Algebraic lattices are complete sublattices of the clone lattice over an infinite set

by

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Abstract. The clone lattice $\text{Cl}(X)$ over an infinite set $X$ is a complete algebraic lattice with $2^{|X|}$ compact elements. We show that every algebraic lattice with at most $2^{|X|}$ compact elements is a complete sublattice of $\text{Cl}(X)$.

1. How complicated is the clone lattice? Fix a base set $X$ and denote for all $n \geq 1$ the set $X^{X^n}$ of all $n$-ary operations on $X$ by $\mathcal{F}^{(n)}$. Then $\mathcal{F} = \bigcup_{n \geq 1} \mathcal{F}^{(n)}$ is the set of all functions on $X$ which have finite arity. A set of finitary functions $\mathcal{C} \subseteq \mathcal{F}$ is called a clone iff it is closed under composition and contains all projections, i.e. for all $1 \leq i \leq n$ the function $\pi^n_i$ satisfying $\pi^n_i(x_1, \ldots, x_n) = x_i$. The set of all clones over $X$ forms a complete algebraic lattice $\text{Cl}(X)$ with respect to inclusion. This lattice is countably infinite and completely known if $|X| = 2$ by a result of Post’s [Pos41]; however, describing the clone lattice completely for larger $X$ is believed impossible.

Several known results suggest this. First, $\text{Cl}(X)$ is large; it is of size continuum if $X$ is finite and has at least three elements, and $|\text{Cl}(X)| = 2^{2^{|X|}}$ if $X$ is infinite. Secondly, the clone lattice does not satisfy any non-trivial lattice identity if $|X| \geq 3$ [Bul93]; it does not satisfy any quasi-identity if $|X| \geq 4$ [Bul94]. Also, if $|X| \geq 4$, then every countable product of finite lattices is a sublattice of $\text{Cl}(X)$ [Bul94]. As for examples on infinite $X$, every completely distributive lattice having not more than $2^{|X|}$ compact elements is a subinterval of a monoidal interval of $\text{Cl}(X)$ [Pin] (a monoidal interval being an interval of clones which have the same unary functions). Moreover, specific complicated parts of $\text{Cl}(X)$ have been exhibited, such as an interval which is isomorphic to the lattice of all filters on $X$ in [GS]. There exist

2000 Mathematics Subject Classification: Primary 08A40; Secondary 08A05.

Key words and phrases: clone lattice, complete sublattice, algebraic lattice.

The author is grateful for support through project P17812 of the Austrian Science Fund.
several examples of parts of $\text{Cl}(X)$ that are still “well-behaved” for finite $X$, but which seem to be hopelessly complicated for infinite $X$: The interval above $O^{(1)}$ is a finite chain for finite $X$ [Bur67] but huge and extremely complex for infinite $X$ ([GS02] and [GSS]), and whereas $\text{Cl}(X)$ is dually atomic with a finite number of dual atoms which are all known if $X$ is finite [Ros70], it is not dually atomic on countably infinite $X$ if the continuum hypothesis holds [GS05], and there exist as many dual atoms as there are clones on all infinite $X$ [Ros76]. A recent survey of clones on infinite sets is [GP].

We are interested in which lattices can be embedded into the clone lattice over an infinite set. Assume henceforth $X$ to be infinite. The compact elements of $\text{Cl}(X)$ are easily seen to be exactly the clones which are generated by a finite number of functions. Since $|O| = 2^{|X|}$, this implies that $\text{Cl}(X)$ has at most $2^{|X|}$ compact elements, and it is readily verified that the compact elements really amount to this number. We are going to prove that $\text{Cl}(X)$ is in some sense the most complicated algebraic lattice with this property.

**Theorem 1.** Let $X$ be infinite. Then every algebraic lattice with at most $2^{|X|}$ compact elements can be completely embedded into $\text{Cl}(X)$.

We remark that the corresponding statement does not hold on finite $X$: There, $\text{Cl}(X)$ has countably infinitely many compact (finitely generated) elements, but as has been proven in [Bul01], the countably infinite lattice $M_\omega$ (consisting of a countably infinite antichain plus a smallest and a largest element) does not embed into the clone lattice over any finite set. Observe also that our result implies that the clone lattice on infinite $X$ does not satisfy any non-trivial properties such as the infinite quasi-identity given in [Bul01] which holds for $\text{Cl}(X)$ if $X$ is finite.

