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N-determined 2-compact groups. I

by

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Abstract. This is the first part of a paper that classifies 2-compact groups. In this first part we formulate a general classification scheme for 2-compact groups in terms of their maximal torus normalizer pairs. We apply this general classification procedure to the simple 2-compact groups of the A-family and show that any simple 2-compact group that is locally isomorphic to $\mathrm{PGL}(n+1,\mathbf{C})$ is uniquely N-determined. Thus there are no other 2-compact groups in the A-family than the ones we already know. We also compute the group of automorphisms of any member of the A-family and show that it consists of unstable Adams operations only.

Contents

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1. INTRODUCTION

A *p-compact group*, where p is a prime number, is a p-complete space BX whose loop space $X = \Omega BX$ has finite mod p singular cohomology. For instance, if G is a Lie group and $\pi_0(G)$ is a finite p-group then the p-completed classifying space, $BG^\wedge_p$, of G is a p-compact group. These homotopy Lie groups were defined and explored by W. G. Dwyer and C. W. Wilkerson in a series of papers [11], [12], [13], [10]. (Consult the survey articles [7], [27], [42], [32] for a quick overview.) Dwyer and Wilkerson show that any p-compact group BX has a maximal torus $BT \to BX$ with a Weyl group acting on it. The Weyl group is defined as the group of components of the Weyl space which is the associative topological monoid of self-maps of BT over BX. They also show that if we let BN denote the Borel construction for the Weyl space action on BT, then BN is the total space of a fibration

$$BT \to BN \to BW$$

whose fiber is the classifying space of the maximal torus and whose base space is the classifying space of the Weyl group. Moreover, the monomorphism $BT \to BX$ extends to a monomorphism $BN \to BX$, called the *normalizer* of the maximal torus [11, 9.2, 9.7, 9.8]. (Strictly speaking, BN is in general not a p-compact group as its fundamental group may not be a finite p-group, but BN is what is called an *extended p-compact torus* [12, 3.12].) In case G is a compact connected Lie group, the maximal torus normalizers in the Lie and in the p-compact group sense are essentially identical. It has been conjectured, as suggested by the analogous situation for connected compact Lie groups [6] (and some nonconnected ones [19], [20]), that BN determines BX. Indeed, this classification conjecture has been verified for p-compact groups at all odd primes [35], [38], [2]. The aim of this paper is to deal with the only remaining case, $p = 2$.

The first obstacle for a classification of 2-compact groups in terms of their maximal torus normalizers is, in contrast to the odd p case, that the maximal torus normalizer does not retain information about component groups. For instance, the nonconnected 2-compact group $B\mathrm{O}(2n)^\wedge_2$ and the connected 2-compact group $B\mathrm{SO}(2n+1)^\wedge_2$ have identical maximal torus normalizers. Thus the maximal torus normalizer itself is simply not strong enough as a complete 2-compact group invariant. That is why we replace maximal torus normalizers by maximal torus normalizer *pairs*.

Consider a 2-compact group BX with identity component $BX_0 \to BX$ and component group $\pi_0(X) = X/X_0$. There is a map of fibrations

$$\begin{tikzcd}
BX_0 \arrow[d] & BN_0 \arrow[l, "Bj_0"'] \arrow[d] \\
BX \arrow[d] & BN \arrow[l, "Bj"'] \arrow[d] \\
B\pi_0(X) & B\pi \arrow[l, "\cong"']
\end{tikzcd}$$

where the top two horizontal arrows are maximal torus normalizers and the bottom horizontal arrow is an isomorphism between $\pi = N/N_0$ and the component group of X. We say that the map of pairs $(BN, BN_0) \to (BX_0, BX)$ is a *maximal torus normalizer pair* for the 2-compact group BX (2.1). Accordingly, two 2-compact groups, BX and BX', *have the same maximal torus normalizer pair* if there exists a commutative diagram

$$\begin{tikzcd}
BX_0 \arrow[d] & BN_0 \arrow[l, "Bj_0"'] \arrow[r, "Bj_0'"] \arrow[d] & BX_0' \arrow[d] \\
BX \arrow[d] & BN \arrow[l, "Bj"'] \arrow[r, "Bj'"] \arrow[d] & BX' \arrow[d] \\
B\pi_0(X) & B\pi \arrow[l, "\cong"'] \arrow[r, "\cong"] & B\pi_0(X')
\end{tikzcd}$$

in which the top four horizontal morphisms are maximal torus normalizers of 2-compact groups and the bottom two horizontal morphisms are isomorphisms of finite 2-groups.

We can now give a precise formulation of the classification theorem for 2-compact groups. We shall say (2.10) that a 2-compact group BX with maximal torus normalizer pair $(BN, BN_0) \to (BX_0, BX)$ is *totally N-determined* if

(1) automorphisms of BX are determined by their restrictions to BN,
(2) for any other 2-compact group, BX', that has the same maximal torus normalizer pair as BX, there exist an isomorphism $Bf\colon BX \to BX'$ and an automorphism $B\alpha\colon BN \to BN$, inducing the identity map on homotopy groups, making the diagram

$$\begin{tikzcd}
BN \arrow[r, "B\alpha", "\cong"'] \arrow[d] & BN \arrow[d] \\
BX \arrow[r, "\cong", "Bf"'] & BX'
\end{tikzcd}$$

commutative.

The role of $B\alpha$ is to compensate for the automorphisms of BN that do not extend to automorphisms of BX; such automorphisms do exist when $p = 2$ but they do not occur at odd primes.

We shall say that BX is *uniquely N-determined* if in addition the automorphisms of BX are determined by their effect on the (two nontrivial) homotopy groups of BN. (See 2.12 for a justification of the terminology.)

The main result of this paper is the classification of 2-compact groups.

1.1. THEOREM.

(1) *All connected* 2*-compact groups are uniquely N-determined.*
(2) *All LHS* 2*-compact groups are totally N-determined.*

In the second part of the theorem, dealing with possibly nonconnected 2-compact groups, we have to impose the extra LHS condition which is a cohomological requirement whose precise formulation can be found in Definition 2.27. It is an open question if in fact there are nonconnected 2-compact groups that are not LHS.

It is known from [15, 1.12] that the maximal torus normalizer of an arbitrary connected 2-compact group is isomorphic to the maximal torus normalizer of $BG \times B\mathrm{DI}(4)^t$ where G is some compact connected Lie group, $\mathrm{DI}(4)$ is the exotic connected 2-compact group from [10], and $t \geq 0$. We may now conclude that the isomorphism between the maximal torus normalizers extends to an isomorphism between the 2-compact groups as conjectured in [7, 5.1, 5.2].

1.2. COROLLARY. *Any connected* 2*-compact group is isomorphic to*

$$BG \times B\mathrm{DI}(4)^t$$

for some compact connected Lie group G and some integer $t \geq 0$.

With the classification completed also at $p = 2$ we are able to strengthen the conclusions or weaken the assumptions of [35, 1.6], [40, 1.6], [2, 1.10] and finally arrive at a proof of the maximal torus conjecture [54].

1.3. COROLLARY. *Any connected finite loop space with a maximal torus* [40, 1.1] *is isomorphic to a compact Lie group.*

The proof of the classification theorem follows the inductive principle of [12, 9.1] as implemented in [35] and can be divided into two stages. The first stage, which was presented at the Algebraic Topology meeting at the Max Planck Institute for Mathematics in Bonn, March 2001, consists in the reduction of the problem to the case of connected, simple and centerless 2-compact groups. The second stage consists of an inductive case-by-case verification that all connected, simple 2-compact groups are uniquely N-determined.

The proof can be reduced to the case of connected, centerless, and simple 2-compact groups because N-determinism is to a large extent hereditary. It turns out that, under certain extra conditions, a 2-compact group with an N-determined identity component is itself N-determined (2.35, 2.40). Also, a connected 2-compact group whose adjoint form, the quotient by the center, is N-determined is itself N-determined (2.38, 2.42). Now, since the adjoint form of any connected 2-compact group is a product of simple 2-compact groups [13], [43], and products of N-determined 2-compact groups are N-determined (2.39, 2.43), this reduces the classification problem to the case of a connected, simple 2-compact group with trivial center. This concludes the first stage in the classification procedure.

Furthermore, any connected and centerless 2-compact group with trivial center can be decomposed as a homotopy colimit of a system of 2-compact groups of smaller dimension [9] and, under certain hypotheses (2.48, 2.51), N-determinism is hereditary also under homotopy colimits. Thus we may prove the classification theorem for simple and centerless connected 2-compact groups by induction over the dimension. This is the second stage in the classification procedure.

In this paper we first show that the simple 2-compact groups associated to the simple Lie groups of the infinite A-, B-, C- and D-families, and to the exceptional simple Lie groups G_2, F_4, E_6, PE_7 and E_8, are uniquely N-determined. We also show that the exotic simple 2-compact group DI(4) from [10] is uniquely N-determined. The main difficulty is to show that the obstruction groups of 2.48 and 2.51 vanish. For this we use the computer algebra program *magma*. Together with the general reduction of the problem to the simple and centerfree case explained above we obtain a proof of Theorem 1.1.

In this first part of the paper we concentrate on the 2-compact groups of the A-family and establish the following basic result.

1.4. THEOREM (The A-family). *The connected, simple 2-compact group* $\mathrm{PGL}(n+1,\mathbf{C})$, $n \geq 1$, *is uniquely N-determined, and its automorphism group is*

$$\mathrm{Aut}(\mathrm{PGL}(n+1,\mathbf{C})) = \begin{cases} \mathbf{Z}^\times \backslash \mathbf{Z}_2^\times, & n = 1, \\ \mathbf{Z}_2^\times, & n > 1. \end{cases}$$

As already noted, any connected Lie group G has an associated 2-compact group which is the 2-completion of the classifying space of G. We shall denote the 2-compact group associated to G also by G. As connected 2-compact groups, $\mathrm{PGL}(n+1,\mathbf{C})$, $\mathrm{PU}(n+1)$, and $\mathrm{PSU}(n+1)$, for instance, are synonyms because their classifying spaces are homotopy equivalent.

The methods are not limited to simple nor even to connected 2-compact groups. Here is an example of the type of consequences that can be obtained for nonsimple 2-compact groups.

1.5. COROLLARY ([40, 1.9]). *The connected* 2*-compact group* $\mathrm{GL}(n, \mathbf{C})$, $n \geq 2$, *is uniquely* N*-determined and its automorphism group is*

$$\mathrm{Aut}(\mathrm{GL}(n, \mathbf{C})) = \begin{cases} \mathbf{Z}^\times \backslash \mathrm{Aut}_{\mathbf{Z}_2 \Sigma_2}(\mathbf{Z}_2^2), & n = 2, \\ \mathrm{Aut}_{\mathbf{Z}_2 \Sigma_n}(\mathbf{Z}_2^n), & n > 2. \end{cases}$$

Related uniqueness results can be found in the papers [41], [44], [45], [52], [50], [51] by Dietrich Notbohm, Antonio Viruel and Aleš Vavpetič, respectively. There is an alternative approach to the general classification of 2-compact groups due to K. Andersen and J. Grodal.

This paper is divided into two parts. Part I contains the general classification procedure and the application of this procedure to the 2-compact groups of the A-family leading to the proofs of Theorem 1.4 and Corollary 1.5 which are presented in Chapter 4. Part II deals with the infinite B-, C- and D-families, with the exceptional compact Lie groups G_2, F_4, E_6, PE_7 and E_8, and with the exotic simple 2-compact group $\mathrm{DI}(4)$. The proofs of Theorem 1.1, Corollary 1.2, and Corollary 1.3 are in Part II.

2. N-DETERMINED 2-COMPACT GROUPS

This chapter contains the basic definitions and the first general results. Whereas p-compact groups are determined by their maximal torus normalizers [38], [2] when $p > 2$, a finer invariant is needed for 2-compact groups as there are examples (2.2) of distinct 2-compact groups with identical maximal torus normalizers.

1. Maximal torus normalizer pairs. Let $N_0 \to N$ be a maximal rank normal monomorphism between two extended 2-compact tori [12, 3.12], meaning simply that there exists a short exact sequence [11, 3.2] of loop spaces $N_0 \to N \to \pi$ for some finite group π. For a 2-compact group, X, let (X, X_0) be the pair consisting of X and its identity component X_0. Then there is a short exact sequence $X_0 \to X \to \pi_0(X)$ of loop spaces where $\pi_0(X) = X/X_0$ is a finite 2-group, the component group of X.

2.1. DEFINITION. If there exists a morphism of loop space short exact sequences [12, 2.1]

$$\begin{tikzcd} N_0 \arrow[r] \arrow[d, "j_0"'] & N \arrow[r] \arrow[d, "j"] & \pi \arrow[d, "\cong"] \\ X_0 \arrow[r] & X \arrow[r] & \pi_0(X) \end{tikzcd}$$

where $j_0\colon N_0 \to X_0$ and $j\colon N \to X$ are maximal torus normalizers [11, 9.8], and $\pi \to \pi_0(X)$ an isomorphism of finite 2-groups, then we say that

$$(N, N_0) \xrightarrow{(j,j_0)} (X, X_0)$$

is a *maximal torus normalizer pair* for X.

A maximal torus normalizer pair for X determines the maximal torus $T(X)$, isomorphic to the identity component of N, the Weyl groups, $W(X) = \pi_0(N)$ and $W(X_0) = \pi_0(N_0)$, of X and X_0, the component group $\pi_0(X) = N/N_0 = W(X)/W(X_0)$ [39, 3.8], and [12, 7.5] the center $Z(X_0) \to X_0$ of X_0 [39], [12].

2.2. Example. (1) $\mathrm{GL}(2n, \mathbf{R})$ and $\mathrm{SL}(2n+1, \mathbf{R})$ have the same maximal torus normalizer

$$\mathrm{GL}(2, \mathbf{R}) \wr \Sigma_n = N(\mathrm{SL}(2n+1, \mathbf{R})) \subset \mathrm{GL}(2n, \mathbf{R}) \subsetneq \mathrm{SL}(2n+1, \mathbf{R}).$$

Their maximal torus normalizer pairs are distinct, however, as their component groups are distinct.

(2) More generally [19], let G be any compact connected Lie group and $N(G)$ its maximal torus normalizer. If $N(G)$ is not maximal, there exists a compact Lie group H such that $N(G) \subseteq H \subsetneq G$. The two compact Lie groups, G and H, have isomorphic maximal torus normalizers but distinct maximal torus normalizer pairs as H is nonconnected [3].

(3) The Weyl groups for $\mathrm{SL}(2n+1, \mathbf{R})$ and $\mathrm{GL}(n, \mathbf{H})$, $n \geq 3$, are isomorphic as reflection groups but $N(\mathrm{SL}(2n+1, \mathbf{R}))$ is a split and $N(\mathrm{GL}(n, \mathbf{H}))$ a nonsplit extension [6], [28] of the Weyl group by the maximal torus. Thus connected 2-compact groups cannot be classified by their Weyl group alone.

2.3. *The Adams–Mahmud homomorphism.* For a 2-compact group (or extended 2-compact torus) X, let $\mathrm{End}(X) = [BX, *; BX, *]$ denote the monoid of pointed homotopy classes of endomorphisms of X. The *automorphism group* $\mathrm{Aut}(X) \subset \mathrm{End}(X)$ of X is the group of invertible elements in $\mathrm{End}(X)$, and the *outer automorphism group* $\mathrm{Out}(X) = \pi_0(X)\backslash\mathrm{Aut}(X) \subset [BX; BX]$ is the group of conjugacy classes (free homotopy classes [12, 2.1]) of automorphisms of X.

Let X be a 2-compact group with maximal torus normalizer pair (N, N_0). Turn the maximal torus normalizer $Bj\colon BN \to BX$ into a fibration. Any automorphism $f\colon X \to X$ of the 2-compact group X restricts to an automorphism $\mathrm{AM}(f)\colon N \to N$ of the maximal torus normalizer, unique up to the action of the Weyl group $W(X_0) = \pi_1(X/N)$ [39, 3.8, 5.6(1)] of the identity component X_0 of X, such that the diagram

$$\begin{tikzcd} BN \arrow[r, "B(\mathrm{AM}(f))"] \arrow[d, "Bj"'] & BN \arrow[d, "Bj"] \\ BX \arrow[r, "Bf"] & BX \end{tikzcd}$$

commutes up to based homotopy [35, §3]. The *Adams–Mahmud homomorphism* is the resulting homomorphism

$$\mathrm{AM}\colon \operatorname{Aut}(X) \to W(X_0)\backslash \operatorname{Aut}(N) \tag{2.4}$$

of automorphism groups.

The automorphism group of N sits [33, 5.2] in a short exact sequence ($\check{T}(X)$ is the discrete approximation to $T(X)$ [11, 6.5])

$$0 \to H^1(W(X);\check{T}(X)) \to \operatorname{Aut}(N) \xrightarrow{\pi_*} \operatorname{Aut}(W(X),\check{T}(X),e(X)) \to 1 \tag{2.5}$$

where the normal subgroup on the left consists of all automorphisms of N that induce the identity on homotopy groups and the group on the right consists of all pairs $(\alpha,\theta) \in \operatorname{Aut}(W(X)) \times \operatorname{Aut}(\check{T}(X))$ such that θ is α-linear and the induced automorphism $H^2(\alpha^{-1},\theta)$ [53, 6.7.6] preserves the extension class $e(X) \in H^2(W(X);\check{T}(X))$ for the discrete approximation $\check{T}(X) \to N \to W(X)$ to the fibration $BT(X) \to BN \to BW(X)$. The image of $W(X_0)$ in $\operatorname{Aut}(N)$ does not intersect the subgroup $H^1(W(X);\check{T}(X))$ (as $W(X_0)$ is represented faithfully in $\operatorname{Aut}(\check{T}(X))$ [11, 9.7]) so there is an induced short exact sequence

$$\begin{aligned} 0 \to H^1(W(X);\check{T}(X)) &\to W(X_0)\backslash\operatorname{Aut}(N) \\ &\xrightarrow{\pi_*} W(X_0)\backslash\operatorname{Aut}(W(X),\check{T}(X),e(X)) \to 1 \end{aligned} \tag{2.6}$$

whose middle term is the target of the Adams–Mahmud homomorphism. In particular, if X is *connected*, this short exact sequence

$$\begin{aligned} 0 \to H^1(W(X);\check{T}(X)) &\to \operatorname{Out}(N) \\ &\xrightarrow{\pi_*} W(X)\backslash\operatorname{Aut}(W(X),\check{T}(X),e(X)) \to 1 \end{aligned} \tag{2.7}$$

has the group $\operatorname{Out}(N) = W(X)\backslash\operatorname{Aut}(N)$ of outer automorphisms of N as its middle term. The group $\operatorname{Aut}(W(X),\check{T}(X),0)$ may also be described as the normalizer $N_{\mathrm{GL}(L(X))}(W(X))$ of $W(X)$ in $\mathrm{GL}(L(X))$ where $L(X) = \pi_2(BT(X))$. This group evidently fits into an exact sequence [31, §2]

$$\begin{aligned} 1 \to Z(W(X))\backslash\operatorname{Aut}_{\mathbf{Z}_2W(X)}(L(X)) &\to W(X)\backslash N_{\mathrm{GL}(L(X))}(W(X)) \\ &\to \operatorname{Out}_{\mathrm{tr}}(W(X)) \end{aligned} \tag{2.8}$$

where $\operatorname{Aut}_{\mathbf{Z}_2W(X)}(L(X)) = \mathbf{Z}_2^\times$ if X is simple by Schur's lemma, and the right hand term, $\operatorname{Out}_{\mathrm{tr}}(W(X))$, is the group of outer automorphisms of $W(X)$ that preserve the trace taken in $L(X)$.

2.9. *Totally N-determined* 2*-compact groups.* We are now ready to formulate the concept of N-determinism that will be used in this paper. The extra complications compared to the odd p case [35, 7.1] stem from the fact that $H^1(W;\check{T})$, the first cohomology group of the Weyl group with coefficients in the discrete maximal torus, is trivial for any connected p-compact group when p is odd [1] but when $p = 2$ it may very well be nontrivial [21].

2.10. Definition. Let X be a 2-compact group with maximal torus normalizer pair $(N, N_0) \xrightarrow{(j,j_0)} (X, X_0)$ (2.1). Then $W = \pi_0(N)$ is the Weyl group of X and $W_0 = \pi_0(N_0)$ the Weyl group of X_0.

(1) The 2-compact group X *has N-determined automorphisms* if
$$\mathrm{AM}\colon \mathrm{Aut}(X) \to W_0\backslash\mathrm{Aut}(N)$$
is injective, and X *has $\pi_*(N)$-determined automorphisms* if
$$\pi_* \circ \mathrm{AM}\colon \mathrm{Aut}(X) \to W_0\backslash\mathrm{Aut}(W, \check{T}, e)$$
is injective. (See (2.4) and (2.6) for the definitions of AM and π_*.)

