FUNDAMENTA
MATHEMATICAE
224 (2014)

Effective decomposition of σ-continuous Borel functions

by

Gabriel Debs (Paris)

Abstract. We prove that if a Δ_1^1 function f with Σ_1^1 domain X is σ-continuous then one can find a Δ_1^1 covering $(A_n)_{n\in\omega}$ of X such that $f_{|A_n}$ is continuous for all n. This is an effective version of a recent result by Pawlikowski and Sabok, generalizing an earlier result of Solecki.

1. Introduction. Let X and Y be two separable and metrizable spaces. A function $f : X \to Y$ is said to be σ-*continuous* if there exists a countable covering (A_n) of X such that $g_n = f_{|A_n}$ is continuous for all n. Note that in that case each g_n admits a continuous extension $\tilde{g}_n : \tilde{A}_n \to Y$ to a $\mathbf{\Pi}_2^0$ (i.e. $\mathbf{G}_\delta$) relative subset $\tilde{A}_n$ of X and f is then also continuous on the set $B_n = \{x \in \tilde{A}_n : \tilde{g}_n(x) = f(x)\} \supset A_n$. Hence if f is Borel of class ξ (i.e. the inverse image of any open set in Y is a $\mathbf{\Sigma}_{1+\xi}^0$ subset of X) then replacing A_n by B_n if necessary one can require each A_n to be a $\mathbf{\Pi}_{1+\xi}^0$ set. However this does not give any information on the complexity of the family (A_n) which is our main concern in this work.

Any function with countably many points of discontinuity is σ-continuous. Such a function is of course of the first Baire class, but as reported by Keldysh [3], Novikov gave an example of a first Baire class function (actually an upper semicontinuous real valued function) which is not σ-continuous (see [7]). In fact "most" first Baire class functions are (in some natural sense) non-σ-continuous (see [5]). Also generalizing Novikov's original example, Keldysh constructed (see [3]) non-σ-continuous functions which are Borel of arbitrarily high rank. More recently, several authors (see [1], [2], [5]) gave alternative examples of first Baire class functions which are not σ-continuous. Probably the simplest such example is given by the *Pawlikowski function* P (see [1]): this is the infinite product function $P = p^\omega$ from $(\omega+1)^\omega$ to ω^ω where $\omega+1$ is endowed with its natural (compact) order topology and

2010 *Mathematics Subject Classification*: Primary 03E15; Secondary 26A21.
Key words and phrases: σ-continuous, Borel functions, Wadge classes, product spaces.

DOI: 10.4064/fm224-2-4

$p : \omega + 1 \to \omega$ is the "circular" permutation defined by

$$\begin{cases} p(n) = n + 1 & \text{if } n < \omega, \\ p(\omega) = 0. \end{cases}$$

The function P plays a central role and is much more than a simple example, as shown by the early work of Solecki [9] which was recently extended by Pawlikowski and Sabok [8]. To state simply these results we shall introduce a notation.

Given two functions $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ we shall write $f_0 \sqsubseteq f_1$ if there exist topological embeddings $\varphi : X_0 \to X_1$ and $\psi : Y_0 \to Y_1$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle\varphi}\big\uparrow & & \big\uparrow{\scriptstyle\psi} \\ X_0 & \xrightarrow{f_0} & Y_0 \end{array}$$

commutes. We can then state the main result of [8] as follows:

THEOREM 1.1 (J. Pawlikowski and M. Sabok). *Let $f : X \to Y$ be a Borel function between analytic spaces. If f is not σ-continuous then $P \sqsubseteq f$.*

Note that the converse is trivially true since the function P is not σ-continuous, hence Theorem 1.1 gives a caracterization of Borel σ-continuous functions with analytic domain. Let us mention that in the particular case of functions of Baire class 1 this result was proved by Solecki [9] by totally different methods. Also Zapletal [10] derived from this particular case the same result for Borel functions with Borel domain in ω^ω.

Our goal in the present work is to give an effective version of Theorem 1.1. We should point out that in the case of functions of Baire class 1 such a version follows from Solecki's proof in [9]. To state the new result properly one needs to fix a recursive setting. For this we shall work in the framework of Σ^1_1 *topological spaces*. By this we mean a Σ^1_1 subset X of some recursively presented Polish space $\tilde{X}$, equipped with the subspace topology inherited from $\tilde{X}$. In this context a set $A \subset X$ is said to be Δ^1_1 *in* X if both A and $X \setminus A$ are Σ^1_1, equivalently if $A = \tilde{A} \cap X$ where $\tilde{A}$ is an Δ^1_1 subset of $\tilde{X}$. Note that the topology of X has then a basis consisting of Δ^1_1 open subsets, and the closure $\overline{A}$ of any Σ^1_1 subset A of X is also Σ^1_1.

Similarly a function $f : X \to Y$ between Σ^1_1 spaces will be said to be Δ^1_1 if its graph is Σ^1_1, equivalently if it admits a Δ^1_1 extension $\tilde{f} : \tilde{X} \to \tilde{Y}$ to some recursively presented Polish spaces $\tilde{X} \supset X$ and $\tilde{Y} \supset Y$. And we shall say that such a function f is Δ^1_1-*σ-continuous* if there exists a Δ^1_1-sequence $(B_n)_{n\in\omega}$ such that $X = \bigcup_n B_n$ and $f_{|B_n}$ is continuous for all n. So a

Δ^1_1-σ-continuous function is both Δ^1_1 and σ-continuous. We can now state our strengthening of Theorem 1.1:

THEOREM 1.2. *If a Δ^1_1 function $f : X \to Y$ between Σ^1_1 spaces is not Δ^1_1-σ-continuous then $P \sqsubseteq f$.*

COROLLARY 1.3. *A function between Σ^1_1 spaces is Δ^1_1-σ-continuous if and only if it is Δ^1_1 and σ-continuous.*

The same results hold if one replaces in the statements above the "descriptive effective classes" Δ^1_1 and Σ^1_1, by any "relative version" $\Delta^1_1(\alpha)$ and $\Sigma^1_1(\alpha)$ for an arbitrary real α. Let us also recall that by a well known result of Louveau [4] a $\mathbf{\Pi}^0_{1+\xi}$ set which is Δ^1_1 is actually $\Pi^0_{1+\xi}(\alpha)$ for some Δ^1_1 real α. Hence combining Louveau's result with Theorem 1.2 one gets the following:

COROLLARY 1.4. *Let $f : X \to Y$ be a Δ^1_1 function between Σ^1_1 spaces of Borel class ξ for some recursive ordinal ξ. If f is σ-continuous then there exists a Δ^1_1 real α and a $\Pi^0_{1+\xi}(\alpha)$-sequence $(B_n)_{n\in\omega}$ such that $X = \bigcup_n B_n$ and $f_{|B_n}$ is continuous for all n.*

The proof of Theorem 1.2 will make use of various methods and results from descriptive set theory in recursively presented Polish spaces in the sense of [6]. Also we shall assume the reader is familiar with all classical and folklore results in this area. The general scheme of proof is quite close to the scheme of proof of Theorem 1.1 as given in [8]. However, as we shall see, the use of effective descriptive set theory will bring significant simplifications even when the effective part of the conclusion is a posteriori ignored. Another source of simplification in our proof is due to the introduction of some adapted topological game, which we shall first introduce in a general setting.