**1.1. Notation.** We denote the unary projection $\pi_1$ by the somewhat simpler symbol id, and use $\mathcal{J}$ for the set of projections on $X$. If $\mathcal{F} \subseteq \mathcal{O}$, then we write $\langle \mathcal{F} \rangle$ for the clone generated by $\mathcal{F}$. Three lattices will appear in the proof, the clone lattice $\text{Cl}(X)$, the lattice $\mathcal{L}$ to be embedded into the clone lattice, and the lattice of join-semilattice ideals of compact elements of $\mathcal{L}$: For all of them, we use the symbols $\wedge, \vee, \bigwedge, \bigvee$ with their standard meanings, and confusion shall be carefully avoided. If $\Phi \subseteq \mathcal{O}^{(1)}$ is a set of unary operations, then $\Phi^*$ will stand for all those functions which arise from functions of $\Phi$ by the addition of any finite number of dummy variables. Such functions will remain essentially unary, i.e. although possibly non-unary they depend on only one variable, as opposed to essentially at least binary functions, which are functions that depend on at least two of their variables.
2. Proof of the main theorem. Let $\mathcal{L}$ be the lattice to be embedded into $\text{Cl}(X)$ and denote by $\mathfrak{P}$ the set of all compact elements of $\mathcal{L}$. Then $\mathfrak{P}$ is a join-semilattice (cf. the textbook [Grä78]). By an ideal $I \subseteq \mathfrak{P}$ we mean a lower subset of $\mathfrak{P}$ closed under (finite) joins. The set of all ideals of $\mathfrak{P}$ is a complete algebraic lattice, and in fact

**Fact 2.** $\mathcal{L}$ is isomorphic to the lattice of ideals of $\mathfrak{P}$.

We are going to assign a clone $\mathcal{C}_I$ to every ideal $I \subseteq \mathfrak{P}$ in such a way that the resulting mapping is a complete embedding of $\mathcal{L}$ into $\text{Cl}(X)$. Fix four elements $0, 1, 2, 4 \in X$ and set $A = X \setminus \{0, 1, 2, 4\}$. Let $\mathcal{A} = (A_p)_{p \in \mathfrak{P}}$ be a family of subsets of $A$ indexed by the elements of $\mathfrak{P}$ with the following property: Whenever $A_p, A_{q_1}, \ldots, A_{q_k} \in \mathcal{A}$ and $p \neq q_i$ for all $1 \leq i \leq k$, then $A_p \nsubseteq A_{q_1} \cup \cdots \cup A_{q_k}$. Such a family exists: For example, there exist independent families of size $2^{|X|}$, where a family $\mathcal{F}$ of subsets of $A$ is called independent iff for all finite disjoint $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{F}$,

$$\bigcap \{F : F \in \mathcal{F}_1\} \cap \bigcap \{A \setminus F : F \in \mathcal{F}_2\} \neq \emptyset.$$  

See the textbook [Jec03, Lemma 7.7]. If $|X| = \aleph_0$, then one could also take $\mathcal{A}$ to be almost disjoint, meaning that all members of $\mathcal{A}$ are infinite and the intersection of any two distinct sets from $\mathcal{A}$ is finite (cf. [Jec03, Lemma 9.21]).

Define for all $p \in \mathfrak{P}$ a unary function $\phi_p \in \mathcal{G}^{(1)}$ by

$$\phi_p(x) = \begin{cases} 0, & x \in A \setminus A_p, \\ 1, & x \in A_p, \\ 2, & x = 2, \\ 4, & x \in \{0, 1, 4\}. \end{cases}$$

so on $A$, $\phi_p$ is the characteristic function of $A_p$. Set $\Phi = \{\phi_p : p \in \mathfrak{P}\}$. Now define for all $p, q_1, q_2 \in \mathfrak{P}$ with $p \leq q_1 \lor q_2$ a ternary function $m_{p}^{q_1,q_2}$ by

$$m_{p}^{q_1,q_2}(x, y, z) = \begin{cases} \phi_p(x), & y = \phi_{q_1}(x) \land z = \phi_{q_2}(x), \\ 2, & (x = 2 \lor y = 2 \lor z = 2) \land (y \notin \{1, 4\}) \land (z \notin \{1, 4\}), \\ 4, & \text{otherwise}. \end{cases}$$

The function is well-defined: We only have to check that there is no conflict between the conditions for $m_{p}^{q_1,q_2}(x, y, z)$ to yield $\phi_p(x)$ and 2, respectively. If both conditions are satisfied, then one of the components of the tuple $(x, y, z)$ equals 2; since $y = \phi_{q_1}(x)$ and $z = \phi_{q_2}(x)$, this implies $x = y = z = 2$, making the function value $m_{p}^{q_1,q_2}(x, y, z) = 2 = \phi_p(x)$ unique.

