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# $\ell_\infty$-sums and the Banach space $\ell_\infty/c_0$

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**Abstract.** This paper is concerned with the isomorphic structure of the Banach space $\ell_\infty/c_0$ and how it depends on combinatorial tools whose existence is consistent with but not provable from the usual axioms of ZFC. Our main global result is that it is consistent that $\ell_\infty/c_0$ does not have an orthogonal $\ell_\infty$-decomposition, that is, it is not of the form $\ell_\infty(X)$ for any Banach space $X$. The main local result is that it is consistent that $\ell_\infty(c_0(\mathfrak{c}))$ does not embed isomorphically into $\ell_\infty/c_0$, where $\mathfrak{c}$ is the cardinality of the continuum, while $\ell_\infty$ and $c_0(\mathfrak{c})$ always do embed quite canonically. This should be compared with the results of Drewnowski and Roberts that under the assumption of the continuum hypothesis $\ell_\infty/c_0$ is isomorphic to its $\ell_\infty$-sum and in particular it contains an isomorphic copy of all Banach spaces of the form $\ell_\infty(X)$ for any subspace $X$ of $\ell_\infty/c_0$.

**1. Introduction.** Drewnowski and Roberts proved in [4] that, assuming the Continuum Hypothesis (abbreviated CH), the Banach space $\ell_\infty/c_0$ is isomorphic to its $\ell_\infty$-sum denoted $\ell_\infty(\ell_\infty/c_0)$. They concluded that under the assumption of CH the Banach space $\ell_\infty/c_0$ is primary, that is, given a decomposition $\ell_\infty/c_0 = A \oplus B$, one of the spaces $A$ or $B$ must be isomorphic to $\ell_\infty/c_0$. The proof relies on the Pełczyński decomposition method and on another striking result from [4] (not requiring CH) which says that one of the factors $A$ or $B$ as above must contain a complemented subspace isomorphic to $\ell_\infty/c_0$. Another conclusion was that $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$ is isomorphic to $\ell_\infty/c_0$ under the assumption of CH.

In this paper we show that some of the above statements cannot be proved without some additional set-theoretic assumptions. Namely, for any cardinal $\kappa \geq \omega_2$, the following statements all hold in the Cohen model obtained by adding $\kappa$-many Cohen reals to a model of CH ($\mathfrak{c}$ denotes the cardinality of the continuum):

(a) $\ell_\infty(c_0(\omega_2))$ does not embed isomorphically into $\ell_\infty/c_0$,
(b) $\ell_\infty(c_0(\mathfrak{c}))$ does not embed isomorphically into $\ell_\infty/c_0$,

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(c) $\ell_\infty(\ell_\infty/c_0)$ does not embed isomorphically into $\ell_\infty/c_0$,
(d) $\ell_\infty/c_0$ is not isomorphic to $\ell_\infty(X)$ for any Banach space $X$,
(e) $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$ is not isomorphic to $\ell_\infty/c_0$.

Below we show that (a) easily implies the other statements and so later we will focus on proving (a). Indeed, (a) implies (b) simply because $\mathfrak{c} \geq \omega_2$ in those models. (c) follows from (b) and the fact that $\ell_\infty/c_0$ contains an isometric copy of $c_0(\mathfrak{c})$ (e.g., the closure of the space spanned by the classes of the characteristic functions of elements of a family $\{A_\xi : \xi < \mathfrak{c}\}$ of infinite subsets of $\mathbb{N}$ whose pairwise intersections are finite). To deduce (d) from (c), notice that if $\ell_\infty/c_0$ were isomorphic to $\ell_\infty(X)$ for some Banach space $X$, then $\ell_\infty(\ell_\infty/c_0)$ would be isomorphic to $\ell_\infty(\ell_\infty(X))$ which in turn is isomorphic to $\ell_\infty(X)$ and hence to $\ell_\infty/c_0$, contradicting (c). Finally (e) follows from (c) alone, because $\ell_\infty(\ell_\infty/c_0)$ embeds isometrically into $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$. Indeed, consider a partition of $\mathbb{N}$ into pairwise disjoint infinite sets $(A_i : i \in \mathbb{N})$ and for each $x \in \ell_\infty(\ell_\infty/c_0)$ consider $\bar{x} \in \ell_\infty(\ell_\infty/c_0)$ such that $\bar{x}(n) = x(i)$ if and only if $n \in A_i$. Note that $T : \ell_\infty(\ell_\infty/c_0) \to \ell_\infty(\ell_\infty/c_0)$ given by $T(x) = \bar{x}$ is an isometric embedding. Moreover it gives an isometric embedding while composed with the quotient map from $\ell_\infty(\ell_\infty/c_0)$ onto $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$.

We emphasize an interesting phenomenon that follows from the gap which may exist between the number of added Cohen reals and $\omega_2$: even when $\mathfrak{c}$ is very large, meaning that $\ell_\infty/c_0$ has large density, still it may not contain an isomorphic copy of $\ell_\infty(c_0(\omega_2))$ while it always contains quite canonical copies of both $\ell_\infty$ and $c_0(\omega_2)$.