2. Hausdorff distance. Given any bounded metric d on a space X we shall also denote by d the classical associated Hausdorff distance on $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$ defined by

$$d(A,B) = \max\Big(\sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\Big).$$

So when writing $d(A,B)$ we implicitly assume that the sets A and B are both nonempty. We recall that d is only a pseudo-metric on $\mathcal{P}^*(X)$ since $d(A,\overline{A}) = 0$, but induces a genuine metric on the set $\mathcal{F}^*(X)$ of all nonempty closed subsets of X. However we shall deal with d on arbitrary subsets. In fact the use of this pseudo-metric is merely a matter of language. For example the density in $\mathcal{P}^*(X)$ of the set of all (nonempty) finite subsets of X expresses simply that the metric space (X,d) is *precompact*. In particular we shall need the following elementary approximation property:

Proposition 2.1. *For any subsets $A' \subset A$ and B in X, and any $\varepsilon > 0$, there exists a set $B' \subset B$ such that $d(A', B') < d(A, B) + \varepsilon$.*

Moreover if X is a Σ^1_1 space, the function d is Δ^1_1 on $X \times X$, and the sets A' and B are Σ^1_1 then we can require B' to be Σ^1_1 too.

Proof. Take $B' = \{y \in B : d(y, A') < r\}$ where r is any rational number such that $d(A, B) < r < d(A, B) + \varepsilon$. ∎

3. A topological game on Σ^1_1 sets. As announced above, one of the main simplifications brought to the arguments of [8] in the proof of Theorem 1.2 will rely on the introduction of some game. We emphasize that this is a topological game in the sense that the winning condition is purely topological and can be considered in a classical setting with no recursive structure, though the particular version we will deal with will refer to Σ^1_1 sets. Actually the game we shall consider is a particular instance of a general family of games which rely on the following (strong) notion of convergence.

Definition 3.1. Let X be a topological space. We shall say that a sequence $(A_n)_{n\in\omega}$ of nonempty subsets of X *converges compactly* if any sequence $(x_n)_{n\in\omega}$ with $x_n \in A_n$ for all n admits a cluster point $x \in \bigcap_n A_n$; we shall then write $\lim_n A_n = \bigcap_n A_n$.

It is not difficult to see that if (X, d) is a bounded metric space then the sequence $(A_n)_{n\in\omega}$ converges compactly if and only if the set $A = \bigcap_n A_n$ is *nonempty, compact*, and $\lim_n d(A_n, A) = 0$. In particular when the sets A_n are compact (which will never be the case in what follows), this notion coincides with the standard topological convergence with respect to the Hausdorff distance.

Coming back to the general case note that if the sequence (A_n) is decreasing ($A_{n+1} \subset A_n$) then *any* cluster point of any sequence $(x_n)_n$ with $x_n \in A_n$ belongs to $A = \bigcap_n A_n$: indeed, if $x = \lim_j x_{n_j}$ with $n_j \uparrow \infty$ then $x_{n_j} \in A_{n_j} \subset A_j$ for all j, hence x which is the unique cluster point of the sequence $(x_{n_j})_j$ is automatically in $\bigcap_n A_n$. Moreover the limit set $\bigcap_n A_n$ is a singleton if and only if $\lim_n \operatorname{diam}(A_n) = 0$. We will use the following elementary property:

Proposition 3.2. *Suppose that $\varphi : X \to Y$ is continuous. If a sequence (A_n) converges compactly in X then the sequence $(\varphi(A_n))$ converges compactly in Y; in particular $\varphi(\bigcap_n A_n) = \bigcap_n \varphi(A_n)$.*

The game $G_d(X, \Sigma^1_1)$. Let X be a Σ^1_1 space and d the Hausdorff distance associated to an arbitrary bounded metric on X (*with no connection with the topology of X*). We denote by $G_d(X, \Sigma^1_1)$ the game in which at each of its moves Player I chooses a Σ^1_1 set A and some $\varepsilon > 0$, and Player II chooses just a Σ^1_1 set A' as follows:

$$\begin{array}{lllll} \mathrm{I}: & (A_0,\varepsilon_0) & & (A_2,\varepsilon_1) & \dots \\ \mathrm{II}: & & A_1 & & A_3 \quad \dots \end{array} \quad \text{with} \begin{cases} A_{2n}\subset A_{2n-1}, \\ A_{2n+1}\subset A_{2n}, \\ d(A_{2n+1},A_{2n})<\varepsilon_n, \end{cases}$$

and Player II wins such an infinite run if the sequence (A_n) converges compactly.

THEOREM 3.3. *If X is a Σ^1_1 space and d is the Hausdorff distance associated to an arbitrary precompact metric on X then Player II has a winning strategy in the game $G_d(X,\Sigma^1_1)$.*

We postpone the proof of this result to Section 6.

4. Analyzing the Pawlikowski function. Let $\omega_\circ$ denote the set ω endowed with the unique compact topology such that $0=\lim_{n\to\infty} n$. So $\omega_\circ\approx\omega+1$ and the Pawlikowski function P can be viewed as the identity mapping from $\omega_\circ^\omega$ to ω^ω. Now to prove that $P\sqsubseteq f$ one needs some finite or countable "approximation" of the function P. Such an approximation is naturally provided by considering the identity mappings from $\omega_\circ^k$ to ω^k for $k<\omega$. As a matter of fact the proof of Theorem 1.1 in [8] relies on a specific analysis of the metric spaces $\omega_\circ^k$ that we present next in a more structured and synthetic way.

Let us first recall some basic general terminology concerning abstract partially ordered sets. A *tree relation* $\prec$ on a set E is a partial strict ordering on E with the property that for any element $a\in E$ the initial segment $E_a=\{x\in E: x\prec a\}$ is finite and totally ordered by $\prec$. Note that the set E_a might be empty, which means that the element a is minimal. A member of E_a is called a *predecessor* of a and, as usual, we shall refer to the largest member of E_a, which we will denote by a°, as *the* predecessor of a. Also given any element $a\in E$ there exists a unique minimal element $u\preceq a$, called the *root* of a. If all elements of a given set $A\subset E$ have the same root u, we shall say that A is a set of *root* u. Notice that since we do not require any connectedness condition, a tree relation $\prec$ on E induces a tree relation $\prec_A$ on any subset A of E; and we shall say that A is a *subtree* of E if the predecessors of any element $a\in A$ are the same in both structures, that is, all predecessors of a in E belong to A.

We now come back to the analysis of the metric spaces $\omega_\circ^k$. For this fix an arbitrary metric $d_\circ$ on $\omega_\circ$; for any $k\le\omega$ we endow the set $\omega_\circ^k$ with the corresponding metric $d_\circ^k$ defined by

$$d_\circ^k(s,t)=\sum_{i<k}2^{-i}d_\circ(s(i),t(i)).$$

The following result is simply a reformulation in the language of tree relations of Lemma 2.5 from [8].

