11]. Also, being n-ample implies $(n-1)$-ampleness: by construction, if $a_0,\dots,a_n$ witness that T is n-ample, the sequence $a_0,\dots,a_{n-1}$ witnesses that T is $(n-1)$-ample. In order to see this, we need only show that $a_{n-1} \not\mathop{\smile\!\!\!\!\!\!\;|} a_0$, which follows from

$$a_n \not\mathop{\smile\!\!\!\!\!\!\;|} a_0 \quad \text{and} \quad a_n \mathop{\smile\!\!\!\!\!\!\;|}_{a_{n-1}} a_0,$$

by transitivity.

In order to prove that the N-dimensional free pseudospace is not $(N+1)$-ample, we need only consider some of the consequences of the conditions listed above. Therefore, we will isolate such conditions for Section 8.

REMARK 2.5. If the (possibly infinite) tuples $a_0,\dots,a_n$ witness that T is n-ample, they satisfy the following conditions:

(a) $a_n \mathop{\smile\!\!\!\!\!\!\;|}_{a_i} a_{i-1}$ for every $1 \le i < n$.
(b) $\mathrm{acl}^{\mathrm{eq}}(a_i,a_{i+1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_i,a_n) = \mathrm{acl}^{\mathrm{eq}}(a_i)$ for every $0 \le i < n-1$.
(c) $a_n \not\mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{acl}^{\mathrm{eq}}(a_i)\cap\mathrm{acl}^{\mathrm{eq}}(a_{i+1})} a_i$ for every $0 \le i < n-1$.

If the tuples $a_0,\dots,a_n$ witness that T is n-ample over some set A of parameters, by adding all elements of A to each of the tuples, we may assume that all the conditions hold with $A = \emptyset$.

Proof. Let $a_0,\dots,a_n$ witness that T is n-ample.

First, the intersection $\mathrm{acl}^{\mathrm{eq}}(a_1) \cap \mathrm{acl}^{\mathrm{eq}}(a_2)$ is contained in $\mathrm{acl}^{\mathrm{eq}}(a_0)$ by property (1). For $i \le 2$, we have

$$\mathrm{acl}^{\mathrm{eq}}(a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_{i+1}) \subset \mathrm{acl}^{\mathrm{eq}}(a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_0,\dots,a_{i-1})$$

again by (1). Now, (2) implies that $\mathrm{acl}^{\mathrm{eq}}(a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_{i-1})$ is contained in $\mathrm{acl}^{\mathrm{eq}}(a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_{i-1})$. By induction, we have

$$\mathrm{acl}^{\mathrm{eq}}(a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_{i+1}) \subset \mathrm{acl}^{\mathrm{eq}}(a_0).$$

The independence $a_n \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_{i-1}$ follows directly from property (2) and yields (a). Since $a_n \mathop{\smile\!\!\!\!\!\!|}_{a_{i+2}} a_0, \ldots, a_{i+1}$, we have

$$a_n \mathop{\smile\!\!\!\!\!\!|}_{a_i, a_{i+2}} a_{i+1}.$$

Hence,

$$\mathrm{acl}^{\mathrm{eq}}(a_i, a_{i+1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_i, a_n) \subset \mathrm{acl}^{\mathrm{eq}}(a_i, a_{i+1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_i, a_{i+2}),$$

and thus it is contained in $\mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_i)$ by (1). Since $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0, \ldots, a_{i-1}$, we get (b).

If

$$a_n \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}^{\mathrm{eq}}(a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_{i+1})} a_i$$

for some $0 \leq i < n-1$, then $i > 0$ by (3). Since $a_n \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0, \ldots, a_{i-1}$, transitivity gives

$$a_n \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}^{\mathrm{eq}}(a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_{i+1})} a_0, \ldots, a_i.$$

Thus, we obtain $a_n \mathop{\smile\!\!\!\!\!\!|}_{a_0} a_0, \ldots, a_i$ and in particular $a_n \mathop{\smile\!\!\!\!\!\!|}_{a_0} a_1$. Since $a_n \mathop{\smile\!\!\!\!\!\!|}_{a_1} a_0$ by (2) and $\mathrm{acl}^{\mathrm{eq}}(a_0) \cap \mathrm{acl}^{\mathrm{eq}}(a_1) = \emptyset$ by (1), this implies that $a_n \mathop{\smile\!\!\!\!\!\!|} a_0$, which contradicts (3). ■

In [3], a weakening of CM-triviality was introduced, following the spirit of [8], where some of the consequences for definable groups in 1-based theories were extended to type-definable groups in theories with the *Canonical Base Property*. For the purpose of this article, we extend the definition to all values of n. However, we do not know of any definability properties for groups that may follow from the general definition.

Let Σ be an $\emptyset$-invariant family of partial types. Recall that a type p over A is *internal to Σ*, or *Σ-internal*, if for every realisation a of p there is some superset $B \supset A$ with $a \mathop{\smile\!\!\!\!\!\!|}_A B$, and there are realisations $b_1, \ldots, b_r$ of types in Σ based on B such that a is definable over $B, b_1, \ldots, b_r$. If we replace definable by algebraic, then p is *almost internal to Σ* or *almost Σ-internal.*

Definition 2.6. A stable theory T is called *n-tight* (possibly working over parameters) with respect to the family Σ if, whenever there are $n+1$ real tuples $a_0, \ldots, a_n$ satisfying:

(1) $\mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_{i-1}, a_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_{i-1})$ for every $0 \leq i < n$,

(2) $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0, \ldots, a_{i-1}$ for every $1 \leq i < n$,

then $\mathrm{Cb}(a_n/a_0)$ is almost Σ-internal over a_1.

Remark 2.7. As before, we may assume that all tuples are models. In particular, the theory T is n-tight if and only if T^{eq} is.

A theory T is 2-tight with respect to Σ if for any sets $A \subset B$ and every tuple c, if $\mathrm{acl}^{\mathrm{eq}}(Ac) \cap \mathrm{acl}^{\mathrm{eq}}(B) = \mathrm{acl}^{\mathrm{eq}}(A)$, then $\mathrm{Cb}(c/A)$ is almost Σ-internal over $\mathrm{Cb}(c/B)$. In particular, this notion agrees with [3, Definition 3.1].

If T is not n-ample, it is n-tight with respect to any family Σ. Furthermore, if T is $(n-1)$-tight, it is n-tight.

Proof. The equivalence between both definitions is a standard reformulation by setting $a_0 = A$, $a_1 = \mathrm{Cb}(c/B)$ and $a_2 = c$ for one direction (working over $\mathrm{acl}^{\mathrm{eq}}(a_0) \cap \mathrm{acl}^{\mathrm{eq}}(a_1)$), and $A = a_0$, $B = a_0 \cup \mathrm{Cb}(a_2/a_1)$ and $c = a_2$ for the other.

If T is not n-ample, it is clearly n-tight, since algebraic types are always almost Σ-internal for any Σ.

Suppose now that T is $(n-1)$-tight, and consider $n+1$ tuples $a_0, \ldots, a_n$ witnessing (1) and (2). So do $a_0, \ldots, a_{n-1}$ as well. Hence, the canonical base $\mathrm{Cb}(a_{n-1}/a_0)$ is almost Σ-internal over a_1.

Since $a_n \mathop{\smile\!\!\!\!\!\!|}_{a_{n-1}} a_0$, by transitivity $\mathrm{Cb}(a_n/a_0)$ is algebraic over $\mathrm{Cb}(a_{n-1}/a_0)$ and therefore the former is also almost Σ-internal over a_1. ∎

In this article, we will show that the free N-dimensional pseudospace is N-ample yet not $(N+1)$-ample. Furthermore, if $N \geq 2$, it is N-tight with respect to the family of Lascar rank 1 types.

3. Fraïssé limits. The results in this section were obtained by the third author in an unpublished note [15] (in a slightly more general context). We include them here for completeness.

Throughout this section, let $\mathcal{K}$ denote a class of structures closed under isomorphisms in a fixed language L. We assume that the empty structure 0 is in $\mathcal{K}$. Furthermore, a class $\mathcal{S}$ of embeddings between elements of $\mathcal{K}$ is given, called *strong embeddings*, containing all isomorphisms and closed under composition. We also assume that the empty map $0 \to A$ is in $\mathcal{S}$ for every $A \in K$.

We call a substructure A of B *strong* if the inclusion map is in $\mathcal{S}$. We denote this by $A \leq B$.

Definition 3.1. An increasing chain $\{A_i\}_{i<\omega}$ of strong substructures is *rich* if, for all $i < \omega$ and all strong $f : A_i \to B$, there is some $i \leq j < \omega$ and a strong $g : B \to A_j$ such that $g \circ f : A_i \to A_j$ is the inclusion map.

A *Fraïssé limit* of $(\mathcal{K}, \mathcal{S})$ is the union of a rich sequence.

Theorem 3.2. *Suppose* $(\mathcal{K}, \mathcal{S})$ *satisfies the following conditions:*

(1) *There are at most countably many isomorphism types in* $\mathcal{K}$.

(2) *For each A and B in $\mathcal{K}$, there are at most countably many strong embeddings $A \to B$.*
(3) *$\mathcal{K}$ has the amalgamation property with respect to strong embeddings.*

Then rich sequences exist and all Fraïssé limits are isomorphic.

The existence of rich sequences is easy to show. Uniqueness will follow from the next lemma. For that, let us say that A is *r-strong in a Fraïssé limit M*, denoted by $A \leq_{\mathrm{r}} M$, if M is the union of a rich sequence starting with A.

Lemma 3.3. *A Fraïssé limit M has the following properties:*

(a) *$\emptyset \leq_{\mathrm{r}} M$.*
(b) *For every finite $A \leq_{\mathrm{r}} M$ and every B in $\mathcal{K}$ such that $A \leq B$, there is an r-strong subset B' of M containing A and isomorphic to B over A.*

Proof. We observe first that if $A_0 \leq A_1 \leq \cdots$ is a rich sequence and $B \leq A_0$, then $B \leq A_0 \leq A_1 \leq \cdots$ is also rich. This implies (a). For (b), choose a rich sequence $A = A_0 \leq A_1 \leq \cdots$ with union M. If $B \geq A$ is given, there exists, by richness, some index j and $B' \leq A_j$ isomorphic to B over A. The set B' is r-strong in M, since the subsequence $B' \leq A_j \leq A_{j+1} \leq \cdots$ is again rich. ■

The lemma implies that Fraïssé limits are isomorphic by a standard back-and-forth argument: given two Fraïssé limits M and M' with rich sequences $A_0 \leq A_1 \leq \cdots$ and $A'_0 \leq A'_1 \leq \cdots$, consider an isomorphism $B \to B'$, where B is strong in A_i, and B' is strong in A'_i. Then there is an extension to an isomorphism $C \to C'$ such that $A_i \leq C \leq A_j$ and $A'_i \leq C' \leq A'_j$ for some $j > i$. This results in an ascending sequence of isomorphisms whose union yields an isomorphism $M \to M'$.

Corollary 3.4. *Assume that M and M' are Fraïssé limits. Given sets $B \leq_{\mathrm{r}} M$ and $B' \leq_{\mathrm{r}} M'$, every isomorphism $B \to B'$ extends to an isomorphism $M \to M'$.*

The convention that $\mathcal{S}$ contains all isomorphisms and is closed under composition represents no obstacle, thanks to the following easy remark.

Remark 3.5. Let $\mathcal{S}$ be a set of embeddings between elements of $\mathcal{K}$ with the amalgamation property. The closure of $\mathcal{S}$ together with all isomorphisms under composition has again the amalgamation property.

4. The free pseudospace. In this section, we will construct and axiomatise the N-dimensional free pseudospace, which is a generalisation of [2], based on the free pseudoplane. An alternative axiomatisation, in terms of *flags*, may be found in [1].

REMARK 4.1. Recall that the (free) pseudoplane is a bicolored graph with infinite branching and no loops. These elementary properties describe a complete ω-stable theory of Morley rank ω.

Quantifier elimination is obtained after adding the collection of binary predicates:

$$d_n(x,y) \Leftrightarrow \text{ the distance between } x \text{ and } y \text{ is exactly } n.$$

In particular, since there are no loops, the set $d_1(x,a)$ is strongly minimal. Morley rank for this theory is additive and agrees with Lascar rank. Given the type of an element c over an algebraically closed set A, its canonical base $\mathrm{Cb}(c/A)$ is the unique point a in A whose distance to c is smallest possible (or empty if there is no path between c and A). It follows that the theory has weak elimination of imaginaries and is moreover CM-trivial but not 1-based.

The idea behind the construction of the free pseudospace [2] is to take a free pseudoplane, whose vertices of one color are called *planes* and vertices of the other are referred to as *lines*, and on each line put an infinite set of *points*, such that, for each plane, the lines which are incident with it, together with the points on them form again a free pseudoplane. Nevertheless, the actual construction was rather combinatorial and therefore less intuitive. Instead, our approach consists in building a model out of some basic operations and study the complete theory of such a structure, in order to show that it agrees with the free pseudospace in [2] for dimension $N = 2$.

DEFINITION 4.2. For $N \geq 1$, a *colored N-space* A is a colored graph with colors (or *levels*) $\mathcal{A}_0, \ldots, \mathcal{A}_N$ such that an element in $\mathcal{A}_i$ can only be linked to vertices in $\mathcal{A}_{i-1} \cup \mathcal{A}_{i+1}$. We will furthermore consider two (invisible) levels $\mathcal{A}_{-1}$ and $\mathcal{A}_{N+1}$, consisting of a single *imaginary* element a_{-1} and a_{N+1} respectively, which are connected to all vertices in $\mathcal{A}_0$ and $\mathcal{A}_N$ respectively. Given such a graph A and a subset s of $\{0, \ldots, N\}$, we set

$$\mathcal{A}_s(A) = \bigcup_{i \in s} \mathcal{A}_i(A).$$

Given x and y in $\mathcal{A}_s(A)$, their distance in $\mathcal{A}_s(A)$ is denoted by $\mathrm{d}_s^A(x,y)$.

Given a colored N-space A and vertices a in $\mathcal{A}_l(A)$ and b in $\mathcal{A}_r(A)$, we say that b *lies over* a (or a *lies beneath* b) if $l < r$ and there is a path of the form $a = a_l, a_{l+1}, \ldots, a_r = b$. Note that a_k must be in $\mathcal{A}_k(A)$. By convention, the point a_{N+1} lies over all other vertices (including a_{-1}) and a_{-1} lies beneath all other vertices.

With A, a and b as above, we denote by A_a the subgraph of A consisting of all the elements of A lying over a. Similarly A^b denotes the subgraph of all the elements lying beneath b. The subgraph $A_a^b = (A_a)^b$ consists of all the elements of A lying between a and b, if a lies beneath b.

Observe that, after a suitable renumbering of levels, the subgraph A_a becomes a colored $(N-l-1)$-space, whereas A^b becomes a colored $(r-1)$-space and A_a^b a colored $(r-l-2)$-space.

Convention. Intervals are assumed to be non-empty.

DEFINITION 4.3. Given an interval $s=(l_s,r_s)$ (where -1 and $N+1$ are possible values) in $\{0,\dots,N\}$ and a colored N-space A with two distinguished vertices a_{l_s} in $\mathcal{A}_{l_s}(A)$ beneath a_{r_s} in $\mathcal{A}_{r_s}(A)$, we say that $B=A\cup\{b_i \mid i\in s\}$ with $b_i\in\mathcal{A}_i(B)$ is obtained from A by applying *the operation* α_s on (a_{l_s},a_{r_s}) if

(a) The sequence $a_{l_s},b_{l_s+1},\dots,b_{r_s-1},a_{r_s}$ is a path in B.
(b) B has no new edges besides the aforementioned (and those of A).

If either $l_s=-1$ or $r_s=N+1$, then a_{l_s} lies automatically beneath a_{r_s}.

The N-dimensional pseudospace will now be obtained by iterating countably many times all operations α_s for s varying over all intervals in $[0,N]$. Clearly, we have the following.

REMARK 4.4. If both B_1 and B_2 are obtained from A by applying respectively α_{s_1} and α_{s_2}, then the graph-theoretic amalgam $C=B_1\otimes_A B_2$ is obtained by applying α_{s_1} to B_2 and α_{s_2} to B_1.

DEFINITION 4.5. Given two colored N-spaces A and B, we say that A a *strong subspace* of B if A is a subgraph of B and B can be obtained from A by a (possibly infinite) sequence of operations α_s for varying s. We denote this by $A\le B$.

A *strong embedding* $A\to B$ is an isomorphism of A with a strong subspace of B. Let $\mathcal{K}_\infty$ be the class of all finite colored N-spaces A with $\emptyset\le A$. By the last remark and Remark 3.5, the class $\mathcal{K}_\infty$ has the amalgamation property with respect to strong embeddings. Clearly, there are only countably many isomorphism types in $\mathcal{K}_\infty$ and only finitely many maps between two structures of $\mathcal{K}_\infty$. We may consider the subclass $\mathcal{K}_0$, where by a 0-strong embedding we only allow operations α_s for singleton s. Again, the class $\mathcal{K}_0$ has the amalgamation property.

By Theorem 3.2, we define the following structures:

DEFINITION 4.6. Let M_∞^N be the Fraïssé limit of $\mathcal{K}_\infty$ with strong embeddings and M_0^N be the Fraïssé limit of $\mathcal{K}_0$ with 0-strong embeddings, starting from a given (fixed) path $a_0-\cdots-a_N$, where $a_i\in\mathcal{A}_i$.

We will drop the superscript N in M_∞^N or M_0^N when it is clear from the context.

In particular, the structure M_0^2 so obtained agrees with the prime model constructed in [2], as Theorem 4.14 will show.

REMARK 4.7. Let p be either 0 or ∞. Consider a in $\mathcal{A}_l(\mathrm{M}_p^N)$ and b be in $\mathcal{A}_r(\mathrm{M}_p^N)$ lying over a. Then

$$(\mathrm{M}_p^N)_a \cong \mathrm{M}_p^{N-l-1}, \quad (\mathrm{M}_p^N)^b \cong \mathrm{M}_p^{r-1}, \quad (\mathrm{M}_p^N)_a^b \cong \mathrm{M}_p^{r-l-2}.$$

Furthermore, given $-1 \leq l < r \leq N+1$, we have $\mathcal{A}_{[l,r]}(\mathrm{M}_p^N) \cong \mathrm{M}_p^{r-l-1}$.

Proof. Given a colored N-space M and corresponding vertices a and b, every operation in M_a can be extended to an operation on M. Moreover, if an operation on M has no meaning restricted to M_a, then M_a does not change. The other statements can be proved in a similar fashion. ∎

We will now introduce a notion, *simple connectedness*, which traditionally implies path-connectedness topologically. Despite this abuse of language, we will use this term since it implies that loops are not punctured (cf. Remark 4.9(2) and Corollary 6.16).

DEFINITION 4.8. A colored N-space M is *simply connected* if, whenever we are given $l < r$ in $[-1, N+1]$, an interval $t \subset [l, r]$, vertices a in $\mathcal{A}_l(M)$ beneath b in $\mathcal{A}_r(M)$ and x and y in $\mathcal{A}_t(M)$ lying between a and b which are t-connected by a path of length k not passing through a nor b, then there is a path in $\mathcal{A}_t(M)$ of length at most k connecting x and y such that every vertex in the path lies between a and b.

Note that simple connectedness is an empty condition for $l = -1$ and $r = N+1$.

REMARK 4.9. Let M be a simply connected connected colored N-space. Then

(1) The subgraph $\mathcal{A}_{[l,l+1]}(M)$ has no closed paths with no repetitions.
(2) In a closed path P in $\mathcal{A}_{[l,r]}(M)$, all elements in $P \cap \mathcal{A}_{[l,r)}$ are connected (in $\mathcal{A}_{[l,r)}(M)$), and likewise for the dual statement.

Proof. For (1), set $r = N+1$, $l = l$ and take $t = [l, l+1]$ in the definition of simple connectedness.