(2) X is *N-determined* if, for any other 2-compact group X' with maximal torus normalizer pair $(N, N_0) \xrightarrow{(j',j_0')} (X', X_0')$, there exist an isomorphism $Bf\colon BX \to BX'$ and an automorphism $\alpha \in H^1(W;\check{T}) \subset W_0\backslash\mathrm{Aut}(N)$ (2.6) such that the diagram

$$\begin{tikzcd} BN \arrow[r, "B\alpha", "\cong"'] \arrow[d, "Bj"'] & BN \arrow[d, "Bj'"] \\ BX \arrow[r, "\cong", "Bf"'] & BX' \end{tikzcd} \tag{2.11}$$

commutes up to based homotopy.

Furthermore, we say that

- X is *totally N-determined* if X has N-determined automorphisms and is N-determined,
- X is *uniquely N-determined* if X is totally N-determined and X has $\pi_*(N)$-determined automorphisms.

If X is a totally N-determined 2-compact group then
$$X \text{ is uniquely } N\text{-determined} \Leftrightarrow H^1(W;\check{T}) \cap \mathrm{Aut}(X) = 0$$
as we see from the short exact sequence (2.6).

2.12. Lemma. *Let X be a* 2*-compact group as in Definition* 2.10.

(1) *X has N-determined automorphisms if and only if for any given $\alpha \in W_0\backslash\mathrm{Aut}(N)$ with $\pi_*(B\alpha) = \mathrm{Id}$ and for any given* 2*-compact group X' as in* 2.10(2), *there is at most one isomorphism $f\colon X \to X'$ such that diagram* (2.11) *commutes up to based homotopy.*

(2) *X has $\pi_*(N)$-determined automorphisms if and only if for any given 2-compact group X' as in* 2.10(2), *diagram* (2.11) *has at most one solution (f,α) with $\pi_*(B\alpha) = \mathrm{Id}$.*

Proof. (1) Suppose that X has N-determined automorphisms. Let (f_1,α) and (f_2,α) be two solutions to diagram (2.11) with the same $\alpha \in H^1(W;\check{T}) \subset W(X_0)\backslash\mathrm{Aut}(N)$. Then $\mathrm{AM}(f_2^{-1}f_1)$ is the identity automorphism of $W(X_0)\backslash\mathrm{Aut}(N)$ and since $\mathrm{AM}\colon \mathrm{Aut}(X) \to W(X_0)\backslash\mathrm{Aut}(N)$ is injective, $f_1 = f_2$. For the converse, take $B\alpha$ to be the identity of BN and take X' to be X. Then the assumption is precisely that AM is injective.

(2) Suppose that X has $\pi_*(N)$-determined automorphisms and let (f_1,α_1) and (f_2,α_2) be two solutions to diagram (2.11). Then $\mathrm{AM}(f_2^{-1}f_1) = \alpha_2^{-1}\alpha_1 \in \mathrm{AM}(\mathrm{Aut}(X)) \cap H^1(W(X);\check{T}(X))$ and this intersection is trivial by hypothesis. Thus $\mathrm{AM}(f_2^{-1}f_1) = 1$ and $f_2 = f_1$ as AM is injective. If X does not have $\pi_*(N)$-determined automorphisms, then $\mathrm{AM}(f)$ lies in $H^1(W(X);\check{T}(X)) \subset W(X_0)\backslash\mathrm{Aut}(N)$ for some nontrivial $f \in \mathrm{Aut}(X)$ so that $(f,\mathrm{AM}(f))$ and $(1,0)$ are two solutions to diagram (2.11) with $X' = X$ and $j' = j$. ∎

2.13. Example. For (the 2-compact group associated to) a connected Lie group G, the cohomology group $H^1(W(G);\check{T}(G))$ is always an elementary abelian 2-group [29, 1.1] (2.21). For instance, this first cohomology group has order two for $G = \mathrm{PGL}(4,\mathbf{C})$ [28, Appendix B]. Let α be an isomorphism of $N(\mathrm{PGL}(4,\mathbf{C}))$ representing the nontrivial element of $H^1(W;\check{T})$. The unique solution (2.12(2)) to diagram (2.11) is

$$\begin{tikzcd} N(\mathrm{PGL}(4,\mathbf{C})) \arrow[r, "\alpha"] \arrow[d, "j"'] & N(\mathrm{PGL}(4,\mathbf{C})) \arrow[d, "j'"] \\ \mathrm{PGL}(4,\mathbf{C}) \arrow[r, equal] & \mathrm{PGL}(4,\mathbf{C}) \end{tikzcd}$$

when we use the morphisms j, induced by an inclusion of Lie groups, and $j' = j\alpha$ for maximal torus normalizers. This example demonstrates that, in contrast with the p odd case [35, 7.1], [38], [2], diagram (2.11) cannot always be solved with α equal to the identity.

2.14. Lemma. *Let X be a connected 2-compact group with maximal torus normalizer $Bj\colon BN \to BX$ and maximal torus $BT \hookrightarrow BN \xrightarrow{Bj} BX$.*

(1) *X is N-determined if and only if for any connected 2-compact group X' with maximal torus normalizer $Bj'\colon BN \to BX'$ there exists a morphism $Bf\colon BX \to BX'$ such that*

$$\begin{tikzcd} & BT \arrow[dl, "Bj|BT"'] \arrow[dr, "Bj'|BT"] & \\ BX \arrow[rr, "Bf"] & & BX' \end{tikzcd} \tag{2.15}$$

commutes up to homotopy.

(2) *X is uniquely N-determined if and only if for any connected 2-compact group X' with maximal torus normalizer $Bj' \colon BN \to BX'$ there exists a unique morphism $Bf \colon BX \to BX'$ such that* (2.15) *commutes up to homotopy.*

Proof. (1) Suppose the connected 2-compact group X is N-determined and let X' be another connected 2-compact group with the same maximal torus normalizer. Then X and X' have the same maximal torus normalizer pair, (N, N), and therefore (2.11) admits a solution (f, α) such that $\pi_*(B\alpha)$ is the identity. In particular, $\pi_2(B\alpha)$ is the identity of $\pi_2(BT)$, which means that $B\alpha$ restricts to the identity on the identity component BT of BN.

Conversely, under the existence assumption of point (1), we shall show that X is N-determined. Let X' be another 2-compact group with the same maximal torus normalizer pair as X. Since the maximal torus normalizer pair informs about component groups (see the remarks just below 2.1), X' is connected. By assumption, there exists a morphism, and in fact [13, 5.6], [36, 3.11] an isomorphism, $Bf \colon BX \to BX'$ under BT. Let $B\alpha \colon BN \to BN$, $B\alpha \in \mathrm{Out}(N) = W \backslash \mathrm{Aut}(N)$, be the restriction of Bf to BN [38, §3], so that

$$\begin{tikzcd} BN \arrow[r, "B\alpha", "\cong"'] \arrow[d, "Bj"'] & BN \arrow[d, "Bj'"] \\ BX \arrow[r, "\cong", "Bf"'] & BX' \end{tikzcd}$$

commutes up to based homotopy as in the definition of the Adams–Mahmud homomorphism (§2.3). The further restriction of $B\alpha$ to the maximal torus BT agrees with the restriction of Bf to BT, the identity of BT, up to the action of a Weyl group element $w \in W$ because $W \backslash [BT, BT] = [BT, BX']$ [34, 3.4], [13, 3.4]. Since $\pi_1(BN) = W$ is faithfully represented in $\pi_2(BT)$ for the connected 2-compact group X' [11, 9.7], it follows that $\pi_1(B\alpha)$ is conjugation by w. Thus $B\alpha$ belongs (2.7) to the subgroup $H^1(W; \check{T})$ of $\mathrm{Out}(N)$ so that (f, α) is a legitimate solution to (2.11).

(2) Suppose that X is uniquely N-determined and let X' be another connected 2-compact group with the same maximal torus normalizer as X. From point (1) we already know that there exists at least one isomorphism $f \colon X \to X'$ under T. Suppose $f_1, f_2 \colon X \to X'$ are two such isomorphisms under T. Then $f_2^{-1} f_1$ is an automorphism of X under T, i.e. $\pi_*(B\mathrm{AM}(f_2^{-1} f_1))$

$\in W\backslash \operatorname{Aut}(W,T)$ is the identity. As $\pi_*\circ\mathrm{AM}$ is injective, $f_2^{-1}f_1$ is the identity of X, so $f_1=f_2$.

Conversely, under the existence and uniqueness assumption of point (2), we shall show that X is uniquely N-determined. By point (1), X is N-determined, so we only need to show that $\pi_*\circ\mathrm{AM}$ is injective. Let $f\colon X\to X$ be an automorphism of X such that

$$\pi_*(B\mathrm{AM}(f))\in W\backslash\operatorname{Aut}(W,T)$$

is the identity. Since $B\mathrm{AM}(f)$ is determined only up to conjugacy, we may assume that $\pi_*(B\mathrm{AM}(f))$ *is* the identity of $\pi_*(BN)$. In particular, $\pi_2(B\mathrm{AM}(f))$ is the identity of $\pi_2(BT)$, meaning that f is an automorphism under T. The identity of X is also an automorphism under T, so f is the identity automorphism of X by the uniqueness hypothesis. This shows that $\pi_*\circ\mathrm{AM}$ is injective. ∎

2.16. Lemma. *Let X be a connected 2-compact group with maximal torus normalizer $N\to X$.*

(1) $\operatorname{Out}(N)=H^1(W(X);\check{T}(X))\cdot\mathrm{AM}(\operatorname{Aut}(X))$ *if X is N-determined.*
(2) $\operatorname{Out}(N)\cong H^1(W(X);\check{T}(X))\rtimes\operatorname{Aut}(X)$ *and*

$$\operatorname{Aut}(X)\cong W(X)\backslash\operatorname{Aut}(W(X),\check{T}(X),e(X))$$

if X is uniquely N-determined. The group $\operatorname{Aut}(W(X),\check{T}(X),e(X))$ *is a subgroup of* $N_{\mathrm{GL}(L(X))}(W(X))$ (2.8) *and is isomorphic to this group if* $e(X)=0$.

Proof. (1) For any $\beta\in\operatorname{Out}(N)$ there exist an automorphism $\alpha\in H^1(W(X);\check{T}(X))\subset\operatorname{Out}(N)$ and an automorphism $f\in\operatorname{Aut}(X)$ such that the diagram

$$\begin{tikzcd}
BN \arrow[r, "B\alpha"] \arrow[d, "Bj"'] & BN \arrow[d, "Bj\circ B\beta"] \\
BX \arrow[r, "Bf"] & BX
\end{tikzcd}$$

commutes up to homotopy (2.10(2)). Thus $\mathrm{AM}(f)=\beta\alpha$ in $\operatorname{Out}(N)$ (2.3).

(2) If the connected 2-compact group X is uniquely N-determined, then there is a commutative diagram

$$\begin{tikzcd}
H^1(W(X);\check{T}(X)) \arrow[r] & \operatorname{Out}(N) \arrow[r, "\pi_*"] & W(X)\backslash\operatorname{Aut}(W(X),\check{T}(X),e(X)) \\
 & \operatorname{Aut}(X) \arrow[u, hook, "\mathrm{AM}"] \arrow[ur, hook] &
\end{tikzcd}$$

where the top row is the short exact sequence (2.7). The composite homomorphism $\pi_*\circ\mathrm{AM}$ is injective by assumption (2.10(1)).It is surjective since

$\mathrm{Out}(N)$ is generated by $H^1(W(X);\check{T}(X))$ and the image of AM by item (1) of this lemma. Thus $\pi_* \circ \mathrm{AM}$ is an isomorphism and AM is a splitting of the short exact sequence (2.7). ∎

As evidence of the conjecture that all connected 2-compact groups are uniquely N-determined we note that all compact connected Lie groups have $\pi_*(N)$-determined automorphisms [26, 2.5] and satisfy the above two formulas for automorphism groups [19, 3.10].

With a view to the situation for possibly nonconnected 2-compact groups, let $\mathrm{Aut}(N, N_0)$ denote the subgroup of $\mathrm{Aut}(N)$ consisting of all automorphisms $\phi \in \mathrm{Aut}(N)$ such that $\pi_0(\phi)$ takes $\pi_0(N_0)$ to itself inducing an isomorphism

$$\begin{tikzcd} \pi_0(N_0) \arrow[r] \arrow[d, "\cong"] & \pi_0(N) \arrow[r] \arrow[d, "\cong"', "\pi_0(\phi)"] & \pi \arrow[d, "\cong"] \\ \pi_0(N_0) \arrow[r] & \pi_0(N) \arrow[r] & \pi \end{tikzcd}$$

of short exact sequences. Since $H^1(W;\check{T})$ is contained in $\mathrm{Aut}(N, N_0)$, there are short exact sequences similar to (2.5) and (2.6) except that $\mathrm{Aut}(W, \check{T}, e)$ has been replaced by its subgroup $\mathrm{Aut}(\check{T}, W, W_0, e)$ consisting of all $(\alpha, \theta) \in \mathrm{Aut}(W, \check{T}, e)$ for which $\alpha(W_0) = W_0$. (In case $N = N_0$, we find that $\mathrm{Aut}(N) = \mathrm{Aut}(N, N_0)$.) Observe that the Adams–Mahmud homomorphism for a nonconnected 2-compact group actually takes values in the subgroup $W(X_0)\backslash\mathrm{Aut}(N, N_0)$ of the group $W(X_0)\backslash\mathrm{Aut}(N)$.

2.17. Lemma. *Let X be a 2-compact group with maximal torus normalizer pair $(N, N_0) \to (X, X_0)$.*

(1) $W(X_0)\backslash\mathrm{Aut}(N, N_0) = H^1(W(X);\check{T}(X)) \cdot \mathrm{AM}(\mathrm{Aut}(X))$ *if X is N-determined.*
(2) $W(X_0)\backslash\mathrm{Aut}(N, N_0) \cong H^1(W(X);\check{T}(X)) \rtimes_{H^1(\pi_0(X);\check{Z}(X_0))} \mathrm{Aut}(X)$ *if X is totally N-determined.*

Proof. The first item is proved like the first item in 2.16. The claim of the second item is that

$$\begin{tikzcd} H^1(\pi_0(X);\check{Z}(X_0)) \arrow[r, hook] \arrow[d] & \mathrm{Aut}(X) \arrow[d, hook, "\mathrm{AM}"] \\ H^1(W(X);\check{T}(X)) \arrow[r, hook] & W(X_0)\backslash\mathrm{Aut}(N, N_0) \end{tikzcd}$$

is a push-out diagram. This will be proved in 2.37. ∎

2.18. Remark. When the 2-compact group X has N-determined automorphisms, also the unbased Adams–Mahmud homomorphism

$$\begin{aligned}\mathrm{Out}(X) = \pi_0(X)\backslash\mathrm{Aut}(X) \to \mathrm{Out}(N) &= \pi_0(N)\backslash\mathrm{Aut}(N)\\ &= \pi_0(X)\backslash W(X_0)\backslash\mathrm{Aut}(N)\end{aligned}$$

is injective.

2.19. *Regular* 2*-compact groups.* For a *connected* 2-compact group X with maximal torus $T \to X$ and Weyl group W, let

$$\theta = \theta(X)\colon\ \mathrm{Hom}(W, \check{T}^W) = H^1(W; \check{T}^W) \to H^1(W; \check{T}) \tag{2.20}$$

be the homomorphism induced by the inclusion $\check{T}^W \hookrightarrow \check{T}$. Following [21, 5.3] we say that X is *regular* if (2.20) is surjective. See [29] for a thorough investigation of θ.

2.21. Lemma ([29]). *Let X be the connected 2-compact group associated to a connected Lie group. Assume that X contains no direct factors isomorphic to an odd orthogonal group* $\mathrm{SO}(2n+1)$, $n \geq 1$. *Consider the homomorphism $\theta = \theta(X)$ (2.20) associated to X.*

(1) $\mathrm{Hom}(W, \check{T}^W)$ *and* $H^1(W; \check{T})$ *are* $\mathbf{F}_2$*-vector spaces, and the kernel of θ, consisting of those homomorphisms $W \to \check{T}^W$ that are principal crossed homomorphisms $W \to \check{T}$, is an $\mathbf{F}_2$-vector space of dimension equal to the number of direct factors of PX isomorphic to* $\mathrm{SO}(2n+1)$, $n \geq 1$.

(2) *Suppose that the projective group PX contains no direct factors isomorphic to an odd orthogonal group* $\mathrm{SO}(2n+1)$, $n \geq 1$, $\mathrm{PSU}(4)$, $\mathrm{PSp}(3)$, $\mathrm{PSp}(4)$, *or* $\mathrm{PS0}(8)$. *Then X is regular.*

Proof. (1) $\mathrm{Hom}(W, \check{T})$ and its subgroup $\mathrm{Hom}(W, \check{T}^W)$ are elementary abelian 2-groups since the abelianization W_{ab} of W is an elementary abelian 2-group of finite rank. The cohomology group $H^1(W; \check{T})$ is isomorphic to $H^2(W; L \otimes \mathbf{Z}_2)$ where L is the fundamental group of the Lie group maximal torus of the Lie group underlying the 2-compact group X. Homological algebra shows that $H^2(W; L \otimes \mathbf{Z}_2) \cong H^2(W; L) \otimes \mathbf{Z}_2$ where $H^2(W; L)$ is an elementary abelian 2-group by [29, 1.1]. The injection $\check{T}^W \to \check{T}$ of W-modules gives a coefficient group long exact sequence

$$\begin{aligned}0 \to (\check{T}/\check{T}^W)^W \to \mathrm{Hom}(W, \check{T}^W) \xrightarrow{\theta} H^1(W; \check{T}) \to H^1(W; \check{T}/\check{T}^W)&\\ \to H^2(W; \check{T}^W) \to \cdots&\end{aligned}$$

in cohomology. Thus the kernel of θ is isomorphic to $(\check{T}/\check{T}^W)^W$ in general. If X is without direct factors isomorphic to $\mathrm{SO}(2n+1)$, then $\check{T}^W$ is the center of X, $\check{T}/\check{T}^W$ is the maximal torus of the adjoint 2-compact group PX, and $(\check{T}/\check{T}^W)^W$ is isomorphic to $(\mathbf{Z}/2)^s$ where s is the number of direct factors isomorphic to an odd special orthogonal group in the adjoint 2-compact group PX [29, 1.6], [39, 4.6, 4.7]. (See 2.25 for the general case.)

(2) The discrete maximal torus of $PX = X/Z(X)$ is $\check{T}(PX) = \check{T}/\check{T}^W$ for $\check{Z}(X) = \check{T}^W$ as X contains no direct factors isomorphic to an odd orthogonal group. The projective group $PX = \prod G_i$ splits as a product of simple and centerfree compact Lie groups G_i all of which satisfy $\check{T}^W(G_i) = 0$ since they are not odd orthogonal groups. Therefore $H^1(W; \check{T}/\check{T}^W) = H^1(\prod W(G_i); \prod \check{T}(G_i)) = \prod H^1(W(G_i); \check{T}(G_i))$ and we note that these cohomology groups are trivial except in the excluded cases [21]. By the above exact sequence, θ is surjective. ∎

For a compact connected Lie group X, let $s(X)$ denote the number of direct factors of X isomorphic to $\mathrm{SO}(2n+1)$ with $n \geq 1$. (Keep in mind the low degree identifications (5.25).)

2.22. Lemma. *Let X be a compact connected Lie group and PX its adjoint form. The kernel of $H^1(W; \check{Z})(X) \to H^1(W; \check{T})(X)$ is an $\mathbf{F}_2$-vector space of dimension $s(PX) - s(X)$.*

Proof. In the exact sequence

$$0 \to \check{Z} \to \check{T}^W \to (\check{T}/\check{Z})^W \to H^1(W; \check{Z}) \to H^1(W; \check{T})$$

induced from the inclusion $\check{Z} \to \check{T}$ of W-modules, the fixed point groups are $\check{T}^W = \check{Z}(X) \times 2^{s(X)}$ and $(\check{T}/\check{Z})^W = \check{Z}(X/Z) \times 2^{s(X/Z)} = 2^{s(X/Z)}$ [29, 1.6]. ∎

2.23. Lemma. *Let X be a connected 2-compact group with maximal torus $T \to X$ and Weyl group W, and let $Z \to T \to X$ be a central monomorphism. If X is regular and $H^2(W; \check{Z}) \to H^2(W; \check{T})$ is injective, then the quotient 2-compact group X/Z is regular.*

Proof. Since the hypothesis implies that $H^1(W; \check{T}) \to H^1(W; \check{T}/\check{Z})$ is surjective, the claim follows from the commutative square

$$\begin{tikzcd}
\mathrm{Hom}(W, \check{T}^W) \arrow[r] \arrow[d, "\theta(X)"'] & \mathrm{Hom}(W, (\check{T}/\check{Z})^W) \arrow[d, "\theta(X/Z)"] \\
H^1(W; \check{T}) \arrow[r, two heads] & H^1(W; \check{T}/\check{Z})
\end{tikzcd}$$

induced by the projection $\check{T} \to \check{T}/\check{Z}$ of W-modules [39, 4.6]. ∎

2.24. Example. (1) $\mathrm{GL}(m, \mathbf{C})$ is regular for all $m \geq 1$. For $m = 1$, this is obvious. For $m > 2$, the restriction homomorphism ($\check{S} = \mathbf{Z}/2^\infty$)

$$\mathrm{Hom}(\Sigma_m, \check{S}) = H^1(\Sigma_m; \check{S}) \xrightarrow{\mathrm{res}=\theta(\mathrm{GL}(m,\mathbf{C}))} H^1(\Sigma_m; \check{S}^m) \overset{\mathrm{Shapiro}}{=} H^1(\Sigma_{m-1}; \check{S}) = \mathrm{Hom}(\Sigma_{m-1}, \check{S})$$

is bijective and for $m = 2$ it is surjective. It now follows [21, 5.7] that all products $\prod \mathrm{GL}(m_j, \mathbf{C})$ are regular.