PROPOSITION 4.1. *For any $k < \omega$ there exists a tree relation $<_\circ^k$ on ω^k satisfying:*

(a) *If $s <_\circ^k t <_\circ^k u$ then $d_\circ^k(s,u) = d_\circ^k(s,t) + d_\circ^k(t,u) \leq 2\, d_\circ(0,k)$.*
(b) *For any $s \in \omega^k$ there exists $R_s > 0$ such that any $d_\circ^k$-open ball $B(s,R)$ of center s and radius $R < R_s$ is a subtree of ω^k of root s.*

Proof. The tree relation $<_\circ^k$ is entirely defined by the following two conditions:

- If $\max(s) < k$ then s is declared to be $<_\circ^k$-minimal.
- If not then *the* $<_\circ^k$-predecessor s° of s is defined for all $j < k$ by

$$s^\circ(j) = \begin{cases} 0 & \text{if } j \text{ is the first } i < k \text{ such that } s(i) = \max(s), \\ s(j) & \text{otherwise.} \end{cases}$$

Part (a) follows from the definition above. For (b) note that for all $n, m \in \omega$,

$$d_\circ(n,m) < \delta_n^p := \begin{cases} \min\{d_\circ(n,q);\ q \neq n\} & \text{if } n > 0 \\ \min\{d_\circ(0,q);\ q \leq p\} & \text{if } n = 0 \end{cases} \Rightarrow \begin{cases} m = n & \text{if } n > 0, \\ m > p & \text{if } n = 0, \end{cases}$$

hence putting $R_s = 2^{-k} \min\{\delta_{s(i)}^p;\ i < k\}$ with $p = \max(k, \max(s))$ we have for all $t \in \omega^k$,

$$d_\circ^k(s,t) < R_s \ \Rightarrow\ \forall i < k,\ \begin{cases} t(i) = s(i) & \text{if } s(i) > 0, \\ t(i) > \max(k, \max(s)) & \text{if } s(i) = 0. \end{cases}$$

In particular if $d_\circ^k(s,t) < R_s$ with $t \neq s$ then $d_\circ^k(s,t^\circ) < d_\circ^k(s,t) < R_s$, which by a straightforward induction shows that for any $R < R_s$ the ball $B(s,R)$ is a subtree of ω^k of root s. ∎

Note that unlike in [8] where s° is defined for all s (and equal to s when s is minimal) we reserve this notation only to denote the $<_\circ^k$-predecessor of s when it exists. Also setting $s^* = s_{|k-1}$ (the $\subset$-predecessor of $s \in \omega^k$) when $k > 0$, we have the following coherence property which follows readily from the definition of s°.

PROPOSITION 4.2. *If $s \in \omega^k$ is not $<_\circ^k$-minimal and $s^{\circ *} \neq s^*$ then s^* is not $<_\circ^{k-1}$-minimal with $s^{*\circ} = s^{\circ *}$; moreover $d_\circ(s^*, s^{*\circ}) = d_\circ(s, s^\circ)$.*

We shall write $d_\circ$ and $<_\circ$ instead of $d_\circ^k$ and $<_\circ^k$. We shall also assume for simplicity that $d_\circ(0,k) = 2^{-k-1}$ so that $d_\circ(s,t) \leq 2^{-|s|}$ whenever $s <_\circ t$.

5. Proof of the main result. So suppose that the Δ_1^1 function $f : X \to Y$ is not Δ_1^1-σ-continuous and consider the sets

$$\mathcal{A} := \{A\ \Sigma_1^1\text{-subset of } X : f_{|A} \text{ is continuous}\}, \qquad X' := X \setminus \bigcup \mathcal{A}.$$

LEMMA 5.1. *X' is a nonempty Σ^1_1 subset of X.*

Proof. By a straightforward computation one can check that $\mathcal{A}$ is a Π^1_1 family in the codes of Σ^1_1 sets, and it follows that $\bigcup \mathcal{A}$ is a Π^1_1 set, hence X' is a Σ^1_1 set. Moreover by Δ^1_1 reflection for any Σ^1_1 set $A \in \mathcal{A}$ there exists a Δ^1_1 set $B \in \mathcal{A}$ such that $B \supset A$. Hence if X' were empty then $\mathcal{A}$ would be a Π^1_1 covering of X by Σ^1_1 sets, hence by Δ^1_1-selection there would exist a Δ^1_1-sequence (B_n) in $\mathcal{A}$ such that $X = \bigcup_n B_n$, showing that f is Δ^1_1- σ-continuous, in contradiction with the hypothesis. ∎

The proof of Theorem 1.2 now reduces to the following result applied to X'.

LEMMA 5.2. *Suppose that for any nonempty Σ^1_1 subset A of X the function $f_{|A}$ is not continuous. Then $P \sqsubseteq f$.*

Plan of proof of Lemma 5.2. Fix a Δ^1_1 *precompact* metric d_X on X (provided by any Δ^1_1 embedding of X in some recursively presented compact space) and an arbitrary Δ^1_1 metric d_Y on Y, and let δ_X and δ_Y denote the corresponding diameter functions. We shall construct a family $(A_s)_{s\in\omega^{<\omega}}$ of nonempty Σ^1_1 subsets of X with the following properties, where $G_s \subset X \times Y$ denotes the graph of the restriction of f to A_s:

(1) *For all $s \in \omega^{<\omega}$, $A_s \subset A_{s^*}$, hence $f(A_s) \subset f(A_{s^*})$.*
(2) *For each $k \ge 1$ the family $(f(A_s))_{s\in\omega^k}$ is* discrete*; in particular, for all $s \neq t$, $f(A_s) \cap f(A_t) = \emptyset$, hence $A_s \cap A_t = \emptyset$.*
(3) *For all $\sigma \in \omega^\omega$, $(G_{\sigma_{|n}})_n$ converges compactly to a singleton $\{(x_\sigma, y_\sigma)\}$.*
(4) *For each $k \ge 1$ the function $s \mapsto \overline{A}_s$ is continuous (in fact locally Lipschitz) from $(\omega^k_\circ, d_\circ)$ to $\mathcal{F}^*(X)$ endowed with the Hausdorff metric.*

Clearly such a construction yields one-to-one mappings $\varphi : \sigma \to x_\sigma$ and $\psi : \sigma \to y_\sigma$ from ω^ω into X and Y respectively, satisfying $f \circ \varphi = \psi$. Note that by Proposition 3.2 we have $\{x_\sigma\} = \lim_n A_{\sigma_{|n}}$ and $\{y_\sigma\} = \lim_n f(A_{\sigma_{|n}})$. It follows then from condition (4) that φ is continuous with respect to $d_\circ$, hence φ is an embedding of the compact space $\omega^\omega_\circ$ into X. Similarly ψ is continuous on ω^ω, and by condition (2), ψ is open, hence ψ is a topological embedding of ω^ω into Y.