For (2), given x and y in $P \cap \mathcal{A}_{[l,r)}$, if they are connected using an arc of P in $\mathcal{A}_{[l,r)}(M)$, there is nothing to prove. Otherwise, replace successively every occurrence of a vertex z in $P \cap \mathcal{A}_r(M)$ by a subpath in $\mathcal{A}_{[l,r)}(M)$ connecting the immediate neighbors of z in P. ∎

As the following lemma shows, simple connectedness is preserved under application of the operations α_s.

LEMMA 4.10. *Let A be a simply connected colored N-space. If B is obtained from A by applying α_s on (a_{l_s}, a_{r_s}), then B is simply connected as well.*

Proof. By hypothesis, the set B equals $A \cup S_B$, where S_B is the path

$$a_{l_s}, b_{l_s+1}, \ldots, b_{r_s-1}, a_{r_s}.$$

Let now $t \subset [l, r]$ be given, as well as a in $\mathcal{A}_l$ beneath b in $\mathcal{A}_r$ and vertices x and y in $\mathcal{A}_t$ lying between a and b connected by a path P in $\mathcal{A}_t(B)$ of length k. We consider the following cases:

(a) Both a and b lie in $B \setminus A$. Take the direct path between x and y.

(b) Both a and b lie in A. We consider the following mutually exclusive subcases:

  (i) Both x and y lie in A. We can replace all repetitions in P to transform it into a path fully contained in A of length at most k. Since A is simply connected, the result follows.

  (ii) Both x and y lie in S_B. Again, take the direct path between x and y.

  (iii) Exactly one vertex, say y, lies in A. The path P must contain either a_{l_s} or a_{r_s}. Suppose that P contains a_{r_s}. Hence, we can decompose P into the direct connection (which lies between a and b) from x to a_{r_s} and a path P' in $\mathcal{A}_t(A)$ from a_{r_s} to y. As A is simply connected, we obtain a path in $\mathcal{A}_t(A)$ between a and b connecting y and a_{r_s} whose length is bounded by the length of P'. This yields a path from y to x between a and b of the appropriate length.

(c) Exactly one vertex in $\{a, b\}$ lies in A. Suppose that a lies in $A \setminus B$ and b lies in $S_B \setminus A$. In particular, the vertex a lies beneath a_{l_s}. Consider the following mutually exclusive cases:

  (i) Both x and y lie in S_B. The direct path between them in S_B again yields the result.

  (ii) Both x and y lie in A. If either x or y equals a_{l_s}, then one of them lies over the other and the direct connection between them yields the result. Otherwise, we may assume that both x and y lie beneath a_{l_s}. Let Q be the path consisting of the direct connection from x to a_{l_s} and from a_{l_s} to y. If the path P connecting x and y necessarily passes through a_{l_s}, then its length is at least the length of Q and the result follows. Otherwise, since A is simply connected, there is a path connecting x and y of length at most k between a and a_{l_s}, and thus between a and b.

  (iii) Exactly one, say y, is in A. Then y must lie beneath x and the direct path between them yields the result. ∎

Since the only moment a vertex from $\mathcal{A}_{l_t} \cup \mathcal{A}_{r_t}$ was added was in case (c)(ii), namely a_{l_s} (though only if the original path passed through it), a careful analysis of the previous proof yields the following, which corresponds to Axiom ($\Sigma 4$) in [2]; though we will not require its full strength.

Corollary 4.11. *A colored N-space B with $\emptyset \leq B$ has the following property. Given $t = [l_t, r_t] \subset [l, r]$, as well as a in $\mathcal{A}_l(B)$ beneath b in $\mathcal{A}_r(B)$, vertices x and y in $\mathcal{A}_t(B)$ lying between a and b and a path in $\mathcal{A}_t(B)$ of length k connecting them, there is a path P in $\mathcal{A}_t(B)$ between a and b connecting x and y of length at most k such that all vertices in P with levels $\mathcal{A}_{l_t} \cup \mathcal{A}_{r_t}$ come from the original path.*

By iterating Lemma 4.10, we obtain the following:

Corollary 4.12. *If A is simply connected, then so is every strong extension of A.*

The following observation can be easily shown.

Lemma 4.13. *Let B be obtained from A by applying the operation α_s. Then, for every $t \subset \{0, \ldots, N\}$ and every x and y in $\mathcal{A}_t(A)$,*

$$\mathrm{d}_t^A(x, y) = \mathrm{d}_t^B(x, y).$$

Theorem 4.14 (Axioms). *Both Fraïssé limits M_∞ and M_0 have the following elementary properties:*

(1) *Simple connectedness.*
(2) *Given a finite subset A and a non-empty interval $s = (l, r)$, for any two elements a_l and a_r in A with a_r over a_l, there are paths*

$$a_l, b_{l+1}, \ldots, b_{r-1}, a_r$$

such that the s-distance of b_i to $\mathcal{A}_s(A)$ is arbitrarily large. In particular, if $s = \{i\}$, there is a new vertex b_i not contained in A.

Proof. (1) This follows from Corollary 4.12.

(2) After enlarging A, we may assume that $A \leq \mathrm{M}_\infty$. One single application of α_s on (a_l, a_r) implies that the s-distance of b_i to A is infinite and remains so at the end of the construction by Lemma 4.13.

If we are considering M_0, we may assume as well that $A \leq \mathrm{M}_0$. Furthermore, we may suppose that in order to build up M_0 from A, each of the operations α_i, for i in s, was applied k times consecutively on each of the new vertices in $\mathcal{A}_{i+1}$ and $\mathcal{A}_{i-1}$ between a_l and a_r. Lemma 4.13 now yields the desired result. ∎

Definition 4.15. We will denote by PS_N the collection of sentences expressing properties (1) and (2) in Theorem 4.14.

Definition 4.16. A *flag* is a subgraph of a colored N-space M of the form

$$a_0 - \cdots - a_N,$$

where a_i belongs to $\mathcal{A}_i(M)$ and they form a path.

A subset D of a colored N-space M is *complete* if every point in D is contained in a flag in D.

Observe that, if D satisfies Axiom (2), it is complete.

Definition 4.17. A subset D of a colored N-space M is *nice* if it satisfies the following conditions:

(1) For any two (possibly imaginary) points a and b in D,
$$D_a^b = D \cap M_a^b.$$
(2) For all intervals $t \subset \{0, \ldots, N\}$ and all x and y in $\mathcal{A}_t(D)$,
$$\mathrm{d}_t^M(x, y) < \infty \ \Rightarrow \ \mathrm{d}_t^D(x, y) < \infty.$$

A set D is *wunderbar* in M if it satisfies the following:

(1) For any two (possibly imaginary) points a and b in D,
$$D_a^b = D \cap M_a^b.$$
(2) For all intervals $t \subset \{0, \ldots, N\}$ and all x and y in $\mathcal{A}_t(D)$,
$$\mathrm{d}_t^M(x, y) = \mathrm{d}_t^D(x, y).$$

Clearly, wunderbar sets are nice. Since an application of the operation α_s on A does not yield connections between points of A unless there was already one, the following result follows immediately from Lemma 4.13.

Lemma 4.18. *If $A \leq B$, then A is wunderbar in B.*

Lemma 4.19. *Let M be a simply connected colored N-space and D nice in M. Given an interval $s = [l, r]$ in $\{-1, \ldots, N+1\}$ and $a_l \in \mathcal{A}_l(D)$ beneath $a_r \in \mathcal{A}_r(D)$, the set $D_{a_l}^{a_r}$ is nice in $\mathcal{A}_s(M)$.*

Proof. Since $D_a^b = D \cap M_a^b$ for any a and b in D, the first condition of niceness holds for $D_{a_l}^{a_r}$.

For the second condition, we may assume that $a_l = -1$ by Remark 4.7. Let $t \subset (-1, r]$ be an interval, and x and y vertices in $\mathcal{A}_t(D)$ beneath a_r. We need only show that, if x and y are connected in $\mathcal{A}_t(D)$, then they are connected in $\mathcal{A}_t(D)$ beneath a_r. Let P be a path in $\mathcal{A}_t(D)$ connecting x and y, but not necessarily running beneath a_r. We call a vertex in P *avoidable* if it does not lie beneath a_r. Let $\mathcal{A}_n$ be the largest level containing an avoidable vertex in P. Let m be the number of avoidable vertices in P of level n. Choose P such that the pair (n, m) is minimal for the lexicographical order.

Given an avoidable vertex b in $\mathcal{A}_n \cap P$, denote by a_1' in $\mathcal{A}_{l_1}$ the first non-avoidable vertex in P between b and x. Likewise, let a_2' in $\mathcal{A}_{l_2}$ be the first non-avoidable vertex in P between b and y. Note that l_1 and l_2 are both smaller than n, by maximality of n. Furthermore, since every avoidable direct neighbor of a non-avoidable vertex necessarily lies in a higher level, by definition, it follows that both l_1 and l_2 are strictly smaller than n. Hence, the subpath P' of P between a'_1 and a'_2 yields a connection in $\mathcal{A}_{t'}$, where $t' = t \cap (-1, n]$, not passing through a_r. As M is simply connected, there is a path Q (with no repetitions) connecting a'_1 and a'_2 running beneath a_r. Now, the paths Q and P' have only a'_1 and a'_2 as common vertices and they induce a loop. Remark 4.9(2) implies that a'_1 and a'_2 are t_1-connected, where $t_1 = t \cap (-1, n)$. Since D is nice, there is also a t_1-connection R in D. Replacing P' by R, we have a path whose avoidable vertices are still contained in $(-1, n]$ and with fewer avoidable vertices of level n. Minimality of (n, m) shows that this path runs beneath a_r, as desired. ∎

COROLLARY 4.20. *Let D be nice in a colored N-space M. If M is simply connected, then so is D.*

LEMMA 4.21. *Let A be a nice subset of a simply connected colored N-space M. Consider a non-empty interval $s = (l, r)$ and two vertices a_{l_s} in $\mathcal{A}_{l_s}(A)$ and a_{r_s} in $\mathcal{A}_{r_s}(A)$ such that a_{r_s} lies over a_{l_s}. Let $B \subset M$ be an extension of A given by new vertices $b_{l_s+1}, \ldots, b_{r_s-1}$ such that the sequence*

$$a_l, b_{l+1}, \ldots, b_{r-1}, a_r$$

is a path. The following are equivalent:

(a) *The set B is nice and obtained from A by applying α_s on (a_{l_s}, a_{r_s}).*
(b) *For some (equivalently, all) i in s, we have $\mathrm{d}_s^M(b_i, A) = \infty$.*
(c) *For some (equivalently, all) i in s, we have $\mathrm{d}^{M_{a_l}^{a_r}}(b_i, A) = \infty$.*

Note that simple connectedness yields

$$\mathrm{d}^{M_{a_l}^{a_r}}(b_i, A) = \mathrm{d}_{(l,r)}^M(b_i, A_{a_l}^{a_r}).$$

We say that B is obtained from A by a *global application of* α_s if it satisfies (any of) the above conditions. In particular, the set B is nice.

Proof. (a)→(b): By the definition of α_s the distance $\mathrm{d}_s^B(b_i, A)$ is infinite for every i in s. Since B is nice in M, so is $\mathrm{d}_s^M(b_i, A) = \infty$.

(b)→(c): Obvious.

(c)→(b): If both a_l and a_r are imaginary, then there is nothing to prove. Thus, we may assume that a_r is real. Furthermore, suppose that there is a path P in $\mathcal{A}_s(M)$ connecting some b_i to some a in $\mathcal{A}_s(A)$. Take P of shortest possible length.

We need to show that

$$\mathrm{d}^{M_{a_l}^{a_r}}(b_i, A) < \infty.$$

Note that a and a_r are connected in $\mathcal{A}_{(l,r]}(M)$ and, since A is nice, there is a shortest path Q in $\mathcal{A}_{(l,r]}(A)$ witnessing this. In particular, let a_{r-1} be the direct neighbor of a_r in Q. Connecting Q and P, we find that a_{r-1} and b_i lie beneath a_r and are connected in $\mathcal{A}_{(l,r]}$ by a path disjoint from a_r. Simple connectedness yields a path Q_1 beneath a_r in $\mathcal{A}_{(l,r)}$ connecting them. If a_l is imaginary, we are done. Otherwise, the vertices a_{r-1} and a_l are connected through b_i. Again by simple connectedness, there is a path Q' connecting them below a_r in $[l,r)$. Let now a_{l+1} be the direct neighbor in Q' above a_l. Note that a_{l+1} and b_i lie between a_l and a_r. Simple connectedness of M implies that there is a path in $M_{a_l}^{a_r}$ between b_i and a_{l+1}. Hence

$$\mathrm{d}^{M_{a_l}^{a_r}}(b_i, A) < \infty.$$

(b)→(a): If both a_l and a_r are imaginary, then there are clearly no new connections between any b_i and A, and thus B is obtained by applying $\alpha_{[0,N]}$ to A. Hence, we may assume that a_r is real.

We first need to show that no b_i is in relation to an element in A besides a_r and a_l. This implies that B is obtained from A by an application of α_s. Assume first that b_{r-1} is connected to some other element a'_r in $\mathcal{A}_r(A)$. Since A is nice, there is a path in $\mathcal{A}_{\{r-1,r\}}(A)$ connecting a_r and a'_r. This, together with the extra connection to b_{r-1} yields a loop in $\mathcal{A}_{\{r-1,r\}}$, which contradicts Remark 4.9(2). Likewise for b_{l+1}. Finally, by assumption, no b_i in $\mathcal{A}_{(l+1,r-1)}$ is in relation with an element in $\mathcal{A}_s(A)$.

Now, in order to show that B is nice, consider x and y in B with finite t-distance in M. If both x and y lie in A, we are done, since A is nice. Likewise, if both x and y lie in the path $a_l, b_{l+1}, \ldots, b_{r-1}, a_r$, the direct connection works as well. Therefore, assume that x lies in A and y does not. By assumption, $t \nsubseteq s$. Suppose that l lies in t. Since y and a_l are t-connected (in M), so are x and a_l. As A is nice, there is a connection between x and a_l in $\mathcal{A}_t(A)$. In particular, there is a connection between x and y in $\mathcal{A}_t(B)$. ■

Theorem 4.22. *Let M be complete and simply connected. Given a nice subset A and b in M, there is a nice subset B of M containing b such that $A \leq B$ in finitely many steps.*

Proof. We may clearly assume that b does not lie in A.

Let r be minimal such that there exists an element a_r in $\mathcal{A}_r(A)$ lying over b (if $r = N+1$, set $a_r = a_{N+1}$). Likewise, choose l maximal such that there exists an element a_l in $\mathcal{A}_l(A)$ beneath b (if $l = -1$, then set $a_l = a_{-1}$). We call the interval $s = (l,r)$ the *width* of b over A. Define as well the *distance* from b to A as $\mathrm{d}_s(b, A_{a_l}^{a_r})$.

We prove the theorem by induction on the width and the distance from b to A. If the distance is infinite, by completeness of M, choose a path

$$a_l, b_{l+1}, \ldots, b_{r-1}, a_r,$$

passing through b. By Lemma 4.21, the set $A \cup \{b_{l+1}, \ldots, b_{r-1}\}$ obtained from A by applying α_s is nice and contains b.

Otherwise, let P be a path of minimal length lying between a_l and a_r connecting b to A. Let b' be the last element in P before b. By assumption, the distance from b' to A is strictly smaller than the length of P. Thus, there is a nice set $B' \geq A$ containing b'. Either the width or the distance of b to B' has become smaller and we can now finish by induction. ∎

In particular, we can now prove that the notions of nice and wunderbar agree.

COROLLARY 4.23. *A nice subset A of a complete simply connected set M is wunderbar.*

Proof. Suppose we are given two points a and b in A and an s-path P in M of length n connecting them. By Theorem 4.22, we can obtain a nice set B such that $A \leq B$ and B contains the path P. By Lemma 4.18, the set A is wunderbar in B, so there is an s-path of length n in A connecting a and b. Thus, the set A is wunderbar. ∎

Combining the previous results, we obtain the following.

COROLLARY 4.24. *Let M be complete and simply connected and A be a nice subset. The following hold:*

(a) *If $M \setminus A$ is countable, then $A \leq M$.*
(b) *A is simply connected.*
(c) *A is wunderbar.*
(d) *If A is countable, then $\emptyset \leq A$.*

Proof. Theorem 4.22 yields (a). Now, Corollary 4.20 yields (b). In order to prove (c), it is sufficient to consider countable nice subsets A. Replace M by a countable elementary substructure M' that contains A. Then A is nice in M' and $A \leq M'$ by (a). Lemma 4.18 implies that A is wunderbar in M' and hence in M. Since $\emptyset$ is nice, clearly (d) follows from (a) and (b). ∎

It follows that, for countable A, we have $\emptyset \leq A$ if and only if A is simply connected and complete. And for simply connected complete countable B, we have $A \leq B$ if and only if A is nice in B. This yields

COROLLARY 4.25. *The model M_∞ is the Fraïssé limit of the class of finite complete simply connected colored N-spaces together with nice embeddings.*

The construction is actually simpler than the general construction given in Section 3, since if a finite set B satisfies $B_a^b = B \cap M_a^b$ for all a and b in B, then B is r-strong in M_∞ if and only it is nice in M_∞. Indeed, consider a rich sequence $A_0 \leq A_1 \leq \cdots$ with union M_∞. Then B is contained in some A_i. But B is also nice in A_i, which implies $B \leq A_i$, and therefore B is r-strong in M_∞.

Having M_∞ as a model, the theory PS_N is consistent. It will follow from the next proposition that it is complete. In particular, the stronger version of Axiom (1) stated in Corollary 4.11 follows formally from our axioms.

PROPOSITION 4.26. *Any two ω-saturated models of* PS_N *have the back-and-forth property with respect to partial isomorphisms between finite nice substructures.*

Proof. Let M and M' be two ω-saturated models and consider a partial isomorphism $f : A \to A'$, where A is nice in M and A' is nice in M'.

Given b in M, Theorem 4.22 yields a nice finite subset $B \geq A$ containing it. Thus, we may assume that B is obtained from A by applying α_s on (a_l, a_r). Since M' is an ω-saturated model of Axiom (2), there is a path $a'_l, b'_{l+1}, \ldots, b'_{r-1}, a'_r$ in M' such that the s-distance of b'_i to A' is infinite. By Lemma 4.21 the set $B' = A' \cup \{b'_{l+1}, \ldots, b'_{r-1}\}$ is nice and f extends to an isomorphism between B and B'. ∎

THEOREM 4.27. *Any partial isomorphism $f : A \to A'$ between two finite nice subsets of two models of* PS_N *is elementary.*

Proof. Replace the models M and M' by two ω-saturated extensions M_1 and M'_1. Note that A and A' remain nice in the corresponding extensions. Lemma 4.26 implies that f is elementary with respect to M_1 and M'_1 and thus its restriction to M and M' is elementary as well. ∎

COROLLARY 4.28. *The theory* PS_N *is complete.*

Proof. Note that the set $\emptyset$ is nice in any colored N-space and apply Theorem 4.27. ∎

COROLLARY 4.29. *The type of a nice set A is determined by its quantifier-free type.*

COROLLARY 4.30. *The model* M_∞ *is ω-saturated.*

Proof. Let M be any ω-saturated model of PS_N. It follows from Lemma 3.3 and the equality of nice and r-strong that the family of isomorphisms between finite nice subsets of M and M_∞ has the back-and-forth property. This implies that M_∞ is also ω-saturated. ∎

COROLLARY 4.31. *The Fraïssé limit M_0 is the prime model of* PS_N.

Proof. Consider any finite $A \subset M$ which can be obtained from some fixed flag by a sequence of applications of $\alpha_{\{i\}}$ for varying $i \in [0, N]$. Since the $\mathrm{d}_{\{i\}}$-distances are either 0 or ∞, it follows inductively from Lemma 4.21 that all intermediate sets are nice. So the quantifier-free type of A implies that A is nice and therefore implies the type of A. Hence A is atomic. This shows that M_0 is atomic. ∎

COROLLARY 4.32. *Nice sets are algebraically closed.*

Proof. By Corollary 4.30, we may assume that the nice set A is a subset of M_∞. By Corollary 4.24(a), the model M is an increasing union of nice sets containing A. Thus, we may reduce the statement to showing that if $B = A\cup\{b_{l_s+1}, \dots, b_{r_s-1}\}$ is obtained by applying the operation α_s on a_{l_s}, a_{r_s} in A, then the tuple $(b_{l_s+1}, \dots, b_{r_s-1})$ has infinitely many A-conjugates. This is now clear, as any two sets resulting from applying the operation α_s on a_{l_s}, a_{r_s} in A have the same type over A, by Lemma 4.21 and Corollary 4.29. ∎

5. Words and letters. In this section, we will study the semigroup $\mathrm{Cox}(N)$ generated by the operations α_s, where s stands for a non-empty interval in $[0, N]$. Such intervals will then be called *letters*. We will exhibit a normal reduced form for words in $\mathrm{Cox}(N)$ and describe the possible interactions between words when multiplying them.