It remains unknown if $\ell_\infty/c_0$ is primary in the above models and in general if the primariness of $\ell_\infty/c_0$ can be proved without additional set-theoretic assumptions. It would also be interesing to derive the above statements in a more axiomatic way as in [12] or [10].

Another problem mentioned in [4] remains open as well (including in the Cohen model), namely if $\ell_\infty/c_0$ has the Schroeder–Bernstein property, that is, if there exists a complemented subspace $X$ of $\ell_\infty/c_0$, nonisomorphic to $\ell_\infty/c_0$ but which contains a complemented isomorphic copy of $\ell_\infty/c_0$. The Pełczyński decomposition method and the existence of an isomorphism between $\ell_\infty/c_0$ and $\ell_\infty(\ell_\infty/c_0)$ imply that $\ell_\infty/c_0$ has the Schroeder–Bernstein property assuming CH. On the other hand, the nonprimariness of $\ell_\infty/c_0$ would imply that it does not have the Schroeder–Bernstein property as observed in [4]. It could be noted that after the first example of a Banach space without the Schroeder–Bernstein property was given in [6], an example of the form $C(K)$, like all the spaces considered in this paper, was constructed as well (see [9]).

Our results (a)–(c) can also be seen in a different light. It is well-known that assuming CH the space $\ell_\infty/c_0$ is isometrically universal for all Banach spaces of density not greater than $\mathfrak{c}$. It has been proved by the present authors in [1] that this is not the case in the Cohen model, even in the isomorphic sense. The results (a)–(c) show that $\ell_\infty(c_0(\mathfrak{c}))$ or $\ell_\infty(\ell_\infty/c_0)$ can be added to a recently growing list of Banach spaces that consistently do not embed into $\ell_\infty/c_0$ (see [2], [10], [12] or [1, Section 3]). A new feature of the examples provided in this paper is that they are neither obtained from a well-ordering of the continuum nor a generically constructed object like those in the above mentioned papers.

The paper exploits the isometry $\ell_\infty/c_0 \equiv C(\mathbb{N}^*)$ where $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ and $\beta\mathbb{N}$ is the Čech–Stone compactification of the integers, and the isomorphism between the Boolean algebras of clopen subsets of $\mathbb{N}^*$ and $\wp(\mathbb{N})/\mathrm{Fin}$. The latter is a structure that can be well handled by infinitary combinatorial methods.

In Section 2 we present some consequences of the assumption that $\ell_\infty/c_0$ contains an isomorphic copy of $\ell_\infty(c_0(\lambda))$ for some cardinal $\lambda \geq \omega_2$. They can be understood as a reduction of the behaviour of a linear operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ to a topological phenomenon in terms of separations of a rich family of clopen sets of $\mathbb{N}^*$. This can be seen as a property of the Boolean algebra $\wp(\mathbb{N})/\mathrm{Fin}$. Section 3 contains the key lemma (Lemma 3.1) which shows that in the Cohen model the above property of $\wp(\mathbb{N})/\mathrm{Fin}$ does not hold. The proof of this key lemma is inspired by the proof of A. Dow [3, Theorem 4.5] that the boundary of a zero set in $\mathbb{N}^*$ is not a retract of $\mathbb{N}^*$ in the Cohen model. The reader unfamiliar with forcing may skip the proof of the lemma and read the rest of the paper.

The undefined notation of the paper is fairly standard. Undefined notions related to set theory and independence proofs can be found in [11] and those related to Banach spaces in [5]. In particular we will be using the following version of the Hajnal free-set lemma (cf. [8, Theorem 19.2]): *For every cardinal $\lambda > \omega_1$ and a function $F$ from $\lambda$ into countable subsets of $\lambda$ there is $S \subseteq \lambda$ of cardinality $\lambda$ such that $F(\alpha) \cap S \subseteq \{\alpha\}$ for all $\alpha \in S$.*

Let us now introduce some particular notation concerning the spaces we consider here. Given $A \subseteq \mathbb{N}$, let us denote by $[A]$ its equivalence class in $\wp(\mathbb{N})/\mathrm{Fin}$, by $A^*$ the corresponding clopen set of $\beta\mathbb{N}$ and by $[A]^*$ the clopen set of $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ corresponding to $[A]$.

Given $x \in \ell_\infty$, let us denote by $[x]$ its equivalence class in $\ell_\infty/c_0$. We will use the isometries $\ell_\infty \equiv C(\beta\mathbb{N})$ and $\ell_\infty/c_0 \equiv C(\mathbb{N}^*)$ and identify each bounded sequence with its extension to $\beta\mathbb{N}$ and each class $y = [x]$ of bounded sequences in $\ell_\infty/c_0$ with the restriction to $\mathbb{N}^*$ of an extension of $x$ to $\beta\mathbb{N}$.

Given a cardinal $\lambda$, by a *partial function* from $\mathbb{N}$ into a subset $\Gamma$ of $\lambda$ we mean a function whose domain is a (nonempty) subset $A$ of $\mathbb{N}$ and whose image is contained in $\Gamma$. We will identify such a function $\sigma : A \to \Gamma$ with its graph inside $\mathbb{N}\times\lambda$. Given two such functions $\sigma,\tau$, we say they are *disjoint* if their graphs are disjoint.