(2) $\mathrm{PGL}(m,\mathbf{C})$, $2 \le m$, is regular for $m \neq 4$ since (2.23)

$$\mathrm{Hom}(H_2(\Sigma_m),\check{S}) = H^2(\Sigma_m;\check{S}) \xrightarrow{\mathrm{res}} H^2(\Sigma_m;\check{S}^m) \overset{\mathrm{Shapiro}}{=} H^2(\Sigma_{m-1};\check{S}) = \mathrm{Hom}(H_2(\Sigma_{m-1}),\check{S})$$

is then an isomorphism. The 2-compact group $\mathrm{PGL}(4,\mathbf{C})$ is not regular as $H^1(W;\check{T}) = \mathbf{Z}/2$ is nontrivial while the discrete center $\check{T}^W$ is trivial.

2.25. Remark. If $X = \mathrm{SO}(2n+1)$, $n \ge 1$, then $\check{T}^W = \mathbf{Z}/2$, W_{ab} is $\mathbf{Z}/2$ for $n=1$ and $(\mathbf{Z}/2)^2$ for $n \ge 2$, $\theta\colon \mathrm{Hom}(W,\check{T}^W) \to H^1(W;\check{T})$ is surjective [21, 5.5], and $H^1(W;\check{T})$ is trivial for $n=1$, $\mathbf{Z}/2$ for $n=2$, and $(\mathbf{Z}/2)^2$ for $n>2$ [21, Main Theorem, 5.5]. Thus the kernel of θ is

$$(\check{T}/\check{T}^W)^W = \begin{cases} \mathbf{Z}/2, & n = 1,2, \\ 0, & n > 2. \end{cases}$$

In general, write the connected Lie group $X = X_1 \times X_2$ where X_1 is the product of all direct factors of X isomorphic to $\mathrm{SO}(2n+1)$ for some $n \ge 1$ and X_2 is without such direct factors. Then

$$(\check{T}/\check{T}^W)^W = (\check{T}_1/\check{T}_1^{W_1})^{W_1} \times (\check{T}_2/\check{T}_2^{W_2})^{W_2} = (\mathbf{Z}/2)^{s_{\le 2}(X)} \times (\mathbf{Z}/2)^{s(PX_2)}$$

where $s_{\le 2}(X)$ is the number of direct factors of X isomorphic to $\mathrm{SO}(3)$ or $\mathrm{SO}(5)$ and $s(PX_2)$ is the number of direct factors of PX_2 isomorphic to $\mathrm{SO}(2n+1)$ for some $n \ge 1$.

2.26. *LHS 2-compact groups.* Let $N_0 \to N$ be a maximal rank normal monomorphism between two extended 2-compact tori, i.e. a commutative diagram with rows and columns that are short exact sequences of loop spaces [11, 3.2]

$$\begin{tikzcd} T \arrow[r, equal] \arrow[d] & T \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ N_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & W/W_0 \arrow[d, equal] \\ W_0 \arrow[r] & W \arrow[r] & W/W_0 \end{tikzcd}$$

where T is a 2-compact torus and $W_0 = \pi_0(N_0)$ a normal subgroup of the finite group $W = \pi_0(N)$. The 5-term *fundamental exact sequence*

$$0 \to H^1(W/W_0;\check{T}^{W_0}) \xrightarrow{\mathrm{inf}} H^1(W;\check{T}) \xrightarrow{\mathrm{res}} H^1(W_0;\check{T})^{W/W_0} \xrightarrow{d_2} H^2(W/W_0;\check{T}^{W_0}) \xrightarrow{\mathrm{inf}} H^2(W;\check{T})$$

is part of the Lyndon–Hochschild–Serre spectral sequence [22] converging to $H^*(W;\check{T})$.

2.27. Definition. A 2-compact group with maximal torus normalizer pair (N, N_0) is *LHS* if the restriction homomorphism

$$\mathrm{res}\colon H^1(W; \check{T}) \to H^1(W_0; \check{T})^W$$

is surjective.

Equivalently, X is LHS if and only if the initial segment of the Lyndon–Hochschild–Serre spectral sequence

$$0 \to H^1(W/W_0; \check{T}^{W_0}) \xrightarrow{\mathrm{inf}} H^1(W; \check{T}) \xrightarrow{\mathrm{res}} H^1(W_0; \check{T})^{W/W_0} \to 0$$

is exact. If $\check{T}^{W_0} = 0$ or $W = W_0 \times W/W_0$ is a direct product, then X is LHS. Note that the Weyl group of a compact Lie group G is always the semidirect product $W(G) = W(G_0) \rtimes \pi_0(G)$ for the action of the component group $\pi_0(G)$ on the Weyl group $W(G_0)$ of the identity component [19, §2.5]. (In fact, it is not so easy to find a nonconnected compact Lie group G for which the extension $G_0 \to G \to G/G_0 = \pi$ is nonsplit [49], [20].)

2.28. Lemma. *Let $W = W(X)$ be the Weyl group of the 2-compact group X, $W_0 = W(X_0)$ the Weyl group of the identity component, and $\pi = W/W_0$ the component group of X. If*

$$\theta(X_0)^\pi\colon \mathrm{Hom}(W_0, \check{T}^{W_0})^\pi \to H^1(W_0; \check{T})^\pi$$

is surjective, then X is LHS.

Proof. Assume that the group $G = H \rtimes Q$ is the semidirect product for a group action $Q \to \mathrm{Aut}(H)$, and let A be a G-module. We show that the image of the restriction homomorphism $\mathrm{res}\colon H^1(G; A) \to H^1(H; A)^Q$ contains the image of $\theta^Q\colon \mathrm{Hom}(H, A^H)^Q \to H^1(H; A)^Q$. Let $\phi \in \mathrm{Hom}(H, A^H)^Q$ be a Q-equivariant homomorphism of H into the fixed point module A^H. Then $\theta(\phi) \in H^1(H; A)^Q$ is the cohomology class represented by the crossed homomorphism $\phi\colon H \to A^H \subset A$. If we define $\overline{\phi}\colon H \rtimes Q \to A$ by $\overline{\phi}(nq) = \phi(n)$, $n \in H$, $q \in Q$, then

$$\begin{aligned}\overline{\phi}(n_1q_1n_2q_2) &= \overline{\phi}(n_1(q_1n_2q_1^{-1})q_1q_2) = \overline{\phi}(n_1(q_1 \cdot n_2)q_1q_2)\\ &= \phi(n_1(q_1 \cdot n_2)) = \phi(n_1) + \phi(q_1 \cdot n_2) = \phi(n_1) + q_1\phi(n_2)\end{aligned}$$

and also

$$\overline{\phi}(n_1q_1) + n_1q_1\overline{\phi}(n_2q_2) = \phi(n_1) + n_1q_1\phi(n_2) = \phi(n_1) + q_1\phi(n_2)$$

as $q_1\phi(n_2) \in A^H$. This shows that the crossed homomorphism ϕ defined on H extends to a crossed homomorphism $\overline{\phi}$ defined on $G = H \rtimes Q$. (I do not know if the LHS spectral sequence differential $d_2\colon H^1(H; A)^Q \to H^2(Q; A^H)$ is always trivial for a semidirect product $H \rtimes Q$ of finite groups.) ∎

2.29. Example. (1) $X = \mathrm{PGL}(6, \mathbf{R}) = \mathrm{PSL}(6, \mathbf{R}) \rtimes C_2$ does not satisfy the condition of 2.28 but it is still LHS for $H^1(W_0; \check{T}) = \mathbf{Z}/2$ [21, Main

Theorem] and $\check{T}^{W_0} = \check{Z}(X_0) = 0$. Thus 2.28 gives a sufficient but not necessary condition for X to be LHS.

(2) $X = \mathrm{PGL}(8,\mathbf{R}) = \mathrm{PSL}(8,\mathbf{R}) \rtimes C_2$ does not satisfy the condition of 2.28 but it is still LHS for $H^1(W_0;\check{T}) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$ [21, Main Theorem] and $\check{T}^{W_0} = \check{Z}(X_0) = 0$.

(3) When $X_0 = \mathrm{SL}(2,\mathbf{C})$, the Weyl group $W_0 = \Sigma_2$ has order two, the center $\check{Z} = \check{T}^{W_0}$ also has order two, and $H^1(W_0;\check{T}) = 0$ is trivial, so the homomorphism $\theta(X_0)$ is trivial as well, of course. Indeed, the nontrivial homomorphism $W_0 \to \check{Z} \subset \check{T}$ is the principal crossed homomorphism corresponding to the element $\mathrm{diag}(i,-i)$ of the maximal torus. More generally, the direct product $X_0^r = \mathrm{SL}(2,\mathbf{C})^r$ is regular [21, 5.7], has Weyl group W_0^r, center $\check{Z}^r$, and 2.21(1) identifies the kernel of $\theta(X_0)$ enabling us to conclude that

$$H^1(W;\check{T})(X_0^r) = \frac{\mathrm{Hom}(W_0^r,\check{Z}^r)}{\mathrm{Hom}(W_0,\check{Z})^r} \tag{2.30}$$

is an $\mathbf{F}_2$-vector space of dimension $r^2 - r$ as in [21, 5.8]. Let $X = X_0 \rtimes C_2$ be the semidirect product for the nontrivial outer automorphism of X_0. The component group C_2^r of X^r acts trivially on (2.30) and as $H^1(W;\check{T})(X^r)$ has dimension $2r^2 - r$ (by induction) and $H^1(C_2^r;\check{Z}^r)$ has dimension r^2, the direct product X^r is LHS for all $r \geq 1$.

(4) When $X = \mathrm{SL}(4,\mathbf{R})$, the Weyl group $W = \langle \sigma, c_1c_2\rangle = \mathbf{Z}/2 \times \mathbf{Z}/2$ is elementary abelian generated by $\sigma = \left(\begin{smallmatrix} 0 & E \\ E & 0 \end{smallmatrix}\right)$ and $c_1c_2 = \mathrm{diag}(-1,1,-1,1)$. The center $\check{Z} = \check{T}^W = \langle \mathrm{diag}(-1,-1,-1,-1)\rangle = \mathbf{Z}/2$ has order 2, and the first cohomology group $H^1(W;\check{T}) = 0$ is trivial, so the homomorphism $\theta\colon \mathrm{Hom}(W;\check{T}^W) \to H^1(W;\check{T})$ is also trivial, of course. Indeed, the principal homomorphism $\varphi(w) = (w \cdot t) \cdot t^{-1}\colon W \to \check{T}$ is the first coordinate function $W \to \check{Z}(X)$ when $t = \mathrm{diag}(-E,E)$ and the second coordinate function when $t = \mathrm{diag}(I,I)$. The outer automorphism, conjugation with $D = \mathrm{diag}(-1,1,1,1) \in \mathrm{GL}(4,\mathbf{R})$, sends σ to $\sigma^D = \sigma(c_1c_2)$ and c_1c_2 to itself.

More generally, when X^r is a product of r copies of $\mathrm{SL}(4,\mathbf{R})$, the Weyl group W^r is a product of r copies of $W = W(\mathrm{SL}(4,\mathbf{R})) = \mathbf{Z}/2 \times \mathbf{Z}/2$, the center $\check{Z}(X) = \check{Z}^r$ is a product of r copies of $\check{Z} = \check{Z}(\mathrm{SL}(4,\mathbf{R})) = \mathbf{Z}/2$, and as $\theta\colon \mathrm{Hom}(W^r,\check{Z}^r) \to H^1(W;\check{T})(X^r)$ is surjective [21, 5.5, 5.7], the first cohomology group

$$H^1(W;\check{T})(X^r) = \frac{\mathrm{Hom}(W^r,\check{Z}^r)}{\mathrm{Hom}(W,\check{Z})^r}$$

has dimension $2r^2 - 2r$ over $\mathbf{F}_2$ (2.21). The component group $\pi_0(\mathrm{GL}(4,\mathbf{R})^r)$ $= C_2^r$ acts on this $\mathbf{F}_2$-vector space in such a way that the space of fixed vectors has dimension $r^2 - r$. By induction we see that $H^1(W;\check{T})(\mathrm{GL}(4,\mathbf{R})^r)$

is an $\mathbf{F}_2$-vector space of dimension $2r^2 - r$ and clearly $H^1(C_2^r; \check{Z}^r)$ has dimension r^2. Thus $\mathrm{GL}(4, \mathbf{R})^r$ is LHS for all $r \geq 1$.

(5) The homomorphism θ is surjective for $\mathrm{SL}(2n, \mathbf{R})$ for all $n \geq 1$ [21, Main Theorem, 5.4] and

$$H^1(W; \check{T})(\mathrm{SL}(2n, \mathbf{R})) = \begin{cases} 0, & n = 1, 2, \\ \mathbf{Z}/2, & n > 2. \end{cases}$$

Hence $\mathrm{GL}(2n, \mathbf{R})$ is LHS for all $n \geq 1$ by 2.28.

I do not know any examples of 2-compact groups that are not LHS.

The short exact sequence $0 \to L \to L \otimes \mathbf{Q} \to \check{T} \to 0$ of abelian groups with a W-action induces an exact sequence

$$0 \to H^0(W; L) \to H^0(W; L \otimes \mathbf{Q}) \to H^0(W; \check{T}) \to H^1(W; L) \to 0,$$

from which we see that

$$H^i(W; \check{T}) = \begin{cases} H^0(W; L \otimes \mathbf{Q})/H^0(W; L) \oplus H^1(W; L), & i = 0, \\ H^{i+1}(W; L), & i > 0. \end{cases} \tag{2.31}$$

2.32. *The center of the maximal torus normalizer.* We need criteria to ensure that the center of the 2-compact group X agrees with the center of its maximal torus normalizer. (This is automatic when $p > 2$ [35, 3.4] but not when $p = 2$ [12, §7].)

2.33. Proposition. *Let X be a 2-compact group with identity component X_0. If $Z(X_0) = Z(N(X_0))$ and X_0 has N-determined automorphisms, then $Z(X) = Z(N(X))$.*

Proof. This is proved in [38, 4.12] for p-compact groups where p is odd. If we replace the assumption that p is odd by the assumption that $Z(X_0) = Z(N(X_0))$ (which always holds when $p > 2$ [12, 7.1]), then the same proof works also for 2-compact groups. ∎

Assume now that X is a *connected* 2-compact group. Then $\check{Z}(N(X)) = \check{T}(X)^{W(X)}$ and there is an injection $\check{Z}(X) \hookrightarrow \check{Z}(N(X))$ which is not necessarily an isomorphism [12, §7].

Inspection shows that $Z(G) = ZN(G)$ for any *simply connected* compact Lie group G; see [14, 1.4] for a conceptual proof of this fact. In fact, $Z(G) = ZN(G)$ for any connected compact Lie group G containing no direct factors isomorphic to $\mathrm{SO}(2n+1)$ [29, 1.6].

Let $Z \to N(X)$ be a central monomorphism such that also the composition $Z \to N(X) \to X$ is central. Under these assumptions, the quotient loop spaces $N(X)/Z$ and X/Z can be defined [12, 2.8]. The action map [11, 8.6] $BZ \times BN(X) \to BN(X)$ induces an action $[BN(X), BZ] \times \mathrm{Out}(N(X)) \to \mathrm{Out}(N(X))$ of the group $[BN(X), BZ] \cong H^1(\check{N}(X); \check{Z})$ on the set $\mathrm{Out}(N(X))$. Let $[BN(X), BZ]_{(1)}$ denote the isotropy subgroup at $(1) \in \mathrm{Out}(N(X))$.

2.34. Lemma. *If* $Z(X) = Z(N(X))$ *and* $[BN(X), BZ]_{(1)} = 0$, *then* $Z(X/Z) = ZN(X/Z)$.

Proof. Using [39, 4.6.4], the assumption of the lemma, and [38, 5.11], we get $Z(X/Z) = Z(X)/Z = Z(N(X))/Z = Z(N(X)/Z) = ZN(X/Z)$. ∎

2. Reduction to the connected, centerless (simple) case. In this section we reduce the general classification problem first to the connected case and next to the connected and centerless case. We first show (2.35, 2.40) that if X is any nonconnected 2-compact group with identity component X_0 then

$$\left.\begin{array}{l} X_0 \text{ is uniquely } N\text{-determined,} \\ X \text{ is LHS,} \\ H^i(W/W_0; \check{Z}(X_0)) \to H^i(W/W_0; \check{T}^{W_0}) \text{ is injective for } i = 1, 2 \end{array}\right\} \Rightarrow X \text{ is totally } N\text{-determined.}$$

The H^i-injectivity condition holds when X_0 is a connected Lie group [29, 1.6] or equals DI(4) [10], [45]. To see this observe that the condition obviously holds when $\check{Z}(X_0) = \check{T}(X_0)^{W_0}$ or $\check{Z}(X_0)$ is trivial. We shall see later that any connected 2-compact group splits as a product of a compact connected Lie group and a finite number of DI(4), and from this it follows that this condition is always satisfied. Indeed, let $X_0 = G' \times G'' \times \mathrm{DI}(4)^s$ where G' is a connected compact Lie group with no direct factors isomorphic to $\mathrm{SO}(2n+1)$, G'' is a direct product of $\mathrm{SO}(2n+1)$s, and $s \geq 0$. The $\pi_0(X)$-equivariant group homomorphism

$$\check{Z}(G') = \check{Z}(X_0) \to \check{T}(G')^{W(G')} \times \check{T}(G'')^{W(G'')}$$

has a left inverse since it takes $\check{Z}(G')$ isomorphically to the $\pi_0(X)$-subgroup $\{1\} \times \check{T}(G'')^{W(G'')}$ of the left hand side. The induced map on cohomology therefore also has a left inverse. However, it is not at present clear if all nonconnected 2-compact groups are LHS.

Next we consider a connected 2-compact group X with adjoint form $PX = X/Z(X)$ [12, 2.8] and show (2.38, 2.42) that

$$PX \text{ is uniquely } N\text{-determined} \Rightarrow X \text{ is uniquely } N\text{-determined.}$$

This reduces in principle the problem to the connected and centerless case. One can go a little further since connected, centerless 2-compact groups split into products of simple factors [13], [43]. We show (2.39, 2.43) that

$$X_1 \text{ and } X_2 \text{ are uniquely } N\text{-determined} \Rightarrow X_1 \times X_2 \text{ is uniquely } N\text{-determined}$$

when X_1 and X_2 are connected. Therefore it suffices to show that all *connected*, *centerless and simple* 2-compact groups are uniquely N-determined. It is already known that all connected compact Lie groups as well as DI(4) have $\pi_*(N)$-determined automorphisms [26], [45]. We are going to reprove this statement here.

Let X be a 2-compact group and $(N, N_0) \xrightarrow{(j,j_0)} (X, X_0)$ a maximal torus normalizer pair for X (2.1). Let $\check{T}$ be the discrete approximation to the common identity component for N and N_0 and let $W = \pi_0(N)$, $W_0 = \pi_0(N_0)$ be the two component groups.