We write $\mathcal{M} = \{m_{p}^{q_1,q_2} : p, q_1, q_2 \in \mathfrak{P} \land p \leq q_1 \lor q_2\}$ and $\mathcal{C} = \langle \Phi \cup \mathcal{M} \rangle$. The clones $\mathcal{C}_I$ will all be subclones of $\mathcal{C}$ and will all contain $\mathcal{M}$. They
will essentially consist of those \( \phi_p \) for which \( p \in I \), plus the functions from \( \mathcal{M} \); the exact definition can only be given later. One can think of the \( \phi_p \) as functions that represent the elements of \( \mathfrak{P} \) in such a way that they are in some sense “independent” of each other, and of the \( m^{q_1,q_2}_p \) as functions representing the order of \( \mathfrak{P} \), since \( m^{q_1,q_2}_p(\text{id}, \phi_{q_1}, \phi_{q_2}) = \phi_p \) and since \( m^{q_1,q_2}_p \) is defined only if \( p \leq q_1 \lor q_2 \). The following lemma follows easily by induction over terms in \( \mathcal{C} \).

**Lemma 3.** The only functions in \( \mathcal{C} \) which take values in \( A \) are the projections.

**Definition 4.** We call a function \( f \in \mathcal{O}(1) \) distracted iff there exists \( a \in A \) such that \( f(a) \in \{2,4\} \).

**Lemma 5.** Let \( t \in \mathcal{C}(n) \) and \( t_1, \ldots, t_n \in \mathcal{O}(1) \). If \( t \) depends on its \( i \)-th variable, where \( 1 \leq i \leq n \), and if \( t_i \) is distracted, then \( t(t_1, \ldots, t_n) \) is distracted.

**Proof.** We use induction over terms in \( \mathcal{C} \). First, let \( t \in \mathcal{J} \cup \Phi \cup \mathcal{M} \). There is nothing to show if \( t \) is a projection. If \( t \in \Phi \) and \( t_1 \in \mathcal{O}(1) \) is distracted, then there exists \( a \in A \) such that \( t_1(a) \in \{2,4\} \), so \( t(t_1(a)) \in \{2,4\} \) and \( t(t_1) \) is distracted. If \( t = m^{q_1,q_2}_p \in \mathcal{M} \) and \( t_i \) is distracted for some \( i \in \{1,2,3\} \), then \( t_i(a) \in \{2,4\} \) for some \( a \in A \) implies that \( m^{q_1,q_2}_p(t_1,t_2,t_3)(a) \in \{2,4\} \): Indeed, if \( m^{q_1,q_2}_p(t_1,t_2,t_3)(a) \in \{0,1\} \), then the definition of \( m^{q_1,q_2}_p \) would allow us to conclude \( t_1(a) \in A \) and \( t_2(a) = \phi_{q_1}(t_1(a)) \in \{0,1\} \) and \( t_3(a) = \phi_{q_2}(t_1(a)) \in \{0,1\} \), which is clearly impossible as \( t_i(a) \in \{2,4\} \).

For the induction step, assume that \( t = f(s_1, \ldots, s_m) \), where \( f \in \mathcal{J} \cup \Phi \cup \mathcal{M} \) and \( s_j \) satisfies the induction hypothesis, \( 1 \leq j \leq m \). Now there exists \( 1 \leq j \leq m \) such that \( f \) depends on its \( j \)-th variable and \( s_j \) depends on its \( i \)-th variable. By induction hypothesis \( s_j(t_1, \ldots, t_n) \) is distracted and so is \( f(s_1(t_1, \ldots, t_n), \ldots, s_m(t_1, \ldots, t_n)) \), by the same proof as for the induction beginning.

**Lemma 6.** Let \( m^{q_1,q_2}_p \in \mathcal{M} \) and \( t_1, t_2, t_3 \in \Phi \cup \{\text{id}\} \). Then \( f = m^{q_1,q_2}_p(t_1,t_2,t_3) \) is distracted unless \( t_1 = \text{id}, t_2 = \phi_{q_1}, \) and \( t_3 = \phi_{q_2} \). In the latter case we have \( f = \phi_p \).

**Proof.** If \( t_2 = \text{id} \) or \( t_3 = \text{id} \), then \( f(a) \in \{2,4\} \) for all \( a \in A \), since \( m^{q_1,q_2}_p \) can yield 0 or 1 only if its second and third argument is in the range of a function in \( \Phi \); hence \( f \) is distracted in that case. Assume henceforth \( t_2,t_3 \in \Phi \) and write \( t_2 = \phi_r \) and \( t_3 = \phi_s \), where \( r, s \in \mathfrak{P} \).