Note that condition (3) is equivalent to the following (a priori weaker) condition:

$$(3')\ \textit{For all } \sigma \in \omega^\omega,\ \begin{cases} \text{(a) } \textit{the sequence } (G_{\sigma_{|n}}) \textit{ converges compactly,} \\ \text{(b) } \lim_n \delta_X(A_{\sigma_{|n}}) = 0. \end{cases}$$

Indeed, if $G_\sigma = \lim_n G_{\sigma_{|n}}$ then by Proposition 3.2, $A_\sigma = \lim_n A_{\sigma_{|n}} = \pi_X(G_\sigma)$; but by condition (3′)(b), A_σ is necessarily a singleton, hence G_σ too.

It is interesting to point out here that in the process of constructing the sets A_s below we will be able to ensure explicit uniform bounds on $\delta_X(A_s)$ (something like $\delta_X(A_s) \le 2^{-|s|}$) but we shall never be in a position to impose any control on $\delta_Y(f(A_s))$ at a finite level (recall that the function f is highly discontinuous). However we shall manage to ensure condition (3′)(a) independently, which will provide a posteriori the asymptotic condition $\lim_n \delta_Y(f(A_{\sigma_{|n}})) = 0$. We emphasize that the argument above collapses completely if one weakens condition (3) by requiring only that $\lim_n \delta_X(A_{\sigma_{|n}}) = 0$ since there is no reasonable way to derive from this that $\lim_n \delta_Y f(A_{\sigma_{|n}}) = 0$, which is the fundamental condition ensuring the continuity of the mapping ψ.

Let us recall that a family $(A_i)_{i\in I}$ of pairwise disjoint subsets of a topological space X is said to be *discrete* if each A_i is relatively open (hence closed) in the subspace $\bigcup_{j\in I} A_j$, equivalently if for all $i \in I$ the set A_i can be separated from $\bigcup_{j\in I,\, j\neq i} A_j$ by an open subset of X. The following notion introduced in [8] will play a central role.

Definition 5.3. Let $f : X \to Y$ and $S \subset \omega^{<\omega}$ be a finite tree. A family $(A_s)_{s\in S}$ of subsets of X is said to be an *f-severing scheme* if the family $(f(A_s))_{s\in S}$ is discrete, and $A_s \subset \overline{A_{s^*}}$ for all $s \neq \emptyset$ in S.

Note that if $(A_s)_{s\in S}$ is an f-severing scheme then for all $s \neq \emptyset$ in S the sets A_s and A_{s^*} are disjoint while $A_s \subset \overline{A_{s^*}}$.

The game G. Let $\theta : x \mapsto (x, f(x))$ be the canonical bijection from X onto $\mathrm{Gr}(f)$ and $\mathcal{T}$ be the topology on X obtained by transferring via θ the topology of $\mathrm{Gr}(f)$ (induced by the product topology on $X \times Y$). To avoid any ambiguity we shall denote by Z the topological space $(X, \mathcal{T})$.

We shall denote by G the game $G_d(Z, \Sigma^1_1)$ (see Section 3) where Z is above and d is the Hausdorff distance associated to the fixed precompact metric d_X on X. Since θ is a Δ^1_1 isomorphism, the space Z is Σ^1_1 and Theorem 3.3 applies. From now on we fix:

- τ, a winning strategy for Player II in the game G,
- $\nu : \omega^{<\omega} \to \omega$, a one-to-one enumeration of $\omega^{<\omega}$ which is nondecreasing with respect to both partial orderings $<_\circ$ and $\subset$.

Lemma 5.4. *For all $n \ge 0$ there exists a Σ^1_1 f-severing scheme $(A^n_s)_{\nu(s)\le n}$ and a finite run π_n of even length in the game G compatible with the strategy τ, satisfying:*

(i) *If $\nu(s) \le n$ and $s' <_\circ s$ then $d(A^n_{s'}, A^n_s) < d_\circ(s', s)$.*
(ii) *If $\nu(s) < n$ then $A^n_s \subset A^{n-1}_s$ and $d_X(A^n_s, A^{n-1}_s) < 2^{-n}$.*
(iii) *If $\nu(s) = n$ then $A^n_s \subset A^{n-1}_{s^*}$; and if moreover s is $<_\circ$-minimal then $\delta_X(A^n_s) < 2^{-|s|}$.*
(iv) *If $\nu(s) = n$ and $\nu(s^*) = m$ then π_n extends π_m and $A^n_s = \tau(\pi_n)$.*

Proof of Lemma 5.2 from Lemma 5.4. Assume that $(A^n_s)_{\nu(s)\le n}$ is as in Lemma 5.4 and set $A_s = A_s^{\nu(s)}$ for all $s \in \omega^{<\omega}$. We shall prove that the family $(A_s)_{s\in\omega^{<\omega}}$ satisfies conditions (1)–(4) of the plan of proof presented above.

Since $\nu(s) > \nu(s^*)$, by (ii) we have $A_s = A_s^{\nu(s)} \subset A_{s^*}^{\nu(s)-1} \subset A_{s^*}^{\nu(s^*)} = A_{s^*}$ which ensures condition (1). Also by (ii) and (iii), if $\nu(s) \le n$ then for all $n' < n$ there exists a unique $s' \subset s$ with $\nu(s') \le n'$ such that $A^n_s \subset A^{n'}_{s'}$. In particular if $|t| = |s|$ and $\nu(t) > \nu(s) = n$ then there exists a unique $t' \subset t$ with $\nu(t') \le n$ such that $A_t = A_t^{\nu(t)} \subset A^n_{t'}$ and since s and t are incomparable then necessarily $\nu(t') < n$. Hence if $\nu(s) = n$ then $\bigcup\{A_t : |t| = |s| \text{ and } \nu(t) > n\} \subset \bigcup\{A^n_{t'} : \nu(t') < n\}$; and since the family $(f(A^n_u))_{\nu(u)\le n}$ is discrete it follows that $f(A_s)$ is $\mathbf{\Sigma}^0_1$ (and $\mathbf{\Pi}^0_1$) separated from $\bigcup\{f(A_t) : |t| = |s| \text{ and } \nu(t) > n\}$. In particular if $|u| = |s|$ and $\nu(u) < n$ then $f(A_u)$ is $\mathbf{\Pi}^0_1$ separated from $f(A_s)$ too. It follows that $f(A_s)$ is $\mathbf{\Sigma}^0_1$ separated from $\bigcup\{f(A_t) : |t| = |s| \text{ and } t \ne s\}$, that is, for all k the family $(f(A_t))_{t\in\omega^k}$ is discrete. This ensures condition (2).