Two letters s and t in $[0, N]$ *commute* if their distance is at least 2. That is, either $r_s \leq l_t$ or $r_t \leq l_s$, where $s = (l_s, r_s)$ and $t = (l_t, r_t)$. By definition, no letter commutes with itself nor with any proper subletter.

Definition 5.1. We define $\mathrm{Cox}(N)$ to be the monoid generated by all letters in $[0, N]$ modulo the following relations:

- $ts = st = s$ if $t \subset s$,
- $ts = st$ if s and t commute.

We denote by 1 the empty word.

The *inversion* $u \mapsto u^{-1}$ of words defines an antiautomorphism of $\mathrm{Cox}(N)$. All concepts introduced from now on will be invariant under inversion.

The *centraliser* $\mathrm{C}(u)$ of a word u in $\mathrm{Cox}(N)$ is the collection of all indices in $[0, N]$ commuting with every letter in u. Clearly, a letter s commutes with u in $\mathrm{Cox}(N)$ if and only if $s \subset C(u)$.

In order to obtain a normal form for elements in $\mathrm{Cox}(N)$, we say that a *word* $s_1 \cdots s_n$ is *reduced* if there is no pair $i \neq j$ of indices such that $s_i \subset s_j$ and s_i commutes with all s_k with k between i and j.

Definition 5.2. The word u can be *reduced to* v, denoted by $u \to v$, if v is obtained from u by finitely many iterations of the following rules:

Commutation: Replace an occurrence of $s \cdot t$ by $t \cdot s$ if s and t commute.

Cancellation: Replace an occurrence of $s \cdot t$ or $t \cdot s$ by s if $t \subset s$.

Two words u and v are *equivalent* (or u is a *permutation* of v), denoted by $u \approx v$, if $u \to v$ by exclusively applying the commutation rule.

It is easy to see that permutations of reduced words remain reduced. In particular, a word is reduced if and only if the cancellation rule cannot be applied to any permutation.

Clearly, two words u and v represent the same element in $\mathrm{Cox}(N)$ if $u \to v$. The following proposition implies in particular that the converse is true: Two words have a common reduction if they represent the same element in $\mathrm{Cox}(N)$ (cf. Corollary 5.4).

PROPOSITION 5.3. *Every word u can be reduced to a unique* (*up to equivalence*) *reduced word v. We refer to v as the* reduct *of u.*

Proof. Among all possible reductions of the word u, choose v of minimal length. Clearly, cancellation cannot be applied any further to a permutation of v, thus v is reduced. We need only show that v is unique such.

For that, we first introduce the following rule:

GENERALISED CANCELLATION: Given a word $s_1 \cdots s_n$ and a pair of indices $i \neq j$ such that $s_i \subset s_j$ and s_i commutes with all s_k's with k between i and j, then delete the letter s_i.

If the situation described above occurs, we say that s_i *is absorbed by* s_j. Note that a generalised cancellation is obtained by successive commutations and one single cancellation. Furthermore, one single cancellation applied to some permutation of u can be obtained as some permutation of a generalised cancellation applied to u. This implies that every reduct can be obtained by a sequence of generalised cancellations followed by a permutation.

Assume now that $u \to v_1$ and $u \to v_2$, where both v_1 and v_2 are reduced. We will show, by induction on the length of u, that v_2 is a permutation of v_1. If u is itself reduced, then v_1 and v_2 are permutations of u and hence the result follows. Otherwise, there are two words u_1 and u_2 obtained from u by one single generalised cancellation such that $u_i \to v_i$ for $i = 1, 2$.

We claim that there is a word u' such that $u_i \to u'$ for $i = 1, 2$, either by a permutation or by a single generalised cancellation. This is immediate except for the case where there are indices i, j and k (with $i \neq k$) such that u_1 is obtained from u because the letter s_i is absorbed by s_j and u_2 is obtained from u in which the same letter s_j is absorbed by s_k. In this case, set u' to be the word obtained from u by having both s_i and s_j absorbed by s_k. Clearly, $u_1 \to u'$. Also, since $s_i \subset s_j$, it follows that s_i commutes also with all letters between s_j and s_k. Hence, the word u' is obtained from u_2 in which s_k absorbs s_i. Let v' be a reduct of u'. Induction applied to u_1 and u_2 implies that v' is a permutation of both v_1 and v_2. So v_1 is a permutation of v_2. ∎

COROLLARY 5.4. *Every element of* $\mathrm{Cox}(N)$ *is represented by a reduced word, which is unique up to equivalence.*

Proof. Let C be the collection of equivalence classes of reduced words. By Proposition 5.3, there is a natural surjection $C \to \mathrm{Cox}(N)$. Write $[u]$ for the equivalence class of the word u. Set

$$[u] \cdot [v] = [w] \quad \text{iff} \quad u \cdot v \to w.$$

Then C has a natural semigroup structure. Since C satisfies the defining relations of $\mathrm{Cox}(N)$, the map $C \to \mathrm{Cox}(N)$ is an isomorphism. ∎

In order to exhibit a canonical representative of the equivalence class $[u]$, we introduce the following partial ordering on letters:

$$(l_s, r_s) < (l_t, r_t) \quad \text{iff} \quad r_s \leq l_t.$$

A reduced word $s_1 \cdots s_n$ is in *normal form* if for all $i < n$, if s_i and s_{i+1} commute, then $s_i < s_{i+1}$.

Remark 5.5. Every reduced word is equivalent to a unique word in normal form.

Proof. We will actually prove a more general result: Let S be any set equipped with a partial order $<$. We say that s and t commute if either $s < t$ or $t < s$. Let S^* be the semigroup generated by S modulo commutation. Two words in S^* are equivalent if they can be transformed into each other by successive commutations of adjacent elements. A word $s_1 \cdots s_n$ is in normal form if $s_i \not> s_{i+1}$ for all $i < n$. We have the following.

Claim. *Every word u in S^* is equivalent to a unique word v in normal form.*

For existence, start with u and swap successively every pair $s_i > s_{i+1}$. This process must stop since the number of inversions $\{(i,j) \mid i < j$ and $s_i > s_j\}$ is decreased by 1 at every step. The resulting v is in normal form.

For uniqueness, consider two equivalent words in normal form $u = s_1 \cdots s_n$ and $v = t_1 \cdots t_n$. Let π be some permutation transforming u into v. Suppose for a contradiction that $\pi(1) = k \neq 1$. Then $t_k = s_1$ commutes with t_i for $i < k$. By hypothesis, $t_{k-1} < t_k$. Note that there is no $i < k$ with $t_i < t_k$ and $t_k < t_{i-1}$. Hence, for all $i < k$, we have $t_i < t_k$ and thus $t_1 < t_k$, that is, $t_1 < s_1$. By means of the permutation π^{-1}, we conclude that $s_1 < t_1$, which yields a contradiction. Thus $\pi(1) = 1$ and hence $s_2 \cdots s_n$ is equivalent to $t_2 \cdots t_n$. Induction on n yields the desired result. ∎

It is an easy exercise to show that, for S and S^* as before, we have

$$r \cdot t_2 \cdots t_n \approx r \cdot s_2 \cdots s_n \;\Rightarrow\; t_2 \cdots t_n \approx s_2 \cdots s_n.$$

Therefore, we obtain the following result.

Remark 5.6. $u \cdot v \approx u \cdot v'$ implies $v \approx v'$.

Given two reduced words $u = s_1 \cdots s_m$ and $v = t_1 \cdots t_n$, their product $u \cdot v$ is not reduced if and only if one of the following two cases occurs:

- There are $i \leq m$ and $j \leq n$ such that s_i commutes with $s_{i+1} \cdots s_m$ and with $t_1 \cdots t_{j-1}$ and it is contained in t_j.
- There are $j \leq n$ and $i \leq m$ such that t_j commutes with $t_1 \cdots t_{j-1}$ and with $s_{i+1} \cdots s_m$ and it is contained in s_i.

Based on the previous observation, we introduce the following definition.

Definition 5.7. Given two words $u = s_1 \cdots s_m$ and $v = t_1 \cdots t_n$ words, we say that:

(1) s_i belongs to the *final segment* of u if s_i commutes with $s_{i+1} \cdots s_m$.
(2) The letter s is (properly) *left-absorbed* by v if it commutes with $t_1 \cdots t_{j-1}$ and is a (proper) subset of t_j for some $j \leq n$. A word is (properly) left-absorbed by v if all its letters are (properly) left absorbed by v.
(3) *v bites u from the right* if v left-absorbs some element in the final segment of u.

The concepts of *initial segment*, *right-absorbed* and *left-biting* are defined likewise.

Clearly, these notions depend only on the equivalence class of u and v. Thus, the following lemma follows.

Lemma 5.8. *Given two reduced words u and v, the product $u \cdot v$ is reduced if and only if none of them bites the other one* (*in the corresponding directions*).

If both u and v are reduced and u is absorbed by v, then $u \cdot v$ reduces to v. Corollary 5.14 will show that the converse also holds.

The following observations will be often used throughout this article.

Lemma 5.9 (Absorption Lemma). *Let v be a* (*possibly non-reduced*) *word.*

(1) *If a letter s is left-absorbed by v, then there is a unique letter in v witnessing it.*
(2) *If two non-commuting letters are absorbed by v, then they are absorbed by the same letter in v.*
(3) *Suppose $v = v_1 \cdot v_2$ and let u be a word left-absorbed by v but not bitten from the right by v_1, then u and v_1 commute and u is left-absorbed by v_2.*

Proof. Assume $v = t_1 \cdots t_n$. Let $r \subset t_i$ commute with $t_1 \cdots t_{i-1}$ and $s \subset t_j$ commute with $t_1 \cdots t_{j-1}$. Assume $i \leq j$. Then either $i = j$ or s commutes with t_i, which implies that s commutes with r. This yields both (1) and (2).

For (3), we apply induction on the length m of $u = s_1 \cdots s_m$. If $m = 0$, there is nothing to prove. Otherwise, the subword $u' = s_2 \cdots s_m$ is not bitten by v_1 by assumption. Induction shows that u' commutes with v_1 and is absorbed by v_2. The letter s_1 cannot be absorbed by v_1, for otherwise s_1 would also commute with u' and thus it would belong to the final segment of u. The word u would then be bitten by v_1. Since s_1 is absorbed by v but not by v_1, it must commute with v_1 and hence it is absorbed by v_2 as well. ■

Based on the the previous result, we introduce the following notions.

DEFINITION 5.10. The *left stabiliser* $\mathcal{S}_{\mathrm{L}}(v)$ of a word $v = t_1 \cdots t_n$ is the union of the sets

$$\mathcal{S}^j_{\mathrm{L}}(v) = t_j \cap \mathrm{C}(t_1 \ldots t_{j-1}).$$

The *right stabiliser* $\mathcal{S}_{\mathrm{R}}(v)$ is defined likewise or, alternatively, as $\mathcal{S}_{\mathrm{L}}(v^{-1})$.

By Lemma 5.9(2), the sets $\mathcal{S}^j_{\mathrm{L}}(v)$ are either empty or intervals commuting with each other. Equivalent words have the same stabilisers. In fact, if $u \to v$ then $\mathcal{S}_{\mathrm{L}}(u) \subset \mathcal{S}_{\mathrm{L}}(v)$.

LEMMA 5.11. *The letter s is absorbed by v if and only if $s \subset \mathcal{S}_{\mathrm{L}}(v)$.*

Set

$$|s_1 \cdots s_m| = s_1 \cup \cdots \cup s_m.$$

Then u is absorbed by v if and only if $|u| \subset \mathcal{S}_{\mathrm{L}}(v)$. Furthermore, the word v bites u from the right if and only if some element in the final segment of u is contained in $\mathcal{S}_{\mathrm{L}}(v)$.

LEMMA 5.12. *Given two words u and v, there is a unique decomposition $u = u_1 \cdot u_2$ (up to commutation) such that:*

- *u_2 is left-absorbed by v.*
- *u_1 is not bitten from the right by v.*

The decomposition of u depends only on the set $\mathcal{S}_{\mathrm{L}}(v)$.

Proof. We proceed by induction on the length of u. If u is not bitten by v, we set $u_1 = u$ and $u_2 = 1$. Otherwise, up to permutation, we have $u = u' \cdot s$, where s is absorbed by v. Decompose u' as $u'_1 \cdot u'_2$ and set $u_1 = u'_1$ and $u_2 = u'_2 \cdot s$.

Uniqueness is proved in a similar fashion. ■

We can now describe the general form of the product of two reduced words in $\mathrm{Cox}(N)$.

THEOREM 5.13 (Decomposition Lemma). *Given two reduced words u and v, there are unique decompositions (up to permutation)*

$$u = u_1 \cdot u', \qquad v' \cdot v_1 = v,$$

such that:

(a) *u' is left-absorbed by v_1,*
(b) *v' is properly right-absorbed by u_1,*
(c) *u' and v' commute,*
(d) *$u_1 \cdot v_1$ is reduced.*

It follows that $u \cdot v \to u_1 \cdot v_1$. We call such a decomposition *fine.*

Proof. We apply Lemma 5.12 to u and v to obtain a decomposition

$$u = u_1 \cdot u'$$

such that u' is left-absorbed by v and u_1 is not bitten by v from the right. The same (in the other direction) with u_1 and v yields

$$v' \cdot v_1 = v,$$

where v' is right-absorbed by u_1 and v_1 is not bitten from the left by u_1.

First, we show (c), that is, the words u' and v' commute. If not, let s the first element of u' which does not commute with v'. Since s is left-absorbed by $v' \cdot v_1$, it must be left-absorbed by v'. As u_1 right-absorbs v', it also right-absorbs s, which contradicts that $u_1 \cdot u'$ is reduced. Lemma 5.9(3) implies that u' is absorbed by v_1, showing (a).

Let us now prove (d): the product $u_1 \cdot v_1$ is reduced. Otherwise, as v_1 is not bitten from the left by u_1, it bites u_1 from the right, i.e. it left-absorbs a letter s from the final segment of u_1. The Absorption Lemma 5.9, applied to $u_1 = u_1^1 \cdot s$ and v', which is right-absorbed by u_1, gives (possibly after permutation) a decomposition $v' = x \cdot y$, where $|x| \subset s$ and y commutes with s. There are two cases:

(1) The word $x = 1$. Then s commutes with v' and is absorbed by v_1. This contradicts that u_1 is not bitten by v_1 from the right.
(2) The word x is not trivial. As it is absorbed by s and s is right-absorbed by v_1, we deduce that x is right-absorbed by v_1. This contradicts that $v' \cdot v_1$ is reduced.

The only point left to prove is that v' is properly right-absorbed by u_1. Otherwise, there is a letter t in v' which is absorbed but not properly absorbed by u_1. Then t occurs in the final segment of u_1 and $v' = t \cdot y$ up to commutation. In particular, the word u_1 is bitten from the right by v' and thus by v, which contradicts our choice of u_1.

In order to show uniqueness, assume we are given another fine decomposition

$$u = u_1 \cdot u', \quad v' \cdot v_1 = v.$$

We need only show the following four facts:

(1) The word u' is left-absorbed by v: Since u' commutes with v' and is left-absorbed by v_1, then it is left-absorbed by $v' \cdot v_1$ as well.

(2) The word u_1 is not bitten by v from the right: Suppose not and take a letter s in the final segment of u_1 which is left-absorbed by v. Since $u_1 \cdot v_1$ is reduced, the letter s must be left-absorbed by v'. Let t in v' contain s. However, the word t is right-absorbed by u_1. As u_1 is reduced and s is in the final segment of u_1, the only possibility is that $s = t$. But then t is not properly left-absorbed by u_1, which is a contradiction.

(3) v' is right-absorbed by u_1: By definition.

(4) v_1 is not bitten from the left by u_1: This clearly follows from the fact that $u_1 \cdot v_1$ is reduced. ∎

Corollary 5.14. *Let u and v be reduced words. Then v left-absorbs u if and only if $uv = v$ in* $\mathrm{Cox}(N)$.

Note that $uv = v$ in $\mathrm{Cox}(N)$ if and only if $u \cdot v \to v$.

Proof. Clearly, if v left-absorbs u, then $u \cdot v \to v$. For the converse, apply the Decomposition Lemma 5.13 to u and v to obtain

$$u = u_1 \cdot u', \qquad v' \cdot v_1 = v,$$

such that u' is left-absorbed by v_1, the word v' is properly right-absorbed by u_1, the words u' and v' commute and $u_1 \cdot v_1$ is reduced. By assumption,we have

$$u \cdot v \to u_1 \cdot v_1 \approx v = v' \cdot v_1.$$

Thus $u_1 = v'$. Since u_1 must properly right-absorb itself, this forces u_1 to be trivial. Hence $u = u'$ is left-absorbed by v. ∎

Since in $\mathrm{Cox}(N)$ (or generally, in any semigroup) the identity $uvx = uv$ holds if $vx = v$, we have the following.

Corollary 5.15. *Let u and v be reduced words and w the reduct of $u \cdot v$. Then $\mathcal{S}_{\mathrm{R}}(v) \subset \mathcal{S}_{\mathrm{R}}(w)$.*

Definition 5.16. The *wobbling* between two words is

$$\mathrm{Wob}(u, v) = \mathcal{S}_{\mathrm{R}}(u) \cap \mathcal{S}_{\mathrm{L}}(v).$$

Remark 5.17. If $u \cdot v$ is reduced, then every $s \subset \mathrm{Wob}(u, v)$ is properly right-absorbed by u and properly left-absorbed by v.

Proof. If s is not properly right-absorbed by u, then s belongs to the final segment of u. Since s is left-absorbed by v, the product $u \cdot v$ would not be reduced. ∎

Lemma 5.18. *Assume that $v_1 \cdot v_2$ and $u \cdot v_2$ are reduced. If v_1 is right absorbed by u, then*

$$\mathrm{Wob}(v_1 \cdot v_2, h) \subset \mathrm{Wob}(u \cdot v_2, h).$$

Proof. The word $u \cdot v_2$ is the reduct of $u \cdot (v_1 \cdot v_2)$. Corollary 5.15 shows that $\mathcal{S}_{\mathrm{R}}(v_1 \cdot v_2) \subset \mathcal{S}_{\mathrm{R}}(u \cdot v_2)$. ∎

We will now study the idempotents of $\mathrm{Cox}(N)$.

DEFINITION 5.19. A word is *commuting* if it consists of pairwise commuting letters.

The letters of the final segment of a word u form a commuting word, which we denote by $\tilde{u}$ (up to equivalence).

Commuting words are automatically reduced. Since every subset of $[0, N]$ can be uniquely written as the union of commuting intervals, a commuting word (up to equivalence) can be considered as just a set of numbers. The following is an easy observation:

LEMMA 5.20. *Every word u is equivalent to a word $x \cdot \tilde{u}$, where $\tilde{u}$ is the final segment of u.*

Note that no letter in the final segment of x commutes with $\tilde{u}$.

PROPOSITION 5.21. *Let u and v be reduced words such that v left-absorbs u. Then, up to permutation, there are unique decompositions*

$$u = u' \cdot w, \qquad w \cdot v' = v,$$

such that

(1) *u' is properly left-absorbed by v',*
(2) *w commutes with u',*
(3) *w is a commuting word.*

Proof. Apply the Absorption Lemma 5.9 to v and u, which is completely left-absorbed by v. The letters of u which are not properly left-absorbed by v must commute with all other letters and form the word w. ∎

We next obtain the following consequence, which implies that a word is commuting if and only if it is an idempotent in $\mathrm{Cox}(N)$.