If \( t_1 = \text{id} \), then \( f \) yields 4 on the symmetric differences \( A_{q_1} \triangle A_r \) and \( A_{q_2} \triangle A_s \) by the very definition of \( m^{q_1,q_2}_p \). Hence \( f \) is distracted unless those sets are empty, i.e. \( s = q_1 \) and \( r = q_2 \); in the latter case we have \( f = \phi_p \) as asserted.
If \( t_1 = \phi_i \in \Phi \), then \( m_p^{q_1,q_2}(\phi_i, \phi_r, \phi_s) \) yields by definition either 2, 4, or an element of the form \( \phi_p(\phi(x)) \in \{2, 4\} \), so \( f \) is distracted. ■

**Lemma 7.** All \( t \in \mathcal{C}^{(1)} \setminus (\Phi \cup \{\text{id}\}) \) are distracted.

**Proof.** We prove this by induction over terms in \( \mathcal{C} \). The beginning is trivial since there are no unary functions in the generating set \( J \cup \Phi \cup \mathcal{M} \) of \( \mathcal{C} \) except those from \( \Phi \cup \{\text{id}\} \).

For the induction step, assume that \( t = f(t_1, \ldots, t_n) \), where \( f \in J \cup \Phi \cup \mathcal{M} \) and \( t_i \) satisfies the induction hypothesis for all \( 1 \leq i \leq n \). The case \( f \in J \) is trivial. If \( f \in \Phi \) and \( t_1 \neq \text{id} \), then \( t_1 \) takes only values outside \( A \) by Lemma 3, so \( f(t_1) \) takes only values in \( \{2, 4\} \) and is distracted. The other possibility is that \( f \in \mathcal{M} \), so write \( t = m_p^{q_1,q_2}(t_1, t_2, t_3) \). If any of the \( t_i \) is distracted then so is \( t \), by Lemma 5. We may therefore assume that the \( t_i \) are not distracted and hence are elements of \( \Phi \cup \{\text{id}\} \). But then Lemma 6 tells us that \( t \), not being an element of \( \Phi \cup \{\text{id}\} \) by assumption, must be distracted. ■

**Definition 8.** We say that \( t \in \mathcal{C}^{(n)} \) is unspoilt iff there exist \( t_1, \ldots, t_n \in \mathcal{C}^{(1)} \) such that \( t(t_1, \ldots, t_n) \in \Phi \). Otherwise we call \( t \) spoilt.

**Remark 9.** By Lemmas 5 and 7, \( t_i \) must be in \( \Phi \cup \{\text{id}\} \) if \( t \) depends on its \( i \)-th variable, for all \( 1 \leq i \leq n \).

**Remark 10.** An easy induction using Lemmas 5 and 6 shows that \( t_i \) is uniquely determined if \( t \) depends on its \( i \)-th variable, for all \( 1 \leq i \leq n \).

**Remark 11.** By Lemmas 5 and 7, a unary \( t \in \mathcal{C}^{(1)} \) is distracted iff it is spoilt.

**Lemma 12.** Let \( t \in \mathcal{C}^{(n)} \) be unspoilt, and assume it depends on its first variable. Then \( t(2, x_2, \ldots, x_n) \in \{2, 4\} \) for all \( x_2, \ldots, x_n \in X \).

**Proof.** We use induction on the complexity of \( t \). The lemma is trivial if \( t \in J \cup \Phi \cup \mathcal{M} \). For the induction step, since the range of \( \phi_p(t_1) \) is contained in \( \{2, 4\} \) and since therefore \( \phi_p(t_1) \) is spoilt for all \( \phi_p \in \Phi \) and all \( t_1 \in \mathcal{C} \setminus J \), we may assume \( t = m_p^{q_1,q_2}(t_1, t_2, t_3) \), where \( t_i \) satisfies the induction hypothesis for \( 1 \leq i \leq 3 \). Now one of the \( t_i \) must depend on its first variable, implying \( t_i(2, x_2, \ldots, x_n) \in \{2, 4\} \) by induction hypothesis. Hence, \( m_p^{q_1,q_2}(t_1, t_2, t_3)(2, x_2, \ldots, x_n) \in \{2, 4\} \) by the definition of \( m_p^{q_1,q_2} \). ■

Let \( t(x, y) \in \mathcal{C}^{(2)} \), and consider a concrete representation \( r = r(t) \) of \( t \) as a term over the generating set \( J \cup \Phi \cup \mathcal{M} \) of \( \mathcal{C} \). In the following, we write such representations without the use of projections, using the variables \( x, y \) instead: for example, we write \( m_p^{q_1,q_2}(x, y, y) \) instead of \( m_p^{q_1,q_2}(\pi_1^2, \pi_2^2, \pi_3^2) \). This is no loss of generality and only avoids unnecessary usage of the projections, as in \( \pi_1^2(\pi_2^2, \phi_p(\pi_1^2)) \) (equivalently, we could demand the projections to appear only as innermost arguments in the representation). We say that
a subterm $s$ of $r$ is a leaf of $r$ iff it involves exactly one function symbol from $\Phi \cup M$. For example, the leaves of

$$m^{q_1,q_2}_p(m^{q_1,q_2}_{q_1}(x, \phi_1(y), \phi_2(x)), \phi_3(y), m^{h_1,h_2}_g(x,x,x))$$

are $\phi_1(y), \phi_2(x), \phi_3(y),$ and $m^{h_1,h_2}_g(x,x,x).$ If we think of $r$ as a tree in which the variables are not represented by their own nodes, the leaves of $r$ are really exactly the leaves of the tree.