Finally we prove the equivalent form (3′) of condition (3). Part (a) follows readily from (iv) since the fixed strategy τ is winning in the game G. To prove part (b) pick any $s \in \omega^k$ and let $u \le_\circ s$ be the root of s. Then by (i)–(iii) we have $\delta_X(A_u^{\nu(u)}) \le 2^{-k}$, $d_X(A_u^{\nu(u)}, A_u^{\nu(s)}) \le 2 \times 2^{-\nu(u)} \le 2 \times 2^{-k}$ and $d_X(A_u^{\nu(s)}, A_s^{\nu(s)}) < d_\circ(u, s) \le 2^{-k}$, hence $\delta_X(A_s^{\nu(s)}) \le 7 \times 2^{-k}$. ∎

LEMMA 5.5. *Let S be a finite tree on ω, and fix $t \in \omega^{<\omega} \setminus S$ such that $t^* \in S$. If $(A_s)_{s\in S}$ is a Σ^1_1 f-severing scheme then for all $\varepsilon > 0$ there exists a Σ^1_1 f-severing scheme $(B_s)_{s\in S\cup\{t\}}$ satisfying:*

(1) *$B_s \subset A_s$ and $d(B_s, A_s) < \varepsilon$, for all $s \in S$,*
(2) *$B_t \subset A_{t^*}$ and $d(B_t, A_{t^*}) < \varepsilon$.*

Proof of Lemma 5.4 from Lemma 5.5. We prove Lemma 5.4 by induction on n. For $n = 0$ take $A^0_\emptyset = X$. Given n, set $S = \{\nu \le n\}$ and assume that $(A^n_s)_{s\in S} = (A_s)_{s\in S}$ is already defined. Apply Lemma 5.5 with $t = \nu^{-1}(n+1)$ and an arbitrary ε (to be fixed later on) to get a severing scheme $(B_s)_{s\in S\cup\{t\}}$. Finally we define $A^{n+1}_s = B_s$ for all $s \in S$, and $A^{n+1}_t = \tau(\pi_{n+1})$ with

$$\pi_{n+1} = \pi_m{}^\frown\langle A_{t^*}\rangle^\frown\langle(A', \varepsilon)\rangle$$

where $m = \nu(t^*) \le n$ and A' is a Σ^1_1 subset of B_t to be specified later on. So condition (iv) is automatically satisfied. Now observe that for any choice of A' we always have

$$A^{n+1}_t \subset A' \subset B_t \subset A_{t^*} = A^n_{t^*}$$

and:

(i′) if $\nu(s) \le n$ and $s' <_\circ s$ then $d_X(A^{n+1}_{s'}, A^{n+1}_s) \le d_X(A^n_{s'}, A^n_s) + 2\varepsilon$,
(ii′) if $\nu(s) \le n$ then $A^{n+1}_s = B_s \subset A_s = A^n_s$ and $d_X(A^{n+1}_s, A^n_s,) < \varepsilon$,
(iii′) $\nu(t) = n + 1$ and $A^{n+1}_t \subset A^n_{t^*}$.

Hence if ever t is $<_\circ$-minimal then choosing A' of diameter $< 2^{-|t|}$, we get $\delta_X(A_t^{n+1}) < 2^{-|t|}$, which ensures (iii). Moreover choosing ε small enough and applying the induction hypothesis one can ensure (ii), as well as (i) for $s \neq t$. So to finish we only need to ensure (i) for $s = t$ when t is not $<_\circ$-minimal. Note that in this case we have total freedom in the choice of A'.

Let $s' <_\circ t$; since the distance $d_\circ$ is additive on $<_\circ$ it is enough to treat the case $s' = t^\circ$. By construction we have

$$A_{t^\circ}^{n+1} = B_{t^\circ} \subset \overline{A_{t^{\circ *}}} = \overline{A_{t^{\circ *}}^n}, \qquad A_t^{n+1} \subset A' \subset B_t \subset A_{t^*} = A_{t^*}^n$$

with $d_X(A_t^{n+1}, A') < \varepsilon$ (by the rules of the game G) and $d(B_t, A_{t^*}) < \varepsilon$ (by condition (2) of Lemma 5.5). Then choosing $A' \subset B_t$ such that

$$d_X(A', B_{t^\circ}) < d_X(B_t, \overline{A_{t^{\circ *}}}) + \varepsilon = d_X(B_t, A_{t^{\circ *}}) + \varepsilon$$

(see Proposition 2.1) we get

$$d_X(A_t^{n+1}, A_{t^\circ}^{n+1}) < d_X(A_{t^*}^n, A_{t^{\circ *}}^n) + 3\varepsilon$$

and we distinguish two cases:

- if $t^{\circ *} = t^*$ then $d_X(A_t^{n+1}, A_{t^\circ}^{n+1}) < 3\varepsilon$, and choosing ε small enough we are done,
- otherwise by Proposition 4.2 we get $d_X(A_t^{n+1}, A_{t^\circ}^{n+1}) < d_X(A_{t^*}, A_{t^{*\circ}}) + 3\varepsilon$; and again for ε small enough the induction hypothesis and Proposition 4.2 yield $d_X(A_{t^\circ}^{n+1}, A_t^{n+1}) < d_\circ(t^{*\circ}, t^*) = d_\circ(t^\circ, t)$. ∎

Proof of Lemma 5.5. Recall that S is a given finite tree on ω and $t = t^* {}^\frown \langle n_0 \rangle \in \omega^{<\omega}$ is such that $t \notin S$ and $t^* \in S$. Write $\{s \in S : s \supset t^*\} = \{t^* {}^\frown s' : s' \in S'\}$, so S' is a finite tree too. Then $(A_{t^* \frown s'})_{s' \in S'}$ is an f-severing scheme which satisfies the same hypothesis as $(A_s)_{s \in S}$, and if the conclusion of Lemma 5.5 is ensured for $(A_{t^* \frown s'})_{s' \in S'}$ providing a scheme $(B_{t^* \frown s'})_{s' \in S' \cup \{\langle n_0 \rangle\}}$, then setting $B_s = A_s$ for all $s \in S$ with $s \not\supset t^*$ one gets the same conclusion for the initial scheme $(A_s)_{s \in S}$. Hence without loss of generality we may assume that $t^* = \emptyset$ and $t = \langle 0 \rangle$.

The following partial result is actually the statement of Lemma 5.5 in the particular case $S = \{\emptyset\}$.

Fact 1. *For any nonempty Σ_1^1 set $A \subset X$ there exist nonempty Σ_1^1 subsets B and C of A such that $B \subset \overline{C}$ and $(f(B), f(C))$ is discrete.*

Moreover given any $\varepsilon > 0$ we can require $d(B, A) < \varepsilon$ and $d(C, A) < \varepsilon$.

Proof. Since $g = f_{|A}$ is not continuous, there exists a Δ_1^1 open subset V of Y such that $g^{-1}(V)$ is not open in A. Set $C = g^{-1}(Y \setminus V)$; then since V is the union of all Δ_1^1 open subsets W such that $\overline{W} \subset V$, we can pick such a W so that the set $B = g^{-1}(W) \setminus \mathrm{Int}_A(g^{-1}(V))$ is nonempty. As one can easily check, both B and C are Σ_1^1 sets. For any open set U in A if $U \cap C = \emptyset$ then $U \subset g^{-1}(V)$, hence $U \subset \mathrm{Int}_A(g^{-1}(V))$ and so $U \cap B = \emptyset$; this proves that

$B \subset \overline{C}$. Moreover $g(B) \subset W$ and $g(C) \subset Y \setminus V \subset Y \setminus \overline{W} = W'$ with W and W' open and disjoint, hence $(g(B), g(C)) = (f(B), f(C))$ is discrete.