COROLLARY 5.22. *A reduced word is commuting if and only if it* (*left-, or equivalently, right-*) *absorbs itself.*

Proof. Clearly, if u is commuting, then $|u| = \mathcal{S}_{\mathrm{L}}(u)$, so u absorbs itself. Suppose now that u left-absorbs itself. By the proposition applied to $v = u$ we find $u = w \cdot u' \approx w \cdot v'$ such that u' is properly left-absorbed by v' and w is a commuting word. It follows that $u' = v'$ properly absorbs itself, so the word u' is trivial. ∎

We can now state a symmetric version of the Decomposition Theorem 5.13, combined with Proposition 5.21.

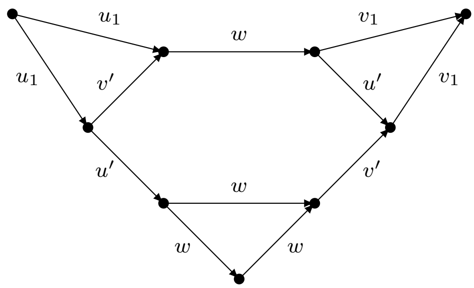
Corollary 5.23 (Symmetric Decomposition Lemma). *Let u and v be two reduced words. Each can be uniquely decomposed (up to commutation) as*

$$u = u_1 \cdot u' \cdot w, \qquad w \cdot v' \cdot v_1 = v,$$

such that:

(a) *u' is properly left-absorbed by v_1,*
(b) *v' is properly right-absorbed by u_1,*
(c) *u', w and v' pairwise commute,*
(d) *w is a commuting word,*
(e) *$u_1 \cdot w \cdot v_1$ is reduced.*

In particular, we have $u \cdot v \to u_1 \cdot w \cdot v_1$.



Proof. Let

$$u = u_1 \cdot \bar{u}', \qquad v' \cdot \bar{v}_1 = v,$$

be a fine decomposition as in Theorem 5.13. Apply Proposition 5.21 to $\bar{u}'$ and $\bar{v}_1$ to obtain

$$\bar{u}' = u' \cdot w, \qquad w \cdot v_1 = \bar{v}_1,$$

where u' is properly left-absorbed by v_1, w commutes with u', and w is a commuting word.

Uniqueness follows similarly. ∎

In order to describe canonical paths between elements (or rather between flags) in the Fraïssé limit M^N_∞, we require a stronger form of reduction, since applying twice the same operation α_s does not necessarily yield a global application of α_s, but rather a finite product of proper subletters.

Definition 5.24. The word u is *strongly reduced to v*, denoted by $u \xrightarrow{*} v$, if v is obtained from u by finitely many iterations of Cancellation, Commutation, and

Splitting: Replace an occurrence of $s \cdot s$ by a (possibly trivial) product $t_1 \cdots t_n$ of letters t_i, each of which is properly contained in s.

If v is reduced, we call v a *strong reduct* of u.

As an example note that $u \cdot u^{-1} \xrightarrow{*} 1$.

Despite the possible confusion for the reader, we will not refer to reductions defined in 5.2 as *weak* reductions.

Related to the notion of strong reduction, we also consider the following partial ordering on words.

DEFINITION 5.25. For words u and v, we define $u \prec v$ if some permutation of u is obtained from v by replacing at least one letter s of v by a (possibly empty) product of proper subletters of s. By $u \preceq v$, we mean $u \prec v$ or $u \approx v$.

LEMMA 5.26.

(1) *$\prec$ is transitive and well-founded.*
(2) *$u' \approx u \prec v \approx v'$ implies $u' \prec v'$.*
(3) *If the strong reduction $u \xrightarrow{*} v$ involves at least one cancellation or splitting, we have $v \prec u$.*

Well-foundedness implies in particular that if $u \prec v$, then $u \not\approx v$. Furthermore, property (2) shows that $\prec$ induces a partial order on $\mathrm{Cox}(N)$, setting $[u] \prec [v]$ if $u \prec v$, where both u and v are reduced. With this notation, the trivial word 1 becomes the smallest element.

Proof. To see that $\prec$ is well-founded, we introduce an ordinal-valued rank function ord. For i in $[0, N]$, set $\mathrm{ord}_i(w)$ to be the number of letters s in w with $i+1$ elements. Define now

$$\mathrm{ord}(w) = \omega^N \mathrm{ord}_N(w) + \omega^{N-1} \mathrm{ord}_{N-1}(w) + \cdots + \mathrm{ord}_0(w).$$

Then $u \prec v$ implies $\mathrm{ord}(u) < \mathrm{ord}(v)$. ∎

The semigroup $\mathrm{Cox}(N)$, equipped with the order function as above, is an ordered semigroup in which left and right cancellation are (almost) order-preserving.

LEMMA 5.27. *Let $w \cdot v$ be reduced and $w \cdot v \preceq w \cdot v'$. Then $v \preceq v'$.*

The condition that $w \cdot v$ is reduced is needed, by taking $v' = t \subsetneq s = w = v$ and $w \cdot v \xrightarrow{*} 1$.

Proof. By induction on the number of letters appearing in w, we need only consider the case where $w = s$ for some interval s.

The assumption implies that $s \cdot v$ is equivalent to a word $u_s \cdot u'$ where $u_s \preceq s$ and $u' \preceq v'$. The word u_s either equals s or is a product of proper subletters of s. If $u_s = s$, we have $v \approx u' \preceq v'$ and are done. Otherwise, since $s \cdot v$ is reduced, it follows that $u_s = 1$. This implies $v \prec s \cdot v \approx u' \preceq v'$. ∎

COROLLARY 5.28. *Given reduced words $w \cdot v$ and v' such that $w \cdot v$ is smaller than some strong reduct of $w \cdot v'$, we have $v \preceq v'$.*

LEMMA 5.29. *The partial order $\preceq$ is compatible with the semigroup operation in* $\mathrm{Cox}(N)$.

Proof. Given reduced words u,v and w, we have to show the following:

$$[u] \preceq [v] \;\Rightarrow\; [w][u] \preceq [w][v],$$
$$[u] \preceq [v] \;\Rightarrow\; [u][w] \preceq [v][w].$$

By symmetry, it is sufficient to show the first implication. By induction on $|w|$, it is enough to consider the case where w is a single letter s.

Suppose first that s is left-absorbed by v. By Corollary 5.14,

$$[s][v] = [v].$$

If s is also left-absorbed by u, we are clearly done. Otherwise, by Theorem 5.13, decompose u (up to permutation) as $u = u' \cdot u_1$, where $s \cdot u_1$ is the reduct of $s \cdot u$. Also, write $v = \bar{v} \cdot t \cdot v_1$ such that $s \subset t$ and $\bar{v}$ is in $\mathrm{C}(s)$. Now, the word $u_1 \preceq u \preceq v$, so write $u_1 = \bar{u}_1 \cdot u_1^t \cdot \bar{u}_1^1$, where $\bar{u}_1 \preceq \bar{v}$, $u_1^t \preceq t$ and $u_1^1 \preceq v_1$. Since $s \cdot u_1$ is reduced, so is $s \cdot \bar{u}_1 \cdot u_1^t = \bar{u}_1 \cdot s \cdot u_1^t$.

This forces u_1^t to be either trivial or different from t (and $s \neq t$ as well). In both cases, we have $s \cdot u_1^t \preceq t$, which implies $s \cdot u_1 \preceq v$, so we are done.

If s is not left-absorbed by v, by Theorem 5.13 we can write (up to permutation) $v = v' \cdot v_1$, where v' is properly absorbed by s and $s \cdot v_1$ is reduced. So $[s][v] = [s \cdot v_1]$. If s is left-absorbed by u, then

$$[s][u] = [u] \preceq [v' \cdot v_1] \prec [s \cdot v_1].$$

Otherwise, write $u = \bar{u} \cdot u' \cdot u_1$ as above such that $s \cdot u \to \bar{u} \cdot s \cdot u_1$. Since $\bar{u}$ and s commute, note that $\bar{u} \cdot u_1$ is irreducible, since u is. Decompose $\bar{u} \cdot u_1 = u_1' \cdot u_{11}$ with $u_1' \preceq v'$ and $u_{11} \preceq v_1$. Since $s \cdot \bar{u} \cdot u_1 = \bar{u} \cdot s \cdot u_1$ is reduced, the word u_1' must be trivial. Therefore $s \cdot \bar{u} \cdot u_1 = s \cdot u_{11} \preceq s \cdot v_1$. ■

In particular, since $1 \preceq v$ for any word v, we obtain the following result.

COROLLARY 5.30. *Let u be reduced. Given any word v, the reduction w of $u \cdot v$ is $\preceq$-larger than u.*

In contrast to Proposition 5.3, uniqueness of strong reductions no longer holds, e.g. $s \cdot s \xrightarrow{*} s$ and $s \cdot s \xrightarrow{*} 1$. However, we get the following result, which allows us to permute the steps of the strong reduction:

PROPOSITION 5.31 (Commutation Lemma). *If x is a strong reduct of $u \cdot v \cdot w$, then there is a strong reduct y of v such that $u \cdot y \cdot w \xrightarrow{*} x$.*

Proof. Consider first the case where $u = t$ has length 1, the word v has length 2 and w is empty. Suppose furthermore that in the first step of the reduction $t \cdot v \xrightarrow{*} x$, the letter t is deleted. It is easy to check that setting y as the reduct of v, the results follows, except if $v = s \cdot s$, the letter t is contained in s and the strong reduction is $t \cdot (s \cdot s) \xrightarrow{*} s \cdot s \xrightarrow{*} x$, where x is a product of letters which are properly contained in s. Then:

- If $t = s$, set $y = s$.
- If $t \cdot x \xrightarrow{*} x$, set $y = x$.
- Otherwise, apply Theorem 5.13 to x and t and decompose $x = x' \cdot x_1$ such that $|x'|$ is properly contained in t and $t \cdot x_1$ is reduced. Set $y = t \cdot x_1$.

In all three cases, the strong reductions hold:

$$t \cdot (s \cdot s) \xrightarrow{*} t \cdot y \xrightarrow{*} x.$$

In order to show the proposition for the general case, motivated by the proof of 5.3, let us introduce the following rule:

Generalised Splitting: Given a word $s_1 \cdots s_n$ and a pair of indices $i \neq j$ such that $s_i = s_j$ and s_i commutes with all s_k's with k between i and j, delete s_j and replace s_i by a product of letters which are properly contained in s.

Note that a strong reduction consists of finitely many generalised cancellations and generalised splittings, followed by commutation (if needed).

If v is reduced, set $y = v$. Otherwise, we will apply induction on the $\prec$-order type of v. Suppose therefore that the assertion holds for all $v' \prec v$ and consider x a strong reduct of $u \cdot v \cdot w$. If $2 < |v|$, then (after permutation) write $v = v_1 \cdot a \cdot v_2$, where a is a non-reduced word of length 2. Note that by assumption, the subword $a \prec v$, so there is a strong reduct b of a such that $u \cdot v_1 \cdot b \cdot v_2 \cdot w \xrightarrow{*} x$. Since a is not reduced, we have $b \prec a$ and thus $v_1 \cdot b \cdot v_2 \prec v$. Induction yields the existence of a strong reduct y of $v_1 \cdot b \cdot v_2$ such that

$$u \cdot y \cdot w \xrightarrow{*} x.$$

Note that $v = v_1 \cdot a \cdot v_2 \xrightarrow{*} v_1 \cdot b \cdot v_2 \xrightarrow{*} y$. Therefore, we may assume that v has length 2 and it is non-reduced. By the above discussion, the first step in the strong reduction

$$u \cdot v \cdot w \xrightarrow{*} x$$

is either a generalised cancellation or a generalised splitting. If it involves only letters from v, its strong reduction is $\preceq$-smaller and one step shorter than the output x, so we are done by induction on the number of steps in the strong reduction. Likewise if the letters involved are in $u \cdot w$. Thus, we may assume that there are two letters t and r witnessing the reduction in the first step and, say, the letter t occurs in u and r in v.

We have two cases:

- The letter t is absorbed by v. In particular, the letter lies in the final segment $\tilde{u}$. Write $u = u_1 \cdot t$. If it was a generalised splitting, the resulting word v' is strictly $\prec$-smaller than v and $u_1 \cdot v' \cdot w \xrightarrow{*} x$. Induction gives a strong reduct x' of v' such that $u_1 \cdot x' \cdot w \xrightarrow{*} x$. In particular, we are now in the case $t \cdot v \xrightarrow{*} x'$ and thus, by the discussion at the beginning of the proof, there exists a strong reduction y of v such that $t \cdot y \xrightarrow{*} x'$. Note that

$$u \cdot v \cdot w = u_1 \cdot (t \cdot v) \cdot w \xrightarrow{*} u_1(t \cdot y) \cdot w \xrightarrow{*} u_1 \cdot x' \cdot w \xrightarrow{*} x,$$

so we are done.

If the first step was a generalised cancellation, the word v does not change and now $u_1 \cdot v \cdot w \xrightarrow{*} x$ in one step less. We obtain a strong reduct x' of v with $u_1 \cdot x' \cdot w \xrightarrow{*} x$. Again, note that $t \cdot v \xrightarrow{*} v \xrightarrow{*} x'$ so, again by the previous discussion, there is a strong reduct y of v which does the job.

- Otherwise, the occurrence r in v is deleted. If $r = t$, we are in the previous case. Suppose hence $r \subsetneq t$ and write $u = u_1 \cdot t \cdot u_2$, where u_2 commutes with r. We may assume that $v = r \cdot s$. Note that r and s are comparable, since v is not reduced. If $r \subseteq s$, then set $y = s$, which is a strong reduct of v. We conclude that $u \cdot y \cdot w \xrightarrow{*} x$.

  If $s \subsetneq r$, then s and u_2 commute as well. Note that $u_1 \cdot (t \cdot s) \cdot u_2 \cdot w = u \cdot s \cdot w \xrightarrow{*} x$ in one step less. We have $u_1 \cdot t \cdot u_2 \cdot w \xrightarrow{*} x$ and setting $y = r$ does the job. ∎

Despite the apparent arbitrariness of strong reductions, they are orthogonal to the reduction without splitting, as the following result shows.

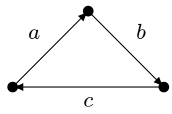
PROPOSITION 5.32. *Let u and v be reduced words and consider the reduct x of $u \cdot v$ and some strong reduct x^* of $u \cdot v$, where splitting occurs. Then $x^* \prec x$.*

Note that this is not true for the product of three reduced words: $s \cdot s \cdot s$ can be strongly reduced to s by one splitting operation.

Proof. Observe first that, if $w = s_1 \cdots s_n$ is a commuting word and y^* is a strong reduct of $w \cdot w$, then $y^* = t_1 \cdots t_n$, where each t_i is a strong reduct of $s_i \cdot s_i$. If splitting ever occurred in the reduction, then $y^* \prec w$.

To prove the proposition, choose decompositions $u = u_1 \cdot u' \cdot w$ and $w \cdot v' \cdot v_1 = v$, as in Corollary 5.23. A general cancellation applied to $u_1 \cdot u' \cdot w \cdot w \cdot v' \cdot v_1$ does the following: either the last letter of (a permutation of) u' is deleted, the first letter of v' is deleted or one letter in one of the copies of w is deleted. Hence, after finitely may generalised cancellations, the end result has the form $z = u_1 \cdot u'' \cdot w' \cdot w' \cdot v'' \cdot v_1$, where u'' is a left end of u', the subword v'' is a right end of v', and w' is a subword of w. A generalised splitting for z can only happen inside $w' \cdot w'$. So we obtain a word $z' = u_1 \cdot u'' \cdot a \cdot v'' \cdot v_1$, where a is obtained from $w \cdot w$ by the splitting operation. If we apply the Commutation Lemma 5.31 to $(u_1 \cdot v') \cdot a \cdot (u' \cdot v_1) \approx z'$, we obtain a strong reduct b of a such that $u_1 \cdot b \cdot v_1 \xrightarrow{*} x^*$. The above observation gives $b \prec w$ and thus $x^* \preceq u_1 \cdot b \cdot v_1 \prec u_1 \cdot w \cdot v_1 \approx x$. ∎

Inspired by the following picture:



we deduce strong reductions from a given one, as long as products are involved.

Proposition 5.33 (Triangle Lemma). *Let a, b and c be reduced words. Then $a \cdot b \xrightarrow{*} c^{-1}$ implies $c \cdot a \xrightarrow{*} b^{-1}$ and $b \cdot c \xrightarrow{*} a^{-1}$.*

Proof. By symmetry, it is enough to show that $a \cdot b \xrightarrow{*} c^{-1}$ implies $c \cdot a \xrightarrow{*} b^{-1}$. Suppose hence that $a \cdot b \xrightarrow{*} c^{-1}$. We apply induction on the $\prec$-type of a and b.

If $a \cdot b$ is reduced, then $c = b^{-1} \cdot a^{-1}$ and so $c \cdot a = b^{-1} \cdot a^{-1} \cdot a \xrightarrow{*} b^{-1}$. Thus, assume $a \cdot b$ is not reduced. We distinguish the following cases (up to permutation):

- $a = a_1 \cdot s$, where s is properly left-absorbed by b. Since b is the only strong reduct of $s \cdot b$, the Commutation Lemma 5.31 gives
$$a \cdot b = a_1 \cdot (s \cdot b) \to a_1 \cdot b \xrightarrow{*} c^{-1}.$$
Since $a_1 \prec a$, induction gives $c \cdot a_1 \xrightarrow{*} b^{-1}$, which implies that
$$c \cdot a = (c \cdot a_1) \cdot s \xrightarrow{*} b^{-1} \cdot s \to b^{-1}.$$
- $b = s \cdot b_1$, where s is properly right-absorbed by a. Again $a \cdot b = a \cdot (s \cdot b_1) \to a \cdot b_1 \xrightarrow{*} c^{-1}$, so by induction $c \cdot a \xrightarrow{*} b_1^{-1}$. Thus
$$c \cdot (a \cdot s) \xrightarrow{*} b_1^{-1} \cdot s = b^{-1}.$$
Since a is the only strong reduct of $a \cdot s$, again Proposition 5.31 gives that $c \cdot a \xrightarrow{*} b^{-1}$.
- $a = a_1 \cdot s$ and $b = s \cdot b_1$ Since $a_1 \cdot (s \cdot s) \cdot b_1 \xrightarrow{*} c^{-1}$, Proposition 5.31 provides a strong reduct x of $s \cdot s$ such that $a_1 \cdot x \cdot b_1 \xrightarrow{*} c^{-1} b \xrightarrow{*} c^{-1}$. The word x is either s or a product of proper subletters of x and hence $\prec$-smaller than s. Since $b = s \cdot b_1$ is reduced, apply Theorem 5.13 to decompose $x = x_1 \cdot x'$, where x' is properly left-absorbed by b_1 and $x_1 \cdot b_1$ is reduced (if $x = s$, then $x_1 = s$ and $x' = 1$). Since $x' \cdot b_1 \xrightarrow{*} b_1$, the reduction $(a_1 \cdot x_1) \cdot (x' \cdot b_1) \xrightarrow{*} c^{-1}$ implies $a_1 \cdot x_1 \cdot b_1 \xrightarrow{*} c^{-1}$. Since $a_1 \prec a$ and $x_1 \cdot b_1 \preceq b$, induction gives
$$c \cdot a_1 \xrightarrow{*} b_1^{-1} \cdot x_1^{-1}.$$
In particular,
$$c \cdot a = c \cdot a_1 \cdot s \xrightarrow{*} (b_1^{-1} \cdot x_1^{-1}) \cdot s \to b_1^{-1} \cdot s \to b^{-1}. \quad \blacksquare$$

We can now easily deduce the following:

Corollary 5.34. *If u and v are both reduced and $u \cdot v \xrightarrow{*} 1$, then $v \approx u^{-1}$.*

Proof. The Triangle Lemma (Proposition 5.33) yields $1 \cdot u \xrightarrow{*} v^{-1}$ and $v \cdot 1 \xrightarrow{*} u^{-1}$. That is, $u^{-1} \xrightarrow{*} v$ and $v \xrightarrow{*} u^{-1}$. Thus

$$u^{-1} \preceq v \preceq u^{-1},$$

and therefore $v \approx u^{-1}$. ∎

Recall by Corollary 5.14 that, if u is the reduct of $u \cdot v$, then v is right-absorbed by u. This is no longer true for strong reductions: take for example

$$(s \cdot t) \cdot (t \cdot s \cdot t) = s \cdot (t \cdot t) \cdot (s \cdot t) \xrightarrow{*} s \cdot (s \cdot t) \xrightarrow{*} s \cdot t.$$

However, in certain situations we are still able to conclude the same for strong reductions as for reductions with no splitting.