We call the representation $r(t)$ reduced iff it has no subterms of the form $m^{q_1,q_2}_p(x, \phi_{q_1}(x), \phi_{q_2}(x))$. Such subterms can be replaced by $\phi_p(x)$ by virtue of Lemma 6, so every term $t$ has a reduced representation. We are only interested in representations of unspoil functions that depend on both variables, so all unary subterms of any representation correspond to elements of $\Phi$, by Lemmas 5 and 7; working with reduced terms means that we demand those unary subterms to be represented by only one function symbol.

Let $r(t)$ be reduced. We set Leaf$(r)$ to consist of all leaves of $r(t)$. Note that Leaf$(r)$ depends on the representation of the function $t$.

**Lemma 13.** Let $r(x,y)$ be a reduced representation of a binary function in $C$ that is unspoilt and depends on both of its variables. Let $a \in A$. Then $r(2,a) = 4$ iff $a \in \bigcup\{A_v : \phi_v(y) \in \text{Leaf}(r)\}$.

**Proof.** We use induction on the complexity of $r$. The beginning is trivial as there are no binary functions depending on both variables in the generating set of $C$. For the induction step, write $r = f(r_1, \ldots, r_n)$, where $f \in \Phi \cup M$, and where $r_i$ satisfies the induction hypothesis for $1 \leq i \leq n$. If $f \in \Phi$, then using Lemma 3 it is readily verified that $f(r_1)$ is spoilt unless $r_1$ is a projection, in which case $r \in \Phi^*$, contradicting the assumption that $r$ depends on both variables. Assume henceforth that $f = m^{q_1,q_2}_p \in M$.

Observe that all $r_i$ must be unspoilt, for otherwise $r$ would be spoilt as well by Lemmas 5 and 7. Since $r$ is unspoilt, there exist $s_1, s_2 \in C^{(1)}$ such that $m^{q_1,q_2}_p(r_1(s_1,s_2), r_2(s_1,s_2), r_3(s_1,s_2)) \in \Phi$. By Lemmas 5–7, this is only possible if $r_1(s_1,s_2)$ is the identity, which together with Lemma 3 implies that $r_1$ is a projection. Suppose that $r_2 = r_1 = \pi_i^2$, where $i \in \{1,2\}$. Then $r(s_1,s_2) = m^{q_1,q_2}_p(s_i, s_i, r_3(s_1,s_2)) \in \Phi$ and Lemma 6 implies that the first argument in $m^{q_1,q_2}_p$ must be the identity, while the second must equal $\phi_{q_1}$, an obvious contradiction. The same contradiction occurs upon assuming $r_3 = r_1$, and hence we have $r_i \neq r_1$, $i = 2,3$. We now distinguish six cases.

Assume first $r_2, r_3 \in J$. Then $r = m^{q_1,q_2}_p(x,y,y)$ or $r = m^{q_1,q_2}_p(x,x,x)$. In either case we have $r(2,a) = 2 \neq 4$, in accordance with our assertion as $r$ does not have any leaves of the form $\phi_v(y)$.

Consider the case where $r_2 \in J$ and $r_3 \in \Phi^*$ (by symmetry, this also covers the case $r_3 \in J$ and $r_2 \in \Phi^*$). Keeping Lemma 6 in mind we conclude that $r = m^{q_1,q_2}_p(x,y,\phi_{q_2}(x))$ or $r = m^{q_1,q_2}_p(y,x,\phi_{q_2}(y))$ or...
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The last two possibilities, however, are spoilt, as substitution of \( \phi_{q_1} \) for \( y \) and \( x \), respectively, yields a distracted third argument \( \phi_{q_2}(\phi_{q_1}) \) of \( m^p_{q_1,q_2} \). The first possibility gives us \( r(2, a) = m^p_{q_1,q_2}(2, a, 2) = 2 \neq 4 \). In accordance with our assertion.

Finally, for the second term we have \( r(2, a) = m^p_{q_1,q_2}(a, 2, \phi_{q_2}(a)) \), which equals 4 iff \( \phi_{q_2}(a) \in \{1, 4\} \) iff \( a \in A_{q_2} \).