For the second part write $A = \bigcup_{j<k} A_j$ as a finite union of Σ^1_1 subsets of diameter $< \varepsilon$ and apply the first part in each A_j to get Σ^1_1 subsets B_j and C_j, and then take $B = \bigcup_{j<k} B_j$ and $C = \bigcup_{j<k} C_j$. ∎

FACT 2. *For any nonempty Σ^1_1 set $A \subset X$ and any $\varepsilon > 0$ there exists an infinite sequence $(F^n)_{n\in\omega}$ of pairwise disjoint finite sets such that $d_X(F^n, A) < \varepsilon$ for all n, and f is one-to-one on $\bigcup_n F^n$.*

Proof. Observe first that any nonempty Σ^1_1 set $A' \subset A$ has an uncountable image: otherwise all members of $f(A')$ would be Δ^1_1 points and picking any $b \in f(A')$ the set $f^{-1}(b)$ would be a nonempty Σ^1_1 set on which f is constant, which contradicts the assumption that $A \subset X$.

Now given ε write $A = \bigcup_{j<k} A_j$ as a finite union of nonempty Σ^1_1 subsets of diameter $< \varepsilon$. Then basing on the previous observation construct inductively an infinite sequence $(a_m)_{m\in\omega}$ such that $a_m \in A_j$ if $m \equiv j \pmod k$ with $f(a_m) \neq f(a_p)$ for all $p < m$; then set $F^n = \{a_{kn+j} : j < k\}$. ∎

FACT 3. *If (A, d) is a precompact metric space then for all $\varepsilon > 0$ there exists an integer N such that if $(B^n)_{n<N}$ is any family of N subsets of A satisfying $d(B^m \cup B^n, A) < \varepsilon/2$ for all $m < n < N$, then there exists some $n < N$ such that $d(B^n, A) < \varepsilon$.*

Proof. Fix a finite set $F \subset A$ such that $d(F, A) < \varepsilon/2$ and let $N > \mathrm{card}(F)$. Given $(B^n)_{n<N}$ it is enough to find some $n < N$ such that $d(F, B^n) < \varepsilon/2$. Indeed, if for all $n < N$ there exists $a_n \in F$ such that $d(a_n, B^n) \geq \varepsilon/2$, then since $N > \mathrm{card}(F)$ we have $a_m = a_n = a$ for some $m < n < N$, hence $d(a, B^m \cup B^n) \geq \varepsilon/2$; a contradiction. ∎

FACT 4. *Let $(A_i, d_i)_{i\in I}$ be a finite family of precompact metric spaces. Then for all $\varepsilon > 0$ there exists an integer N such that if for all $i \in I$, $(B^n_i)_{n<N}$ is a family of subsets of A_i such that $d_i(B^m_i \cup B^n_i, A_i) < \varepsilon/2$ for all $m < n < N$, then there exists some $n < N$ such that $d_i(B^n_i, A_i) < \varepsilon$ for all $i \in I$.*

Proof. Consider the integer N associated by Fact 3 to the precompact metric space $(A, d) = \sum_{i\in I}(A_i, d_i)$ (discrete metric sum) and apply Fact 3 to the family $(B^n)_{n<N}$ defined by $B^n = \sum_{i\in I} B^n_i$. ∎

Given any set $E \subset A_\emptyset$ we define inductively for all $s \in S$:

$$E_\emptyset = E, \qquad E_s = A_s \cap \overline{E_{s^*}} \qquad \text{if } s \neq \emptyset.$$

We shall refer to $(E_s)_{s\in S}$ as the *severing scheme of root E*. Note that if E is Σ^1_1 then each E_s is Σ^1_1 too.

Fact 5. *If $E \cup E'$ is dense in $A_\emptyset$ then $E_s \cup E'_s$ is dense in A_s for all $s \in S$.*

Proof. By induction on s: if $\overline{E_{s^*}} \cup \overline{E'_{s^*}} \supset A_{s^*}$ then $\overline{E_{s^*}} \cup \overline{E'_{s^*}} \supset \overline{A_{s^*}} \supset A_s$. ∎

Also given any set $F \subset A_\emptyset$ and any $\delta > 0$ define

$$F^{(\delta)} = \{x \in A_\emptyset : d_Y(f(x), f(F)) > \delta\}.$$

Fact 6. *If $2\delta < \min\{d_Y(f(x), f(x')) : (x, x') \in F \times F'\}$ then $F^{(\delta)} \cup F'^{(\delta)} = A_\emptyset$.*

Proof. Otherwise there exist $a \in A_\emptyset$, $x \in F$ and $x' \in F'$ such that $d_Y(f(a), f(x)) \le \delta$ and $d_Y(f(a), f(x')) \le \delta$, hence $d_Y(f(x), f(x')) \le 2\delta$. ∎

Given any $F \subset A_\emptyset$ we denote by $(F_s^{(\delta)})_{s \in S}$ the severing scheme of root $F^{(\delta)}$. Note that F is disjoint from $F_\emptyset^{(\delta)} = F^{(\delta)}$.

Fact 7. *For all $\varepsilon > 0$ there exist $\delta > 0$ and a finite subset F of $A_\emptyset$ such that $d_X(F, A_\emptyset) < \varepsilon$ and $d_X(F_s^{(\delta)}, A_s) < \varepsilon$ for all $s \in S$.*

Proof. Starting from ε and the family $(A_s, d)_{s \in S}$ of precompact metric spaces fix N as in Fact 4. Let $(F^n)_{n<N}$ be the corresponding initial segment of the sequence given by Fact 2 applied to $A = A_\emptyset$ and fix $\delta < \frac{1}{2}\min\{d_Y(f(x), f(x')) : (x, x') \in \bigcup_{m<n<N} F^m \times F^n\}$. Then by Facts 5 and 6 for all $s \in S$, $F_s^{m(\delta)} \cup F_s^{n(\delta)}$ is dense in A_s, so $d(F_s^{m(\delta)} \cup F_s^{n(\delta)}, A_s) = 0 < \varepsilon/2$; hence by the choice of N one of the F_n's satisfies the conclusion of Fact 7. ∎

Fact 8. *For all $\varepsilon > 0$ there exist Σ_1^1 subsets D, E of $A_\emptyset$ such that $(f(D), f(E))$ is discrete, $d_X(D, A_\emptyset) < \varepsilon$ and $d_X(E_s, A_s) < \varepsilon$ for all $s \in S$.*

Proof. Let (F, δ) be as in Fact 7. For any $a \in F$ fix in Y a Δ_1^1 open neighborhood V_a of $f(a)$ of diameter $< \delta/2$ and set $D = \bigcup_{a \in F} f^{-1}(V_a)$ and $E = \bigcap_{a \in F}\{x \in A_\emptyset : \exists b \in f^{-1}(V_a),\ d_Y(f(x), f(b)) > \delta\}$.