2.35. Lemma ([35, 4.2]). *Suppose that X_0 has N-determined automorphisms. Then*

$$X \text{ has } N\text{-determined automorphisms} \\ \Leftrightarrow\ H^1(W/W_0; \check{Z}(X_0)) \to H^1(W/W_0; \check{T}^{W_0}) \text{ is injective.}$$

Proof. The restriction of the homomorphism AM (2.4) to the subgroup $H^1(W/W_0; \check{Z}(X_0))$ of $\mathrm{Aut}(X)$ [33, 5.2] is the homomorphism

$$H^1(W/W_0; \check{Z}(X_0)) \to H^1(W/W_0; \check{T}^{W_0}) \xrightarrow{\inf} H^1(W; \check{T}) \tag{2.36}$$

where inf is the inflation monomorphism. If the first homomorphism has a nontrivial kernel, X does not have N-determined automorphisms. Conversely, assume that the first homomorphism is injective, and let $f \in \mathrm{Aut}(X)$ be an automorphism such that $\mathrm{AM}(f) \in W_0\backslash\mathrm{Aut}(N)$ is the identity. Then $\mathrm{AM}(f_0) \in W_0\backslash\mathrm{Aut}(N_0)$ and $\pi_0(f)$ equal the respective identity maps. Since X_0 has N-determined automorphisms by assumption, f_0 is the identity. Thus f belongs to the subgroup $H^1(W/W_0; \check{Z}(X_0))$ of $\mathrm{Aut}(X)$ where AM is injective, so f is the identity automorphism of X. (The description of the kernel in the short exact sequence of [33, 5.2] holds for all p-compact groups, not just those with a completely reducible identity component.) ∎

2.37. Lemma. *Suppose that X has N-determined automorphisms and that X_0 has $\pi_*(N)$-determined automorphisms. Then* $\mathrm{Aut}(X) \cap H^1(W; \check{T}) = H^1(W/W_0; \check{Z}(X_0))$ *so that*

$$X \text{ has } \pi_*(N)\text{-determined automorphisms } \Leftrightarrow\ H^1(W/W_0; \check{Z}(X_0)) = 0.$$

Proof. Let $f \in \mathrm{Aut}(X)$ be an automorphism such that $\pi_* \mathrm{AM}(f)$ is the identity. Then also $\pi_* \mathrm{AM}(f_0)$ and $\pi_0(f)$ equal the respective identity maps. Since X_0 is assumed to have $\pi_*(N)$-determined automorphisms, f_0 is the identity. Thus f belongs to the subgroup $H^1(W/W_0; \check{Z}(X_0))$ of $\mathrm{Aut}(X)$ [33, 5.2]. This shows that $\mathrm{Aut}(X) \cap H^1(W; \check{T}) \subset H^1(W/W_0; \check{Z}(X_0))$. The opposite inclusion is immediate from (2.36). ∎

2.38. Lemma ([35, 4.8]). *Suppose that X is connected. If the adjoint form $PX = X/Z(X)$ has $\pi_*(N)$-determined automorphisms, so does X.*

Proof. If $f \in \mathrm{Aut}(X)$ is an automorphism under $T(X)$, the induced automorphism $Pf \in \mathrm{Aut}(PX)$ is an automorphism under $T(PX)$, hence equals the identity, and the induced automorphism $Z(f) \in \mathrm{Aut}(ZX)$ is also the identity since the center $ZX \to X$ factors through the maximal torus $T(X) \to X$ [12, 7.5], [39, 4.3]. But then f itself is the identity, for $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(PX) \times \mathrm{Aut}(ZX)$ [34, 4.3]. ■

2.39. Lemma ([38, 9.4]). *If the two connected 2-compact groups X_1 and X_2 have N-determined (resp. $\pi_*(N)$-determined) automorphisms, so does the product $X_1 \times X_2$.*

Proof. Since the statement concerning N-determined automorphisms is proved in [38, 9.4] we deal here only with the case of $\pi_*(N)$-determined automorphisms. Let f be an automorphism under $T_1 \times T_2$ of the product 2-compact group $X_1 \times X_2$. Then

$$f_1 \colon X_1 \to X_1 \times X_2 \xrightarrow{f} X_1 \times X_2 \to X_1,$$
$$f_2 \colon X_2 \to X_1 \times X_2 \xrightarrow{f} X_1 \times X_2 \to X_2$$

are endomorphisms under the maximal tori and therefore conjugate to the respective identity maps. But f is [38, 9.3] in fact conjugate to the product morphism (f_1, f_2), which is the identity. ■

2.40. Lemma (cf. [35, 7.8]). *Suppose that*

(1) *X_0 is uniquely N-determined.*
(2) *X is LHS.*
(3) *$H^2(W/W_0; \check{Z}(X_0)) \to H^2(W/W_0; \check{T}^{W_0})$ is injective.*

Then X is N-determined.

Proof. Let X' be another 2-compact group with maximal torus normalizer pair $(N, N_0) \xrightarrow{(j', j_0')} (X', X_0')$. The assumption on the identity component X_0 means (2.14) that there exists an isomorphism $f_0 \colon X_0 \to X_0'$ under T. For any ξ in

$$W/W_0 = N/N_0 = X/X_0 = X'/X_0',$$

the isomorphism $\xi f_0 \xi^{-1}$ is also an isomorphism under T and thus $\xi f_0 = f_0 \xi$ as X_0 is uniquely N-determined. By the second assumption, the automorphism $\alpha_0 = \mathrm{AM}(f_0) \colon N_0 \to N_0$ with $\pi_*(B\alpha_0) = \mathrm{Id}$ extends to an isomorphism $\alpha \colon N \to N$ with $\pi_*(B\alpha) = \mathrm{Id}$.

The situation is now as shown in the commutative diagram

$$\begin{tikzcd}
BX_0 \arrow[d] & BN_0 \arrow[l, "Bj_0"'] \arrow[r, "B\alpha_0"] \arrow[d] & BN_0 \arrow[r, "Bj_0'"] \arrow[d] & BX_0' \arrow[d] \\
BX \arrow[d] & BN \arrow[l, "Bj"'] \arrow[r, "B\alpha"] \arrow[d] & BN \arrow[r, "Bj'"] \arrow[d] & BX' \arrow[d] \\
B\pi_0(X) & B(W/W_0) \arrow[l, "\cong"'] \arrow[r, equal] & B(W/W_0) \arrow[r, "\cong"] & B\pi_0(X')
\arrow[from=1-1, to=1-4, bend left, "Bf_0"]
\end{tikzcd}$$

Our aim is to find an isomorphism $f\colon X \to X'$ to fill in the based homotopy commutative diagram

$$\begin{tikzcd}
BX_0 \arrow[r, "Bf_0", "\cong"'] \arrow[d] & BX_0' \arrow[d] \\
BX \arrow[r, dotted] \arrow[d] & BX' \arrow[d] \\
B\pi_0(X) \arrow[r, "\cong"] & B\pi_0(X')
\end{tikzcd}$$

where the isomorphism between the base 2-compact groups is given by the isomorphisms $\pi_0(X) \leftarrow N/N_0 \to \pi_0(X')$. Since f_0 is $\pi_0(X)$-equivariant up to homotopy, $\mathrm{map}(BX_0, BX_0')_{Bf_0}$ is a $\pi_0(X)$-space in the sense that there exists a fibration

$$\mathrm{map}(BX_0, BX_0'; Bf_0) \to \mathrm{map}(BX_0, BX_0'; Bf_0)_{h\pi_0(X)} \to B\pi_0(X)$$

over $B\pi_0(X)$ with $\mathrm{map}(BX_0,BX_0')_{Bf_0}$, here written as $\mathrm{map}(BX_0,BX_0';Bf_0)$, as fiber. The space of sections of this fibration, $\mathrm{map}(BX_0, BX_0'; Bf_0)^{h\pi_0(X)}$, is a space of fiber maps of BX to BX'. This fibration sits on the left in the commutative diagram

$$\begin{tikzcd}
\mathrm{map}(BX_0,BX_0';Bf_0) \arrow[r] \arrow[d] & \mathrm{map}(BN_0,BX_0';B(j_0'\alpha)) \arrow[d] & \mathrm{map}(BN_0,BN_0;B\alpha_0) \arrow[l, "\simeq"'] \arrow[d] \\
\mathrm{map}(BX_0,BX_0';Bf_0)_{h\pi_0(X)} \arrow[r] \arrow[d] & \mathrm{map}(BN_0,BX_0';B(j_0'\alpha))_{h(W/W_0)} \arrow[d] & \mathrm{map}(BN_0,BN_0;B\alpha_0)_{h(W/W_0)} \arrow[l] \arrow[d] \\
B\pi_0(X) & B(W/W_0) \arrow[l] \arrow[r, equal] & B(W/W_0)
\end{tikzcd}$$

where the columns are fibrations and the horizontal maps are defined as composition with Bj and Bj', respectively. The fiber map from the right column to the central one is actually a fiber homotopy equivalence because the centralizers of the maximal torus in X_0' and in N_0 are isomorphic in that they are both isomorphic to the maximal torus.

The middle fibration admits a section corresponding to the fiberwise map $Bj' \circ B\alpha$. But then the left fibration also admits a section: the obstruction to a section of the left fibration is a cohomology class in $H^2(\pi_0(X); \check{Z}(X_0))$. Since the middle fibration does admit a section, this obstruction class is in the kernel of the coefficient group homomorphism $H^2(\pi_0(X); \check{Z}(X_0)) \to H^2(W/W_0, \check{T}^{W_0})$. But the assumption is that this is an injection and therefore the obstruction must vanish. (We are here tacitly replacing the three fibrations above by their fiberwise discrete approximations [33, 4.3].)

A section of the left fibration corresponds to a morphism $Bf\colon BX \to BX'$ under the isomorphism $Bf_0\colon BX_0 \to BX_0'$ and over $B\pi_0(X) \xrightarrow{\cong} B\pi_0(X')$ such that $Bf \circ Bj$ and $Bj \circ B\alpha$ are homotopic over $B(N/N_0) \to B\pi_0(X')$. But since the fiber BX_0' of $BX' \to B\pi_0(X')$ is simply connected this means that $Bf \circ Bj$ and $Bj \circ B\alpha$ are based homotopic maps $BN \to BX'$. ∎

2.41. Example. (1) Any 2-compact torus T is uniquely N-determined, for if $j\colon T \to X$ is the maximal torus normalizer for the connected 2-compact group X, then j is an isomorphism. Indeed, $H^*(BT; \mathbf{Q}_2) \cong H^*(BX; \mathbf{Q}_2)$ [11, 9.7(3)] and the connected space X/T has cohomological dimension $\mathrm{cd}_{\mathbf{F}_2}(X/T) = 0$ [12, 4.5, 5.6] so it is a point.

(2) Any 2-compact toral group G is *totally* N-determined: G clearly has N-determined automorphisms as G is its own maximal torus normalizer. If the 2-compact group X has the same maximal torus normalizer pair (G, T) as G, then X is a 2-compact toral group and $j'\colon G \to X$ is an isomorphism. G is *uniquely* N-determined if and only if $H^1(\pi_0(G); \check{T}) = 0$. In particular, $\mathrm{GL}(2, \mathbf{R})$ is uniquely N-determined.

2.42. Lemma ([35, 7.10]). *Suppose that X is connected. If the adjoint form $PX = X/Z(X)$ is N-determined, so is X.*

Proof. Let $j\colon N \to X$ be the maximal torus normalizer for X and $j'\colon N \to X'$ the maximal torus normalizer for some other connected 2-compact group X'. It suffices (2.14) to find a morphism $f\colon X \to X'$ under the maximal tori $X \xleftarrow{i} T \xrightarrow{i'} X'$. The 2-discrete center $\check{Z}$ of X and X' is contained in the 2-discrete maximal torus $\check{T}$ [12, 7.5]. Factoring out [11, 8.3] these central monomorphisms we obtain the commutative diagram

$$\begin{tikzcd}
B\check{X} \arrow[d] & B\check{T} \arrow[l, "Bi"'] \arrow[r, "Bi'"] \arrow[d] & B\check{X}' \arrow[d] \\
B(X/Z) \arrow[rr, bend right, "B(f/Z)"'] & B(T/Z) \arrow[l, "B(i/Z)"'] \arrow[r, "B(i'/Z)"] & B(X'/Z)
\end{tikzcd}$$

where the vertical maps are fibrations with fiber $B\check{Z}$, the total spaces, such as $B\check{X}$, are the fiberwise discrete approximations, and $f/Z\colon X/Z \to X'/Z$

is the isomorphism under T/Z that exists because X/Z is N-determined. Construct the fibration

$$\mathrm{map}(B\check{Z}, B\check{Z}; B1) \to B\check{Z}_{h(X/Z)} \to B(X/Z)$$

whose sections are maps $BX \to BX'$ over $B(f/Z)$ and under $B\check{Z}$. There are two other such fibrations related to this one as shown in the commutative diagram

$$\begin{tikzcd}
\mathrm{map}(B\check{Z}, B\check{Z}; B1) \arrow[r, equal] \arrow[d] & \mathrm{map}(B\check{Z}, B\check{Z}; B1) \arrow[r, equal] \arrow[d] & \mathrm{map}(B\check{Z}, B\check{Z}; B1) \arrow[d] \\
B\check{Z}_{h(X/Z)} \arrow[d] & B\check{Z}_{h(T/Z)} \arrow[l] \arrow[r, "Bi^*"] \arrow[d] & B\check{Z}_{h(T/Z)} \arrow[d] \\
B(X/Z) & B(T/Z) \arrow[l, "B(i/Z)"'] \arrow[r, equal] & B(T/Z)
\end{tikzcd}$$

where the middle fibration is the pull-back along $B(i/Z)$ of the left fibration and the fiber over $b \in B(T/Z)$ of the right fibration consists of one component of the space of maps of the fiber $B\check{T}_b$ over b into the fiber $B\check{X}'_{B(i'/Z)(b)}$ over $B(i'/Z)(b)$. The fiber equivalence Bi^* is induced by $Bi\colon B\check{T} \to B\check{X}$. The middle fibration has a section u' such that $Bi^* \circ u'$ is the section $Bi'\colon B\check{T} \to B\check{X}'$ of the right fibration. We now have fiber maps

$$\begin{tikzcd}
X/T \arrow[r, "u|X/T"] \arrow[d] & B\check{Z} \arrow[d] \\
B\check{T} \arrow[r] \arrow[dr, "B(i/Z)"'] & B\check{Z}_{h(X/Z)} \arrow[d] \\
 & B(X/Z)
\end{tikzcd}$$

where u is the composition of u' and $B\check{Z}_{h(T/Z)} \to B\check{Z}_{h(X/Z)}$. The canonical map $B\check{Z} \to \mathrm{map}(X/T, B\check{Z})$, given by constants, is a homotopy equivalence since X/T is simply connected [39, 5.6] and hence a version [35, 6.6] of the Zabrodsky lemma implies that $u = v \circ B(i/Z)$ for some section $v\colon B(X/Z) \to B\check{Z}_{h(X/Z)}$ of the left fibration. The section v is, after fiberwise completion, a fiber map $BX \to BX'$ under BT. ∎

Let $j_1\colon N_1 \to X_1$ and $j_2\colon N_2 \to X_2$ be maximal torus normalizers for the connected 2-compact groups X_1 and X_2 and suppose that X' is some connected 2-compact group that admits a maximal torus normalizer of the form $j'\colon N_1 \times N_2 \to X'$. The Splitting Theorem [13, 1.4], more explicitly in the form of [43, 5.5], says that there exist 2-compact groups X_1' and X_2' and an isomorphism $X_1' \times X_2' \to X'$ such that

$$\begin{tikzcd} & BN_1 \times BN_2 \arrow[dl, "Bj_1' \times Bj_2'"'] \arrow[dr, "Bj'"] & \\ BX_1' \times BX_2' \arrow[rr, "\simeq"] & & BX' \end{tikzcd}$$

commutes, where $j_1'\colon N_1 \to X_1'$ and $j_2'\colon N_2 \to X_2'$ are maximal torus normalizers. The following lemma is an immediate consequence.

2.43. Lemma. *The product of two N-determined connected 2-compact groups is N-determined.*

Proof. Since the connected 2-compact groups X_1, X_2 are N-determined there exist isomorphisms $f_1\colon X_1 \to X_1'$, $f_2\colon X_2 \to X_2'$ and automorphisms $\alpha_1 \in H^1(W_1; \check{T}_1) \subset \mathrm{Out}(N_1)$, $\alpha_2 \in H^1(W_2; \check{T}_2) \subset \mathrm{Out}(N_2)$ such that

$$\begin{tikzcd} BN_1 \times BN_2 \arrow[r, "B\alpha_1 \times B\alpha_2"] \arrow[d, "Bj_1 \times Bj_2"'] & BN_1 \times BN_2 \arrow[d, "Bj_1' \times Bj_2'"'] \arrow[dr, "Bj'"] & \\ BX_1 \times BX_2 \arrow[r, "Bf_1 \times Bf_2"] & BX_1' \times BX_2' \arrow[r, "\simeq"] & BX' \end{tikzcd}$$

commutes up to based homotopy. ■

3. N-determined connected, centerless 2-compact groups. In this section we formulate inductive criteria that, at least in favorable cases, can be used to show total N-determinism for connected, centerless (simple) 2-compact groups X. The key tool is the homology decomposition

$$\operatorname{hocolim}_{\mathbf{A}(X)^{\mathrm{op}}} BC_X \to BX \tag{2.44}$$

of BX in terms of centralizers of elementary abelian subgroups [12, 8.1]. Since X has no center, the cohomological dimension of each centralizer $C_X(V,\nu)$ is smaller than the cohomological dimension of X. As part of an inductive argument we will therefore assume that all centralizers are totally N-determined and formulate criteria (2.48, 2.51) that imply that also X is totally N-determined.

2.45. Definition ([12, §8]). The objects of the *Quillen category* $\mathbf{A}(X)$ are conjugacy classes of monomorphisms $\nu\colon V \to X$ of nontrivial elementary abelian 2-groups into X; the morphisms $\alpha\colon (V_1,\nu_1) \to (V_2,\nu_2)$ are injective group homomorphisms $\alpha\colon V_1 \to V_2$ such that ν_1 and $\nu_2\alpha$ are conjugate monomorphisms $V_1 \to X$. We shall write $\mathbf{A}(X)(V_1,V_2)$ for the set of morphisms $V_1 \to V_2$ and $\mathbf{A}(X)(V)$ for the group of all endomorphisms (which are all isomorphisms) of V.

The functor

(2.46) $$BC_X\colon \mathbf{A}(X)^{\mathrm{op}} \to \mathbf{Top} \quad \text{(topological spaces)}$$

takes an object (V,ν) of the Quillen category $\mathbf{A}(X)$ to its centralizer

$$BC_X(V,\nu) = \operatorname{map}(BV, BX)_{B\nu}.$$

The functor

(2.47) $$\pi_i(BZC_X)\colon \mathbf{A}(X) \to \mathbf{Ab} \quad \text{(abelian groups)}$$

takes (V,ν) to the abelian homotopy group

$$\pi_i(\operatorname{map}(BC_X(V,\nu), BX), e(\nu))$$

based at the evaluation map $e(\nu)\colon BC_X(V,\nu) \to BX$. (The mapping space $\operatorname{map}(BC_X(V,\nu), BX)_{e(\nu)}$ is homotopy equivalent to $BZC_X(V,\nu)$ [8]).

2.48. Lemma ([35, 4.9]). *Suppose that X is connected and centerless. If*

(1) *$C_X(L,\lambda)$ has N-determined ($\pi_*(N)$-determined) automorphisms for each rank one object (L,λ) of $\mathbf{A}(X)$,*
(2) $\lim^1(\mathbf{A}(X);\pi_1(BZC_X)) = 0 = \lim^2(\mathbf{A}(X);\pi_2(BZC_X))$,

then X has N-determined (resp. $\pi_*(N)$-determined) automorphisms.

Proof. Suppose first that each line centralizer has $\pi_*(N)$-determined automorphisms. Let $f\colon X \to X$ be an automorphism under the maximal torus $T \to X$. Since any monomorphism $\lambda\colon L \to X$, $L = \mathbf{Z}/2$, factors through the maximal torus, the commutative diagram

$$\begin{tikzcd} & & N \arrow[r] \arrow[d, "\mathrm{AM}(f)"] & X \arrow[d, "f"] \\ L \arrow[r, "\lambda^T"] & T \arrow[ur] \arrow[dr] & & \\ & & N \arrow[r] & X \end{tikzcd}$$

shows that $f\lambda = \lambda$ and gives a commutative diagram

$$\begin{tikzcd} & C_N(L) \arrow[r] \arrow[d, "C_{\mathrm{AM}(f)}(L)"] & C_X(L) \arrow[d, "C_f(L)"] \\ T \arrow[ur] \arrow[dr] & & \\ & C_N(L) \arrow[r] & C_X(L) \end{tikzcd}$$

of automorphisms under T. Thus $\mathrm{AM}(C_f(L)) = C_{\mathrm{AM}(f)}(L)\colon C_N(L) \to C_N(L)$. Now, $\pi_*(C_N(L))$ is a subgroup of $\pi_*(N)$ (for $\pi_1(C_N(L)) = \pi_1(N)$ and $\pi_0(C_N(L)) = W(X)(L)$ is [12, 7.6], [34, 3.2(1)] the stabilizer subgroup at $L < \check{T}$ for the action of $W(X)$ on $\check{T}$) so $\pi_*(C_{\mathrm{AM}(f)}(L)) = 1$ and $C_f(L) \simeq 1_{C_X(L)}$ since $C_X(L)$ has $\pi_*(N)$-determined automorphisms. For any other object (V,ν) of $\mathbf{A}(X)$ of rank > 1, choose a line L in V. Since the monomorphism $\nu\colon V \to X$ canonically factors through $C_X(L)$ [11, 8.2], [38, 3.18], the commutative diagram

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ V \arrow[r] \arrow[urr, bend left, "\nu"] \arrow[drr, bend right, "\nu"'] & C_X(L) \arrow[ur] \arrow[dr] & X \end{tikzcd}$$

shows that $f\nu = \nu$ and the induced diagram

$$\begin{tikzcd} & C_X(V) \arrow[dd, "C_f(V)"] \\ C_{C_X(L)}(V) \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & \\ & C_X(V) \end{tikzcd}$$

that $C_f(V)\colon C_X(V) \to C_X(V)$ is conjugate to the identity. The third assumption of the lemma ensures that there are no obstructions to conjugating f to the identity now that we know that the restriction of f to each of the centralizers is conjugate to the identity (see [35, 4.9]).