For $m,n\in\mathbb{N}$, $\alpha,\beta\in\lambda$ and a partial function $\sigma$ from $\mathbb{N}$ into a subset $\Gamma$, let

$$1_{n,\alpha}(m)(\beta)=\begin{cases}1 & \text{if } (n,\alpha)=(m,\beta),\\ 0 & \text{otherwise},\end{cases}$$
$$1_{\sigma}(m)(\beta)=\begin{cases}1 & \text{if } (m,\beta)\in\sigma,\\ 0 & \text{otherwise},\end{cases}$$

and notice that $1_{n,\alpha},1_\sigma\in\ell_\infty(c_0(\lambda))$ and they can be thought of as the characteristic functions of $\{(n,\alpha)\}$ and of the graph of $\sigma$ inside $\mathbb{N}\times\lambda$ respectively.

Some of the problems addressed in this paper were considered in [7] under different set-theoretic assumptions. Unfortunately the forthcoming paper announced there which was to contain the proofs of the statements instead of their sketches has not appeared yet. Also the statements and arguments outlined in [7, p. 303] concerning the Cohen model contradict our results.

## 2. Facts on isomorphic embeddings of $l_\infty(c_0(\lambda))$ into $\ell_\infty/c_0$

LEMMA 2.1. *Suppose $y\in\ell_\infty/c_0\setminus\{0\}$ and $A\subseteq\mathbb{N}$ is infinite such that $y|[A]^*\not\equiv 0$. Then there is an infinite $B\subseteq A$ and $r\in\mathbb{R}\setminus\{0\}$ such that $\|y|[A]^*\|\le 2|r|$ and $y|[B]^*\equiv r$.*

*Proof.* Let $x=(x_n)_{n\in\mathbb{N}}\in\ell_\infty$ be such that $y=[x]$. Since $B'=\{n\in A:\|y|[A]^*\|\le 2|x_n|\}$ is infinite and $\{x_n:n\in B'\}$ is bounded, there is an infinite $B\subseteq B'$ such that $(x_n)_{n\in B}$ converges to some $r\in\mathbb{R}\setminus\{0\}$. Notice that $\|y|[A]^*\|\le 2|r|$ and $y|[B]^*\equiv r$. ■

THEOREM 2.2. *Assume $\lambda\ge\omega_2$ and $T:\ell_\infty(c_0(\lambda))\to\ell_\infty/c_0$ is an isomorphic embedding. Then there is $\Gamma\in[\lambda]^\lambda$ and for each $(n,\alpha)\in\mathbb{N}\times\Gamma$ there is an infinite set $E_{n,\alpha}\subseteq\mathbb{N}$ and $r_{n,\alpha}\in\mathbb{R}$ such that*

$$|r_{n,\alpha}|\ge\|T(1_{n,\alpha})\|/2$$

*and if $\sigma:\mathbb{N}\to\Gamma$ is one-to-one, then*

$$T(1_\sigma)|[E_{n,\alpha}]^*\equiv\begin{cases}r_{n,\alpha} & \text{if } (n,\alpha)\in\sigma,\\ 0 & \text{if } \alpha\in\Gamma\setminus\mathrm{Im}(\sigma).\end{cases}$$

*Proof.* For each $n\in\mathbb{N}$ and each $\alpha\in\lambda$, by Lemma 2.1 there is $r_{n,\alpha}\in\mathbb{R}$ and an infinite set $E'_{n,\alpha}\subseteq\mathbb{N}$ such that

$$|r_{n,\alpha}|\ge\|T(1_{n,\alpha})\|/2$$

and $T(1_{n,\alpha})|[E'_{n,\alpha}]^*\equiv r_{n,\alpha}$.

CLAIM. *For every $(n,\alpha)\in\mathbb{N}\times\lambda$, there is a countable set $X_{n,\alpha}\subseteq\lambda$ and an infinite set $E_{n,\alpha}\subseteq^* E'_{n,\alpha}$ such that whenever $\sigma$ is a partial function from*

$\mathbb{N}$ *into* $\lambda \setminus X_{n,\alpha}$, *then*

$$T(1_\sigma)|[E_{n,\alpha}]^* \equiv 0.$$

*Proof of the claim.* Suppose the claim fails for $(n,\alpha) \in \mathbb{N} \times \lambda$. We will carry out a transfinite inductive construction of length $\omega_1$ that will lead to the conclusion that the operator $T$ is not bounded, which is a contradiction. We construct for each $\xi < \omega_1$ an infinite set $F_\xi \subseteq \mathbb{N}$, $r_\xi \in \mathbb{R} \setminus \{0\}$ and a partial function $\sigma_\xi$ from $\mathbb{N}$ into $\lambda$ such that

(1) $F_\eta \subseteq^* F_\xi \subseteq^* E'_{n,\alpha}$ for all $\xi < \eta < \omega_1$,
(2) $\sigma_\xi \cap \sigma_\eta = \emptyset$ for all $\xi < \eta < \omega_1$,
(3) $T(1_{\sigma_\xi})|[F_\xi]^* \equiv r_\xi$ for all $\xi < \omega_1$.