Now assume that \( r_2 \in J \) and \( r_3 \notin J \cup \Phi^* \). Then \( r_3 \) depends on both of its variables by Lemma 7, and therefore satisfies the assertion of this lemma by induction hypothesis. By Lemma 12, we find that \( r(2, a) = 4 \) iff \( r(2, a) \neq 2 \); the definition of \( m^p_{q_1,q_2} \) tells us that this is the case iff \( 2 \notin \{r_1(2, a), r_2(2, a), r_3(2, a)\} \) or \( r_2(2, a) \in \{1, 4\} \) or \( r_3(2, a) \in \{1, 4\} \).

Again, \( r_3(2, a) \in \{2, 4\} \) by Lemma 12, and \( r_3(2, a) \in \{0, 1, 2\} \) as \( r_2 \in \Phi^* \), implying \( r(2, a) = 4 \) iff \( r_2(2, a) = 1 \) or \( r_3(2, a) = 4 \). Now if \( r_2(x, y) = \phi_{q_1}(x) \), then \( r_2(2, a) = 2 \); and so \( r(2, a) = 4 \) iff \( r_2(3, a) = 4 \) iff \( a \in A \cap \{A_v : \phi_v(y) \in Leaf(r_3)\} \) by induction hypothesis. This is in accordance with our assertion since then \( \phi_v(y) \in Leaf(r_3) \) iff \( \phi_v(y) \in Leaf(r) \). If on the other hand \( r_2(x, y) = \phi_{q_1}(y) \), then \( r_2(2, a) = 1 \) iff \( a \in A_{q_1} \), and hence \( r(2, a) = 4 \) iff \( a \in A_{q_1} \cup \{A_v : \phi_v(y) \in Leaf(r_3)\} \); this is the case iff \( a \in \{A_v : \phi_v(y) \in Leaf(r)\} \).

If \( r_2, r_3 \in \Phi^* \), then up to symmetry we have \( r = m^p_{q_1,q_2}(x, \phi_{q_1}(x), \phi_{q_2}(y)) \) or \( r = m^p_{q_1,q_2}(x, \phi_{q_1}(y), \phi_{q_2}(y)) \) or \( r = m^p_{q_1,q_2}(y, \phi_{q_1}(x), \phi_{q_2}(x)) \) or \( r = m^p_{q_1,q_2}(y, \phi_{q_1}(y), \phi_{q_2}(y)) \). Therefore \( r(2, a) = 4 \) iff \( a \in A_{q_2} \) in the first case, iff \( a \in A_{q_1} \cup A_{q_2} \) in the second case, and iff \( a \in A_{q_2} \) in the fourth case; in the third case, \( r(2, a) = 2 \neq 4 \).

Finally, consider \( r_2, r_3 \notin J \cup \Phi^* \). By Lemma 12, \( \{r_2(2, a), r_3(2, a)\} \subseteq \{2, 4\} \); thus, \( r(2, a) = 4 \) iff \( r(2, a) \neq 2 \) iff \( r_2(2, a) = 4 \) or \( r_3(2, a) = 4 \) or \( r_2, r_3 \notin J \cup \Phi^* \). Using the induction hypothesis, we find that \( r(2, a) \) yields 4 iff \( a \in \{A_v : \phi_v(y) \in Leaf(r_2)\} \) or \( a \in \{A_v : \phi_v(y) \in Leaf(r_3)\} \); hence, \( r(2, a) = 4 \) iff \( a \in \{A_v : \phi_v(y) \in Leaf(r)\} \).

Set \( J = \{t \in C : t \text{ spoilt}\} \). For all \( I \subseteq \Phi \) define sets of functions \( \Phi_I = \{\phi_p \in \Phi : p \in I\} \) and \( G_I = \Phi_I \cup M \cup J \), and a clone \( C_I = \langle G_I \rangle \). Write \( \langle I \rangle \) for the ideal of \( \Phi \) generated by \( I \).

**Lemma 14.** Let \( p \in \Phi \) and \( I \subseteq \Phi \). Then \( \phi_p \in C_I \) iff \( p \in \langle I \rangle \).