Lemma 5.35. *Let u and v be reduced. If every letter in v which is right-absorbed by u is properly absorbed and $u \cdot v \xrightarrow{*} u$, then $u \cdot v \to u$.*

Proof. Apply Theorem 5.13 to obtain fine decompositions $u = u_1 \cdot u'$ and $v' \cdot v_1 = v$ such that u' is properly left-absorbed by v_1, the word v' is right-absorbed by u_1, the words u' and v' commute and $u_1 \cdot v_1$ is reduced.

By hypothesis, the word v' is properly right-absorbed by u_1. The Commutation Lemma 5.31 applied to $(u_1 \cdot v') \cdot (u' \cdot v_1) \xrightarrow{*} u$ gives

$$(u_1 \cdot v') \cdot (u' \cdot v_1) \to u_1 \cdot v_1 \xrightarrow{*} u.$$

Since $u_1 \cdot v_1$ is reduced, we have $u_1 \cdot v_1 = u$. So $v_1 = u'$ must properly absorb itself, which is a contradiction unless $v_1 = 1$ and thus $u \cdot v \to u$. ∎

Let us conclude by giving a criterion for a word to wobble inside two others. This will be useful for determining all possible paths between two given flags.

Proposition 5.36. *Let $u \cdot v$ and w be reduced. If $u \cdot w \xrightarrow{*} u$ and $w^{-1} \cdot v \xrightarrow{*} v$, then $|w| \subset \mathrm{Wob}(u, v)$.*

Proof. By Remark 5.17 and Lemma 5.35, it is enough to prove that w is properly right-absorbed by u (and likewise for v). We proceed by induction on the length of $|v|$.

If $v = 1$, then $w^{-1} \cdot 1 \xrightarrow{*} 1$ implies $w^{-1} = 1$, since w is reduced.

Suppose now that $v = s \cdot v_1$. Set $u \cdot s = u_1$, which is again reduced. So is $u_1 \cdot v_1 = u \cdot v$.

The condition $w^{-1} \cdot v \xrightarrow{*} v$ implies $v^{-1} \cdot w \xrightarrow{*} v^{-1}$ by Proposition 5.33. This implies

$$v_1^{-1} \cdot (s \cdot w \cdot s) \xrightarrow{*} (v_1^{-1} \cdot s) \cdot s \xrightarrow{*} v_1^{-1}.$$

By the Commutation Lemma (Proposition 5.31), there is a strong reduct w_1 of $s \cdot w \cdot s$ with $v_1^{-1} \cdot w_1 \xrightarrow{*} v_1^{-1}$, or equivalently, $w_1^{-1} \cdot v_1 \xrightarrow{*} v_1$.

The Triangle Lemma 5.33 shows that $s \cdot (w \cdot s) \xrightarrow{*} w_1$ implies $w_1^{-1} \cdot s \xrightarrow{*} s \cdot w^{-1}$, that is, $s \cdot w_1 \xrightarrow{*} w \cdot s$.

In particular, $u_1 \cdot w_1 = u \cdot (s \cdot w_1) \xrightarrow{*} u \cdot (w \cdot s) \xrightarrow{*} u \cdot s = u_1$.

By the induction hypothesis applied to u_1, v_1 and w_1, we find that w_1 is properly right-absorbed by $u_1 = u \cdot s$. By Lemma 5.9(3), write w_1 as $w_s \cdot w_u$ where w_s is properly absorbed by s and w_u is properly right-absorbed by u and commutes with s. Note that $s \cdot w_u$ is the only strong reduct of $s \cdot w_1$. Proposition 5.31 shows that the strong reduction $(s \cdot w_1) \cdot s \xrightarrow{*} w \cdot s \cdot s \xrightarrow{*} w$ factors through $s \cdot w_u \cdot s \xrightarrow{*} w$.

Since $s \cdot w_u \cdot s$ is equivalent to $s \cdot s \cdot w_u$, there is a strong reduct x of $s \cdot s$ such that $x \cdot w_u \xrightarrow{*} w$. However, the product $x \cdot w_u$ is already reduced and so $x \cdot w_u = w$. The reduct x is either s or consists of proper subletters of s. Suppose that $x = s$. Then $u \cdot w = u \cdot s \cdot w_u = u \cdot s$, since w_u is properly right-absorbed by u and commutes with s. This contradicts $u \cdot w \xrightarrow{*} u$. Hence, the word x consists of proper subletters of s. By Theorem 5.13, since $u \cdot s$ is reduced, decompose x into $x' \cdot x_1$, where x' is properly right-absorbed by u and $u \cdot x_1$ is reduced. Then $u \cdot x_1$ is the only strong reduct of $u \cdot w = u \cdot x' \cdot x_1 \cdot w_u$. We conclude that $u \cdot x_1 = u$ and thus $x_1 = 1$ by Corollary 5.14. Hence, the word $w = x' \cdot w_u$ is properly right-absorbed by u. ■

6. Flags and paths. Let M be any colored N-space. As in Definition 4.16, recall that a *flag* F in M is a path $a_0 - \cdots - a_N$ of length N, where each a_i belongs to $\mathcal{A}_i(M)$. We call a_i the *i-vertex* of the flag F.

DEFINITION 6.1. Given flags F and G, we say that G is obtained from F by the *weak operation* α_s if s consists of the indices where the vertices of F and G differ. A *weak flag path* P is a sequence of flags $F_0, \ldots, F_n$, where each F_i is obtained from F_{i-1} by a weak operation α_{s_i}. We call $s_1 \cdots s_n$ the *word of* P.

More generally, we define:

DEFINITION 6.2. Let A be a subset of $[0, N]$. Two flags are *equivalent modulo* A if they have the same vertices in all levels outside A. We write F/A for the equivalence class of F modulo A.

Note that F/A is interdefinable with the set of vertices of F with levels outside A. For i in $[0, N]$ and $A_i = [0, N] \setminus \{i\}$, the equivalence class F/A_i is interdefinable with the vertex f_i. We can say that F/A_i and F'/A_j, for i and j immediate successors, are connected in case they belong to a class of a common flag G. This induces a structure bi-interpretable with PS_N.

Any two flags can be connected by a weak flag path: decompose the set I of indices where the vertices of F and G differ as the disjoint union $s_1 \cup \cdots \cup s_n$ of intervals such that s_i and s_j commute for $i \neq j$. Then F and G are connected by a weak path with word $s_1 \cdots s_n$. In particular, we obtain the following.

LEMMA 6.3. *Two flags F and G are equivalent modulo A if and only if they can be connected by a weak path whose word consists of letters contained in A. Furthermore, there is such a path whose word is commuting. In particular, any two flags are connected by a weak path, by taking $A = [0, N]$.*

Commuting letters in a path induce another path whose word is a permutation of the previous one.

LEMMA 6.4. *Let s and t be commuting letters and assume that F and G are connected by a weak flag path with word $s \cdot t$. Then there is a unique weak flag path from F to G with word $t \cdot s$.*

Proof. Given the path $F - H - G$ with word $s \cdot t$, define a new flag H' by replacing the s-part of H by the s-part of F and its t-part by the t-part of G. By construction, the weak path $F - H' - G$ has word $t \cdot s$.

Uniqueness is clear since the s-part and the t-part of H' are determined by those of F and G. ■

Iterating the previous result, since any permutation can be achieved by a sequence of transpositions of adjacent commuting letters, given a weak path P_u from F to G with word u, if v is a permutation of u, we can connect F and G by a weak path P_v with word v. Note that P_v does not depend on the sequence of transpositions and the collection of vertices of flags occurring in P_u agrees with the one of flags in P. We call the path P_v a *permutation of P_u*.

We will now link the words appearing in weak paths with the distance of flags as in Lemma 4.13.

LEMMA 6.5. *Let $t = (l, r)$ and F and G be equivalent modulo t. Let a_l and a_r be the vertices of F (and G) of level l and r, respectively. Given a subletter $s \subset t$, the following are equivalent:*

(a) *The flags F and G have finite s-distance in $M_{a_l}^{a_r}$.*
(b) *The flags F and G are connected by a weak flag path whose letters are contained in t but do not contain s.*

Proof. (a)$\to$(b): Consider a path $b_0, \ldots, b_n$ in $\mathcal{A}_s(M_{a_l}^{a_r})$ connecting two vertices of F and G. For every i in $\{1, \ldots, n-1\}$, pick a flag F_i containing b_i and b_{i+1} which agrees with F and G outside the levels in t. Set $F_0 = F$ and $F_n = G$. If b_{i+1} has level j_i, then F_i and F_{i+1} are equivalent modulo $t \setminus \{j_i\}$. They are thus connected by a weak flag path whose letters are contained in

$t \setminus \{j_i\}$ and therefore none contains s. The concatenation of these flag paths gives the result.

(b)$\to$(a): Let $F = F_0 - \cdots - F_n = G$ be a weak flag path whose letters are in t but do not contain s. For every i in $\{0, n-1\}$, the flags F_i and F_{i+1} have a common vertex in $\mathcal{A}_s(M_{a_l}^{a_r})$. Thus, we can connect F and G by a path whose vertices lie in $\mathcal{A}_s(F_0) \cup \cdots \cup \mathcal{A}_s(F_n)$ and hence between a_l and a_r. ■

In order to distinguish between weak operations between flags and global applications of α_s to nice sets, as in Lemma 4.21, we introduce the following definition, at the level of flags.

Definition 6.6. For $s = (l, r)$, the flag G is obtained by a *global application of* α_s from F if G is obtained by a weak application of α_s from F and its new vertices have infinite distance in $M_{a_l}^{a_r}$ from F, where a_l and a_r are the vertices of F (and G) of level l and r, respectively.

Since a flag is in particular a nice set, these two definitions agree by applying Lemma 6.5 to the case $t = s$:

Corollary 6.7. *Given an interval s and flags F and G, the following are equivalent:*

(a) *The flag G is obtained from F by a global application of α_s, as in Lemma* 4.21.
(b) *The flag G is obtained from F by the weak operation α_s and there is no weak flag path connecting them whose word consists of proper subletters of s.*

Definition 6.8. A *flag path* is a weak flag path where each flag is obtained from its predecessor by a global operation. If F and G are connected by a flag path with word u, we write

$$F \underset{u}{\to} G.$$

A flag path is *reduced* if its word is reduced.

Lemma 6.9. *If there is a weak path from F to G with word u, we have $F \underset{v}{\to} G$ for some v with $v \preceq u$.*

Proof. By Lemma 6.3, choose a weak path $F = F_0 - \cdots - F_n = G$ whose word $v = s_1 \cdots s_n$ is $\preceq$-smaller than u and minimal such. We need only show that this path is a flag path. Otherwise, some operation α_{s_i} is not global and, by Corollary 6.7, we can connect F_{i-1} and F_i by a weak path whose word consists of proper subletters of s_i. The resulting word is $\prec$-smaller than v, contradicting its minimality. ■

Combining the previous result and Corollary 6.7, we obtain the following:

COROLLARY 6.10. *If F and G are equivalent modulo t, then either $F \underset{t}{\rightarrow} G$, or $F \underset{x}{\rightarrow} G$ for some product x whose factors are proper subletters of t.*

Proof. By Lemma 6.3, the flag G is obtained from F by a weak path P whose word x either equals t or consists of letters properly contained in t. By Lemma 6.9, we may assume that P is a flag path. ∎

We can now compose flag paths, using the results of the previous section.

LEMMA 6.11. *Assume $F \underset{s}{\rightarrow} G \underset{t}{\rightarrow} H$.*

(1) *If s and t commute, there is a unique G' with $F \underset{t}{\rightarrow} G' \underset{s}{\rightarrow} H$.*
(2) *If s is a proper subset of t, then $F \underset{t}{\rightarrow} H$. Similarly, if t is a proper subset of s, then $F \underset{s}{\rightarrow} H$.*
(3) *If $s = t$, then either $F \underset{t}{\rightarrow} H$ or $F \underset{x}{\rightarrow} H$, for some product x whose factors are proper subletters of t.*

In particular, a permutation of a flag path yields again a flag path, by (1).

Proof. Property (1) follows easily from Lemma 6.4, since the permutation of a reduced word remains reduced.

For (2), assume $s \subsetneq t$. Then H is equivalent to F modulo t. So by Corollary 6.10, either $F \underset{t}{\rightarrow} H$, or $F \underset{x}{\rightarrow} H$, where x consists of proper subletters of t. The latter implies that $G \underset{s\cdot x}{\longrightarrow} H$, which contradicts the assumption $G \underset{t}{\rightarrow} H$. The proof is similar if t is a proper subset of s.

Property (3) clearly follows from Corollary 6.10, as F and H are equivalent modulo t. ∎

Lemma 6.9 yields the following.

COROLLARY 6.12. *Let F and G be two flags.*

(1) *If $F \underset{u}{\rightarrow} G$, then $F \underset{v}{\rightarrow} G$ for some strong reduct v of u.*
(2) *If u is $\prec$-minimal with $F \underset{u}{\rightarrow} G$, then u is reduced.*

DEFINITION 6.13. Let A be a subset of M and let a_l and a_r be two vertices in A such that a_l lies below a_r in A. The pair (a_l, a_r) is called *open* in A if there are vertices b and c in $A_{a_l}^{a_r}$ whose distance in $M_{a_l}^{a_r}$ is infinite.

A pair as before which is not open is called *closed*.

LEMMA 6.14. *Let $s = (l, r)$ be an interval and M be simply connected. Take a nice subset A of M with two distinguished vertices a_l and a_r of levels l and r, respectively. Given a flag F in A containing a_l and a_r, assume that*

$F \underset{s}{\rightarrow} G$ for some flag G in M. Set $B = A \cup G$. If the pair (a_l, a_r) is closed in A, then:

(1) *The set B is obtained from A by a global application of α_s on (a_l, a_r).*
(2) *The open pairs in B are exactly the open pairs of A together with (a_l, a_r).*

Proof. For the first assertion, by Lemma 4.21, we need only check that

$$\mathrm{d}^{M_{a_l}^{a_r}}(d, A) = \infty,$$

where d is one of the new vertices of G.

Pick any b in $A_{a_l}^{a_r}$ and choose some vertex c in F between a_l and a_r. Since (a_l, a_r) is closed in A, we have $\mathrm{d}^{M_{a_l}^{a_r}}(b, c) < \infty$. Since $F \underset{s}{\rightarrow} G$, Lemma 6.5 shows that $\mathrm{d}^{M_{a_l}^{a_r}}(c, d) = \infty$. In particular,

$$\mathrm{d}^{M_{a_l}^{a_r}}(b, d) = \infty,$$

which gives the desired result.

For the second assertion, clearly (a_l, a_r) is now open in B. We need only show there are no new open pairs in B. Consider an open pair (x, y). If x is one of the new elements of G, then y is either also in $B \setminus A$ or in A and either equal to a_r or above it. If both x and y lie in $B \setminus A$, they form a closed pair. If $y = a_r$, all vertices between x and y lie on $B \setminus A$, and thus the pair (x, y) is closed. If y lies above a_r in A, then all vertices between x and y are connected with a_r and thus their distance is finite, so (x, y) is closed.

Hence, we conclude that both x and y lie in A. Suppose (x, y) is not (a_l, a_r). Either it was already open in A or there is a vertex d in $B \setminus A$ whose distance to some b in A is infinite in M_x^y. In particular, the vertex x lies below a_l, and y lies above a_r. Since (x, y) is closed in A, the distance between b and a_l in M_x^y is finite and thus b and d have finite distance in M_x^y, which is a contradiction. ∎

Flag paths provide scaffolds which are nice sets, as the following lemma shows.

Lemma 6.15. *Let M be simply connected and $F_0 \underset{s_1}{\longrightarrow} F_1 \underset{s_2}{\longrightarrow} \ldots \underset{s_n}{\longrightarrow} F_n$ be a reduced flag path in M. Then:*

(1) *The set $A_n = F_0 \cup F_1 \cup \cdots \cup F_n$ is nice in M.*
(2) *If $a_0 - \cdots - a_N$ are the vertices of F_n, then (a_l, a_r) is open in A_n if and only if the letter (l, r) belongs to the final segment of $s_1 \cdots s_n$.*

Proof. We use induction on n. Let $s_i = (l_i, r_i)$ and $w_i = s_1 \cdots s_i$. If $n = 0$, there is nothing to prove, since any flag is nice and the word w_0 is trivial.

Suppose hence that $n > 0$ and let $F_n = a_0 - \cdots - a_N$. Since w_n is reduced by assumption, the letter s_n does not belong to the final segment of w_{n-1}. Therefore, the pair (a_{l_n}, a_{r_n}) appeared already in F_{n-1} and, by induction, it is closed in A_{n-1}, which is nice. Lemma 6.14 shows that so is A_n.

Furthermore, Lemma 6.14 also implies that (a_l, a_r) is open in A_n if and only if $(a_l, a_r) = (a_{l_n}, a_{r_n})$ or it belongs to A_{n-1} and was already open in A_{n-1}. In particular, the pair (a_l, a_r) belongs to A_{n-1} if and only if either (l, r) commutes with s_n or (l, r) contains s_n. Since s_n is not contained in the final segment of w_{n-1}, induction implies that (a_l, a_r) is open in A_n iff $(l, r) = s_n$ or (l, r) commutes with s_n and belongs to the final segment of w_{n-1}, which means that (l, r) belongs to the final segment of w_n. ■

If the space is simply connected, we shall prove that there are no flag loops, unless they are not reduced.

Corollary 6.16. *If M is simply connected, there are no non-trivial closed reduced flags paths.*

Proof. Let $F_0 \underset{s_1}{\longrightarrow} F_1 \underset{s_2}{\longrightarrow} \cdots \underset{s_n}{\longrightarrow} F_n$ be a non-trivial reduced flag path. By Lemmata 6.14 and 6.15, the flag F_n is obtained by a global application of α_{s_n} to $F_0 \cup \cdots \cup F_{n-1}$. In particular, the flag F_n must differ from F_0. ■

Since there are no loops, the reduced word of a flag path is unique, up to permutation.

Proposition 6.17. *The word of a reduced path between two flags F and G is uniquely determined up to equivalence.*

Proof. If u and v are both reduced and there are two flag paths $F \underset{u}{\rightarrow} G$ and $F \underset{v}{\rightarrow} G$ connecting F and G, composing them we get a weak path $F - F$ with word $u \cdot v^{-1}$. Corollary 6.12 yields a strong reduct w of $u \cdot v^{-1}$ with $F \underset{w}{\rightarrow} F$. Corollary 6.16 implies that $w = 1$ and thus $u \approx v$ by Corollary 5.34. ■

If u is reduced, we will sometimes refer to $F \underset{u}{\rightarrow} G$ by saying that *the reduced word u connects F to G.*

Lemma 6.18. *Let M be simply connected and P be a reduced flag path in M. Denote by A the set of vertices of flags occurring in P. Every flag contained in A appears in some permutation of P.*

Proof. We use induction on the length of P. Let $u = v \cdot s$ be the word of P with $s = (l, r)$. Split P in a path Q from F to G with word v and in the path from G to H with word s. Denote by B the vertices of flags occurring in Q. Consider a flag $K \subset A$. If $K \subset B$, then K occurs in a permutation of Q by induction. Thus, it occurs in a permutation of P. If $K \not\subseteq B$, since u is reduced, the letter s does not belong to the final segment of v, so by Lemma 6.15 the pair (a_l, a_r) in K is closed. Lemma 6.14 shows that H is obtained by applying the operation α_s to the nice set B. So $K \underset{w}{\rightarrow} H$, where the reduced word w commutes with s. By Lemma 6.3, there is a unique $G' \subset B$ such that $G' \underset{w}{\rightarrow} G$ and $G' \underset{s}{\rightarrow} K$. Induction implies that G' is part of a reduced path $F \rightarrow G' \underset{w}{\rightarrow} G$, which is a permutation of Q. Then $F \rightarrow G' \underset{w}{\rightarrow} G \underset{s}{\rightarrow} H$ is a permutation of P. We permute w and s and obtain $F \rightarrow G' \underset{s}{\rightarrow} K \underset{w}{\rightarrow} H$, as desired. ∎

Once the word of a flag path between F and G is fixed, the intermediate flags appearing in the path are unique up to wobbling.