Given $\xi < \omega_1$, suppose we have already constructed $(F_\eta)_{\eta<\xi}$, $(r_\eta)_{\eta<\xi}$ and $(\sigma_\eta)_{\eta<\xi}$ as above. Let $F'_\xi \subseteq \mathbb{N}$ be an infinite set such that $F'_\xi \subseteq^* F_\eta$ for every $\eta < \xi$. Since $\Lambda = \bigcup\{\mathrm{Im}(\sigma_\eta) : \eta < \xi\}$ is a countable subset of $\lambda$, by our hypothesis there is a partial function $\sigma_\xi$ from $\mathbb{N}$ into $\lambda \setminus \Lambda$ such that

$$T(1_{\sigma_\xi})|[F'_\xi]^* \not\equiv 0,$$

and using Lemma 2.1 we find $F_\xi \subseteq F'_\xi$ infinite and $r_\xi \in \mathbb{R} \setminus \{0\}$ such that

$$T(1_{\sigma_\xi})|[F_\xi]^* \equiv r_\xi.$$

This concludes the inductive construction of objects satisfying (1)–(3).

We can now find some $\varepsilon > 0$ for which $R_\varepsilon = \{\xi < \omega_1 : |r_\xi| \geq \varepsilon\}$ is infinite (uncountable, actually), and splitting $R_\varepsilon$ into two sets, we may assume without loss of generality that either $r_\xi \geq \varepsilon$ for every $\xi \in R_\varepsilon$ or $-r_\xi \geq \varepsilon$ for every $\xi \in R_\varepsilon$.

Fix $m \in \mathbb{N}$ such that $m \cdot \varepsilon > \|T\|$. Choose $\xi_1 < \cdots < \xi_m$ in $R_\varepsilon$ and notice that $|\sum_{i\leq m} r_{\xi_i}| \geq m\varepsilon > \|T\|$.

Since the $\sigma_{\xi_i}$'s are pairwise disjoint, we get

$$\Big\|\sum_{i\leq m} 1_{\sigma_{\xi_i}}\Big\| = 1$$

but

$$\Big\|T\Big(\sum_{i\leq m} 1_{\sigma_{\xi_i}}\Big)\Big\| \geq \Big\|T\Big(\sum_{i\leq m} 1_{\sigma_{\xi_i}}\Big)\Big|[F_{\xi_m}]^*\Big\| \geq \Big|\sum_{i\leq m} r_{\xi_i}\Big| > \|T\|,$$

which is a contradiction and completes the proof of the claim.

For each $\alpha \in \lambda$, let $X_\alpha = \bigcup_{n\in\mathbb{N}} X_{n,\alpha}$ and notice that $X_\alpha$ is a countable subset of $\lambda$ such that for every $n \in \mathbb{N}$, there is an infinite set $E_{n,\alpha} \subseteq E'_{n,\alpha}$ such that for every $\sigma : \mathbb{N} \to \lambda \setminus X_\alpha$, we have

$$T(1_\sigma)|[E_{n,\alpha}]^* \equiv 0. \tag{2.1}$$

Now apply the Hajnal free-set lemma [8, Theorem 19.2] to obtain $\Gamma \subseteq \lambda$ of cardinality $\lambda$ such that $X_\alpha \cap \Gamma \subseteq \{\alpha\}$ for each $\alpha \in \Gamma$. This implies that for distinct $\alpha, \beta \in \Gamma$, $\alpha \notin X_\beta$.

Given $\sigma \in \mathbb{N} \to \Gamma$ which is one-to-one, notice that for distinct $n, n' \in \mathbb{N}$, $\sigma(n) \notin X_{\sigma(n')}$, which guarantees that $\mathrm{Im}(\sigma \setminus \{(n, \sigma(n))\}) \cap X_{\sigma(n)} = \emptyset$. For each $(n, \alpha) \in \omega \times \Gamma$ let us consider two cases. If $(n, \alpha) \in \sigma$, then

$$T(1_\sigma)|[E_{n,\alpha}]^* = T(1_{n,\alpha})|[E_{n,\alpha}]^* + T(1_{\sigma\setminus\{(n,\alpha)\}})|[E_{n,\alpha}]^* \equiv r_{n,\alpha},$$

where the last equality follows from (2.1) and the choice of $E_{n,\alpha}$.