Suppose next that each line centralizer has N-determined automorphisms. Let $f\colon X \to X$ be an automorphism such that the diagram

$$\begin{tikzcd} & X \arrow[dd, "f"] \\ N \arrow[ur] \arrow[dr] & \\ & X \end{tikzcd}$$

commutes up to conjugacy. For each line L in T, the induced diagram

$$\begin{tikzcd} & C_X(L) \arrow[dd, "C_f(L)"] \\ C_N(L) \arrow[ur] \arrow[dr] & \\ & C_X(L) \end{tikzcd}$$

also commutes up to conjugacy. By assumption, this means (2.18) that the induced automorphisms $C_f(L)$ of line centralizers are conjugate to the identity. As above, this implies that the induced map $C_f(V)\colon C_X(V) \to C_X(V)$ is conjugate to the identity for any object (V, ν) of the Quillen category for X and that f is conjugate to the identity. ∎

Consider next an extended 2-compact torus N and two connected, centerless 2-compact groups X and X' both having N as their maximal torus normalizer:

$$X \xleftarrow{j} N \xrightarrow{j'} X'. \tag{2.49}$$

Our task is (2.14(1)) to construct an isomorphism $X \to X'$ under the maximal torus.

2.50. Definition. An object (V,ν) of $\mathbf{A}(X)$ is *toral* if the monomorphism $\nu\colon V \to X$ factors through the maximal torus $T \to X$. Let $\mathbf{A}(X)^{\leq t}$ denote the full subcategory of toral objects, and $\mathbf{A}(X)^{\leq t}_{\leq 2}$ the full subcategory of toral objects of rank ≤ 2.

For each toral object (V,ν) of $\mathbf{A}(X)^{\leq t}$, let $\nu^N\colon V \to N$ be the unique preferred lift [36, 4.10] of ν (which is the factorization of ν through the maximal torus, the identity component of N) and let (V,ν') be the toral object of $\mathbf{A}(X')$ defined by $\nu' = j \circ \nu^N\colon V \to X'$ as in the commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "\nu"'] \arrow[d, "\nu^N"] \arrow[dr, "\nu'"] & \\ X & N \arrow[l, "j"'] \arrow[r, "j'"] & X' \end{tikzcd}$$

The functor $\mathbf{A}(X)^{\leq t} \to \mathbf{A}(X')^{\leq t}$ that takes the object (V,ν) to the object (V,ν') and is the identity on morphisms is an equivalence of toral Quillen categories [38, 2.8].

2.51. Theorem (cf. [38, 3.8]). *In the situation of* (2.49), *assume the following*:

(1) *The centralizer $C_X(V,\nu)$ of any*
$$(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 2})$$
has N-determined automorphisms.

(2) *There exists a self-homotopy equivalence $\alpha \in H^1(W;\check{T}) \subseteq \mathrm{Out}(N)$ such that for every object $(L,\lambda) \in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 1})$ the diagram*
$$\begin{tikzcd} C_N(L,\lambda^N) \arrow[r, "\alpha|C_N(\lambda^N)"] \arrow[d, "j|C_N(\lambda^N)"'] & C_N(L,\lambda^N) \arrow[d, "j'|C_N(\lambda^N)"] \\ C_X(L,\lambda) \arrow[r, "f_\lambda"] & C_{X'}(L,\lambda') \end{tikzcd}$$
commutes for some isomorphism f_λ.

(3) *For any nontoral rank two object (V,ν) of $\mathbf{A}(X)$ the composite monomorphism*
$$\nu'_L\colon\ V \xrightarrow{\overline{\nu}(L)} C_X(L,\nu|L) \xrightarrow[\cong]{f_{\nu|L}} C_{X'}(L,(\nu|L)') \xrightarrow{\mathrm{res}} X'$$

and the induced isomorphism $f_{\nu,L}\colon C_X(V,\nu) \to C_{X'}(V,\nu'_L)$ *defined by the commutative diagram*

$$\begin{tikzcd} C_{C_X(L,\nu|L)}(V,\overline{\nu}(L)) \arrow[r, "C_{f_{\nu|L}}"] \arrow[d, "\cong"'] & C_{C_{X'}(L,(\nu|L)')}(V, f_{\nu|L}\circ\overline{\nu}(L)) \arrow[d, "\cong"] \\ C_X(V,\nu) \arrow[r, "f_{\nu,L}"] & C_{X'}(V,\nu'_L) \end{tikzcd}$$

do not depend on the choice of the line $L < V$*. (See* 2.65 *for the definition of the canonical factorization* $\overline{\nu}(L)$*.)*

(4) $\lim^2(\mathbf{A}(X);\pi_1(BZC_X)) = 0 = \lim^3(\mathbf{A}(X);\pi_2(BZC_X))$.

Then there exists an isomorphism $f\colon X \to X'$ *under* T (2.14).

Proof. The idea is that the isomorphisms $f_\lambda\colon C_X(\lambda) \to C_{X'}(\lambda')$ on the line centralizers restrict to isomorphisms $f_\nu\colon C_X(\nu) \to C_{X'}(\nu')$ for all centralizers in the $\mathbf{F}_2$-homology decomposition (2.44) of BX. These locally defined isomorphisms combine to a globally defined isomorphism $BX \to BX'$.

First observe that the isomorphisms $f_\lambda\colon C_X(\lambda) \to C_{X'}(\lambda')$ on the line centralizers are uniquely determined by the cohomology class $\alpha \in H^1(W;\check{T})$ (2.12(1)).

Let now (V,ν) be a rank two object of $\mathbf{A}(X)$ and L a line in the plane V. If (V,ν) is *toral*, define $f_\nu\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ to be the isomorphism induced by $f_{\nu|L}\colon C_X(L,\nu|L) \to C_{X'}(L,(\nu|L)')$. Since f_ν is an isomorphism under $\alpha|C_N(V,\nu^N)$ it does not depend on the choice of L in V (2.12(1)). If (V,ν) is *nontoral*, define ν' to be ν'_L and define $f_\nu\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ to be $f_{\nu,L}$. By assumption (3), the monomorphism ν' and the isomorphism $f_{\nu,L}$ are independent of the choice of L.

This construction respects morphisms in $\mathbf{A}(X)$. Consider first, for instance, a morphism $\beta\colon (L_1,\lambda_1) \to (L_2,\lambda_2)$ between two lines in X. Then $\lambda_1 = \lambda_2\beta$ and $\lambda_1^N = \lambda_2^N\beta$. The commutative diagram of isomorphisms

$$\begin{tikzcd} C_X(\lambda_1) \arrow[ddd, "f_{\lambda_1}"'] & & & C_X(\lambda_2) \arrow[lll, "C_X(\beta)"'] \arrow[ddd, "f_{\lambda_2}"] \\ & C_N(\lambda_1^N) \arrow[ul] \arrow[d, "\alpha|C_N(\lambda_1^N)"'] & C_N(\lambda_2^N) \arrow[l, "C_N(\beta)"'] \arrow[ur] \arrow[d, "\alpha|C_N(\lambda_2^N)"] & \\ & C_N(\lambda_1^N) \arrow[dl] & C_N(\lambda_2^N) \arrow[l, "C_N(\beta)"'] \arrow[dr] & \\ C_{X'}(\lambda'_1) & & & C_{X'}(\lambda'_2) \arrow[lll, "C_{X'}(\beta)"'] \end{tikzcd}$$

shows that $C_{X'}(\beta)^{-1} \circ f_{\lambda_1} \circ C_X(\beta) = f_{\lambda_2}$ for they are both isomorphisms under $C_N(\beta)^{-1} \circ \alpha|C_N(\lambda_1^N) \circ C_N(\beta) = \alpha|C_N(\lambda_2^N)$. Second, by the very definition of f_ν, the diagram

$$\begin{tikzcd} C_X(V,\nu) \arrow[r, "f_\nu"] \arrow[d] & C_{X'}(V,\nu') \arrow[d] \\ C_X(L,\nu|L) \arrow[r, "f_{\nu|L}"] & C_{X'}(L,(\nu|L)') \end{tikzcd}$$

commutes whenever $L < V$ and (V,ν) is a (toral or nontoral) rank 2 object of $\mathbf{A}(X)$.

We have now defined natural isomorphisms $f_\nu\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ for all objects $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X))$ of rank ≤ 2. For any other object (E,ε) of $\mathbf{A}(X)$, choose a line $L < E$ and proceed as for toral rank 2 objects. That is, define $\varepsilon'\colon E \to X'$ to be the monomorphism

$$E \xrightarrow{\overline{\varepsilon}(L)} C_X(E,\varepsilon|L) \xrightarrow{f_{\varepsilon|L}} C_{X'}(E,(\varepsilon|L)') \xrightarrow{\mathrm{res}} X'$$

and define $f_\varepsilon\colon C_X(E,\varepsilon) \to C_{X'}(E,\varepsilon')$ to be the isomorphism

$$\begin{tikzcd} C_{C_X(E,\varepsilon|L)}(\overline{\varepsilon}(L)) \arrow[r, "(f_{\varepsilon|L})_*"] \arrow[d, "\cong"'] & C_{C_{X'}(E,(\varepsilon|L)')}(f_{\varepsilon|L} \circ \overline{\varepsilon}(L)) \arrow[d, "\cong"] \\ C_X(E,\varepsilon) \arrow[r, "f_\varepsilon"] & C_{X'}(E,\varepsilon') \end{tikzcd}$$

induced by $f_{\varepsilon|L}$. If L_1 and L_2 are two distinct lines in E, let $P = \langle L_1, L_2\rangle$ be the plane generated by them. Then the commutative diagram

$$\begin{tikzcd} & C_X(L_1,\varepsilon|L_1) \arrow[r, "f_{\varepsilon|L_1}", "\cong"'] & C_{X'}(L_1,(\varepsilon|L_1)') \arrow[dr, "\mathrm{res}"] & \\ P \arrow[ur, "\overline{\varepsilon}(L_1)"] \arrow[r, "\overline{\varepsilon}(P)"] \arrow[dr, "\overline{\varepsilon}(L_2)"'] & C_X(P,\varepsilon|P) \arrow[u] \arrow[d] \arrow[r, "\cong", "f_{\varepsilon|P}"'] & C_{X'}(P,(\varepsilon|P)') \arrow[u] \arrow[d] \arrow[r, "\mathrm{res}"] & X' \\ & C_X(L_2,\varepsilon|L_2) \arrow[r, "\cong", "f_{\varepsilon|L_2}"'] & C_{X'}(L_2,(\varepsilon|L_2)') \arrow[ur, "\mathrm{res}"'] & \end{tikzcd}$$

shows that neither $(E,\varepsilon') \in \mathrm{Ob}(\mathbf{A}(X'))$ nor the isomorphism f_ε depend on the choice of line in E. Thus we have constructed a collection of centric [8] maps

$$BC_X(V,\nu) \to BX', \quad (V,\nu) \in \mathrm{Ob}(\mathbf{A}(X)), \tag{2.52}$$

that are homotopy invariant under $\mathbf{A}(X)$-morphisms. The vanishing of the obstruction groups (assumption (4)) means [55] that these homotopy $\mathbf{A}(X)$-

invariant maps can be realized by a map

$$Bf\colon BX \xleftarrow{\simeq} \operatorname{hocolim} BC_X \to BX'$$

such that $f \circ \mathrm{res} = \mathrm{res} \circ f_\nu$ for all $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X))$. In particular, f is a map under T and an isomorphism (2.14). ∎

2.53. *Verification of condition 2.51(2).* Let $\mathbf{A}(X)^{\leq t}$ be the toral part of the Quillen category and let $H^1(W_0;\check{T})^{W/W_0}\colon \mathbf{A}(X)^{\leq t} \to \mathbf{Ab}$ be the functor with value $H^1(W(C_X(V,\nu)_0);\check{T})^{\pi_0 C_X(V,\nu)}$ on the object (V,ν). If the 2-compact group C satisfies the conditions of Lemma 2.40 and $\check{Z}(C_0) = \check{T}(C_0)^{W(C_0)}$ we say that C satisfies the conditions of Lemma 2.40 *in the strong sense.*

2.54. Lemma. *Suppose that*

- *the centralizers $C_X(V,\nu)$ of all $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 2})$ satisfy the conditions of Lemma* 2.40 *in the strong sense*,
- $H^1(W;\check{T}) \to \lim^0(\mathbf{A}(X)^{\leq t}_{\leq 2}; H^1(W_0;\check{T})^{W/W_0})$ *is surjective.*

Then conditions 2.51(1) *and* 2.51(2) *are satisfied.*

Proof. Let (V,ν) be an object of $\mathbf{A}(X)^{\leq t}$ of rank ≤ 2. Since (2.40) the centralizer $C_X(V,\nu)$ is N-determined there is a solution $(f(V,\nu),\alpha(V,\nu))$ to the isomorphism problem

$$\begin{array}{ccc} C_N(V,\nu^N) & \xrightarrow{\alpha(V,\nu)} & C_N(V,\nu^N) \\ \big\downarrow & & \big\downarrow \\ C_X(V,\nu) & \xrightarrow{f(V,\nu)} & C_{X'}(V,\nu') \end{array}$$

and the set of all solutions is (2.35, 2.37) an $H^1(W/W_0;\check{T}^{W_0})(C_X(V,\nu))$-coset. Let

$$\overline{\alpha}(V,\nu) \in H^1(W_0;\check{T}^{W/W_0})(C_X(V,\nu))$$

be the restriction of any solution $\alpha(V,\nu) \in H^1(W;\check{T})(C_X(V,\nu))$ to the above isomorphism problem. Then

$$\{\overline{\alpha}(V,\nu)\}_{(V,\nu)\in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 2})} \in \lim{}^0(\mathbf{A}(X)^{\leq t}_{\leq 2}; H^1(W_0;\check{T})^{W/W_0}) \tag{2.55}$$

because the restriction of a solution is a solution. By assumption, there is an element $\alpha \in H^1(W;\check{T})$ that maps to (2.55) and α satisfies 2.51(2). ∎

In case $H^1(W;\check{T}) = 0$, the second point of Lemma 2.54 reduces to $\lim^0(\mathbf{A}(X)^{\leq t}_{\leq 2}; H^1(W_0;\check{T})^{W/W_0}) = 0$. Alternatively, we see that if it happens that $\lim^1(\mathbf{A}(X)^{\leq t}_{\leq 2}; H^1(W/W_0;\check{T}^{W_0})) = 0$, then the short exact sequences (2.27) for $C_X(V,\nu)$, $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 2})$, will produce a short exact sequence

$$0 \to \lim^0(\mathbf{A}(X)^{\le t}_{\le 2}, H^1(W/W_0; \check{T}^{W_0})) \to \lim^0(\mathbf{A}(X)^{\le t}_{\le 2}, H^1(W;\check{T})) \to \lim^0(\mathbf{A}(X)^{\le t}_{\le 2}, H^1(W_0;\check{T})^{W/W_0}) \to 0$$

in the limit. Since $H^1(W;\check{T})$ is isomorphic to the middle term by [10, 8.1], it maps onto the third term.

2.56. *Verification of condition 2.51(3).* In this subsection we assume that conditions 2.51(1) and 2.51(2) are satisfied. The following observations can sometimes be useful in the verification of condition 2.51(3).

Let (V,ν) be a nontoral rank two object of $\mathbf{A}(X)$ and $L < V$ a rank one subgroup. The commutative diagram

$$(2.57)\qquad \begin{tikzcd} & N \arrow[rr, "\alpha"] & & N \arrow[dr, bend left, "j'"] & \\ V \arrow[ur, bend left, "\nu_L^N"] \arrow[r, "\overline{\nu_L^N}(L)"] \arrow[dr, bend right, "\overline{\nu}(L)"'] & C_N(L,\nu_L^N|L) \arrow[u] \arrow[d] \arrow[rr, "C_\alpha"] & & C_N(L,\nu_L^N|L) \arrow[u] \arrow[d] & X' \\ & C_X(L,\nu|L) \arrow[rr, "f_{\nu|L}"] & & C_{X'}(L,(\nu|L)') \arrow[ur, bend right, "\mathrm{res}"'] & \end{tikzcd}$$

shows that ν'_L, which is defined to be $\mathrm{res}\circ f_{\nu|L}\circ\overline{\nu}(L)$, is equal to the composite $\nu'_L = j'\circ\alpha\circ\nu_L^N$. Moreover, we see, by taking the centralizer of $\overline{\nu}(L)$, that

$$(2.58)\qquad \begin{tikzcd} & V \arrow[dl, "\overline{\nu}(V)"'] \arrow[dr, "\overline{\nu}'_L(V)"] & \\ C_X(V,\nu) \arrow[rr, "\cong", "f_{\nu,L}"'] & & C_{X'}(V,\nu') \end{tikzcd}$$

commutes.

We are looking for criteria that ensure that $\nu'_L\colon V \to X'$ is independent of the choice of $L < V$.

2.59. LEMMA. *Let (V,ν) be a nontoral rank two object of $\mathbf{A}(X)$ and $L < V$ a line in V. Write C_3 for the Sylow 3-subgroup of $\mathrm{GL}(V)$. Suppose that*

(1) $C_3 \subseteq \mathbf{A}(X)(V,\nu)\cap\mathbf{A}(X')(V,\nu'_L)$,
(2) $f_{\nu,L}\colon C_X(V,\nu)\to C_{X'}(V,\nu'_L)$ *is C_3-equivariant.*

Then condition 2.51(3) *is satisfied.*

Proof. Let β be an automorphism of V. For general reasons, $\nu_L^N\beta = (\nu\beta)^N_{\beta^{-1}L}$ and the diagram

$$\begin{tikzcd} C_X(V,\nu) \arrow[r, "f_{\nu,L}"] \arrow[d, "C_X(\beta)"', "\cong"] & C_{X'}(V,\nu'_L) \arrow[d, "\cong"', "C_{X'}(\beta)"] \\ C_X(V,\nu\beta) \arrow[r, "f_{\nu\beta,\beta^{-1}L}"] & C_{X'}(V,\nu'_L\beta) \end{tikzcd}$$

commutes. Now, if $\beta \in \mathbf{A}(X)(V,\nu) \cap \mathbf{A}(X')(V,\nu'_L)$, then $\nu\beta = \nu$, $\nu'_L\beta = \nu'_L$, and $f_{\nu\beta,\beta^{-1}L} = f_{\nu,\beta^{-1}L}$ so that $f_{\nu,\beta^{-1}L} = C_{X'}(\beta) \circ f_{\nu,L} \circ C_X(\beta)^{-1}$ according to the above diagram. If also $f_{\nu,L}$ commutes with the action of β, we conclude that $f_{\nu,L} = f_{\nu,\beta^{-1}L}$. ∎

The following lemma ensures that condition 2.59(1) holds.

2.60. Lemma. *Let L and V denote elementary abelian 2-groups of rank one and two, respectively. Suppose that*

(1) *there is (up to conjugacy) a unique monomorphism $\lambda\colon L \to X$ with nonconnected centralizer,*
(2) *there is (up to conjugacy) a unique monomorphism $\nu\colon V \to X$ that is nontoral.*

Then the same holds for X', and $\mathbf{A}(X)(V,\nu) = \mathrm{GL}(V) = \mathbf{A}(X')(V,\nu')$ for the unique nontoral rank two objects (V,ν) of $\mathbf{A}(X)$ and (V,ν') of $\mathbf{A}(X')$.

Proof. Let $\nu'\colon V \to X'$ be a nontoral monomorphism and $i\colon L \to V$ an inclusion. Then $(L,\nu' i) = (L,\lambda')$ for $C_{X'}(L,\nu' i)$ is nonconnected so that $\nu' i$ and λ must correspond to each other under the bijection $\mathbf{A}(X)^{\le t} \to \mathbf{A}(X')^{\le t}$ between toral categories. Moreover, the diagram

(2.61)
$$\begin{tikzcd} X & C_X(L,\lambda) \arrow[l, "\mathrm{res}"'] \arrow[r, "f_\lambda", "\cong"'] & C_{X'}(L,\lambda') \arrow[r, "\mathrm{res}"] & X' \\ & & V \arrow[ull, bend left, "\nu"] \arrow[ul, "\overline{\nu}(L)"'] \arrow[u, "\overline{\nu}'(L)"] \arrow[ur, bend right, "\nu'"'] & \end{tikzcd}$$

is commutative. To see this, observe that $(V, \mathrm{res} \circ f_\lambda^{-1} \circ \overline{\nu}'(L))$ is a nontoral rank two object of $\mathbf{A}(X)$ (its centralizer is isomorphic to $C_{C_{X'}(L,\lambda')}(V,\overline{\nu}'(L))$ $= C_{X'}(V,\nu'))$ so that $(V,\nu) = (V, \mathrm{res} \circ f_\lambda^{-1} \circ \overline{\nu}'(L))$ by uniqueness of (V,ν). Also, we see from the commutative diagram

$$\begin{tikzcd} & & L \arrow[dll, bend right, "\lambda"'] \arrow[dl, "\overline{\lambda}(L)"'] \arrow[d, "i"] \\ X & C_X(L,\lambda) \arrow[l, "\mathrm{res}"] & V \arrow[l, shift right, "\overline{\nu}(L)"'] \arrow[l, shift left, "f_\lambda^{1}\circ\overline{\nu}'(L)"] \end{tikzcd}$$

that $\overline{\nu}(L) = f_\lambda^{-1} \circ \overline{\nu}'(L)$ by uniqueness of canonical factorizations under L [37, 3.9]. We conclude that $\nu' = \mathrm{res} \circ \overline{\nu}'(L) = \mathrm{res} \circ f_\lambda \circ \overline{\nu}(L)$. This means

(2.57) that $\nu' = \nu'_L$ for any choice of line $L < V$. Since ν' is thus unique up to conjugacy, $\nu'\beta = \nu'$ for any automorphism β of V. ∎

Note in connection with the verification of condition 2.59(2) that if 2.59(1) is satisfied so that $\nu'_L = \nu'$ is independent of L, then (2.58) shows that $f_{\nu,L}$ is a map under V in the sense that

(2.62)
$$\begin{tikzcd} & V \arrow[dl, "\overline{\nu}(V)"'] \arrow[dr, "\overline{\nu}'(V)"] & \\ C_X(V,\nu) \arrow[rr, "\cong", "f_{\nu,L}"'] & & C_{X'}(V,\nu') \end{tikzcd}$$

commutes. Since the canonical monomorphisms, $\overline{\nu}(V)$ and $\overline{\nu}'(V)$, are $\mathrm{GL}(V)$-equivariant, the restriction of $f_{\nu,L}$ to V is C_3-equivariant.