LEMMA 6.19 (Wobbling Lemma). *Given two paths between F and G with reduced word $s_1 \cdots s_i \cdots s_n$,*

$$\begin{tikzcd} & H_1 \arrow[r] & \cdots \arrow[r] & H_{n-1} \arrow[dr, "s_n"] & \\ F \arrow[ur, "s_1"] \arrow[dr, "s_1"'] & & & & G \\ & H'_1 \arrow[r] & \cdots \arrow[r] & H'_{n-1} \arrow[ur, "s_n"'] & \end{tikzcd}$$

the flags H_i and H'_i are equivalent modulo $\mathrm{Wob}(s_1 \cdots s_i, s_{i+1} \cdots s_n)$ *for every i in $\{1, \ldots, n-1\}$.*

Proof. Write $u = s_1 \cdots s_i$ and $v = s_{i+1} \cdots s_n$. Suppose we are given flags H_i and H'_i as in the previous picture. Hence

$$F \underset{u}{\rightarrow} H_i \underset{v}{\rightarrow} G, \qquad F \underset{u}{\rightarrow} H'_i \underset{v}{\rightarrow} G.$$

Let w be some reduced word with $H_i \underset{w}{\rightarrow} H'_i$. By Corollary 6.12 and Proposition 6.17, the word u is a strong reduct of $u \cdot w$. Likewise, the word v is a strong reduct of $w^{-1} \cdot v$. Proposition 5.36 gives $|w| \subset \mathrm{Wob}(u, v)$, which yields the result. ∎

We finish this section by observing that nice sets are flag-connected.

PROPOSITION 6.20. *Let M be simply connected and A some union of flags from M. The set A is nice if and only if any two flags in A can be connected by a reduced flag path which belongs to A.*

Proof. Clearly, any union of flags satisfies $A_a^b = A \cap M_a^b$.

Suppose A is nice. Consider two flags F and G in A and connect them in M by some weak path. Since A is nice, we can find a weak path P belonging to A which is reduced *in the sense of A*. In order to show that P is a flag path (in the sense of M), we need only show that if G is obtained from F by a global application of α_s in A, then it remains a global application of α_s

in M. Equivalently, for any b in $G \setminus F$, if $\mathrm{d}_s^A(b, F) = \infty$ then $\mathrm{d}_s^M(b, F) = \infty$. This is exactly the definition of niceness.

Assume now that any two flags in A are connected in A by a reduced flag path. Consider two vertices b and c in $\mathcal{A}_s(A)$ with finite s-distance in M and choose two flags F and G in A containing b and c, respectively. Lemma 6.5 (with $t = [0, N]$) and Lemma 6.9 imply that we can connect F and G by a reduced path P with word u whose letters do not contain s. By assumption, there is a reduced flag path P' in A connecting F and G as well. Thus, the word of P' is a permutation of u by Proposition 6.17. So, again by Lemma 6.5, the points b and c are s-connected in A and hence A is nice. ■

7. Forking in the free pseudospace. In this section we provide a detailed description of non-forking over nice sets and canonical bases. In particular, we obtain weak elimination of imaginaries. The theory PS_N has trivial forking and is totally trivial, as in [2].

We will work inside a sufficiently saturated model M. We start with an easy observation which follows immediately from Theorem 4.22.

PROPOSITION 7.1. *The theory* PS_N *is* ω*-stable.*

Proof. Work over a countable subset A, which we may assume to be nice. Theorem 4.22 shows that every 1-type over A lies in some nice set B, obtained from A by a finite number of applications of α_s. In particular, there are countably many quantifier-free types of such B's over A and thus countably many types by Corollary 4.29. The theory PS_N is therefore ω-stable. ■

The following result will allow us to determine the type of a flag over a nice set.

PROPOSITION 7.2. *Let X be a nice set and F a flag which is connected to a flag G in X by a reduced flag path P with word u. The following are equivalent:*

(a) *Let v be a reduced word connecting G to another flag G' in X. Then F is connected to G' by the reduct of $u \cdot v$.*
(b) *u is the $\preceq$-smallest word connecting F to a flag in X.*
(c) *u is $\preceq$-minimal among words connecting F to a flag in X.*

Proof. (a)→(b) follows from Corollary 5.30.

(b)→(c) is trivial.

(c)→(a): Let G' be any flag in X. Then G is connected to G' by a flag path P with word v. By Proposition 6.20, we may assume that P is in X. Choose a decomposition $u = u_1 \cdot u' \cdot w$ and $w \cdot v' \cdot v_1 = v$ as in Corollary

5.23, with corresponding paths

$$F \underset{u_1 \cdot u'}{\longrightarrow} F^* \underset{w}{\to} G \underset{w}{\to} G^* \underset{v' \cdot v_1}{\longrightarrow} G',$$

where G^* is a flag in X.

Let b be a strong reduct of $w \cdot w$ connecting F^* to G^*. If $b \not\approx w$, consider the reduced word c which connects F to G^*. Since c is a strong reduct of $u_1 \cdot u' \cdot b$, we have $c \preceq u_1 \cdot u' \cdot b \prec u$, a contradiction. So b is equivalent to w. We obtain a path from F to G' with word $u_1 \cdot u' \cdot w \cdot v' \cdot v_1$. Up to permutation, its only possible strong reduct is $u_1 \cdot w \cdot v_1$. So F connects to G' by the word $u_1 \cdot w \cdot v_1$, which is the reduct of $u \cdot v$. ∎

Definition 7.3. Given a nice set X. We call a flag G in X a *base point* of F over X if the conditions of Proposition 7.2 hold: The word connecting F to G is $\preceq$-minimal among words which connect F with flags in X.

Lemma 7.4. *Let X be a nice set and $F_0 \underset{s_1}{\longrightarrow} \cdots \underset{s_n}{\longrightarrow} F_n$ be a reduced flag path with $F_n \in X$. Then F_n is a base point of F_0 over X if and only if the flag F_{i-1} is obtained from $F_i \cup \cdots \cup F_n \cup X$ by a global application of α_{s_i} for all $i \geq 1$.*

In particular, if F_n is a base point of F_0 over X, then $F_0 \cup \cdots \cup F_n \cup X$ is nice.

Proof. The equivalence for $n = 1$ is clear, since F_0 is obtained by a global application of α_{s_1} from $F_1 \cup X = X$ if and only if there is no connection of F_0 to X by a product of proper subletters of s by Lemma 6.5.

We proceed now by induction over n and assume first that each F_{i-1} is obtained from $F_i \cup \cdots \cup F_n \cup X$ by a global application of α_{s_i}. Lemma 4.21 implies that $Y = F_1 \cup \cdots \cup F_n \cup X$ is nice. Furthermore, the flag F_1 is a base point of F_0 over Y. We will show that property 7.2(a) holds for F_0 and F_n over X. Let G be a flag in X. Choose reduced words x, y and v with

$$F_0 \underset{x}{\to} G, \qquad F_1 \underset{y}{\to} G, \qquad F_n \underset{v}{\to} G.$$

Then x is the reduct of $s_1 \cdot y$ and, by induction, the word y is the reduct of $s_2 \cdots s_n \cdot v$. So x is the reduct of $s_1 \cdots s_n \cdot v$. Therefore, the flag F_n is a base point of F_0 over X.

For the other direction, note first that F_{n-1} is obtained from $F_n \cup X = X$ by a global application of α_{s_n}. So $Y = F_{n-1} \cup F_n \cup X$ is nice. If we can show that F_{n-1} is a base point of F_0 over Y, we can conclude by induction. For that, we will verify 7.2(b). Consider any flag G in Y and let x be the reduced word which connects F_0 to G. If G belongs to X, we have $s_1 \cdots s_{n-1} \prec s_1 \cdots s_n \preceq x$. Otherwise, there are a flag G' in X and a word w commuting with s_n such the following diagram holds:

$$\begin{tikzcd}
& F_{n-1} \arrow[r, "s_n"] & F_n \\
F_0 \arrow[ur, "\prod_{j=1}^{n-1} s_j"] \arrow[dr, "x"'] & & \\
& G \arrow[r, "s_n"] & G' \arrow[uu, "w"']
\end{tikzcd}$$

The reduced word x' connecting F_0 to G' is a strong reduct of $x \cdot s_n$. Minimality of $u = s_1 \cdots s_n$ yields $u \preceq x'$. Corollary 5.28 gives $s_1 \cdots s_{n-1} \preceq x$. ∎

Corollary 7.5. *Let G be a flag in a nice set X. Given a reduced word u, there is a flag F and a path P from F to G with word u such that G is a base point of F over X. The set $X \cup P$ is nice. The type of F over G (and thus over X) is uniquely determined.*

Denote these types by $\mathrm{p}_u(G)$ and $\mathrm{p}_u(G)|X$.

In order to describe the regular types and the dimensions of PS_N, we will need a characterisation of non-forking over nice sets in terms of the reduction of the corresponding words connecting the paths.

Lemma 7.6. *Let F and G be flags, where G lies in a nice set X. The independence $F \mathop{\smile\hspace{-0.9em}\mid}_G X$ holds if and only if G is a base point of F over X.*

Proof. Let u be the reduced word which connects F to G. Then the type $\mathrm{p}_u(G)$ of F over G has a canonical extension $\mathrm{p}_u(G)|Y$ to every nice set Y which contains G. Since PS_N is stable, it follows that $\mathrm{p}_u(G)|X$ is the only non-forking extension of $\mathrm{p}_u(G)$ to X. ∎

Proposition 7.7. *Given three flags with reduced paths $F \underset{u}{\to} G$, $G \underset{v}{\to} H$ and $F \underset{w}{\to} H$, we have $F \mathop{\smile\hspace{-0.9em}\mid}_G H$ if and only if $u \cdot v \to w$.*

Proof. If $F \mathop{\smile\hspace{-0.9em}\mid}_G H$, there is a nice set X containing G and H such that $F \mathop{\smile\hspace{-0.9em}\mid}_G X$. But then G is a base point of F over X and $u \cdot v \to w$ follows.

Assume now $u \cdot v \to w$. Let P be the reduced path from G to H with word v. The set P is nice. It is enough to show $F \mathop{\smile\hspace{-0.9em}\mid}_G P$ by verifying 7.2(a). Given any flag G' in P, by Lemma 6.18 we may assume that G' occurs in P. Thus, write $v_1 \cdot v_2 = v$ with $G \underset{v_1}{\longrightarrow} G' \underset{v_2}{\longrightarrow} H$. If x is reduced with $F \underset{x}{\to} G'$, then

$$u \cdot v = (u \cdot v_1) \cdot v_2 \overset{*}{\to} x \cdot v_2 \overset{*}{\to} w.$$

By assumption $u \cdot v \to w$, so Proposition 5.32 shows that no splitting occurs in the strong reductions above. This implies that $u \cdot v_1 \to x$, which completes the proof. ∎

Note that the previous proof also yields $x \cdot v_2 \to w$, which will be used in the proof of Lemma 7.19. Furthermore, we have the following:

Corollary 7.8. *Given flags F, G and H with $F \mathop{\smile\!\!\!\!\!\!\;\mid}_G H$, we have*

$$F \mathop{\smile\!\!\!\!\!\!\;\mid}_G P,$$

where P is the reduced flag path connecting G to H.

We will now compute the Morley rank $\mathrm{MR}(p)$ and Lascar rank $\mathrm{U}(p)$ of certain types in PS_N.

Definition 7.9. Given reduced words u and v, we say that u is a *proper left-divisor* of v if $u \not\approx v$ and there is a reduced w such that $uw = v$ in $\mathrm{Cox}(N)$.

Note that $uw = v$ in $\mathrm{Cox}(N)$ is equivalent to $u \cdot w \to v$.

If u is a proper left-divisor of v, it follows by Corollary 5.30 that $u \prec v$. In particular, Lemma 5.26 shows that being a proper left-divisor is well-founded. Let $\mathrm{R}_{\mathrm{div}}$ be its foundation rank and likewise let $\mathrm{R}_{\prec}$ denote the foundation rank with respect to $\prec$.

Lemma 7.10. *For every flag G and every reduced word u,*

$$\mathrm{U}(\mathrm{p}_u(G)) = \mathrm{R}_{\mathrm{div}}(u).$$

Proof. We show $\mathrm{U}(\mathrm{p}_u(G)) \leq \mathrm{R}_{\mathrm{div}}(u)$ by induction on $\mathrm{R}_{\mathrm{div}}(u)$. Assume that $\alpha < \mathrm{U}(\mathrm{p}_u(G))$. Then there is is a nice extension X of G and a realisation F of $\mathrm{p}_u(G)$ such that $\alpha \leq \mathrm{U}(F/X)$. Since $F \not\mathop{\smile\!\!\!\!\!\!\;\mid}_G X$, the type of F over X is of the form $\mathrm{p}_v(H)|X$ for a reduced word v and some flag H in X. Proposition 7.2(a) and Lemma 7.6 imply that v is a proper left-divisor of u. By induction, we have

$$\alpha \leq \mathrm{U}(F/X) = \mathrm{U}(\mathrm{p}_v(H)) = \mathrm{R}_{\mathrm{div}}(v) < \mathrm{R}_{\mathrm{div}}(u),$$

which proves $\mathrm{U}(\mathrm{p}_u(G)) \leq \mathrm{R}_{\mathrm{div}}(u)$.

For the other direction, assume $\alpha < \mathrm{R}_{\mathrm{div}}(u)$. Then there is a proper left-divisor v of u such that $\alpha \leq \mathrm{R}_{\mathrm{div}}(v)$. Choose a reduced word w such that $v \cdot w \to u$. It is easy to construct a flag H with

$$F \underset{v}{\to} H \underset{w}{\to} G.$$

Actually, such an H exists whenever $v \cdot w \xrightarrow{*} u$. By Proposition 7.7 we have $F \mathop{\smile\!\!\!\!\!\!\;\mid}_H G$. Let P be a path from H to G with associated word w. Seen as a collection of points, the path P is nice by Lemma 6.15. Corollary 7.8 gives that $F \mathop{\smile\!\!\!\!\!\!\;\mid}_H P$, so $\mathrm{tp}(F/P) = \mathrm{p}_v(H)|P$ and thus $F \not\mathop{\smile\!\!\!\!\!\!\;\mid}_G P$. By induction,

$$\alpha \leq \mathrm{R}_{\mathrm{div}}(v) = \mathrm{U}(\mathrm{p}_v(H)) < \mathrm{U}(\mathrm{p}_u(G). \ \blacksquare$$

Lemma 7.11. *For every flag G and reduced word u, we have*

$$\mathrm{MR}(\mathrm{p}_u(G)) \leq \mathrm{R}_{\prec}(u).$$

Proof. Extend $\mathrm{p}_u(G)$ to $p = \mathrm{p}_u(G)|X$, where X is an ω-saturated model containing G. The type p contains a formula $\varphi(x)$ stating that there is a weak path connecting the flag x to G with word u. If F realizes φ, then either F realizes p or there is a path connecting F to X with word $\prec$-smaller than u. For the latter, induction implies that the Morley rank of F over X is strictly smaller than $\mathrm{R}_\prec(u)$. Since X is ω-saturated, this implies that $\mathrm{MR}(p) \leq \mathrm{R}_\prec(u)$. ∎

LEMMA 7.12. *If $u = s_1 \cdots s_n$ is reduced and $|s_i| \geq |s_{i+1}|$ for $i = 1, \ldots, n-1$, then*

$$\mathrm{R}_{\mathrm{div}}(u) = \mathrm{R}_\prec(u) = \omega^{|s_1|-1} + \cdots + \omega^{|s_n|-1}.$$

Proof. Let ord be the function introduced in the proof of Lemma 5.26. Recall that for any reduced word w,

$$\mathrm{R}_{\mathrm{div}}(w) \leq \mathrm{R}_\prec(w) \leq \mathrm{ord}(w).$$

If u satisfies the above hypotheses, then $\mathrm{ord}(u) = \omega^{|s_1|-1} + \cdots + \omega^{|s_n|-1}$. Hence, we need only that $\mathrm{ord}(u) \leq \mathrm{R}_{\mathrm{div}}(u)$. By induction, it is enough to find, for every $\alpha < \mathrm{ord}(u)$, a proper left-divisor u' of u satisfying the hypotheses of the lemma such that $\alpha \leq \mathrm{ord}(u')$.

There are two cases: If $|s_n| = 1$, set $u' = s_1 \cdots s_{n-1}$. If $|s_n| > 1$, let k be large enough such that

$$\alpha \leq \omega^{|s_1|-1} + \cdots + \omega^{|s_{n-1}|-1} + \omega^{|s_n|-2} \cdot k.$$

Then choose an appropriate sequence $t_1 \cdots t_k$ of subletters of s_n, each of size $|s_n| - 1$, such that $u' = s_1 \cdots s_{n-1} \cdot t_1 \cdots t_k$ is reduced. ∎

COROLLARY 7.13. *For every flag G and every reduced word $u = s_1 \cdots s_n$ with $|s_i| \geq |s_{i+1}|$ for $i = 1, \ldots, n-1$,*

$$\mathrm{U}(\mathrm{p}_u(G)) = \mathrm{MR}(\mathrm{p}_u(G)) = \omega^{|s_1|-1} + \cdots + \omega^{|s_n|-1}.$$

However, Lascar and Morley rank may differ in general, as the following example shows.

REMARK 7.14. Consider the word $u = [0,1][1,3]$. It is easy to see that $\mathrm{R}_{\mathrm{div}}(u) = \omega^2$ and $\mathrm{R}_\prec(u) = \omega^2 + \omega$, since the inversion antiautomorphism $u \to u^{-1}$ preserves $\prec$. In particular, the Lascar rank of $\mathrm{p}_u(G)$ is ω^2. To compute the Morley rank of $\mathrm{p}_u(G)$, consider the following sequence of words:

$$u_k = \underbrace{[1][0] \cdots [1][0]}_{k}[1,3].$$

The Morley rank of u_k is at least $\mathrm{R}_{\mathrm{div}}(u_k) = \omega^2$. Since $\mathrm{p}_u(G)$ is the limit of the types $\mathrm{p}_{u_k}(G)$, its Morley rank of $\mathrm{p}_u(G)$ is at least $\omega^2 + 1$. Actually, it is easy to show that $\mathrm{MR}(\mathrm{p}_u(G)) = \omega^2 + 1$.

The non-orthogonality classes of regular types over a nice set in PS_N are given by global operations of α_s for s varying among all intervals. These types have trivial forking and therefore so does PS_N.

Theorem 7.15. *The theory* PS_N *is* ω*-stable of rank* ω^N*. Every type over a nice set* X *is non-orthogonal to some type* $\mathrm{p}_s(G)|X$*, where* G *lies in* X*. Forking is trivial, that is, any three pairwise independent tuples are independent* (*as a set*).

Proof. By Lemma 7.11, the Morley ranks of a flag cannot exceed $\mathrm{R}_\prec([0,N]) = \omega^N = \mathrm{U}(\mathrm{p}_{[0,N]}(G)) = \mathrm{MR}(\mathrm{p}_{[0,N]}(G))$, by Corollary 7.13. Thus, the Lascar and Morley ranks of a flag over the empty set are both ω^N. Let a be a vertex of F. The Lascar inequalities imply that $\mathrm{U}(F/a)+\mathrm{U}(a) \le \mathrm{U}(F)$. Since $\mathrm{U}(a) > 0$, this implies that $\mathrm{U}(a) = \omega^N$, and therefore $\mathrm{MR}(a) = \omega^N$.