If $\alpha \in \Gamma \setminus \mathrm{Im}(\sigma)$, it follows from (2.1) that $T(1_\sigma)|[E_{n,\alpha}]^* \equiv 0$. ∎

Although the above theorem is sufficient for our applications, let us note that it has the following more elegant version:

COROLLARY 2.3. *Assume $\lambda \geq \omega_2$ and $T : \ell_\infty(c_0(\lambda)) \to \ell_\infty/c_0$ is an isomorphic embedding. Then there is an isomorphic embedding $T' : \ell_\infty(c_0(\lambda)) \to \ell_\infty/c_0$ and for each $(n, \alpha) \in \mathbb{N} \times \lambda$ there is an infinite set $E_{n,\alpha} \subseteq \mathbb{N}$ and $r_{n,\alpha} \in \mathbb{R}$ such that*

$$|r_{n,\alpha}| \geq \|T'(1_{n,\alpha})\|/2$$

*and for all $(n, \alpha) \in \mathbb{N} \times \lambda$, if $\sigma : \mathbb{N} \to \lambda$, then*

$$T'(1_\sigma)|[E_{n,\alpha}]^* \equiv \begin{cases} r_{n,\alpha} & \textit{if } (n,\alpha) \in \sigma, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* Let $\Gamma \subseteq \lambda$ of cardinality $\lambda$ and an infinite set $E_{n,\alpha} \subseteq \mathbb{N}$ and $r_{n,\alpha} \in \mathbb{R}$ for each $(n, \alpha) \in \mathbb{N} \times \Gamma$ be as in Theorem 2.2.

Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a partition of $\Gamma$ into countably many sets of cardinality $\lambda$ and enumerate each $\Gamma_n$ as $\Gamma_n = \{\gamma^n_\beta : \beta < \lambda\}$.

Define $S : \ell_\infty(c_0(\lambda)) \to \ell_\infty(c_0(\lambda))$ by

$$S(f)(n)(\beta) = \begin{cases} f(n)(\alpha) & \text{if } \beta = \gamma^n_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $S$ has the following properties:

- $S$ is an isometric embedding.
- If $\sigma : \mathbb{N} \to \lambda$, then $S(1_\sigma) = 1_{s(\sigma)}$ where $s(\sigma) = \{(n, \gamma^n_\alpha) : (n, \alpha) \in \sigma\}$, so that $s(\sigma) : \mathbb{N} \to \lambda$ is one-to-one, $\mathrm{Im}(s(\sigma)) \subseteq \Gamma$ and $\gamma_{n,\alpha} \in \mathrm{Im}(s(\sigma))$ if and only if $(n, \alpha) \in \sigma$.

Let $T' = T \circ S$, $E'_{n,\alpha} = E_{n,\gamma^n_\alpha}$ and $r'_{n,\alpha} = r_{n,\gamma^n_\alpha}$. Given $\alpha < \lambda$ and $n \in \mathbb{N}$,

$$|r'_{n,\alpha}| = |r_{n,\gamma^n_\alpha}| \geq \|T(1_{n,\gamma^n_\alpha})\|/2 = \|T(S(1_{n,\alpha}))\|/2 = \|T'(1_{n,\alpha})\|/2.$$

Also, given any $\sigma : \mathbb{N} \to \lambda$ and $(n, \alpha) \in \mathbb{N} \times \lambda$, we have:

- if $(n, \alpha) \in \sigma$, then
$$T'(1_\sigma)|[E'_{n,\alpha}]^* = T(S(1_\sigma))|[E_{n,\gamma^n_\alpha}]^* = r_{n,\gamma^n_\alpha} = r'_{n,\alpha},$$
- if $(n, \alpha) \notin \sigma$, then $\gamma^n_\alpha \notin \mathrm{Im}(s(\sigma))$ so that
$$T'(1_\sigma)|[E'_{n,\alpha}]^* = T(1_{s(\sigma)})|[E_{n,\gamma^n_\alpha}]^* = 0.$$

This concludes the proof that $T'$ is an isomorphic embedding with the required properties. ■

**3. The forcing argument.** The next lemma still holds if we replace $\omega_2$ by any regular cardinal $\lambda$ with $\omega_2 \le \lambda \le \kappa$. To simplify the notation we state it in this weaker form, which is sufficient for our purposes. $\mathrm{Fn}_{<\omega}(\kappa, 2)$ denotes the Cohen forcing which adds $\kappa$ many Cohen reals to a model of CH, that is, the forcing formed by partial functions whose domains are finite subsets of $\kappa$ and whose ranges are included in $2 = \{0, 1\}$, ordered by extension of functions.

LEMMA 3.1. *Let $V$ be a model of CH, $\kappa \ge \omega_2$ and $\mathbb{P} = \mathrm{Fn}_{<\omega}(\kappa, 2)$. In $V^{\mathbb{P}}$, if $(E_{n,\alpha} : (n, \alpha) \in \mathbb{N} \times \omega_2)$ are infinite subsets of $\mathbb{N}$ and for each $\sigma \in \omega_2^{\mathbb{N}}$, $B_\sigma$ is a subset of $\mathbb{N}$ such that*

$$\forall (n, \alpha) \in \sigma \quad E_{n,\alpha} \subseteq^* B_\sigma,$$

*then there is a pairwise disjoint subset $\Sigma \subset \omega_2^{\mathbb{N}}$ of cardinality $\omega_2$ such that $\{B_\sigma : \sigma \in \Sigma\}$ has the finite intersection property, that is, for every $\sigma_1, \dots, \sigma_m \in \Sigma$, $B_{\sigma_1} \cap \dots \cap B_{\sigma_m}$ is infinite.*

*Proof.* In $V$, for $(n, \alpha) \in \mathbb{N} \times \omega_2$ let $\dot{E}_{n,\alpha}$ be a nice name for an infinite subset of $\mathbb{N}$.