For any nontoral object (not necessarily of rank two) (V,ν) of $\mathbf{A}(X)$ and any rank one subgroup $L \subset V$, let $\nu^N_L\colon V \to N$ be a preferred lift of ν such that $\nu^N_L|L$ is the preferred lift of $\nu|L$, i.e. $\nu^N_L|L = (\nu|L)^N$. (It is always possible to extend a preferred lift given on the subgroup L to a preferred lift defined on all of V but a preferred lift defined on V may not restrict to a preferred lift on L [36, 4.9].) Also, define $\nu'_L\colon V \to X'$ and $f_{\nu,L}\colon C_X(V,\nu) \to C_{X'}(V,\nu'_L)$ as in 2.51(3).

2.63. Lemma. *Let $\nu\colon V \to X$ be any nontoral object of $\mathbf{A}(X)$.*

(1) *If the centralizer of ν has a nontrivial identity component, then $\nu'_L\colon V \to X'$ is independent up to conjugacy of the choice of $L \subset V$, and $\nu'_L = j' \circ \alpha \circ \nu^N_L$.*

(2) *If there also exist a 2-compact torus T_ν and isomorphisms $T_\nu \to C_N(V,\nu^N_L)_0$ such that the composites $T_\nu \to C_N(V,\nu^N_L)_0 \to T$ are independent of $L < V$ up to conjugacy, then*
$$f_{\nu,L}\colon C_X(V,\nu) \to C_{X'}(V,\nu')$$
are isomorphisms under the maximal torus T_ν for all $L < V$.

Proof. (1) Just as in (2.57) we see that $\nu'_L = \mathrm{res} \circ f_{\nu|L} \circ \overline{\nu}(L) = j' \circ \alpha \circ \nu^N_L$. The hypothesis implies that there exists [11, 5.4, 7.3] a morphism $\phi\colon L_1 \times V \to X$ extending $\nu\colon V \to X$ whose adjoint $L_1 \to C_X(\nu)$ factors through the identity component of $C_X(\nu)$. Let $L_1 \to C_N(V,\nu^N_L)$ be the preferred lift of $L_1 \to C_X(V,\nu)$ as in the commutative diagram

$$\begin{tikzcd} & C_N(V,\nu^N_L) \arrow[r, "\mathrm{res}"] \arrow[d] & N \arrow[d, "j"] \\ L_1 \arrow[ur] \arrow[r] & C_X(V,\nu) \arrow[r, "\mathrm{res}"] & X \end{tikzcd}$$

This preferred lift will factor through the identity component of $C_N(\nu^N_L)$

(and hence its composition with $C_N(\nu_L^N) \to N$ will factor through the identity component of N) since $L_1 \to C_X(V,\nu)$ factors through the identity component of $C_X(\nu)$ [36, 4.10]. Let $\phi_L^N \colon L_1 \times V \to N$ be the adjoint of the preferred lift $L_1 \to C_N(\nu_L^N)$. Then $\phi_L^N|L_1 \colon L_1 \to N$ factors through the identity component of N (the maximal torus) so it is [36, 4.10] the preferred lift of $\phi|L_1 \colon L_1 \to X$. In particular, $\phi_L^N|L_1 = (\phi|L_1)^N$ does not depend on the choice of L.

The adjoints, $\phi_2^N \colon V \to C_N(\phi_L^N|L_1)$ and $\phi_2 \colon V \to C_X(\phi|L_1)$, of ϕ_L^N and ϕ, respectively, with respect to the second factor, give a commutative diagram

$$\begin{tikzcd}
 & C_X(L_1,\phi|L_1) \arrow[r, "f_{\phi|L_1}"] & C_{X'}(L_1,(\phi|L_1)') \arrow[dr, bend left, "\mathrm{res}"] & \\
V \arrow[ur, bend left, "\phi_2"] \arrow[r, "\phi_2^N"] \arrow[dr, bend right, "\nu_L^N"'] & C_N(L_1,\phi_L^N|L_1) \arrow[u] \arrow[d] \arrow[r, "C_\alpha"] & C_N(L_1,\phi_L^N|L_1) \arrow[u] \arrow[d] & X' \\
 & N \arrow[r, "\alpha"] & N \arrow[ur, bend right, "j'"'] &
\end{tikzcd}$$

We conclude that $\nu'_L = j' \circ \alpha \circ \nu_L^N = \mathrm{res} \circ f_{\phi|L_1} \circ \phi_2 \colon V \to X'$ is independent of the choice of $L < V$.

(2) The upper square in the diagram

$$\begin{tikzcd}
T_\nu \arrow[r, equal] \arrow[d] & T_\nu \arrow[d] \\
C_N(V,\nu_L^N) \arrow[r, "\alpha"] \arrow[d] & C_N(V,\alpha\nu_L^N) \arrow[d] \\
C_X(V,\nu) \arrow[r, "f_{\nu,L}"] & C_{X'}(V,\nu')
\end{tikzcd}$$

commutes because α restricts to the identity on the identity component T of N and hence also on T_ν. That the lower square is commutative is a consequence of the commutative diagram

$$\begin{tikzcd}
 & C_N(L,\nu_L^N|L) \arrow[r, "C_\alpha"] \arrow[d] & C_N(L,\alpha\nu_L^N) \arrow[d] \\
V \arrow[ur, bend left, "\overline{\nu}_L^N(L)"] \arrow[r, "\overline{\nu}(L)"] & C_X(L,\nu|L) \arrow[r, "f_{\nu|L}"] & C_{X'}(L,(\nu|L)')
\end{tikzcd}$$

where $\overline{\nu}_L^N(L)$ and $\overline{\nu}(L)$ are the canonical factorizations (2.65). ∎

Let $\mu \colon U \to X$ be a nontrivial elementary abelian 2-group and $\mu \colon U \to X$ a monomorphism whose centralizer $C_X(U,\mu)$ has nontrivial identity component. Suppose that U contains a nontrivial subgroup $V < U$ such that the restriction of μ to V is nontoral. Choose a rank one subgroup $L \subset V \subset U$.

We may choose the preferred lifts μ_L^N and $(\mu|V)_L^N$ such that $\mu_L^N|V = (\mu|V)_L^N$. Since $C_X(U,\mu)$ has nontrivial identity component, the conjugacy classes of the monomorphisms $\mu' = \mu_L'$ and $(\mu|V)_L' = \mu'|L$ are independent of the choice of L by 2.63(1). Then there is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\overline{\mu}(V)"'] \arrow[dr, "\overline{\mu}'(V)"] & \\ C_X(V,\mu|V) \arrow[rr, "f_{\mu|V,L}"] & & C_{X'}(V,\mu'|V) \end{tikzcd} \tag{2.64}$$

similar to (2.62).

2.65. *Canonical factorizations.* Let $\nu\colon V \to X$ be a monomorphism from an elementary abelian p-group to the p-compact group X. The canonical factorization of ν through its centralizer is the central monomorphism $\overline{\nu}(V)\colon V \to C_X(V,\nu)$ whose adjoint is $V \times V \xrightarrow{+} V \xrightarrow{\nu} X$ [11, 8.2]. If $\alpha\colon (V_1,\nu_1) \to (V_2,\nu_2)$ is a morphism in $\mathbf{A}(X)$ then the canonical factorizations are related by a commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "\overline{\nu}_1(V_1)"] \arrow[d, "\alpha"'] & C_X(V_1,\nu_1) \arrow[r, "\mathrm{res}"] & X \arrow[d, equal] \\ V_2 \arrow[r, "\overline{\nu}_2(V_2)"] & C_X(V_2,\nu_2) \arrow[r, "\mathrm{res}"] \arrow[u, "C_X(\alpha)"'] & X \end{tikzcd} \tag{2.66}$$

We write $\overline{\nu}_2(V_1)\colon V_2 \to C_X(V_1,\nu_1)$ for $C_X(\alpha)\circ\overline{\nu}_2(V_2)$ and call it the *canonical factorization* of ν_2 through the centralizer of ν_1. The induced diagram

$$\begin{tikzcd} C_{C_X(V_2,\nu_2)}(V_2,\overline{\nu}_2(V_2)) \arrow[r, "C_{C_X(\alpha)}", "\cong"'] \arrow[d, "\cong"'] & C_{C_X(V_1,\nu_1)}(V_2,\overline{\nu}_2(V_1)) \arrow[r, "C_{C_X(V_1,\nu_1)}(\alpha)"] & C_{C_X(V_1,\nu_1)}(V_1,\overline{\nu}_1(V_1)) \arrow[d, "\cong"] \\ C_X(V_2,\nu_2) \arrow[rr, "C_X(\alpha)"] & & C_X(V_1,\nu_1) \end{tikzcd} \tag{2.67}$$

is a factorization of $C_X(\alpha)$.

4. An exact functor. Let W be a finite group, p a prime, and $\varrho\colon W \to \mathrm{GL}(t)$ a representation of W in an $\mathbf{F}_p$-vector space t of finite dimension. For any nontrivial subgroup $V \subset t$, let

$$W(V) = \{w \in W \mid \forall v \in V\colon wv = v\}$$

be the subgroup of elements of W that act as the identity on V. For any two nontrivial subgroups $V_1, V_2 \subset t$, let

$$\overline{W}(V_1,V_2) = \{w \in W \mid wV_1 \subset V_2\}$$

be the transporter set. (Even though this is suppressed in the notation, this set depends on the representation ϱ.)

Suppose that we are also given a $\mathbf{Z}_pW$-module L.

2.68. Definition ([38, 2.2]). $\mathbf{A}(\varrho,t)$ is the category whose objects are nontrivial subspaces of V and whose morphisms are group homomorphisms induced by the W-action. The functor $L_i\colon \mathbf{A}(\varrho,t)\to\mathbf{Ab}$ takes the object $V\subset t$ to $H^i(W(V);L)$ and the morphism $w\colon V_1\to V_2$ to

$$H^i(W(V_1);L)\xrightarrow{w^*}H^i(W(V_1)^w;L)\xrightarrow{\text{res}}H^i(W(V_2);L)$$

where res is restriction and w^* is induced from conjugation with $w\in W$.

The category $\mathbf{A}(\varrho,t)$ depends only on the image of W in $\mathrm{GL}(t)$ but the functor L_i depends on the actual representation. The morphism set in $\mathbf{A}(\varrho,t)$ is the set of orbits

$$\mathbf{A}(\varrho,t)(V_1,V_2)=\overline{W}(V_1,V_2)/W(V_1)$$

for the action of the group $W(V_1)$ on the set $\overline{W}(V_1,V_2)$. We shall often write $\mathbf{A}(W,t)$ for $\mathbf{A}(\varrho,t)$ when the representation ϱ is clear from the context and $\mathbf{A}(W,t)(V)$ will be used as an abbreviation for the endomorphism group $\mathbf{A}(W,t)(V,V)=\overline{W}(V,V)/W(V)$.

2.69. Lemma ([10, 8.1]). *L_i is an exact functor with limit $H^i(W;L)$:*

$$\lim{}^j(\mathbf{A}(W,t),L_i)=\begin{cases}H^i(W;L), & j=0,\\ 0, & j>0.\end{cases}$$

Proof. The proof of [10, 8.1] also applies to this slightly different setting where the action of W on the $\mathbf{F}_p$-vector space t may not be faithful and L is a $\mathbf{Z}_pW$-module (and not an $\mathbf{F}_pW$-module).

Another possibility is to use the ideas of [25]. It suffices to show that the category $\mathbf{A}(W,t)$ satisfies (the duals of) the conditions of [25, 5.16] and that L_* is a proto-Mackey functor. Define $L^*\colon \mathbf{A}(W,t)\to\mathbf{Ab}$ to be the contravariant functor that agrees with L_* on objects but takes the $\mathbf{A}(W,t)$-morphism $w\colon E_0\to E_1$ to the group homomorphism

$$H^*(W(E_0);L)\xleftarrow{(w^{-1})^*}H^*(W(E_0)^w;L)\xleftarrow{\text{tr}}H^*(W(E_1);L)$$

where tr is transfer. To prove the existence of coproducts and push-outs in the multiplicative extension $\mathbf{A}(W,t)_{\prod}$ we follow [25, 6.3]. Let E_0,E_1,E_2 be elementary abelian subgroups of t where $E_0\subset E_1$ and there is a morphism $E_0\to E_2$ represented by an element $w\in\overline{W}(E_0,E_2)\subset W$. ($E_0$ is possibly empty to allow for the construction of coproducts.) Each coset $gW(E_1)\in$

$W(E_0)/W(E_1)$ has an associated special diagram

$$\begin{tikzcd} E_0 \arrow[r, hook] \arrow[d, "w"'] & E_1 \arrow[d, "wg"] \\ E_2 \arrow[r, hook] & E_2 + wgE_1 \end{tikzcd}$$

where we note that $W(E_2+wgE_1) = W(E_2)\cap W(E_1)^{wg}$. This construction determines a bijection between the double coset $w^{-1}W(E_2)w\backslash W(E_0)/W(E_1)$ and the set of isomorphism classes of special diagrams (cf. [25, 7.3]) and therefore

$$\begin{tikzcd} E_0 \arrow[r, hook] \arrow[d, "w"'] & E_1 \arrow[d, "\prod wg"] \\ E_2 \arrow[r, hook] & \prod(E_2 + wgE_1) \end{tikzcd}$$

where the product is taken over all $g \in w^{-1}W(E_2)w\backslash W(E_0)/W(E_1)$, is a push-out diagram in $\mathbf{A}(W,t)_{\prod}$ [25, 6.3]. By [25, 5.13], we need to show that the diagram

$$\begin{tikzcd} H^*(W(E_0);L) \arrow[d, "L_*(w)"'] & & H^*(W(E_1);L) \arrow[ll, "L^*(E_0\subseteq E_1)"'] \arrow[d, "\prod L_*(wg)"] \\ H^*(W(E_2);L) & & \prod H^*(W(E_2)\cap W(E_1)^{wg};L) \arrow[ll, "\sum L^*(E_2\subseteq E_2+gwE_1)"'] \end{tikzcd}$$

commutes. But this is precisely the content of the Cartan–Eilenberg double coset formula relating the restriction and transfer homomorphisms in group cohomology [5], [16, 4.2.6].

The restriction homomorphism $H^*(W;L) \to \lim^0(\mathbf{A}(W,t);L_*)$ is injective since t contains an elementary abelian subgroup $E \subset t$ such that the index of $W(E)$ in W is prime to p. To show surjectivity, we use the argument from the proof of [25, 7.2]. ∎

3. THE A-FAMILY

The A-family consists of the matrix groups

$$\mathrm{PGL}(n+1,\mathbf{C}) = \frac{\mathrm{GL}(n+1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}, \qquad n \geq 1,$$

where $\mathrm{GL}(n+1,\mathbf{C})$ is the Lie group of complex $(n+1)\times(n+1)$ matrices with center $\mathrm{GL}(1,\mathbf{C})$ consisting of scalar matrices. The maximal torus normalizer for $\mathrm{PGL}(n+1,\mathbf{C})$ is

$$N(\mathrm{PGL}(n+1,\mathbf{C})) = \frac{\mathrm{GL}(1,\mathbf{C})^{n+1}}{\mathrm{GL}(1,\mathbf{C})} \rtimes \Sigma_{n+1}$$

where $\Sigma_{n+1} = W(\mathrm{PGL}(n+1,\mathbf{C})) \subset \mathrm{PGL}(n+1,\mathbf{C})$ is the Weyl group of permutation matrices. It is known [21], [29] that

$$(3.1)\qquad H^0(W;\check{T}) = \begin{cases} \mathbf{Z}/2, & n=1, \\ 0, & n>1, \end{cases} \qquad H^1(W;\check{T}) = \begin{cases} \mathbf{Z}/2, & n=3, \\ 0, & n\neq 3 \end{cases}$$

for $\mathrm{PGL}(n+1,\mathbf{C})$. For all n, $\mathrm{PGL}(n+1,\mathbf{C}) = \mathrm{PSL}(n+1,\mathbf{C})$. When $n+1$ is odd, $\mathrm{PGL}(n+1,\mathbf{C}) = \mathrm{PSL}(n+1,\mathbf{C}) = \mathrm{SL}(n+1,\mathbf{C})$ as 2-compact groups.

1. The structure of $\mathrm{PGL}(n+1,\mathbf{C})$**.** In this and the following section we use the results of Chapter 2 to show that the 2-compact groups $\mathrm{PGL}(n+1,\mathbf{C})$, $n \geq 1$, are uniquely N-determined. This section provides the information about the Quillen category needed for the calculation (3.20) of the higher limit obstruction groups from 2.48 and 2.51.

3.2. *The toral subcategory of* $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$. We consider the full subcategory of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ generated by the toral nontrivial elementary abelian 2-groups in $\mathrm{PGL}(n+1,\mathbf{C})$, $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}$ (2.50) .

3.3. Lemma. *The monomorphism* $\nu\colon V \to \mathrm{PGL}(n+1,\mathbf{C})$ *is toral if and only if it lifts to a morphism* $V \to \mathrm{GL}(n+1,\mathbf{C})$. *If* $n+1$ *is odd, all objects of* $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ *are toral.*

Proof. Any monomorphism $V \to \mathrm{GL}(n+1,\mathbf{C}) \to \mathrm{PGL}(n+1,\mathbf{C})$ is toral since it is toral already in $\mathrm{GL}(n+1,\mathbf{C})$ by complex representation theory. Conversely, any toral monomorphism $V \to \mathrm{GL}(1,\mathbf{C})^{n+1}/\mathrm{GL}(1,\mathbf{C}) \subset \mathrm{PGL}(n+1,\mathbf{C})$ lifts to $\mathrm{GL}(1,\mathbf{C})$ since $\mathrm{GL}(1,\mathbf{C})$ is divisible. When $n+1$ is odd, $\mathrm{PGL}(n+1,\mathbf{C}) = \mathrm{SL}(n+1,\mathbf{C}) \subset \mathrm{GL}(n+1,\mathbf{C})$ as 2-compact groups so all monomorphisms $V \to \mathrm{PGL}(n+1,\mathbf{C})$ are toral. ∎

Let

$$e_i = \mathrm{diag}(+1,\dots,+1,-1,+1,\dots,+1) \in \mathrm{GL}(n+1,\mathbf{C}), \qquad 1 \leq i \leq n+1,$$

be the diagonal matrix with -1 in position i and $+1$ at all other positions. The maximal toral elementary abelian 2-groups

$$\Delta_{n+1} = \langle e_1,\dots,e_{n+1}\rangle = \langle \mathrm{diag}(\pm 1,\dots,\pm 1)\rangle \cong (\mathbf{Z}/2)^{n+1} \subset \mathrm{GL}(n+1,\mathbf{C}),$$
$$P\Delta_{n+1} = \langle e_1,\dots,e_{n+1}\rangle/\langle e_1\cdots e_{n+1}\rangle \cong (\mathbf{Z}/2)^{n} \subset \mathrm{PGL}(n+1,\mathbf{C})$$

have Quillen automorphism groups $\Sigma_{n+1} \cong \mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(\Delta_{n+1}) \cong \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(P\Delta_{n+1})$.

3.4. Lemma. *The inclusion functors*

$$\mathbf{A}(\Sigma_{n+1},\Delta_{n+1}) \to \mathbf{A}(\mathrm{GL}(n+1,\mathbf{C})),$$
$$\mathbf{A}(\Sigma_{n+1},P\Delta_{n+1}) \to \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}$$

are equivalences of categories.

Proof. This is a general fact; the first part of [38, 2.8] also holds for the case $p = 2$. However, it may be more illustrative to prove the lemma directly in this special case.

By complex representation theory, any nontrivial elementary abelian 2-group in $\mathrm{GL}(n+1,\mathbf{C})$ is conjugate to a subgroup of Δ_{n+1} and the automorphism group $\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(\Delta_{n+1}) = \Sigma_{n+1}$. Thus there is a faithful inclusion functor

$$\mathbf{A}(\Sigma_{n+1}, \Delta_{n+1}) \to \mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))$$

which is surjective on the sets of isomorphism classes of objects. It remains to show that this functor is full. Since any morphism in the category $\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))$ is an isomorphism followed by an inclusion, it is enough to show that any conjugation induced isomorphism $V_1 \to V_2$ between nontrivial subgroups $V_1, V_2 \subset \Delta_{n+1}$ is actually induced from conjugation by an element of $N(\mathrm{GL}(n+1,\mathbf{C}))$. But this is a well-known fact from Lie group theory easily derived from e.g. [4, IV.2.5].