Given a type p over X, we may assume it is the type of a flag F and thus determined by some reduced word u connecting F to a base point G over X. In particular, take any s in the final segment of u. The type p is hence non-orthogonal to the type $\mathrm{p}_s(G)|X$, since the connecting word of F over the nice set consisting of G together with a realisation of $\mathrm{p}_s(G)|X$ is $\prec$-smaller than u.

Since the type $\mathrm{p}_s(G)$ has monomial Lascar rank, it is regular. A different way to see this is by taking a non-forking realisation F of $\mathrm{p}_s(G)|X$ and a forking realisation F' to X. Now, since F' forks with X over G, Proposition 7.2(b) gives a flag G' in X such that the word connecting F' to G' is a finite product x of proper subletters of s. Since the reduction $s \cdot x \xrightarrow{*} s$ involves no splitting, the flags F and F' are independent over G by Proposition 7.7. The type $\mathrm{p}_s(G)$ is regular, and so is $\mathrm{p}_s(G)|X$.

Note that the geometry on every type $\mathrm{p}_s(G)$ is trivial: given three pairwise independent realisations F_1, F_2 and F_3 of $\mathrm{p}_s(G)$, note that any flag in $G \cup F_2 \cup F_3$ must be either G, F_2 or F_3, for there are no new s-connections between them. Hence, $F_1 \mathop{\smile\!\!\!\!\!\!\;|}_G F_2 \cup F_3$ and forking is trivial on each $\mathrm{p}_s(G)|X$. Since the theory is superstable, forking is trivial [6, Proposition 2]. ∎

Nice sets are algebraically closed in $\mathrm{PS}_N^{\mathrm{eq}}$.

Remark 7.16. Let X be nice and F be a flag with $F/A \in \mathrm{acl}^{\mathrm{eq}}(X)$ for some set $A \subset [0,N]$. Then the class F/A lies in X^{eq}. That is, all vertices of F with level outside A belong to X.

Since X is nice, this is equivalent to $F/A = G/A$ for some G in X.

Proof. Let u be the reduced word connecting F to a base point G over X. By taking a sufficiently large initial segment of a sequence of X-independent realisations of $\mathrm{tp}(F/X)$, since the class F/A is algebraic, we may find another realisation F' with $F \mathop{\smile\!\!\!\!\!\!\;|}_G F'$ and $F/A = F'/A$. By Lemmata 6.3 and 6.9, there is a path connecting F and F' whose reduced word v satisfies $|v| \subset A$. Proposition 7.7 and the independence $F \mathop{\smile\!\!\!\!\!\!\;|}_G F'$ imply that v is the reduct of $u \cdot u^{-1}$. Thus $|u| = |u \cdot u^{-1}| = |v| \subset A$. In particular, the flags F and G are equivalent modulo A. ∎

Let us now explicitly describe canonical bases of types over nice sets. They are interdefinable with finite sets of real elements and hence PS_N has weak elimination of imaginaries (cf. Corollary 7.24).

THEOREM 7.17. *Let u be a reduced word and G a flag. Then the canonical base of $\mathrm{p}_u(G)$ is interdefinable with $G/\mathcal{S}_{\mathrm{R}}(u)$.*

Observe that $G/\mathcal{S}_{\mathrm{R}}(u)$ is interdefinable with a finite set by Definition 6.2.

Proof. We have to show that $\mathrm{p}_u(G)$ and $\mathrm{p}_u(G')$ have a common nonforking extension if and only if G and G' are equivalent modulo $\mathcal{S}_{\mathrm{R}}(u)$. Or, in other words, given a nice set X, if F is a realisation of $\mathrm{p}_u(G)|X$, then $G' \in X$ is a base point of F over X if and only if $G/\mathcal{S}_{\mathrm{R}}(u) = G'/\mathcal{S}_{\mathrm{R}}(u)$.

If v is a reduced word connecting G and G', then $G/\mathcal{S}_{\mathrm{R}}(u) = G'/\mathcal{S}_{\mathrm{R}}(u)$ means that $|v| \subset \mathcal{S}_{\mathrm{R}}(u)$, or equivalently by Lemma 5.11, v is right-absorbed by u. Let w be the reduced word connecting F to G'. Then w is the reduct of $u \cdot v$ by Proposition 7.2(a). The flag G' is a base point of F if and only if $w \approx u$. By Corollary 5.14, this is equivalent to v being right-absorbed by u. ∎

The following result will be useful in order to prove that the theory PS_N is not $(N+1)$-ample.

LEMMA 7.18 (Base Point Lemma). *Let X be a nice set and F connected by a reduced word u to its base point G in X. Assume $u = w \cdot v$ and pick a flag H with*

$$F \underset{w}{\to} H \underset{v}{\to} G.$$

If $H/A \in X$ for some set $A \subset [0, N]$, then $|v|$ is a subset of A.

Proof. By Remark 7.16 and Corollary 6.12, there is a flag G' in X connected to H by a reduced word $|v'| \subset A$. The flag G is a base point of H over X by Lemma 7.4. Proposition 7.2(b) gives $v \preceq v'$ and therefore $|v| \subset |v'| \subset A$. ∎

We finish the section with a strengthening of triviality, called *totally trivial* [6], that is, given any set of parameters X and tuples a, b and c such that a is both independent from b and c over X, then it is independent of $\{b, c\}$ over X. For theories of finite U-rank, both notions agree [6, Proposition 5].

By Lemma 7.6, recall that, given a nice set X and a distinguished flag F_0 in X, the following are equivalent for any flag F,

- $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} X$,
- $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} H$ for every flag H in X,
- F_0 is a base point of F over X.

Whilst considering flag paths, there is a simpler version of transitivity of non-forking, due to the nature of the reduction with non-splitting.

LEMMA 7.19. *Given flags H, F, H_0 and F_0, then $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H_0$ and $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{H_0} H$ imply $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H$. If there is a reduced path $F_0 \underset{v}{\to} H_0 \underset{w}{\to} H$, the converse also holds: $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H$ implies $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H_0$ and $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{H_0} H$.*

Observe that the condition on the path being reduced is needed for the converse, as the following example shows, where $t \subsetneq s$:

$$\begin{tikzcd}
F_0 \arrow[r, "s"] \arrow[rr, bend left, "t"] & H_0 \arrow[r, "s"] & H \\
& F \arrow[ul, "s"] \arrow[u, "t"] \arrow[ur, "s"'] &
\end{tikzcd}$$

Although $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H$, since no splitting occurs when reducing $s \cdot t$ to s, we see that $F \not\mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H_0$, as t is not the reduct of $s \cdot s$.

Proof. Throughout the proof we will use the characterisation of independence between flags given by Proposition 7.7. It actually follows from the proof of Proposition 7.7 that the above converse holds, by taking F, G, G', H instead of H, F, H_0, F_0 in the proof. Alternatively, we may argue as follows: as H_0 occurs in a reduced path P from F_0 to H, the proof of Proposition 7.7 shows that $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} P$. This implies $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H_0$. Since $F_0 \underset{v}{\to} H_0 \underset{w}{\to} H$, we have $F_0 \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{H_0} H$ by Proposition 7.7. This, together with $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H$, the first part of the lemma and forking symmetry, implies $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{H_0} H$.

Assume now $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H_0$ and $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{H_0} H$. Choose reduced paths $F \underset{u}{\to} F_0$, $F_0 \underset{v}{\to} H_0$, $H_0 \underset{w}{\to} H$ and $F_0 \underset{x}{\to} H$. The word a which connects F to H_0 is the reduct of $u \cdot v$. Also, the word b connecting F to H is the reduct of $u' \cdot w$. Hence, the word b is the reduct of $u \cdot v \cdot w$. If x were the reduct of $v \cdot w$, then b is the reduct of $u \cdot x$, so we are done. Therefore, suppose that splitting occurs in $v \cdot w \overset{*}{\to} x$. We treat first the case $v = w = s$. Then x is a product of proper subintervals of s. By the Decomposition Lemma 5.13, either s is right-absorbed by u, or $u = u_1 \cdot u'$, where u' is properly absorbed by s and $u_1 \cdot s$ is reduced. In the first case, the word x is properly absorbed by u, hence $F \mathop{\smile\hspace{-0.9em}\vert}\nolimits_{F_0} H$.

For the second case, decompose $u = u_1 \cdot u'$ as above. Then b (the word connecting F and H) equals $u_1 \cdot s$. This cannot be a strong reduct of $u_1 \cdot u' \cdot x$, since the latter is $\prec$-smaller, contradicting Proposition 5.32.

For the general case, as in the proof of Proposition 5.32).we may assume that the splitting in $v \cdot w \overset{*}{\to} x$ happens at the first step of the reduction.

Write hence $v = v' \cdot s$ and $w = s \cdot w'$, where

$$F_0 \underset{v'}{\longrightarrow} K_1 \underset{s}{\rightarrow} H_0 \underset{s}{\rightarrow} K_2 \underset{w'}{\longrightarrow} H.$$

The word y connecting K_1 and K_2 consists of proper subletters of s. By the first part of the proof, since $F \mathop{\smile\kern-0.9em\mid}_{F_0} H_0$, we have $F \mathop{\smile\kern-0.9em\mid}_{F_0} K_1$ and $F \mathop{\smile\kern-0.9em\mid}_{K_1} H_0$. Similarly, we obtain $F \mathop{\smile\kern-0.9em\mid}_{H_0} K_2$ and $F \mathop{\smile\kern-0.9em\mid}_{K_2} H$. By the previous discussion, we have $F \mathop{\smile\kern-0.9em\mid}_{K_1} K_2$. This, together with $F \mathop{\smile\kern-0.9em\mid}_{F_0} K_1$, yields $F \mathop{\smile\kern-0.9em\mid}_{F_0} K_2$, by induction on the length of v. Now, the word connecting $F_0 \to K_2$ is a strong reduction of $v' \cdot y$, so $\prec$-smaller than v. Induction on the complexity of v together with $F \mathop{\smile\kern-0.9em\mid}_{K_2} H$ gives $F \mathop{\smile\kern-0.9em\mid}_{F_0} H$, as desired. ∎

In order to prove the total triviality of PS_N, we will use the following lemma, a stronger form of which follows already from total triviality, without the assumption $F_0 \mathop{\smile\kern-0.9em\mid}_A B$, since if

$$A \underset{s}{\rightarrow} B \underset{t}{\rightarrow} C,$$

where s and t commute with each other, then B is definable in $A \cup C$, by Lemma 6.19.

Lemma 7.20. *Let A, B, C, F, F_0 be flags and s and t two commuting letters such that $A \underset{s}{\rightarrow} B \underset{t}{\rightarrow} C$. If the following independencies hold:*

$$F \mathop{\smile\kern-0.9em\mid}_{F_0} A, \qquad F \mathop{\smile\kern-0.9em\mid}_{F_0} C, \qquad F_0 \mathop{\smile\kern-0.9em\mid}_{A} B,$$

then $F \mathop{\smile\kern-0.9em\mid}_{F_0} B$.

Proof. In order to show that $F \mathop{\smile\kern-0.9em\mid}_{F_0} B$, since $F \mathop{\smile\kern-0.9em\mid}_{F_0} A$, by Lemma 7.19 we need only show $F \mathop{\smile\kern-0.9em\mid}_{A} B$. Thus, consider a reduced word z with $F \underset{z}{\rightarrow} B$ and connect the above flags by reduced paths as in the diagram below.

$$\begin{tikzcd}
 & B & \\
A \arrow[ur, "s"] & & C \arrow[ul, "t"'] \\
 & F_0 \arrow[uu, "v"] \arrow[ul, "x"] \arrow[ur, "y"'] & \\
 & F \arrow[u, "u"] \arrow[uul, "a"] \arrow[uur, "b"'] &
\end{tikzcd}$$

Assume for a contradiction that $F \not\mathop{\smile\kern-0.9em\mid}_{A} B$. Then z, which is a strong reduct of $a \cdot s$, is not the reduct of $a \cdot s$. This has two consequences: first, the letter s does not occur in the final segment of z. Secondly, up to permutation, the path $F \underset{a}{\to} A$ ends with a flag $A' \underset{s}{\to} A$ such that A' is connected to B by a word consisting of proper subletters of s. Since $F_0 \mathop{\smile\!\!\!\!\!\!|}_A B$, such a flag A' cannot occur in any permutation of x. Thus, as a is a reduct of $u \cdot x$, it follows that s commutes with x and is in the final segment of u. In particular, the word $x \cdot s$ is reduced, which implies that v is (up to permutation) the word $x \cdot s$.

On the other hand, the word $v = x \cdot s$ is a strong reduct of $y \cdot t$. It is easy to see that this can only be possible if (after permutation) y has the form $y' \cdot s$, where y' and s commute. The independence $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} C$ implies that b is the reduct of $u \cdot y$. Hence s still belongs to the final segment of b. Finally, since z is a strong reduct of $b \cdot t$, the word s must belong to the final segment of z, which contradicts $F \not\mathop{\smile\!\!\!\!\!\!|}_A B$. ■

In order to ensure the independence of a flag with respect to a whole flag path over a nice set, it is enough to check the independence with respect to the set itself and the end flag of the path.

Lemma 7.21. *Let A be a nice set and a reduced path P connecting a flag H to a base point in A. Given a flag F_0 in A and a flag F, we have $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} A \cup P$ if and only if $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} A$ and $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} H$.*

Proof. Left-to-right is clear. Assume now that $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} A$ and $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} H$. Since $A \cup P$ is nice by Lemma 7.4, in order to check that $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} A \cup P$, we need to check that $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} H'$ for any flag H' in $A \cup P$ by the remark above Lemma 7.19. This is clear for flags in A, so let H' be in $A \cup P$ but not in A.

We treat first the case where H' is in P. Let H_0 be the base point of H in A. We then have $F_0 \mathop{\smile\!\!\!\!\!\!|}_{H_0} H$ and $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} H$ by assumption, which implies $F \mathop{\smile\!\!\!\!\!\!|}_{H_0} H$ by Lemma 7.19. Since the path P is reduced, Lemma 7.19 gives $F \mathop{\smile\!\!\!\!\!\!|}_{H_0} H'$, which together with $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} H_0$ implies $F \mathop{\smile\!\!\!\!\!\!|}_{F_0} H'$.

For the general case, we will proceed by induction on the length of P, based on the above paragraph. Thus, it suffices to consider the case where P has length 1 and let s be its letter:

$$H_0 \underset{s}{\to} H.$$

If H' is a flag in $A \cup P$ not completely contained in A, it differs from H only on the indices outside s. As in the proof of Lemma 6.18, we can find a reduced word w commuting with s such that $H' \underset{w}{\to} H$. Furthermore, there is some flag H'_0 in A with $H'_0 \underset{w}{\to} H_0$ and $H'_0 \underset{s}{\to} H'$.

Note that H'_0 is again a base point of H' over A, so in particular $F_0 \mathop{\smile\!\!\!\!\!\!|}_{H'_0} H'$. By induction on the length of w, we may assume that w is a letter t. Setting

$A = H'_0$, $B = H'$ and $C = H$, the hypotheses of Lemma 7.20 are satisfied. We conclude that $F \mathop{\smile\!\!\!\!\!\!\,|}_{F_0} H'$, which gives the desired result. ∎

We have now all the ingredients to prove total triviality of forking.

PROPOSITION 7.22. *The theory* PS_N *is totally trivial, that is, given any set of parameters* X *and tuples* a, b *and* c *such that* a *is independent of both* b *and* c *over* X, *then it is independent of* $\{b, c\}$ *over* X. *In particular, the canonical base of a tuple is the union of the canonical bases of each singleton.*

Proof. We may assume that our parameter set X is nice, by choosing a small model containing it, independent of a, b, c.

Suppose first that the tuples a, b and c consist of singletons: By transitivity, choose flags H_1 and H_2 independently of a over X containing b and c respectively. Choose now a flag F containing a independently of H_1 and of H_2 over X. We need only show that

$$F \mathop{\smile\!\!\!\!\!\!\,|}_{X} H_1 \cup H_2.$$

Let F_0 and H_0 be base points of F and H_1 respectively over X. Since $F \mathop{\smile\!\!\!\!\!\!\,|}_{F_0} X$ and $F \mathop{\smile\!\!\!\!\!\!\,|}_{X} H_1$, we have $F \mathop{\smile\!\!\!\!\!\!\,|}_{F_0} X \cup P_1$ by Lemma 7.21, where P_1 denotes the reduced flag path (connecting H_1 to H_0) determined by H_1 over X. The set $X \cup P_1$ is again nice by Lemma 7.4. We work now over $X \cup P_1$ in order to show that $F \mathop{\smile\!\!\!\!\!\!\,|}_{F_0} X \cup P_1 \cup P_2$, where P_2 is the flag path given by H_2 over $X \cup P_1$. Lemma 7.21 shows that F is independent of $H_1 \cup H_2$ over X.

Transitivity of forking allows us to work with finite tuples by choosing accordingly non-forking extensions for each coordinate. The result now follows by local character. ∎

Since PS_N is superstable, [6, Proposition 7] yields the following.

COROLLARY 7.23. *The theory* PS_N *is perfectly trivial, that is, given any set of parameters* X *and tuples* a, b *and* c *such that* a *and* b *are both independent over* X, *then they are so over* $X \cup \{c\}$.

COROLLARY 7.24. *The theory* PS_N *has weak elimination of imaginaries.*

Proof. By Proposition 7.22, in order to study the canonical base of a real tuple $\bar{a}$ over an algebraically closed set B (in $\mathrm{PS}_N^{\mathrm{eq}}$), we may assume that $\bar{a}$ is an enumeration of a flag F. Furthermore, we may suppose that B is nice. By Theorem 7.17, the canonical base is interdefinable with a finite set, thus we get weak elimination of imaginaries. ∎

Although the theory PS_N is not 1-based, being N-ample by Proposition 8.1, it is 2-based, i.e. the canonical base of a type is determined by two independent realisations.

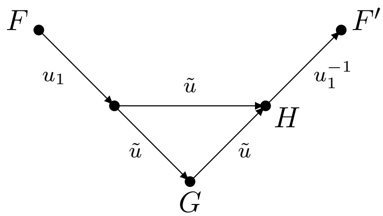
Proposition 7.25. *Let u be a reduced word and X a nice set. The canonical base of* $\mathrm{p}_u(G)|X$ *is algebraic over two independent realisations.*

Proof. Let F and F' be realisations of $\mathrm{p}_u(G)|X$, which are X-independent. Since the base point is only determined up to $\mathcal{S}_{\mathrm{R}}(u)$-equivalence, pick a common base point G in X for both F and F'.

As $F \mathop{\smile\!\!\!\!\!\!|}_X F'$ and $F \mathop{\smile\!\!\!\!\!\!|}_G X$, combining Lemmata 7.19 and 7.21, we conclude that $F \mathop{\smile\!\!\!\!\!\!|}_G F'$. Therefore, the word connecting F and F' is the reduction of $u \cdot u^{-1}$. Write $u = u_1\tilde{u}$, where $\tilde{u}$ is the final segment of u. Hence,

$$u \cdot u^{-1} \to u_1 \cdot \tilde{u} \cdot u_1^{-1},$$

as the diagram shows:



Note that G and H are equivalent modulo $|\tilde{u}| \subset \mathcal{S}_{\mathrm{R}}(u)$. By Lemma 6.19, the flag H is determined by F and F' modulo $\mathcal{S}_{\mathrm{R}}(u) \cap \mathcal{S}_{\mathrm{L}}(u_1^{-1})$ and thus modulo $\mathcal{S}_{\mathrm{R}}(u)$. In particular, the canonical base $G/\mathcal{S}_{\mathrm{R}}(u)$ is algebraic over F, F'. ∎

8. Ample yet not wide ample. This last section shows that the ample hierarchy defined in 2.2 is proper, since the theory of the free N-dimensional pseudospace PS_N is N-ample but not $(N+1)$-ample. We will furthermore show that it is N-tight with respect to the family Σ of Lascar rank 1 types, if $N \geq 2$.

The proof that PS_N is N-ample is a direct translation of the proof in [2], which we nonetheless include for completeness.