**Proof.** Let \( t \in C_I \); using induction over the complexity of \( t \) as a term over the generating set \( G_I \), we show that \( t = \phi_p \) implies \( p \in \langle I \rangle \). The
beginning is trivial, since if \( t \in \mathcal{G}_1 \), then \( t \in \Phi_1 \) and so \( p \in I \). For the induction step, write \( t = f(t_1, \ldots, t_n) \), with \( f \in \mathcal{G}_1 \) and \( t_i \in \mathcal{G}_1 \) satisfying the induction hypothesis, \( 1 \leq i \leq n \). Clearly, \( f \in \mathcal{I} \) is impossible. \( f \in \Phi_1 \) implies that \( t_1 \) is the identity and so \( f = \phi_v \); hence \( p \in I \). Assume therefore that \( f = m_u^{q_1,q_2} \in \mathcal{M} \). Then \( u = p \), \( t_1 = \text{id} \), \( t_2 = \phi_{q_1} \) and \( t_3 = \phi_{q_2} \) by Lemmas 5–7. By induction hypothesis, \( q_1, q_2 \in \langle I \rangle \). Hence, \( p \leq q_1 \lor q_2 \in \langle I \rangle \).

For the other direction, it is enough to show that if \( \phi_{q_1}, \phi_{q_2} \in \mathcal{G}_1 \), then \( \phi_u \in \mathcal{G}_I \) for all \( u \leq q_1 \lor q_2 \). But this is clear since \( \phi_u = m_u^{q_1,q_2}(\text{id}, \phi_{q_1}, \phi_{q_2}) \in \mathcal{G}_I \).

**Lemma 15.** Let \( \mathcal{I} \) be a family of ideals of \( \mathfrak{P} \). Then \( \bigvee \{ \mathcal{G}_I : I \in \mathcal{I} \} = \mathcal{G}_{\bigvee \mathcal{I}} \).

**Proof.** Trivially, \( \mathcal{G}_{\bigvee \mathcal{I}} \) contains all \( \mathcal{G}_I \), where \( I \in \mathcal{I} \), hence it contains \( \bigvee \{ \mathcal{G}_I : I \in \mathcal{I} \} \). For the other inclusion we have to show that \( \mathcal{G}_{\bigvee \mathcal{I}} \) is contained in \( \bigvee \{ \mathcal{G}_I : I \in \mathcal{I} \} \); clearly, it is enough to show that \( \Phi_{\bigvee \mathcal{I}} \subseteq \bigvee \{ \mathcal{G}_I : I \in \mathcal{I} \} \). Indeed, if \( \phi_p \in \Phi_{\bigvee \mathcal{I}} \), then \( p \in \bigvee \mathcal{I} \). Since \( \bigvee \mathcal{I} = \langle \bigcup \mathcal{I} \rangle \), the preceding lemma implies \( \phi_p \in \mathcal{G}_{\bigvee \mathcal{I}} \). Now it is enough to observe that \( \mathcal{G}_{\bigvee \mathcal{I}} \) equals \( \langle \bigcup \{ \mathcal{G}_I : I \in \mathcal{I} \} \rangle \), which is exactly \( \bigvee \{ \mathcal{G}_I : I \in \mathcal{I} \} \).

**Lemma 16.** Let \( \mathcal{I} \) be a family of ideals of \( \mathfrak{P} \). Then \( \bigwedge \{ \mathcal{G}_I : I \in \mathcal{I} \} = \mathcal{G}_{\bigwedge \mathcal{I}} \).

**Proof.** \( \mathcal{G}_{\bigwedge \mathcal{I}} \) is a subclone of all \( \mathcal{G}_I \), where \( I \in \mathcal{I} \), so trivially \( \mathcal{G}_{\bigwedge \mathcal{I}} \subseteq \bigwedge \{ \mathcal{G}_I : I \in \mathcal{I} \} \). For the other direction, let \( t \in \bigwedge \{ \mathcal{G}_I : I \in \mathcal{I} \} = \bigcap \{ \mathcal{G}_I : I \in \mathcal{I} \} \). If \( t \) is spoilt, then \( t \in \mathcal{G}_{\bigwedge \mathcal{I}} \) by definition, so assume that \( t \) is unspoiled. If \( t \) is essentially unary, then \( t \) is a projection or an element of \( \Phi^* \), by Lemma 7. In the latter case, \( t \in \bigcap \{ \Phi^*_I : I \in \mathcal{I} \} \) by Lemma 14, so \( t \in \mathcal{G}_{\bigwedge \mathcal{I}} \). So let \( t \) be essentially at least binary, and assume without loss of generality that it depends on all of its variables. Because \( t \) is unspoiled, there exist \( t_1, \ldots, t_n \in \Phi \cup \{ \text{id} \} \) such that \( t(t_1, \ldots, t_n) \in \Phi \). Set \( s_i(x,y) = t(t_1(x), \ldots, t_{i-1}(x), y, t_{i+1}(x), \ldots, t_n(x)) \) for all \( 1 \leq i \leq n \). Obviously, all \( s_i \) are unspoiled. They also depend on both variables: indeed, let without loss of generality \( i = 1 \). Then \( s_1(2, t_1(a)) = t(t_1(a), 2, \ldots, 2) \in \{2,4\} \) by Lemma 12 but \( s_1(a, t_1(a)) = t(t_1, \ldots, t_n)(a) \in \{0,1\} \) for all \( a \in A \), so \( s_1 \) depends on the first variable. For the second variable, observe that \( s_1(a,2) = t(2, t_2(a), \ldots, t_n(a)) \in \{2,4\} \) and \( s_1(a, t_1(a)) \neq s_1(a,2) \).