Any toral nontrivial elementary abelian 2-group in $\mathrm{PGL}(n+1,\mathbf{C})$ is the image of an elementary abelian 2-group in $\mathrm{GL}(n+1,\mathbf{C})$ and hence conjugate to subgroup of $P\Delta_{n+1}$. Since any $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$-morphism between subgroups of $P\Delta_{n+1}$ is induced from conjugation with an element of $N(\mathrm{PGL}(n+1,C))$, it follows that $\mathbf{A}(\Sigma_{n+1}, P\Delta_{n+1}) \to \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\le t}$ is an equivalence of categories. ∎

For any partition $n+1 = i_0 + i_1 + \cdots + i_r$ of $n+1$ into a sum of $r+1$ positive integers, let $(\pm1)^{i_0}(\pm1)^{i_1}\cdots(\pm1)^{i_r}$ denote the diagonal matrix

$$\mathrm{diag}(\overbrace{\pm1,\ldots,\pm1}^{i_0},\overbrace{\pm1,\ldots,\pm1}^{i_1},\ldots,\overbrace{\pm1,\ldots,\pm1}^{i_r})$$

in $\mathrm{GL}(n+1,\mathbf{C})$.

For any partition (i_0, i_1) of $n+1 = i_0 + i_1$ into a sum of two positive integers $i_0 \ge i_1 \ge 1$, let $L[i_0,i_1] \subset \mathrm{PGL}(n+1,\mathbf{C})$ be the image in $\mathrm{PGL}(n+1,\mathbf{C})$ of the elementary abelian 2-group

$$L[i_0,i_1]^* = \langle (+1)^{i_0}(-1)^{i_1}, (-1)^{n+1}\rangle$$

in $\mathrm{GL}(n+1,\mathbf{C})$. The centralizer of $L[i_0,i_1]$ is

$$C_{\mathrm{PGL}(n+1,\mathbf{C})}L[i_0,i_1] = \begin{cases} \dfrac{\mathrm{GL}(i_0,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2, & i_0 = i_1, \\ \dfrac{\mathrm{GL}(i_0,\mathbf{C}) \times \mathrm{GL}(i_1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}, & i_0 > i_1, \end{cases} \tag{3.5}$$

where the action of

$$C_2 = \left\langle \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \right\rangle$$

interchanges the two $\mathrm{GL}(i_0,\mathbf{C})$-factors. The center of the centralizer of $L[i_0,i_1]$ is

$$ZC_{\mathrm{PGL}(n+1,\mathbf{C})}L[i_0,i_1] = \begin{cases} L[i_0,i_1], & i_0 = i_1, \\ \dfrac{\mathrm{GL}(1,\mathbf{C})\times\mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}, & i_0 > i_1. \end{cases} \tag{3.6}$$

For any partition (i_0,i_1,i_2) of $n+1 = i_0+i_1+i_2$ into a sum of three positive integers $i_0 \geq i_1 \geq i_2 \geq 1$, let $P[i_0,i_1,i_2] \subset \mathrm{PGL}(n+1,\mathbf{C})$ be the image in $\mathrm{PGL}(n+1,\mathbf{C})$ of the elementary abelian 2-group

$$P[i_0,i_1,i_2]^* = \langle (+1)^{i_0}(-1)^{i_1}(+1)^{i_2}, (+1)^{i_0}(+1)^{i_1}(-1)^{i_2}, (-1)^{n+1}\rangle.$$

in $\mathrm{GL}(n+1,\mathbf{C})$. The centralizer of $P[i_0,i_1,i_2]$ is

$$C_{\mathrm{PGL}(n+1,\mathbf{C})}P[i_0,i_1,i_2] = \frac{\mathrm{GL}(i_0,\mathbf{C})\times\mathrm{GL}(i_1,\mathbf{C})\times\mathrm{GL}(i_2,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} \tag{3.7}$$

so that the center of the centralizer,

$$ZC_{\mathrm{PGL}(n+1,\mathbf{C})}P[i_0,i_1,i_2] = \frac{\mathrm{GL}(1,\mathbf{C})\times\mathrm{GL}(1,\mathbf{C})\times\mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}, \tag{3.8}$$

is connected.

For any partition (i_0,i_1,i_2,i_3) of $n+1 = i_0+i_1+i_2+i_3$ into a sum of four positive integers $i_0 \geq i_1 \geq i_2 \geq i_3 \geq 1$, let $P[i_0,i_1,i_2,i_3] \subset \mathrm{PGL}(n+1,\mathbf{C})$ be the image in $\mathrm{PGL}(n+1,\mathbf{C})$ of the elementary abelian 2-group

$$\begin{aligned} &P[i_0,i_1,i_2,i_3]^* \\ &\quad = \langle (+1)^{i_0}(-1)^{i_1}(+1)^{i_2}(-1)^{i_3}, (+1)^{i_0}(+1)^{i_1}(-1)^{i_2}(-1)^{i_3}, (-1)^{n+1}\rangle. \end{aligned}$$

The centralizer of $P[i_0,i_1,i_2,i_3]$ is

$$\begin{aligned} &C_{\mathrm{PGL}(n+1,\mathbf{C})}P[i_0,i_1,i_2,i_3] \\ &= \begin{cases} \dfrac{\mathrm{GL}(i_0,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes (C_2\times C_2), & i_0 = i_1 = i_2 = i_3, \\ \dfrac{\mathrm{GL}(i_0,\mathbf{C})^2\times\mathrm{GL}(i_2,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2, & i_0 = i_1 > i_2 = i_3, \\ \dfrac{\mathrm{GL}(i_0,\mathbf{C})\times\mathrm{GL}(i_1,\mathbf{C})\times\mathrm{GL}(i_2,\mathbf{C})\times\mathrm{GL}(i_3,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}, & \text{otherwise}, \end{cases} \end{aligned} \tag{3.9}$$

where

$$C_2 \times C_2 = \left\langle \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix} \right\rangle,$$

$$C_2 = \left\langle \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix} \right\rangle.$$

The center of the centralizer of $P[i_0, i_1, i_2, i_3]$ is

$$(3.10) \qquad ZC_{\mathrm{PGL}(n+1,\mathbf{C})}P[i_0, i_1, i_2, i_3]$$
$$= \begin{cases} P[i_0, i_1, i_2, i_3], & i_0 = i_1 = i_2 = i_3, \\ \dfrac{\mathrm{GL}(1,\mathbf{C}) \times \mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} \times \langle (+1)^{i_0}(-1)^{i_1}(+1)^{i_2}(-1)^{i_3} \rangle, & i_0 = i_1 > i_2 = i_3, \\ \dfrac{\mathrm{GL}(1,\mathbf{C}) \times \mathrm{GL}(1,\mathbf{C}) \times \mathrm{GL}(1,\mathbf{C}) \times \mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}, & \text{otherwise.} \end{cases}$$

We collect the information about the toral subcategory that we shall need later on in the following proposition. Let $P(m, k)$ denote the number of partitions of m into sums of k natural integers.

3.11. Proposition. *The category* $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ *contains precisely*

- $P(n+1, 2)$ *isomorphism classes of toral rank one objects represented by the lines* $L[i_0, i_1]$,
- $P(n+1, 3) + P(n+1, 4)$ *isomorphism classes of toral rank two objects represented by the planes* $P[i_0, i_1, i_2]$ *and* $P[i_0, i_1, i_2, i_3]$.

The centralizers of these objects are listed in (3.5), (3.7), *and* (3.9).

The automorphism groups are easily computed using complex representation theory because

$$\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(P[i_0, i_1, i_2, i_3]^*) \to \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(P[i_0, i_1, i_2, i_3])$$

is surjective (as in 3.18). One finds that

$$\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))P[i_0, i_1, i_2, i_3] = \begin{cases} \mathrm{GL}(2,\mathbf{F}_2), & \ell(i_0, i_1, i_2, i_3) \geq 3, \\ C_2, & \ell(i_0, i_1, i_2, i_3) = 2, \\ \{1\}, & \ell(i_0, i_1, i_2, i_3) = 1, \end{cases}$$

where $\ell(i_0, i_1, i_2, i_3) = \max_{1 \leq j \leq 4} \#\{k \mid i_k = i_j\}$ is the maximal number of repetitions in the sequence (i_0, i_1, i_2, i_3). This formula also holds for the objects $P[i_0, i_1, i_2]$ when interpreted as $P[i_0, i_1, i_2, 0]$.

2. Centralizers of objects of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\le t}_{\le 2}$ are LHS. In this section we check that all toral objects of rank ≤ 2 have LHS (§2.26) centralizers.

3.12. Lemma. *The centralizers of the objects of* $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\le t}_{\le 2}$,

(1) $\dfrac{\mathrm{GL}(i,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2$ (3.5),

(2) $\dfrac{\mathrm{GL}(i,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes (C_2\times C_2)$ (3.9),

(3) $\dfrac{\mathrm{GL}(i_0,\mathbf{C})^2 \times \mathrm{GL}(i_2,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2$ (3.9),

are LHS.

Proof. (1) Let

$$X = \frac{\mathrm{GL}(i,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2, \qquad i \ge 1,$$

where the C_2-action switches the two $\mathrm{GL}(i,\mathbf{C})$-factors. For $i=1$, X is a 2-compact toral group, hence LHS. For $i=2$ explicit computer computation yields

$\frac{\mathrm{GL}(i,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})^\pi$	$H^1(W_0;\check{T})$
$i=2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$

so X is manifestly LHS in this case (even though X_0 is not regular). For $i>2$, $\theta(X_0)$ is bijective and thus X is LHS by 2.28. ($\theta(X_0)$ is injective by 2.21(1) and surjective by 2.21(2) for $i \ne 4$ and for $i=4$ by inspection, or by 2.23 and 2.24 for all $i>2$.)

(2) Let $X = \frac{\mathrm{GL}(i,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes (C_2\times C_2)$, $i\ge 1$, where $C_2\times C_2 = \langle (12)(34), (13)(24)\rangle$ permutes the four $\mathrm{GL}(i,\mathbf{C})$-factors. For $i=1$, X is a 2-compact toral group, hence LHS. For $i=2$ explicit computer computation yields

$\frac{\mathrm{GL}(i,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes (C_2\times C_2)$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})^\pi$	$H^1(W_0;\check{T})$
$i=2$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^8$

so X is manifestly LHS in this case. (Alternatively, observe that X_0 is regular (2.24, 2.23), the kernel of $\theta(X_0)$ is $(\mathbf{Z}/2)^4$, and $\theta(X_0)^\pi$ is surjective because $H^1(C_2\times C_2;(\mathbf{Z}/2)^4)=0$ for the regular representation.) For $i>2$, we see as in part (1) that $\theta(X_0)$ is bijective and hence X is LHS by Lemma 2.28.

(3) Let $X = \frac{(\mathrm{GL}(i_0,\mathbf{C})\times\mathrm{GL}(i_2,\mathbf{C}))^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2$, $1\le i_0 < i_2$, where C_2 switches the two identical factors. Using 2.23 and 2.24 we see (details omitted) that X_0 is regular. By 2.21(1), $\theta(X_0)$ is in fact bijective except when i_0 or i_2 is 2. In those cases, the kernel of $\theta(X_0)$ is $(\mathbf{Z}/2)^2$ and $\theta(X_0)^{C_2}$ is surjective as

$H^1(C_2;(\mathbf{Z}/2)^2)=0$ for the regular representation. Therefore X is LHS by Lemma 2.28. ∎

3. Limits over the Quillen category of $\mathrm{PGL}(n+1,\mathbf{C})$**.** In this section we show that the problem of computing the higher limits of the functors $\pi_i(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})$, $i=1,2$, (2.47) is concentrated on the nontoral objects of the Quillen category.

3.13. Lemma ([38, 2.8]). *Let $V\subset P\Delta_{n+1}$ be a nontrivial subgroup representing an object of $\mathbf{A}(\Sigma_{n+1},P\Delta_{n+1})=\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\le t}$* (3.4). *Then*

$$\check{Z}C_{\mathrm{PGL}(n+1,\mathbf{C})}(V)=\check{T}^{\Sigma_{n+1}(V)}$$

where $\check{T}=\check{T}(\mathrm{PGL}(n,\mathbf{C}))$ is the discrete approximation [12, §3] *to the maximal torus of* $\mathrm{PGL}(n+1,\mathbf{C})$ *and $\Sigma_{n+1}(V)$ is the pointwise stabilizer subgroup* (2.68).

Proof. Let $\nu^*\colon V\to T(\mathrm{GL}(n+1,\mathbf{C}))$ be a lift to $\mathrm{GL}(n+1,\mathbf{C})$ of the inclusion homomorphism of V into $T(\mathrm{PGL}(n+1,\mathbf{C}))$. Then

$$C_{\mathrm{GL}(n+1,\mathbf{C})}(\nu^*V)=\prod_{\varrho\in V^\vee}\mathrm{GL}(i_\varrho,\mathbf{C}),\qquad \Sigma_{n+1}(\nu^*V)=\prod_{\varrho\in V^\vee}\Sigma_{i_\varrho}$$

where $i\colon V^\vee\to\mathbf{Z}$ records the multiplicity of each linear character $\varrho\in V^\vee$ in the representation ν^*. Using [38, 5.11] and 5.20 below, we get

$$C_{\mathrm{PGL}(n+1,\mathbf{C})}(V)=\frac{C_{\mathrm{GL}(n+1,\mathbf{C})}(\nu^*V)}{\mathrm{GL}(1,\mathbf{C})}\rtimes V^\vee_{\nu^*},\qquad \Sigma_{n+1}(V)=\Sigma_{n+1}(\nu^*V)\rtimes V^\vee_{\nu^*}$$

where $V^\vee_{\nu^*}=\{\zeta\in V^\vee=\mathrm{Hom}(V,\mathrm{GL}(1,\mathbf{C}))\mid\forall\varrho\in V^\vee\colon i_{\zeta\varrho}=i_\varrho\}$. The semidirect products are obtained because the elements of $V^\vee_{\nu^*}$ can be realized by permutations from Σ_{n+1} that fix $V\subset\mathrm{PGL}(n+1,\mathbf{C})$ pointwise. The discrete approximation [12, §3] to the center of the centralizer is therefore

$$\begin{aligned}\check{Z}C_{\mathrm{PGL}(n+1,\mathbf{C})}(V)&=\check{Z}\Big(\frac{\prod\mathrm{GL}(i_\varrho,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}\rtimes V^\vee_{\nu^*}\Big)\overset{(5.14)}{=}\check{Z}\Big(\frac{\prod\mathrm{GL}(i_\varrho,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}\Big)^{V^\vee_{\nu^*}}\\&=\Big(\frac{\prod\check{Z}\mathrm{GL}(i_\varrho,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}\Big)^{V^\vee_{\nu^*}}\\&=\Big(\frac{\check{T}(\mathrm{GL}(n+1,\mathbf{C}))^{\Sigma_{n+1}(\nu^*V)}}{\mathrm{GL}(1,\mathbf{C})}\Big)^{V^\vee_{\nu^*}}\\&=(\check{T}(\mathrm{PGL}(n+1,\mathbf{C}))^{\Sigma_{n+1}(\nu^*V)})^{V^\vee_{\nu^*}}\\&=\check{T}(\mathrm{PGL}(n+1,\mathbf{C}))^{\Sigma_{n+1}(V)}\end{aligned}$$

where the penultimate equality sign is justified by the fact that

$$H^1(\Sigma_{n+1}(\nu^*V);\mathrm{GL}(1,\mathbf{C})) \to H^1(\Sigma_{n+1}(\nu^*V);\check{T}(\mathrm{GL}(n+1,\mathbf{C})))$$

is injective. ■

Define $\pi_i(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t}$ to be the subfunctor of $\pi_i(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})$ (2.47) that vanishes on all toral objects and is unchanged on all nontoral objects of the Quillen category. This means that

$$\text{(3.14)} \quad \pi_i(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t}(V,\nu) = \begin{cases} 0, & (V,\nu) \text{ is toral}, \\ \pi_i(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})(V,\nu), & (V,\nu) \text{ is nontoral}, \end{cases}$$

for all objects (V,ν) of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$. The reason for introducing this subfunctor is that in the computation of the higher limits, we can ignore the toral objects, as shown below.

3.15. Corollary. *When $n > 1$ and $i = 1, 2$,*

$$\lim{}^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}));\pi_i(BZC)_{\not\leq t}) \cong \lim{}^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}));\pi_i(BZC))$$

where $\pi_i(BZC) = \pi_i(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})$ (2.47).

Proof. The result of Lemma 3.13 is (2.31) equivalent to

$$\pi_i(BZC)(V) = H^{2-i}(\Sigma_{n+1}(V);L), \qquad V \subset P\Delta_{n+1},$$

where L is the $\mathbf{Z}_2\Sigma_{n+1}$-module $\pi_2 BT(\mathrm{PGL}(n+1,\mathbf{C}))$ and therefore (2.69)

$$\lim{}^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t},\pi_i(BZC)) = \begin{cases} H^{2-i}(\Sigma_{n+1},L), & j = 0, \\ 0, & j > 0 \end{cases}$$

where the cohomology groups $H^{2-i}(\Sigma_{n+1};L)$, $i = 1, 2$, are trivial for $n > 1$ (3.1).

Since the quotient functor $\pi_i(BZC)/\pi_i(BZC)_{\not\leq t}$ vanishes on all nontoral objects,

$$\lim{}^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})),\pi_i(BZC)/\pi_i(BZC)_{\not\leq t}) \overset{[38,\ 13.12]}{\cong} \lim{}^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t},\pi_i(BZC)).$$

We conclude that $\lim^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})),\pi_i(BZC)/\pi_i(BZC)_{\not\leq t}) = 0$. The long exact coefficient functor sequence for higher limits now shows that $\lim^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})),\pi_i(BZC)_{\not\leq t})$ and $\lim^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})),\pi_i(BZC))$ are isomorphic. ■

4. The category $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{[\ ,\ ]\neq 0}$. For any nontrivial elementary abelian 2-group V in $\mathrm{PGL}(n+1,\mathbf{C})$, let $[\ ,\ ]\colon V \times V \to \mathbf{F}_2$ be the symplectic bilinear form [23, II.9.1] given by $[u\mathbf{C}^\times, v\mathbf{C}^\times] = r$ if $[u,v] = (-E)^r$ where $u, v \in \mathrm{GL}(n+1,\mathbf{C})$ are such that $u\mathbf{C}^\times, v\mathbf{C}^\times \in V$. (The elements $[u,v]$ and u^2 lie in the center $\mathbf{C}^\times$ of $\mathrm{GL}(n+1,\mathbf{C})$ so that $E = [u^2, v] =$

$[u,v]^u[u,v] = [u,v]^2$ and thus $[u,v] \in \mathbf{C}^\times$ has order 2. Therefore $[u,v] = [u,v]^{-1} = [v,u]$.)

3.16. Lemma. *V in $\mathrm{PGL}(n+1,\mathbf{C})$ is toral $\Leftrightarrow [V,V]=0$.*

Proof. Let $e_i\mathbf{C}^\times$, $1 \le i \le d$, be a basis for V. Since $\mathbf{C}^\times$ is divisible, we can assume that each $e_i \in \mathrm{GL}(n+1,\mathbf{C})$ has order 2. If $[V,V]=0$, these e_is commute and span a lift to $\mathrm{GL}(n+1,\mathbf{C})$ of $V \subseteq \mathrm{PGL}(n+1,\mathbf{C})$. ∎

An extraspecial 2-group is of *positive type* if it is isomorphic to a central product of dihedral groups D_8 of order 8 [48, pp. 145–146].

3.17. Lemma ([18, 3.1], [38, 5.4]). *Let $\nu\colon V \to \mathrm{PGL}(n,\mathbf{C})$ be a nontoral monomorphism of a nontrivial elementary abelian 2-group V into $\mathrm{PGL}(n+1,\mathbf{C})$. Then there exists a morphism of short exact sequences of groups*

$$\begin{tikzcd} 1 \arrow[r] & Z(P) \arrow[r] \arrow[d, hook] & PE \arrow[r] \arrow[d, hook] & V \arrow[r] \arrow[d, hook, "\nu"] & 1 \\ 1 \arrow[r] & \mathbf{C}^\times \arrow[r] & \mathrm{GL}(n+1,\mathbf{C}) \arrow[r] & \mathrm{PGL}(n+1,\mathbf{C}) \arrow[r] & 1 \end{tikzcd}$$

where PE is the direct product of an extraspecial 2-group $P \subseteq \mathrm{GL}(n+1,\mathbf{C})$ of positive type and an elementary abelian 2-group $E \subseteq \mathrm{GL}(n+1,\mathbf{C})$ with $P \cap E = \{1\} = [P,E]$.

Let $G = \langle P, E, i\rangle = P \circ C_4 \times E$ be the group generated by E and the central product $P \circ C_4$ of P and the cyclic group $C_4 = \langle i \rangle \subseteq \mathbf{C}^\times$ with $C_2 = \langle -E \rangle$ amalgamated. The image of G in $\mathrm{PGL}(n+1,\mathbf{C})$ is V.