For each $n \in \mathbb{N}$ and $\alpha \in \omega_2$, let $S_{n,\alpha} = \mathrm{supp}(\dot{E}_{n,\alpha})$, which are countable subsets of $\kappa$ since $\mathbb{P}$ is ccc and we may assume without loss of generality that they are all infinite.

By CH and the $\Delta$-system lemma, we may find a pairwise disjoint family $(A_n)_{n\in\omega} \subseteq [\omega_2]^{\omega_2}$ such that

- for each $n \in \mathbb{N}$, $(S_{n,\alpha})_{\alpha \in A_n}$ is a $\Delta$-system with root $\Delta_n$.

Let $\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$; since this is a countable set, by a further thinning out of each $A_n$, we may assume that

- for every $\alpha \in A_n$, $\Delta \cap (S_{n,\alpha} \setminus \Delta_n) = \emptyset$, i.e. $\Delta \cap S_{n,\alpha} = \Delta_n$.

Using CH, we may also assume that for each $\alpha < \beta$ in $A_n$ there is a bijection $\pi_{n,\alpha,\beta} : S_{n,\alpha} \to S_{n,\beta}$ such that

- $\pi_{n,\alpha,\beta}|_{\Delta_n} = \mathrm{id}$,
- $\pi_{n,\alpha,\beta}(\dot{E}_{n,\alpha}) = \dot{E}_{n,\beta}$ (here $\pi_{n,\alpha,\beta}$ denotes the automorphism of $\mathbb{P}$ obtained by lifting $\pi_{n,\alpha,\beta}$).

Inductively choose, for $\xi < \omega_2$, functions $\sigma_\xi \in \omega_2^{\mathbb{N}}$ such that

- $\sigma_\xi(n) \in A_n$ for each $n \in \mathbb{N}$,
- for all distinct $(\xi, n), (\eta, m) \in \omega_2 \times \mathbb{N}$,
$$(S_{n,\sigma_\xi(n)} \setminus \Delta_n) \cap (S_{m,\sigma_\eta(m)} \setminus \Delta_m) = \emptyset,$$
- for all $\xi < \eta < \omega_2$, $\sup_{n\in\mathbb{N}} \sigma_\xi(n) < \min_{n\in\mathbb{N}} \sigma_\eta(n)$, so that $\sigma_\xi \cap \sigma_\eta = \emptyset$.

For each $\xi < \omega_2$, let $\dot{B}_\xi$ be a name for a subset of $\mathbb{N}$ as in the hypothesis of the lemma, that is, such that

$$\mathbb{P} \Vdash \forall (n,\alpha) \in \check{\sigma}_\xi \ \ \dot{E}_{n,\alpha} \subseteq^* \dot{B}_{\sigma_\xi},$$

and let $\dot{h}_\xi$ be a nice name such that

$$\mathbb{P} \Vdash \dot{h}_\xi : \mathbb{N} \to \mathbb{N} \text{ is such that } \forall (n,\alpha) \in \check{\sigma}_\xi \ \ \dot{E}_{n,\alpha} \setminus \dot{h}_\xi(n) \subseteq \dot{B}_{\sigma_\xi}.$$

Let $R_\xi = \operatorname{supp}(\dot{h}_\xi)$ and $S_\xi = \bigcup_{n\in\mathbb{N}} (S_{n,\sigma_\xi(n)} \setminus \Delta_n)$ and notice that the $S_\xi$'s are pairwise disjoint countable subsets of $\kappa$. Using CH and the $\Delta$-system lemma, by a further thinning out there is $A \subseteq \omega_2$ of cardinality $\omega_2$ such that

- $(R_\xi)_{\xi\in A}$ is a $\Delta$-system with root $R$,
- for all $\xi \in A$, $\Delta \cap (R_\xi \setminus R) = \emptyset$.

If we apply Hajnal's free set lemma [8, Theorem 19.2] to the family of sets $X_\xi = \{\eta \in A : S_\xi \cap R_\eta \neq \emptyset\}$, for $\xi \in A$, we obtain a subset of $A$ of cardinality $\omega_2$—which we will call $A$ to simplify the notation—such that $X_\xi \cap A \subseteq \{\xi\}$, which implies that

- for all distinct $\xi, \eta \in A$, $S_\xi \cap R_\eta = \emptyset$.