Assume that \( t \) is represented as a reduced term. The \( s_i \) might not be reduced: for example, \( t \) could have a subterm like \( m_{p_1,q_2}^{q_1}(x_2, \phi_{q_1}(x_3), x_4) \), which becomes \( m_{p_1,q_2}^{q_1}(x, \phi_{q_1}(x), \phi_{q_2}(x)) \) when we substitute \( x_2 = x_3 = x \) and \( x_4 = \phi_{q_2}(x) \) upon building, say, \( s_1 \). However, such redundancies will occur only for the variable \( x \). Thus, when simplifying \( s_i \) to a reduced term according to the equation \( m_{p_1,q_2}^{q_1}(x, \phi_{q_1}(x), \phi_{q_2}(x)) = \phi_p(x) \), the leaves of the form \( \phi_p(y) \), which were originally (that is, in \( t \)) leaves of the form \( \phi_p(x_i) \),
do not change. Therefore, \( \phi_p(y) \) is a leaf of the new reduced \( s_i \) iff \( \phi_p(x_i) \) is a leaf of \( t \).

By Lemma 13, for all \( 1 \leq i \leq n \) and for all \( a \in A \) we deduce that \( s_i(2,a) = 4 \) iff \( a \in \bigcup \{ A_v : \phi_v(x_i) \in \text{Leaf}(s_i) \} \). This is the case iff \( a \in \bigcup \{ A_v : \exists i (\phi_v(x_i) \in \text{Leaf}(t)) \} \). Pick arbitrary \( I,J \in \mathcal{F} \) and consider two reduced representations \( t_I,t_J \) of \( t \), where \( t_I \) is a term over \( G_I \) and \( t_J \) one over \( G_J \). Then, since whether or not \( s_i(2,a) = 4 \) does not depend on the representation,

\[
\bigcup \{ A_v : \exists i (\phi_v(x_i) \in \text{Leaf}(s_i)) \} = \bigcup \{ A_v : \exists i (\phi_v(x_i) \in \text{Leaf}(t_J)) \}.
\]

Thus, the latter set is a subset of both \( I \) and \( J \), implying that \( t_I \) actually involves only functions from \( \mathcal{G}_{I \cap J} \) as leaves. Since \( J \) was arbitrary, we may conclude that the term \( t_I \) uses only functions from \( \mathcal{G}_{I \cap J} \) as leaves. Because functions from \( \Phi \) can appear only as leaves in an unspoilt term (\( \phi_v(f) \) is spoilt for all \( \phi_v \in \Phi \) and all \( f \in \mathcal{C} \) unless \( f \) is a projection), this means that \( t_I \) contains only functions from \( \mathcal{G}_{I \cap J} \). Hence, \( t \in \mathcal{C}_{I \cap J} \).

**Proposition 17.** The mapping assigning \( \mathcal{C}_I \) to every ideal \( I \subseteq \mathcal{P} \) is a complete lattice embedding of \( \mathcal{L} \) into \( \text{Cl}(X) \).

**Proof.** The function is injective by Lemma 14 and preserves arbitrary suprema and infima by Lemmas 15 and 16.

3. Concluding remarks and outlook. The only place where we used the infinity of the base set \( X \) is when we claim the existence of a family \( \mathcal{A} \) which is as large as \( \mathcal{P} \) and has the property that whenever \( A_p,A_{q_1},\ldots,A_{q_k} \in \mathcal{A} \) and \( p \neq q_i \) for all \( 1 \leq i \leq k \), then

\[
A_p \not\subseteq A_{q_1} \cup \cdots \cup A_{q_k}.
\]

Therefore surprisingly, the same proof works to show that every finite lattice \( \mathcal{L} \) is a sublattice of the clone lattice over a finite \( X \) for some \( X \) large enough (\( |X| \geq |\mathcal{L}| + 4 \) suffices). However, as mentioned in the introduction, much better results already exist for finite \( X \).

Answering the following question would be a next interesting step in answering the question of how complicated the clone lattice is.

**Problem 18.** Is every algebraic lattice with at most \( 2^{|X|} \) compact elements an interval of \( \text{Cl}(X) \)?
References

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Received 25 September 2005;
in revised form 19 April 2007