Since $D \supset F$ and $E \supset F^{(\delta)}$, clearly $d(D, A_\emptyset) < \varepsilon$ and $d(E_s, A_s) < \varepsilon$ for all $s \in S$. Moreover since $f(D) \subset \bigcup_{a \in F}\{x \in A_\emptyset : d_Y(f(x), f(a)) < \delta/2\}$ and $f(E) \subset \bigcap_{a \in F}\{x \in A_\emptyset : d_Y(f(x), f(a)) > \delta/2\}$, it follows that $(f(D), f(E))$ is discrete. ∎

End of the proof of Lemma 5.5. Consider the Σ_1^1 sets D, E given by Fact 8 and then B, C given by Fact 1 (applied with $A = D$), and define

$$B_0 = B, \quad B_\emptyset = C \cup E, \quad B_s = E_s \quad \text{if } s \in S \text{ and } s \ne \emptyset.$$

To see that $(B_s)_{s \in S \cup \{\langle 0 \rangle\}}$ is a severing scheme note that $B \subset \overline{C}$ and $E_s \subset \overline{E_{s^*}} \subset \overline{E}$. Moreover $(f(B), f(C))$, $(f(D), f(E))$ and $(f(A_s))_{s \in S}$ are discrete with $B, C \subset D$, $D, E \subset A_\emptyset$ and $E_s \subset A_s$; hence $(B_s)_{s \in S \cup \{\langle 0 \rangle\}}$ is discrete. ∎

This finishes the proof of Lemma 5.2, hence of Theorem 1.2.

6. Appendix: Variations around the Choquet game. Given any game G we shall say that one of the players *wins the game* G if this player has a winning strategy in the game.

Let $X = (X, \mathcal{T})$ be a topological space. We recall that the classical *Choquet game* $G_0(X)$ is a topological game in which at each of their moves Player I chooses a pair $(A, a) \in \mathcal{T} \times X$, and Player II chooses a set $A' \in \mathcal{T}$ with the following rules:

$$\begin{array}{lllll} \mathrm{I}: & (A_0, a_0) & (A_2, a_1) & \ldots & \\ & & & & \text{with} \\ \mathrm{II}: & & A_1 & & A_3 \quad \ldots \end{array} \begin{cases} a_n \in A_{2n} \subset A_{2n-1}, \\ \\ a_n \in A_{2n+1} \subset A_{2n}, \end{cases}$$

and Player II wins an infinite run if $\bigcap_n A_n \neq \emptyset$.

Now fix an arbitrary family $\mathcal{A}$ of subsets of the topological space $X = (X, \mathcal{T})$. We shall consider some variants of the game $G_0(X)$ by changing both the *rules* (at finite stages of the game) as well as the *winning condition* (of a given infinite run). Note that in the Choquet game the topology intervenes only in the rules (since the players have to choose open sets) while the winning condition $\bigcap_n A_n \neq \emptyset$ is purely set-theoretical. In the new games we shall consider, the rules will not refer to the topology of X but to the abstract given family $\mathcal{A}$; however the winning condition will be a topological condition. More precisely in an infinite run of any of these games the players will produce (by alternate choices) a decreasing sequence $(A_n)_{n\in\omega}$ in $\mathcal{A}$, and Player II wins this infinite run if *the sequence* (A_n) *converges compactly.* So all games we shall consider will have the same winning condition and we shall only state their specific rules.

(a) *The game* $G_0(X, \mathcal{A})$. In this game Player I chooses a pair $(A, a) \in \mathcal{A} \times X$, and Player II chooses a set $A' \in \mathcal{A}$ with the following rules:

$$\begin{array}{lllll} \mathrm{I}: & (A_0, a_0) & (A_2, a_1) & \ldots & \\ & & & & \text{with} \\ \mathrm{II}: & & A_1 & & A_3 \quad \ldots \end{array} \begin{cases} a_n \in A_{2n} \subset A_{2n-1}, \\ \\ a_n \in A_{2n+1} \subset A_{2n}. \end{cases}$$

So if $\mathcal{A} = \mathcal{T}$, the rules in $G_0(X, \mathcal{T})$ are exactly the same as in the Choquet game $G_0(X)$, but the winning condition for Player II is more restrictive in $G_0(X, \mathcal{T})$ than in $G_0(X)$. Nevertheless in all classical classes of spaces where Player II wins the Choquet game, the "natural" winning strategy in $G_0(X)$ is actually winning in $G_0(X, \mathcal{T})$ too. This is in particular the case for complete metric spaces where Player II can even ensure that $\bigcap_n A_n = \{a\}$ is a singleton.

Another important instance of $G_0(X, \mathcal{A})$ is given by the case where X is a Σ^1_1 space and $\mathcal{A}$ is the family of all Σ^1_1 subsets of X. We recall the fundamental well known fact that if X^* denotes the set X equipped with the so-called *Gandy–Harrington topology* (the topology generated by the family of all Σ^1_1 subsets of X) then Player II wins the Choquet game $G_0(X^*)$. But again one easily checks that the "standard" winning strategy for Player II in $G_0(X^*)$ realizes actually the stronger winning condition of $G_0(X,\Sigma^1_1)$ (this is not true for $G_0(X^*,\Sigma^1_1)$). So:

LEMMA 6.1. *For any Σ^1_1 space X, Player II wins the game $G_0(X,\Sigma^1_1)$.*

(b) *The game $G_1(X,\mathcal{A})$.* This is a slight variation of the game $G_0(X,\mathcal{A})$. In $G_1(X,\mathcal{A})$ Player I chooses a pair (A,F) where $A\in\mathcal{A}$ and $F\subset A$ is a finite set, and Player II chooses some $A'\in\mathcal{A}$ with the following rules:

$$\begin{array}{llll} \mathrm{I}: & (A_0,F_0) \quad\quad (A_2,F_1) & \ldots & \\ & & & \text{with} \begin{cases} F_n\subset A_{2n}\subset A_{2n-1}, \\ \\ F_n\subset A_{2n+1}\subset A_{2n}. \end{cases} \\ \mathrm{II}: & \quad\quad\quad A_1 \quad\quad\quad\quad A_3 & \ldots & \end{array}$$

LEMMA 6.2. *If Player II wins $G_0(X,\mathcal{A})$ then Player II wins $G_1(X,\mathcal{A})$.*

Proof. Fix a winning strategy τ_0 for Player II in G_0. Consider an infinite run in G_1 as above; without loss of generality we may assume that $F_n\neq\emptyset$ for all n. Let T_n denote the set of all finite G_0-runs compatible with τ_0, of the form

$$s=\langle (A_0,a_0),B_1,(A_2,a_1),\ldots,B_{2n-1},(A_{2n},a_n)\rangle$$

with $a_k\in F_k$ for all k. Define inductively a strategy τ in G_1 by setting

$$A_{2n+1}=\bigcup\{\tau_0(s):s\in T_n\}.$$

Note that for any $a\in F_n$ there exists at least one $s\in T_n$ ending in (A_{2n},a) and since $a\in\tau_0(s)\subset A_{2n}$ for any such s, we have $F_n\subset A_{2n+1}\subset A_{2n}$.