Proposition 8.1. *Consider a flag $a_0 - \cdots - a_N$. We have the following:*

(a) $\mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_{i-1}, a_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_{i-1})$ *for every* $0 \leq i < N$.

(b) $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0, \ldots, a_{i-1}$ *for every* $1 \leq i < N$.

(c) $a_N \not\mathop{\smile\!\!\!\!\!\!|} a_0$.

In particular, the theory PS_N *is N-ample.*

Proof. In order to prove (a), fix some $i < N$ and choose parameters $b_i, \ldots, b_N$ independently of a_i, a_{i+1} such that

$$a_0 - \cdots - a_{i-1} - b_i - \cdots - b_N$$

is a flag. Set $X = \{a_0, \ldots, a_{i-1}, b_i, \ldots, b_N\}$, which is nice.

By Fact 2.1, assume for a contradiction that there is an element e in

$$\mathrm{acl}^{\mathrm{eq}}(X, a_i) \cap \mathrm{acl}^{\mathrm{eq}}(X, a_{i+1}) \setminus \mathrm{acl}^{\mathrm{eq}}(X).$$

Choose now a_i' realising $\mathrm{tp}(a_i/X, e)$. Since e lies also in $\mathrm{acl}^{\mathrm{eq}}(X, a_i')$, we have $a_i \not\perp_X a_i'$. As the $\preceq$-minimal word connecting a_i (or rather the flag $a_0 - \cdots - a_N$) to X is $[i, N]$, it follows from Lemma 7.6 that a_i and a_i' (or rather generic flags containing them) are connected through a finite product of proper intervals of $[i, N]$. Compactness (and Lemma 6.5) implies that there exists a natural number n such that

$$\mathrm{tp}(a_i/X, e) \models \mathrm{d}_{[i,N]}(x, a_i) \leq n.$$

Let m be such that $2m > n$. Consider the reduced word

$$u = \underbrace{[i+1, N] \cdot i \cdots [i+1, N] \cdot i}_{2m}.$$

Corollary 7.5 provides us with a flag F and a path P from $G = a_0 - \cdots - a_N$ to F with word u

$$F = F_0 \xrightarrow[{[i+1,N]}]{} F_0' \xrightarrow[i]{} F_1 \xrightarrow[{[i+1,N]}]{} \cdots \xrightarrow[{[i+1,N]}]{} F_{m-1}' \xrightarrow[i]{} F_m = G$$

such that G is a base point of F over the nice set G. Since the F_i and F_i' are connected by the word $[i, N]$ to G, they have all the same type over X. Denote

$$F_r = a_0 - \cdots - a_{i-1} - a_i^r - a_{i+1}^r - \cdots - a_N^r,$$
$$F_r' = a_0 - \cdots - a_{i-1} - a_i^r - a_{i+1}^{r+1} - \cdots - a_N^{r+1}.$$

Since F_0 and F_0' have the same type over X, they have also the same type over Xa_i^0 and therefore over Xe. This implies that e belongs to $\mathrm{acl}^{\mathrm{eq}}(Xa_{i+1}^1)$. Similarly, the flags F_0' and F_1 have the same type over Xa_{i+1}^1 and therefore over Xe, which implies that e belongs to $\mathrm{acl}^{\mathrm{eq}}(Xa_i^1)$. Iterating, we see that a_i^m has the same type over Xe as a_i. Thus $\mathrm{d}_{[i,N]}(a_i^m, a_i) \leq n$, which gives a contradiction since the shortest path between a_i and a_i^m in $\mathcal{A}_{[0,N]}$ is

$$a_i^0 - a_{i+1}^1 - a_i^1 - \cdots - a_{i+1}^m - a_i^m,$$

of length $2m$.

For (b), choose generic flags F containing a_{i+1} and G containing $a_0, \ldots, a_i$. The canonical base $\mathrm{Cb}(a_{i+1}/a_0, \ldots, a_i)$ equals $\mathrm{Cb}(F/G)$. On the other hand, the flags F and G are connected by the reduced word $u = [0, i][i+1, N]$. So

$$\mathrm{Cb}(F/G) = G/\mathcal{S}_{\mathrm{R}}(u) = G/([0, i-1] \cup [i+1, N]) = a_i$$

by Theorem 7.17, which gives the desired independence.

For (c), choose a generic flag F which contains a_N and a generic flag G which contains a_0. Then $\mathrm{Cb}(a_N/a_0)$ equals $\mathrm{Cb}(F/G)$. On the other hand the reduced word connecting F to G is $u = [0, N-1][1, N]$. So

$$\mathrm{Cb}(F/G) = G/\mathcal{S}_{\mathrm{R}}(u) = G/[1, N] = a_0,$$

which is clearly not algebraic over a_1. Thus, $a_N \not\mathop{\smile\!\!\!\!\!\!\mid}\, a_0$. ■

For the proof that PS_N is not $(N+1)$-ample, we need some auxiliary results on the nature of the reduced words arising from the hypothesis on ampleness.

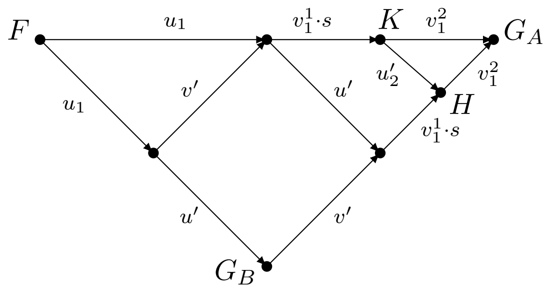
Lemma 8.2. *Consider nice sets A and B and a flag F such that* $\mathrm{acl}^{\mathrm{eq}}(AB) \cap \mathrm{acl}^{\mathrm{eq}}(A, F) = \mathrm{acl}^{\mathrm{eq}}(A)$ *and $F \mathop{\smile\!\!\!\!\!\!\mid}_B A$. Let $u = u_B$ (resp. u_A) be the $\preceq$-minimal word connecting F to a flag G_B in B (resp. G_A in A) and let v be the reduced word connecting G_B to G_A. If*

$$u = u_1 \cdot u', \qquad v' \cdot v_1 = v$$

is the fine decomposition as in Theorem 5.13, *then v_1 is commuting.*

Proof. By hypothesis, $F \mathop{\smile\!\!\!\!\!\!\mid}_{G_B} G_A$, so the product $u_1 \cdot v_1$ is equivalent to u_A. Suppose for a contradiction that v_1 is not commuting. Hence, we may decompose $v_1 = v_1^1 \cdot s \cdot v_1^2$, where v_1^2 is the final segment of v_1 and s does not commute with v_1^2.

By Lemma 5.9, we can write $u' = u_2' \cdot u_1'$, where u_1' is left-absorbed by $v_1^1 \cdot s$, the word u_2' commutes with $v_1^1 \cdot s$ and is left-absorbed by v_1^2. We have the following diagram:



where the path connecting K and H is given by u_2'. So the flags H and K are equivalent modulo $|u_2'|$.

Lemma 5.18 shows that $\mathrm{Wob}(v' \cdot v_1^1 \cdot s, v_1^2)$, the wobbling of v at H, is contained in $W = \mathrm{Wob}(u_1 \cdot v_1^1 \cdot s, v_1^2)$. In particular, by Lemma 6.19, the class H/W lies in $\mathrm{acl}^{\mathrm{eq}}(AB)$. So does $K/(|u_2'| \cup W)$, which also lies in $\mathrm{acl}^{\mathrm{eq}}(AF)$. By assumption, $K/(|u_2'| \cup W)$ lies in $\mathrm{acl}^{\mathrm{eq}}(A)$ since $\mathrm{acl}^{\mathrm{eq}}(AB) \cap \mathrm{acl}^{\mathrm{eq}}(AF) = \mathrm{acl}^{\mathrm{eq}}(A)$, and therefore in A by Remark 7.16. Since u_A is $\preceq$-

minimal connecting F to a flag in A, Lemma 7.18 implies

$$|v_1^2| \subset |u_2'| \cup W.$$

Observe that u_2' centralises s and W is contained in $s \cup \mathrm{C}(s)$. Hence, so does $|v_1^2|$. Since v_1 is reduced and v_1^2 is commuting, no letter of v_1^2 is contained in s. So v_1^2 must commute with s, which contradicts the definition of v_1^2. ■

PROPOSITION 8.3. *Consider nice sets A and B and a flag F such that $\mathrm{acl}^{\mathrm{eq}}(AB) \cap \mathrm{acl}^{\mathrm{eq}}(A, F) = \mathrm{acl}^{\mathrm{eq}}(A)$ and $F \mathop{\smile\!\!\!\!\!\!|}_B A$. Let $u = u_B$ (resp. u_A) be the minimal word connecting F to a flag G_B in B (resp. G_A in A). (These are the same hypotheses as in Lemma 8.2.) Then either $F \mathop{\smile\!\!\!\!\!\!|}_{A\cap B} AB$ or u is non-trivial and its final segment $\tilde{u}$, as a set of indices, is strictly contained in $\tilde{u}_A$, the final segment of u_A.*

In particular, consider the reduced word v which connects G_B to G_A and the associated fine decomposition

$$u = u_1 \cdot u', \qquad v' \cdot v_1 = v,$$

as in Theorem 5.13. If $F \not\mathop{\smile\!\!\!\!\!\!|}_{A\cap B} A$, then $\tilde{u}$ is non-trivial and

$$|v'| \not\subseteq |\tilde{u}| \subsetneq |\tilde{u}_A|.$$

Proof. Since $F \mathop{\smile\!\!\!\!\!\!|}_B A$ and v is reduced connecting G_B to G_A, the word $u \cdot v$ reduces to u_A. If

$$u = u_1 \cdot u', \qquad v' \cdot v_1 = v,$$

is the fine decomposition (cf. Theorem 5.13) applied to u and v, we may thus assume that $u_A = u_1 \cdot v_1$.

Let H be the flag in the path $G_B \xrightarrow[v]{} G_A$ between v' and v_1. Likewise, let K be the flag in the path $F \xrightarrow[u_A]{} G_A$ between u_1 and v_1. Note that H and K are connected through u'. Furthermore, Lemma 5.18 implies that $\mathrm{Wob}(v', v_1)$ is contained in $W = \mathrm{Wob}(u_1, v_1)$. Since H and K are equivalent modulo $|u'|$ and $H/\mathrm{Wob}(v', v_1)$ lies in $\mathrm{acl}^{\mathrm{eq}}(AB)$ by Lemma 6.19, it follows that $K/(W \cup |u'|)$ lies in $\mathrm{acl}^{\mathrm{eq}}(AB) \cap \mathrm{acl}^{\mathrm{eq}}(AF) = \mathrm{acl}^{\mathrm{eq}}(A)$ and hence in A by Remark 7.16. Lemma 7.18 gives now

$$|v_1| \subset |u'| \cup W.$$

Decompose the final segment of u as

$$\tilde{u} = w_1 \cdot w_2,$$

where w_2 is the final segment of u' and w_1 is a subword of the final segment of u_1. In particular $u' = u'' \cdot w_2$ and w_1 and u'' commute. We show first that w_1 and v_1 commute: since $u' \subset \mathrm{C}(w_1)$ and $W \subset \mathcal{S}_{\mathrm{R}}(u_1) \subset |w_1| \cup \mathrm{C}(w_1)$, we have $v_1 \subset |w_1| \cup \mathrm{C}(w_1)$. A letter s of v_1 cannot be contained in $|w_1|$, since $u_1 \cdot v_1$ is reduced. So s belongs to $\mathrm{C}(w_1)$, which gives the desired result. Recall that v_1 is commuting by Lemma 8.2. Thus, the final segment of $u_A = u_1 \cdot v_1$ is

$$\tilde{u}_A = w_1 \cdot v_1,$$

which clearly contains $\tilde{u}$, as $|w_2|$ is a subset of $|v_1|$.

Suppose the inclusion is not strict. Hence, $|w_2| = |v_1|$. Then $|v_1| \subset \mathcal{S}_{\mathrm{R}}(u)$ and hence $|v| \subset \mathcal{S}_{\mathrm{R}}(u)$. So G_B and G_A are equivalent modulo $\mathcal{S}_{\mathrm{R}}(u)$. In particular, the canonical base $\mathrm{Cb}(F/B)$ lies in A and thus $F \mathop{\smile\!\!\!\!\!\!\mid}_{A\cap B} B$. Since $F \mathop{\smile\!\!\!\!\!\!\mid}_{B} A$, transitivity of non-forking implies that $F \mathop{\smile\!\!\!\!\!\!\mid}_{A\cap B} AB$.

Finally, assume that $\tilde{u} = 1$, which forces $u = 1$ and thus $v' = 1$. In particular, since $|v_1| \subset |\tilde{u}_A| \subset \mathcal{S}_{\mathrm{R}}(u_A)$ and G_A and G_B are equivalent modulo $v = v_1$, they are equivalent modulo $\mathcal{S}_{\mathrm{R}}(u_A)$, so $\mathrm{Cb}(F/A) = G_A/\mathcal{S}_{\mathrm{R}}(u_A)$ lies in B and hence $F \mathop{\smile\!\!\!\!\!\!\mid}_{A\cap B} A$.

Similarly, if $|v'| \subset |\tilde{u}| \subset |\tilde{u}_A| \subset \mathcal{S}_{\mathrm{R}}(u_A)$, we conclude as before that $\mathrm{Cb}(F/A) = G_A/\mathcal{S}_{\mathrm{R}}(u_A)$ lies in B and thus $F \mathop{\smile\!\!\!\!\!\!\mid}_{A\cap B} A$. ∎

We can now state and prove the desired result.

THEOREM 8.4. *The theory* PS_N *is not* $(N+1)$*-ample and is* N*-tight with respect to the family of Lascar rank* 1 *types.*

Proof. By Remark 2.5, we need only show that given tuples $b_0, \ldots, b_{N+1}$ with:

(a) $\mathrm{acl}^{\mathrm{eq}}(b_i, b_{i+1}) \cap \mathrm{acl}^{\mathrm{eq}}(b_i, b_{N+1}) = \mathrm{acl}^{\mathrm{eq}}(b_i)$ for every $0 \le i < N$,

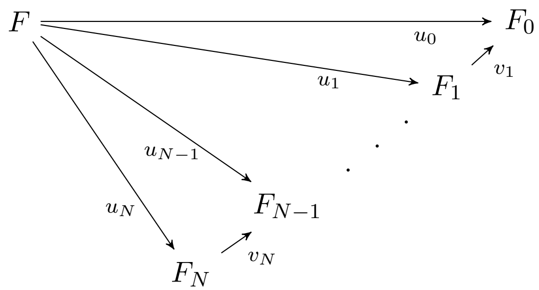
(b) $b_{N+1} \mathop{\smile\!\!\!\!\!\!\mid}_{b_i} b_{i-1}$ for every $1 \le i \le N$,

there is some i in $\{0, \ldots, N-1\}$ such that

$$b_{N+1} \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(b_i)\,\cap\,\mathrm{acl}^{\mathrm{eq}}(b_{i+1})} b_i.$$

By Fact 2.1, it suffices to prove this for tuples $b_0, \ldots, b_N$ which enumerate small models $B_0, \ldots, B_N$, although for the proof, we only require that each B_i is nice. Total triviality (cf. Proposition 7.22) allows us to assume that b_{N+1} consists of a single flag F.

Choose for every $i \le N$ a base point F_i for F over B_i. Note that we obtain the following configuration:



such that $u_i \cdot v_i$ reduces to u_{i-1}, for every i in $\{1, \dots, N\}$, due to (b). Proposition 8.3 implies that either, for some $i < N$,

$$F \underset{B_i \cap B_{i+1}}{\perp} B_i,$$

or the final segment $\tilde{u}_{i+1}$ of u_{i+1} is non-trivial and strictly contained in $\tilde{u}_i$ for all $i < N$.

The second possibility for every $i < N$ provides a strictly increasing sequence of length $N+1$ of non-empty subsets of $\{0, \dots, N\}$, which implies that $\tilde{u}_0$ equals $[0, N]$ and thus $u_0 = [0, N]$. Hence $F \perp B_0$, and thus

$$F \underset{\mathrm{acl}^{\mathrm{eq}}(B_0) \cap \mathrm{acl}^{\mathrm{eq}}(B_1)}{\perp} B_0.$$

The first possibility implies

$$F \underset{\mathrm{acl}^{\mathrm{eq}}(B_i) \cap \mathrm{acl}^{\mathrm{eq}}(B_{i+1})}{\perp} B_i,$$

as desired. This proves that PS_N is not $(N+1)$-ample.

Suppose now that $N \geq 2$. In order to show that PS_N is N-tight with respect to Σ, where Σ denotes the collection of all Lascar rank 1 types, assume we are given tuples $b_0, \dots, b_N$ witnessing the following conditions:

(a) $\mathrm{acl}^{\mathrm{eq}}(b_0, \dots, b_i) \cap \mathrm{acl}^{\mathrm{eq}}(b_0, \dots, b_{i-1}, b_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(b_0, \dots, b_{i-1})$ for every $0 \leq i < N$.
(b) $b_{i+1} \perp_{b_i} b_0, \dots, b_{i-1}$ for every $1 \leq i < N$.

As in Remark 2.5, it follows that:

(c) $\mathrm{acl}^{\mathrm{eq}}(b_{i+1}) \cap \mathrm{acl}^{\mathrm{eq}}(b_i) \subset \mathrm{acl}^{\mathrm{eq}}(b_0)$ for every $1 \leq i < N$.
(d) $b_N \perp_{b_i} b_{i-1}$ for every $1 \leq i < N$.
(e) $\mathrm{acl}^{\mathrm{eq}}(b_i, b_{i+1}) \cap \mathrm{acl}^{\mathrm{eq}}(b_i, b_N) = \mathrm{acl}^{\mathrm{eq}}(b_i)$ for every $0 \leq i < N-1$.

Note that (almost) internality is preserved under taking non-forking restrictions. Furthermore, if a tuple d is (almost) internal over C and e is algebraic over Cd, then e is (almost) internal over C. Thus, we may as before replace every b_i by a nice set B_i by Fact 2.1 and assume that b_N is a flag F by total triviality (cf. Proposition 7.22). In particular, we need to prove that $\mathrm{Cb}(F/B_0)$ is almost Σ-internal over B_1.

As before, let u_i be $\preceq$-minimal connecting F to a flag F_i of F in B_i for $i < N$. Since $N \geq 2$, there is (at least) one triangle to apply Proposition 8.3, and thus either for some $0 \leq i < N-1$ we have

$$F \underset{B_i \cap B_{i+1}}{\perp} B_i,$$

or the final segment $\tilde{u}_{i+1}$ of u_{i+1} is non-trivial and strictly contained in $\tilde{u}_i$ for every $i < N$. The independence $F \perp_{B_i \cap B_{i+1}} B_i$ implies by properties (b) and (c) that $F \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}^{\mathrm{eq}}(B_0)\cap\mathrm{acl}^{\mathrm{eq}}(B_1)} B_0$. So $\mathrm{Cb}(F/B_0)$ is algebraic over B_1, and hence internal over B_1.

Otherwise, if

$$F \underset{B_i \cap B_{i+1}}{\not\mathop{\smile\!\!\!\!\!\!|}} B_i$$

for every $i < N$, then the final segment $\tilde{u}_0$ must have length N. Consider the fine decomposition $u_1 = u_1^1 \cdot u_1'$ and $v_1' \cdot v_1^1 = v_1$ from Theorem 5.13. Proposition 8.3 implies that $|v_1'|$ is not fully contained in $\tilde{u}_1$, which must then have non-trivial centraliser. Since $\tilde{u}_1$ has size $N-1$, it must be either $[2, N]$ or $[0, N-2]$. Let us consider the first case. The canonical base $\mathrm{Cb}(b_N/B_0)$ is F_0 modulo $\mathcal{S}_{\mathrm{R}}(u_0) = [1, N]$, which is the 0-vertex f_0 of F_0. Furthermore, since $v_1 = [0] \cdot [1, N]$, the vertex f_0 is directly connected to B_1 and, by Theorem 7.15, it has rank 1 over B_1, so the canonical base $\mathrm{Cb}(F/B_0)$ is Σ-internal over B_1, which concludes the proof. ■

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