Fix $m \in \mathbb{N}$ and $\xi_1 < \cdots < \xi_m$ from $A$ and let us prove that

$$\mathbb{P} \Vdash \dot{B}_{\sigma_{\xi_1}} \cap \cdots \cap \dot{B}_{\sigma_{\xi_m}} \text{ is infinite,}$$

so that $\{B_{\sigma_\xi} : \xi \in A\}$ has the finite intersection property. Otherwise, there are $p \in \mathbb{P}$ and $l \in \mathbb{N}$ such that

$$p \Vdash \dot{B}_{\sigma_{\xi_1}} \cap \cdots \cap \dot{B}_{\sigma_{\xi_m}} \subseteq \check{l}.$$

Given $n \in \mathbb{N}$ we say that $q \in \mathbb{P}$ is *$n$-symmetric* if

$$\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}(q|S_{n,\sigma_{\xi_i}(n)}) = q|S_{n,\sigma_{\xi_j}(n)} \qquad \text{for all } 1 \le i < j \le m.$$

Fix $n \in \mathbb{N}$ such that $\operatorname{dom}(p) \cap S_{n,\sigma_{\xi_i}(n)} \subseteq \Delta_n$ for all $1 \le i \le m$ and notice that $p$ is $n$-symmetric, because $p|S_{n,\sigma_{\xi_i}(n)} \subseteq p|\Delta_n$ and $\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}|\Delta_n = \mathrm{id}_{\Delta_n}$. Let us find $q \le p$ which is $n$-symmetric and $k_1, \ldots, k_m \in \mathbb{N}$ such that $q \Vdash \dot{h}_{\xi_i}(\check{n}) = \check{k}_i$. To do so, we will construct conditions $p_m \le p_{m-1} \le \cdots \le p_1 \le p$ and $k_1, \ldots, k_m \in \mathbb{N}$ such that each $p_i$ is $n$-symmetric and for all $1 \le i \le m$, $p_i \Vdash \dot{h}_{\xi_i}(\check{n}) = \check{k}_i$. Let $p_0 = p$.

Given $1 \le i \le m$, let $q_i \le p_{i-1}$ and $k_i \in \mathbb{N}$ be such that $q_i \Vdash \dot{h}_{\xi_i}(\check{n}) = \check{k}_i$ and $\operatorname{dom}(q_i) \setminus \operatorname{dom}(p_{i-1}) \subseteq R_{\xi_i}$. This can be done because $\dot{h}_{\xi_i}$ is a nice name with support $R_{\xi_i}$. Let

$$p_i = q_i \cup \bigcup_{1\le j\le m} \pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}(q_i|S_{n,\sigma_{\xi_i}(n)}),$$

where $\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_i}(n)} = \mathrm{id}$.

Claim. *$p_i \in \mathbb{P}$, that is, $p_i$ is a well-defined function.*

*Proof of the claim.* First of all, notice that

$$q_i \leq q_i|S_{n,\sigma_{\xi_i}(n)} = \pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_i}(n)}(q_i|S_{n,\sigma_{\xi_i}(n)})$$

since $\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_i}(n)} = \mathrm{id}$, and

$$\mathrm{dom}(p_i) \subseteq \mathrm{dom}(p_{i-1}) \cup R_{\xi_i} \cup \bigcup\{S_{n,\sigma_{\xi_j}(n)} : 1 \leq j \leq m,\ j \neq i\}$$

since $\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}[S_{n,\sigma_{\xi_i}(n)}] = S_{n,\sigma_{\xi_j}(n)}$ and $\mathrm{dom}(q_i) \subseteq \mathrm{dom}(p_{i-1}) \cup R_{\xi_i}$. Fix $\alpha \in \mathrm{dom}(p_i)$ and let us analyse a few cases.

If $\alpha \in S_{n,\sigma_{\xi_j}(n)} \setminus \Delta_n$ for some $j \neq i$, since $S_{\xi_j} \cap R_{\xi_i} = \emptyset$, we see that $\alpha \notin R_{\xi_i}$. So, if $\alpha \notin \mathrm{dom}(p_{i-1})$, the only possible value for $p_i(\alpha)$ is $\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}(q_i|S_{n,\sigma_{\xi_i}(n)})(\alpha)$, and hence it is well-defined. If $\alpha \in \mathrm{dom}(p_{i-1})$, the possible values for $p_i(\alpha)$ are those of $\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}(q_i|S_{n,\sigma_{\xi_i}(n)})(\alpha)$ and $p_{i-1}(\alpha)$, in which case we have

$$p_{i-1}(\alpha) = \pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}(p_{i-1})(\alpha) = \pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}(q_i|S_{n,\sigma_{\xi_i}(n)})(\alpha),$$

where the first equality follows from the fact that $p_{i-1}$ is $n$-symmetric, and the second, from the fact that $q_i \leq p_{i-1}$.

If $\alpha \in \Delta_n$, then

$$q_i(\alpha) = \pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_i}(n)}(q_i|S_{n,\sigma_{\xi_i}(n)})(\alpha)$$

and

$$\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_i}(n)}(q_i|S_{n,\sigma_{\xi_i}(n)})(\alpha) = \pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}(q_i|S_{n,\sigma_{\xi_i}(n)})(\alpha)$$

for all $j \neq i$, because $\pi_{n,\sigma_{\xi_i}(n),\sigma_{\xi_j}(n)}|\Delta_n = \mathrm{id}_{\Delta_n}$. Hence, $p_i(\alpha)$ is well-defined.