Now if $\bar{x}=(x_n)$ is any sequence with $x_n\in A_n$ for all n, then $T_{\bar{x}}=\{s\in T_n : x_{2n+1}\in\tau_0(s)\}$ is an infinite tree which is finitely branching. So $T_{\bar{x}}$ admits some infinite branch

$$\langle (B_0,a_0),B_1,(B_2,a_1),\ldots,B_{2n-1},(B_{2n},a_n),B_{2n+1},\ldots\rangle$$

with $B_{2n}=A_{2n}$ and $B_{2n+1}\subset A_{2n+1}$, which can be identified to an infinite run in G_0 compatible with τ_0; and since $x_n\in B_n$ for all n, by the winning condition in G_0 the sequence $\bar{x}$ admits a cluster point $x\in\bigcap_n B_n\subset\bigcap_n A_n$. ■

(c) *The game $G_\delta(X,\mathcal{A})$.* Fix a function $\delta:\mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}$ such that for all $A\in\mathcal{P}(X)$ the partial function $\delta(A,\cdot):\mathcal{P}(A)\to\mathbb{R}$ is *decreasing* and *finitely determined*, that is:

(1) If $B'\subset B\subset A$ then $\delta(A,B)\le\delta(A,B')$.
(2) $\delta(A,B)=\sup\{\delta(A,F) : F \text{ finite} \subset B\}$.

A typical example of such a function is given by the Hausdorff distance associated to an arbitrary *precompact* metric on X (with a priori no connection with the topology of X). At each of its moves in the game $G_\delta(X,\mathcal{A})$

Player I chooses a pair $(A,\varepsilon)\in\mathcal{A}\times\mathbb{R}_+^*$, and Player II chooses as usual just some $A'\in\mathcal{A}$ with the following rules:

$$\begin{array}{lllll} \mathrm{I}: & (A_0,\varepsilon_0) & & (A_2,\varepsilon_1) & & \ldots \\ \mathrm{II}: & & A_1 & & A_3 & \ldots \end{array} \quad \text{with} \quad \begin{cases} A_{2n}\subset A_{2n-1}, \\ A_{2n+1}\subset A_{2n}, \\ \delta(A_{2n+1},A_{2n})<\varepsilon_n. \end{cases}$$

LEMMA 6.3. *If Player II wins $G_1(X,\mathcal{A})$ then Player II wins $G_\delta(X,\mathcal{A})$.*

Proof. To any run in G_δ as above one can associate a run in G in which, in his $(n+1)$th move, Player I plays a pair (F_n,A_{2n}) with A_{2n} the same as in the run in G_δ, and F_n a finite subset of A_{2n} such that $\delta(F_{2n},A_{2n})<\varepsilon_n$ so that any set A such that $F_n\subset A\subset A_{2n+1}$ satisfies automatically $\delta(A,A_{2n+1})<\varepsilon_n$. Hence any winning strategy for Player II in G defines canonically a winning strategy for Player II in G_δ. ∎

Theorem 3.3 follows readily from Lemmas 6.1–6.3.

REMARK 6.4. Let $\pi:\tilde X\to X$ be any recursive parametrization of the Σ_1^1 space X by a recursively presented Polish space $\tilde X$. The standard winning strategy for Player II in $G_0(X,\Sigma_1^1)$ associates actually to any infinite run (A_n,a_n) in $G_0(X,\Sigma_1^1)$ infinitely many infinite runs $(\tilde A_n^{(i)},\tilde a_n^{(i)})$ in the Choquet game $G_0(\tilde X)$ on $\tilde X$ such that for all i, $A_n=\pi(\tilde A_n^{(i)})$ and $\bigcap_n\tilde A_n^{(i)}=\{\tilde a^{(i)}\}$ is a singleton with $\pi(\tilde a^{(i)})=a$ constant; hence $\bigcap_n A_n=\bigcap_n\pi(\tilde A_n^{(i)})=\{a\}$ is a singleton too. But this is no longer the case in the game $G_1(X,\Sigma_1^1)$. Indeed, the winning strategy for Player II provided by the proof of Lemma 6.2 associates to any infinite run (A_n,F_n) in $G_1(X,\Sigma_1^1)$ infinitely many sequences $(\tilde A_n^{(i)})$ of subsets of $\tilde X$ such that, as above, $A_n=\pi(\tilde A_n^{(i)})$ and $\bigcap_n\tilde A_n^{(i)}=\tilde K^{(i)}$ is a nonempty compact subset of $\tilde X$ with $\pi(\tilde K^{(i)})=K$ a constant compact set which is not a priori a singleton. Note that if for any reason (for example by the will of Player I) $\operatorname{diam}(A_n)\searrow 0$ then the compact set K will be a singleton; nevertheless, even in this case, Player II has no way to ensure that any of the compact sets $\tilde K^{(i)}$ is a singleton. The same remarks hold for $G_\delta(X,\Sigma_1^1)$ and as a matter of fact the construction of each point $(x_\sigma,y_\sigma)\in\mathrm{Gr}(f)\subset X\times Y$ in the proof of Lemma 5.2 hides somewhere (infinitely many) nonempty compact sets $\tilde K$ such that $\pi(\tilde K)=\{(x_\sigma,y_\sigma)\}$, where π is as above a parametrization of the space $\mathrm{Gr}(f)$.

References

[1] J. Cichoń, M. Morayne, J. Pawlikowski, and S. Solecki, *Decomposing Baire functions*, J. Symbolic Logic 56 (1991), 1273–1283.

[2] S. Jackson and R. D. Mauldin, *Some complexity results in topology and analysis*, Fund. Math. 141 (1992), 75–83.

[3] L. Keldysh, *Sur les fonctions premières mesurables B*, Dokl. Akad. Nauk SSSR 4 (1934), 192–197.

[4] A. Louveau, *Ensembles analytiques et boréliens dans les espaces produits*, Astérisque 78 (1980).

[5] J. van Mill and R. Pol, *Baire 1 functions which are not countable unions of continuous functions*, Acta Math. Hungar. 66 (1995), 289–300.

[6] Y. N. Moschovakis, *Descriptive Set Theory*, Stud. Logic Found. Math. 100, North-Holland, Amsterdam, 1980; 2nd ed.: Math. Surveys Monogr. 155, Amer. Math. Soc., Providence, RI, 2009.

[7] P. S. Novikov and S. I. Adyan, *On a semicontinuous function*, Moskov. Gos. Ped. Inst. Uch. Zapiski 138 (1958), 3–10 (in Russian).

[8] J. Pawlikowski and M. Sabok, *Decomposing Borel functions and structure at finite levels of the Baire hierarchy*, Ann. Pure Appl. Logic 163 (2012), 1748–1764.

[9] S. Solecki, *Decomposing Borel sets and functions and the structure of the Baire class 1 functions*, J. Amer. Math. Soc. 11 (1998), 521–550.

[10] J. Zapletal, *Descriptive set theory and definable forcing*, Mem. Amer. Math. Soc. 167 (2004), no. 793.

Gabriel Debs
Analyse Fonctionnelle
Institut Mathématique de Jussieu
Boîte 186
4, Place Jussieu
75252 Paris Cedex 05, France
E-mail: debs@math.jussieu.fr

Received 17 January 2013;
in revised form 22 November 2013