Finally, if $\alpha \notin \bigcup_{i\neq j} S_{n,\sigma_{\xi_j}(n)}$, then the only possible value for $p_i(\alpha)$ is $q_i(\alpha)$ and it is therefore well-defined.

This concludes the proof of the claim.

Notice that $p_i$ is $n$-symmetric and, since $p_i \leq q_i$, $p_i \Vdash \dot{h}_{\xi_i}(\check{n}) = \check{k}_i$, as we wanted.

Now $p_m$ is an $n$-symmetric condition such that

$$p_m \Vdash \forall 1 \leq i \leq m\ \ \dot{h}_{\xi_i}(\check{n}) = \check{k}_i.$$

Since $\mathbb{P}$ forces that $\dot{E}_{n,\sigma_{\xi_1}(n)}$ is infinite, let $n_0 > \max\{k_1, \ldots, k_m, l\}$ and $r_1 \leq p_m$ be such that $r_1 \Vdash n_0 \in \dot{E}_{n,\sigma_{\xi_1}(n)}$ and $\mathrm{dom}(r_1) \setminus \mathrm{dom}(p_m) \subseteq S_{n,\sigma_{\xi_1}(n)}$. Let

$$r = r_1 \cup \bigcup_{2\leq j\leq m} \pi_{n,\sigma_{\xi_1}(n),\sigma_{\xi_j}(n)}(r_1)$$

and notice that $r \in \mathbb{P}$, $r \leq p$ and

$$r \Vdash \check{n}_0 \in (\dot{E}_{n,\xi_1} \setminus \dot{h}_{\xi_1}(n)) \cap \cdots \cap (\dot{E}_{n,\xi_m} \setminus \dot{h}_{\xi_m}(n)),$$

so that

$$r \Vdash \check{n}_0 \in \dot{B}_{\sigma_{\xi_1}} \cap \cdots \cap \dot{B}_{\sigma_{\xi_m}},$$

which contradicts our assumption since $n_0 > l$. This concludes the proof. ∎

THEOREM 3.2. *Let $V$ be a model of CH, $\kappa \geq \omega_2$ and $\mathbb{P} = \mathrm{Fn}_{<\omega}(\kappa, 2)$. In $V^{\mathbb{P}}$ there is no isomorphic embedding $T : \ell_\infty(c_0(\omega_2)) \to \ell_\infty/c_0$.*

*Proof.* We work in $V^{\mathbb{P}}$ and suppose by contradiction that there is $T$ as above; we will get a contradiction with the fact that $T$ is bounded. Let $\varepsilon > 0$ be such that $1/\|T^{-1}\| > \varepsilon$.

By Corollary 2.3 we may assume that for each $(n, \alpha) \in \mathbb{N} \times \omega_2$ there is an infinite set $E_{n,\alpha} \subseteq \mathbb{N}$ and $r_{n,\alpha} \in \mathbb{R}$ such that

$$|r_{n,\alpha}| \geq \frac{\|T(1_{n,\alpha})\|}{2} \geq \frac{1}{2\|T^{-1}\|} > \frac{\varepsilon}{2}$$

and for all $n \in \mathbb{N}$ and all $\alpha \in \omega_2$, if $\sigma : \mathbb{N} \to \omega_2$, then

$$T(1_\sigma)|[E_{n,\alpha}]^* \equiv \begin{cases} r_{n,\alpha} & \text{if } (n, \alpha) \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

For each $\sigma : \mathbb{N} \to \omega_2$, fix any representative $x_\sigma \in \ell_\infty$ of $T(1_\sigma)$ and let

$$B_\sigma = \{k \in \mathbb{N} : |x_\sigma(k)| > \varepsilon/4\}.$$

Then $B_\sigma$'s are as in the hypothesis of Lemma 3.1.

Given $m \in \mathbb{N}$ such that $\|T\| < m\varepsilon/4$, by Lemma 3.1 there are pairwise disjoint functions $\sigma_1, \ldots, \sigma_{2m} \in \omega_2^{\mathbb{N}}$ such that

$$B = B_{\sigma_1} \cap \cdots \cap B_{\sigma_{2m}} \text{ is infinite.}$$

Given $u \in [B]^*$, let $1 \leq j_1 < \cdots < j_m \leq 2m$ be such that $T(1_{\sigma_{j_i}})(u)$ are either all positive or all negative. Then

$$\left| T\Big( \sum_{i=1}^{m} 1_{\sigma_{j_i}} \Big)(u) \right| \geq m\frac{\varepsilon}{4} > \|T\|,$$

which contradicts the fact that $\| \sum_{i=1}^m 1_{\sigma_{j_i}} \| = 1$ and concludes the proof. ∎

Note that apparently we did not use in the above proof the entire strength of Corollary 2.3, namely we did not use the fact that $T(1_\sigma)|E_{n,\alpha} \equiv 0$ when $(n, \alpha) \notin \sigma$. However this is used within the proof of Corollary 2.3 to conclude that $T(1_\sigma)|E_{n,\alpha} \equiv r_{n,\alpha}$ when $(n, \alpha) \in \sigma$.

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