Let $\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(G)$ be the subgroup, isomorphic to $N_{\mathrm{GL}(n+1,\mathbf{C})}(G)/G \cdot C_{\mathrm{GL}(n+1,\mathbf{C})}(G)$, of $\mathrm{Out}(G)$ consisting of all outer automorphisms of G induced from conjugation in $\mathrm{GL}(n+1,\mathbf{C})$ [38, 5.8]. In other words,

$$\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(G) = \mathrm{Out}_{\mathrm{tr}}(G)$$

is the group of trace preserving outer automorphisms of G.

3.18. Lemma. *$\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(G) \to \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(V)$ is surjective.*

Proof. Suppose that $B \in \mathrm{GL}(n+1,\mathbf{C})$ is such that $V^{B\mathbf{C}^\times} = V$. Then $G^B \subseteq G \cdot \mathbf{C}^\times$: for any $g \in G$ there exist $h \in G$ and $z \in \mathbf{C}^\times$ such that $g^B = hz$. But since G has exponent 4, $z^4 = 1$ so $z \in C_4$ and $g^B \in G$. ∎

A monomorphic conjugacy class $\nu\colon V \to \mathrm{PGL}(n+1,\mathbf{C})$ is said to be a $(2d+r,r)$ *object* of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ if the underlying symplectic vector space of (V,ν) is isomorphic to $V = H^d \times V^\perp$ where H denotes the symplectic plane over $\mathbf{F}_2$ and $\dim_{\mathbf{F}_2} V^\perp = r$ [23, II.9.6] (so that $\dim_{\mathbf{F}_2} V = 2d+r$). An (r,r) object is the same thing as an r-dimensional toral object. We write

$\mathrm{Sp}(V)$ or $\mathrm{Sp}(2d + r, r)$ (abbreviated to $\mathrm{Sp}(2d)$ if $r = 0$) for the group of linear automorphisms of V that preserve the symplectic form.

3.19. COROLLARY. *Suppose that $n+1 = 2^d m$ for some natural numbers $d \geq 1$ and $m \geq 1$.*

(1) *There is up to isomorphism a unique $(2d, 0)$ object, H^d, of the category $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$, and*
$$\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(H^d) = \mathrm{Sp}(2d),$$
$$C_{\mathrm{PGL}(n+1,\mathbf{C})}(H^d) = H^d \times \mathrm{PGL}(m,\mathbf{C})$$
for this object.

(2) *For $r > 0$, isomorphism classes of $(2d+r, r)$ objects, V, of the category $\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))$ correspond bijectively to isomorphism classes of (r, r) objects, $V^\perp$, of $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))$, and*
$$\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))(V) = \begin{pmatrix} \mathrm{Sp}(2d) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V^\perp) \end{pmatrix},$$
$$C_{\mathrm{PGL}(2^d m,\mathbf{C})}(V) = V/V^\perp \times C_{\mathrm{PGL}(m,\mathbf{C})}(V^\perp)$$
for these objects.

Proof. (1) By [24, 7.5] the group $2_+^{1+2d} \circ 4$ has 2^{1+2d} characters of degree 1, and two irreducible characters of degree 2^d (interchanged by the action of $\mathrm{Out}(2_+^{1+2d} \circ 4) \cong \mathrm{Sp}(2d) \times \mathrm{Aut}(C_4)$ [17, pp. 403–404]), given by
$$\chi_\lambda(g) = \begin{cases} 2^d \lambda(g), & g \in C_4, \\ 0, & g \notin C_4, \end{cases}$$
where $\lambda\colon C_4 \to \mathbf{C}^\times$ is an injective group homomorphism ($\lambda(i) = \pm i$). The linear characters vanish on the derived group $2 = [2_+^{1+2d} \circ 4, 2_+^{1+2d} \circ 4]$ but the irreducible characters of degree 2^d do not. Thus the only faithful representations of $2_+^{1+2d} \circ 4$ with central centers are multiples $m\chi_\lambda$ of χ_λ for a fixed λ. Phrased slightly differently, $\mathrm{GL}(m2^d,\mathbf{C})$ contains up to conjugacy a unique subgroup with central center isomorphic to $2_+^{1+2d} \circ 4$. For this group and its image H^d in $\mathrm{PGL}(2^d m,\mathbf{C})$ we have
$$\mathbf{A}(\mathrm{GL}(m2^d,\mathbf{C}))(2_+^{1+2d} \circ 4, 2_+^{1+2d} \circ 4) \cong \mathrm{Sp}(2d) \cong \mathbf{A}(\mathrm{PGL}(m2^d,\mathbf{C}))(H^d, H^d),$$
$$C_{\mathrm{GL}(m2^d,\mathbf{C})}(2_+^{1+2d} \circ 4) \cong \mathrm{GL}(m,\mathbf{C}),$$
$$C_{\mathrm{PGL}(m2^d,\mathbf{C})}(H^d) \cong H^d \times \mathrm{PGL}(m,\mathbf{C}),$$
where the last isomorphism is a consequence of [38, 5.9].

(2) The $(2d + r, r)$ object (V, ν) of $\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))$ and the $(r, 0)$ object $(V^\perp, \nu^\perp)$ of $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))$ correspond to each other iff there is an m-dimensional representation $\mu\colon V^\perp \to \mathrm{GL}(m,\mathbf{C})$ such that $\mathbf{C}^{2^d} \otimes \mu$ is a

lift of $\nu|V^\perp$ and μ a lift of $\nu^\perp$. According to 3.17 any lift of $\nu|V^\perp$ has this form for some μ uniquely determined up to the action of $(V^\perp)^\vee$.

We use 3.18 to calculate the Quillen automorphism group of a $(2d+r,r)$ object $H^d \times V^\perp$ of $\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))$. Let $H^d \times V^\perp$ be covered by the group $P \circ C_4 \times V^\perp$ as in 3.17. Let α be an automorphism of $P \circ C_4$, let β be any homomorphism of the form $P \circ C_4 \to H^d \to V^\perp$, and let γ be any Quillen automorphism of $(V^\perp,\nu^\perp)$. Choose a homomorphism $\zeta_1\colon P\circ C_4 \to H^d \times C_4/C_2 \to C_4$ such that $\lambda(\zeta_1(x)\alpha(x)) = \lambda(x)$ for all x in C_4 and a homomorphism $\zeta_2\colon V^\perp \to C_4$ such that $\lambda(\zeta_2(v))\mu(\gamma(v)) = \mu(v)$ for all $v \in V^\perp$. Then the automorphism of $P \circ C_4$ that takes (x,v) to $(\zeta_1(x)\zeta_2(v)\alpha(x), \beta(x)+\gamma(v))$ preserves the trace of $\chi_\lambda \# \mu$ and therefore the automorphism induced on the quotient is a Quillen automorphism of $H^d \times V^\perp$. Conversely, any automorphism of $P \circ C_4 \times V^\perp$ takes the center $C_4 \times V^\perp$ isomorphically to itself and hence it is of the form $(x,v) \mapsto (\zeta(x,v)\alpha(x), \beta(x)+\gamma(v))$ for some automorphism α of $P \circ C_4$, some homomorphism $\beta\colon P\circ C_4 \to V^\perp$ vanishing on C_4, and some homomorphism $\zeta\colon P \circ C_4 \times V^\perp \to C_4$. Such an automorphism preserves the trace of $\chi_\lambda \# \mu$ iff $\lambda(\zeta(x,v)\alpha(x)) = \mu(\gamma(v))$ for all $(x,v) \in Z(P\circ C_4 \times V^\perp) = C_4 \times V^\perp$. But this means that the induced automorphism of $H^d \times V^\perp$ is of the stated form. ∎

We conclude that the nontoral objects of $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))$ of rank ≤ 4 are

- one $(2,0)$ object H, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(H) = \mathrm{Sp}(2)$,
- $P(m,2)$ $(3,1)$ objects V, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(V) = \mathrm{Sp}(3,1)$,
- $P(m,3)+P(m,4)$ $(4,2)$ objects E,
$$\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(E) = \begin{pmatrix} \mathrm{Sp}(2) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)\end{pmatrix},$$
where $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp) = 1$, C_2, or $\mathrm{GL}(E^\perp)$,
- one $(4,0)$ object H^2 if m is even, $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(H^2) = \mathrm{Sp}(4)$.

This information will be needed in the next section as input for Oliver's cochain complex [46] for computing higher limits.

5. Higher limits of the functors $\pi_i(BZC)$. We compute the higher limits from 2.48(2) and 2.51(4) by means of Corollary 3.15 and Oliver's cochain complex [46].

3.20. Lemma. *The first higher limits of the functors $\pi_i(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})$, $i=1,2$, are:*

(1) $\lim^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})), \pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})) = 0$ *for* $j=1,2$,

(2) $\lim^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})), \pi_2(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})) = 0$ *for* $j=2,3$,

for all $n \ge 1$.

For any elementary abelian 2-group E in $\mathrm{PGL}(n+1,\mathbf{C})$ we shall write

$$[E] = \mathrm{Hom}_{\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E)}(\mathrm{St}(E), \pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})}(E)))$$

for the $\mathbf{F}_2$-vector space of $\mathbf{F}_2\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E)$-module homomorphisms from the Steinberg representation $\mathrm{St}(E)$ over $\mathbf{F}_2$ of $\mathrm{GL}(E)$ to the value, $\pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})(E)$, of the functor $\pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})$ at E.

Oliver's cochain complex for computing the first limits of the functor $\pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t}$ (3.14, 3.15) has the form

$$\text{(3.21)} \qquad 0 \to [H] \xrightarrow{d^1} \prod_{1\le i\le [m/2]} [H \# L[m-i,i]]$$
$$\xrightarrow{d^2} [H \# P[1,1,m-2]] \times \prod_{2<i<[m/2]} [H \# P[1,i-1,m-i]]$$

where we only list some of the nontoral rank four objects. Here,

$$\begin{aligned}
&[H] = \mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H) \cong \mathbf{F}_2,\\
&[H \# L[m-i,i]] = \mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V), V) \cong \mathbf{F}_2, \qquad V = H \# L[m-i,i],\\
&[H \# P[1,1,m-2]] = \mathrm{Hom}_{\left(\begin{smallmatrix}\mathrm{Sp}(2) & 0\\ * & C_2\end{smallmatrix}\right)}(\mathrm{St}(E_2), E_2/E_2^{\perp}) \cong \mathbf{F}_2\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{where } E_2 = H \# P[1,1,m-2],\\
&[H \# P[1,i-1,m-i]] = \mathrm{Hom}_{\left(\begin{smallmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{smallmatrix}\right)}(\mathrm{St}(E_i), E_i) \cong \mathbf{F}_2 \times \mathbf{F}_2\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{where } E_i = H \# P[1,i-1,m-i]
\end{aligned}$$

with $2 < i \le [m/2]$ in the last line. The dimensions of these spaces were found using the computer algebra program *magma*. It suffices to show that the first differential d^1 is injective and that the second differential d^2 has rank $[m/2]-1$.

Let $H = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2$ be a 2-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2\}$ and symplectic inner product matrix

$$\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

Let $\mathbf{F}_2[1]$ be the 3-dimensional $\mathbf{F}_2$-vector space on all length zero flags $[L]$ of nontrivial proper subspaces $L \subset H$. The Steinberg module $\mathrm{St}(H)$ for H is the 2-dimensional $\mathbf{F}_2\mathrm{GL}(H)$-module that is the kernel for the augmentation $d\colon \mathbf{F}_2[1] \to \mathbf{F}_2$ given by $d[L] = 1$ for all L. Let $f\colon \mathrm{St}(H) \to H$ be the restriction to $\mathrm{St}(H)$ of the $\mathbf{F}_2\mathrm{GL}(H)$-module homomorphism $\overline{f}\colon \mathbf{F}_2[1] \to H$ given by $\overline{f}[L] = L$.

Let $V = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2 + \mathbf{F}_2 e_3$ be a 3-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2, e_3\}$ and (degenerate) symplectic inner product matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{F}_2[1]$ be the 21-dimensional $\mathbf{F}_2$-vector space on all length one flags $[P > L]$ and $\mathbf{F}_2[0]$ the 14-dimensional $\mathbf{F}_2$-vector space on all length zero flags, $[P]$ or $[L]$, of nontrivial and proper subspaces of V. The Steinberg module $\mathrm{St}(V)$ over $\mathbf{F}_2$ for V is the $2^3 = 8$-dimensional kernel of the linear map $d\colon \mathbf{F}_2[1] \to \mathbf{F}_2[0]$ given by $d[P > L] = [P] + [L]$. Define $df\colon \mathrm{St}(V) \to V$ to be the restriction to $\mathrm{St}(V)$ of the linear map $\overline{df}\colon \mathbf{F}_2[1] \to V$ given by

$$\overline{df}[P > L] = \begin{cases} L, & P \cap P^{\perp} = \{0\}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.22}$$

on the basis vectors.

Let $E = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2 + \mathbf{F}_2 e_3 + \mathbf{F}_2 e_4$ be a 4-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2, e_3, e_4\}$ and (degenerate) symplectic inner product matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{F}_2[2]$ be the 315-dimensional $\mathbf{F}_2$-vector space on all length two flags $[V > P > L]$ and $\mathbf{F}_2[1]$ the also 315-dimensional $\mathbf{F}_2$-vector space on all length one flags, $[P > L]$ or $[V > L]$ or $[V > P]$, of nontrivial, proper subspaces of E. The Steinberg module $\mathrm{St}(E)$ over $\mathbf{F}_2$ for E is the $2^6 = 64$-dimensional kernel of the linear map $d\colon \mathbf{F}_2[2] \to \mathbf{F}_2[1]$ given by $d[V > P > L] = [P > L] + [V > L] + [V > P]$. Define $F_1 = \overline{F}_1 | \mathrm{St}(E)\colon \mathrm{St}(E) \to E$ as the restriction to $\mathrm{St}(E)$ of the linear map $\overline{F}_1\colon \mathbf{F}_2[2] \to E$ with values

$$\overline{F}_1[V > P > L] = \begin{cases} L, & P \cap P^{\perp} = 0, V \cap V^{\perp} = \mathbf{F}_2 e_3, \\ 0, & \text{otherwise}, \end{cases} \tag{3.23}$$

on the basis elements. Define $F_2 = \overline{F}_2 | \mathrm{St}(E)\colon \mathrm{St}(E) \to E$ similarly but replace the condition $V \cap V^{\perp} = \mathbf{F}_2 e_3$ by $V \cap V^{\perp} = \mathbf{F}_2 e_4$. The linear maps F_1 and F_2 are $\left(\begin{smallmatrix} \mathrm{Sp}(2) & 0 \\ * & 1 \end{smallmatrix}\right)$-equivariant because this group preserves the symplectic inner product on E and preserves $V^{\perp} = \mathbf{F}_2\langle e_3, e_4 \rangle$ pointwise.

3.24. Lemma. *Let f and F_1, F_2 be the linear maps defined above.*

(1) *The vector f is a basis for $[H]$.*
(2) *The vector df is a basis for $[H \# L[m-i, i]]$, $1 \le i \le [m/2]$.*
(3) *The vector F_2 is a basis for $[H \# P[1, 1, m-2]]$.*

(4) *The set* $\{F_1, F_2\}$ *is a basis for* $[H \# P[1, i-1, m-i]]$, $2 < i \leq [m/2]$. *The sum* $F_1 + F_2$ *is the linear map defined as in* (3.23) *but with the condition* $V \cap V^\perp = \mathbf{F}_2 e_3$ *replaced by* $V \cap V^\perp = \mathbf{F}_2(e_3 + e_4)$.

Proof. This can be directly verified by machine computation. ∎

Proof of Lemma 3.20. Since we already know that these higher limits vanish when $n+1$ is odd (3.3, 3.15) we can assume that $n+1 = 2m$ is even.

(1) See Proposition 4.1 for the case $m = 1$ and assume now that $m \geq 2$. The image in $[H \# L[m-i, i]]$ of $f \in [H]$ is

$$df_{L[m-i,i]}[P > L] = \begin{cases} L, & P = H, \\ 0, & \text{otherwise}, \end{cases}$$

which equals df (3.22). For $1 < i \leq [m/2]$, let

$$ddf_{L[m-i,i]}[V > P > L] = \begin{cases} L, & V = H \# L[m-i, i],\ P = H, \\ 0, & \text{otherwise}, \end{cases}$$

The object $H \# P[1, 1, m-2]$ receives morphisms from $H \# L[m-1, 1]$ and (when $m > 2$) $H \# L[m-2, 2]$. Using a computer program one easily checks that $ddf_{L[m-1,1]} = F_2 = ddf_{L[m-2,2]}$ in $[H \# P[1, 1, m-2]]$. The object $H \# P[1, i-1, m-i]$ receives morphisms from $H \# L[m-1, 1]$, $H \# L[m-i+1, i-1]$, and $H \# L[m-i, i]$. Using a computer program one easily checks that $ddf_{L[m-i+1,i-1]} = F_1$, $ddf_{L[m-i,i]} = F_1$, and $ddf_{L[m-1,1]} = F_1 + F_2$ in $[H \# P[1, i-1, m-i]]$. For $m = 2$ or $m = 3$, the cochain complexes (3.21) take the form

$$0 \to [H] \xrightarrow{d^1} [H \# L[1,1]] \xrightarrow{d^2} 0, \quad 0 \to [H] \xrightarrow{d^1} [H \# L[2,1]] \xrightarrow{d^2} [H \# P[1,1,1]]$$

where d^1 is an isomorphism. For $m \geq 4$, and with our choice of basis (3.24), the matrix for the differential d^1 is the injective $(1 \times [m/2])$-matrix

$$(1 \quad 1 \quad \cdots \quad 1)$$

and the matrix for d^2 (or rather, the components of d^2 shown in 3.21) is the $([m/2] \times (2[m/2] - 3))$-matrix

	$[H \# P[1,1,8]]$	$[H \# P[1,2,7]]$	$[H \# P[1,3,6]]$	$[H \# P[1,4,5]]$
$[H \# L[9,1]]$	(1)	$(1 \quad 1)$	$(1 \quad 1)$	$(1 \quad 1)$
$[H \# L[8,2]]$	(1)	$(1 \quad 0)$		
$[H \# L[7,3]]$		$(0 \quad 1)$	$(1 \quad 0)$	
$[H \# L[6,4]]$			$(0 \quad 1)$	$(1 \quad 0)$
$[H \# L[5,5]]$				$(0 \quad 1)$

(shown here for $m = 10$) of rank $[m/2] - 1$.

(2) Oliver's cochain complex for computing these higher limits over the category $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))$ involves the $\mathbf{Z}_2$-modules (3.19(2))

$$\mathrm{Hom}_{\left(\begin{smallmatrix}\mathrm{Sp}(2) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)\end{smallmatrix}\right)}(\mathrm{St}(E), \pi_2(BZC_{\mathrm{PGL}(2m,\mathbf{C})}(E^\perp))), \qquad \dim_{\mathbf{F}_2} E = 3, 4,$$

that are submodules of finite products of $\mathbf{Z}_2$-modules of the form

$$\mathrm{Hom}_{\left(\begin{smallmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{smallmatrix}\right)}(\mathrm{St}(E), \mathbf{Z}_2), \qquad \dim_{\mathbf{F}_2} E = 3, 4,$$

where the action on $\mathbf{Z}_2$ is trivial. According to the computer program *magma*, these latter modules are trivial. ∎

4. PROOFS OF THE MAIN RESULTS OF PART I

This chapter contains the proofs of Theorem 1.4 and Corollary 1.5.

We show that $\mathrm{PGL}(n+1,\mathbf{C})$ is uniquely N-determined by induction on n. The start of the induction is provided by the following proposition.

4.1. Proposition. $\mathrm{PGL}(2,\mathbf{C})$ *is uniquely N-determined.*

Proof. The centralizer cofunctor $C_{\mathrm{PGL}(2,\mathbf{C})}$ takes the Quillen category of $\mathrm{PGL}(2,\mathbf{C})$, consisting (3.11, 3.19) of one toral line, L, and one nontoral plane, H,

$$L \to H \circlearrowleft \mathrm{GL}(H) \tag{4.2}$$

to the diagram

$$\mathrm{GL}(1,\mathbf{C})^2/\mathrm{GL}(1,\mathbf{C}) \rtimes C_2 \leftarrow H \circlearrowleft \mathrm{GL}(H)^{\mathrm{op}} \tag{4.3}$$

of uniquely N-determined 2-compact groups. The 2-compact toral group to the left is uniquely N-determined because (2.41) $H^1(C_2;\mathbf{Z}/2^\infty) = 0$ for the nontrivial action of C_2 on $\mathbf{Z}/2^\infty$. The center cofunctor takes this diagram back to the starting point (4.2) for which the higher limits vanish (2.69). $\mathrm{PGL}(2,\mathbf{C})$ is thus uniquely N-determined by 2.48 and 2.51. ∎

4.4. Lemma. *Suppose that* $\mathrm{PGL}(r+1,\mathbf{C})$ *is uniquely N-determined for all* $0 \le r < n$. *Then* $\mathrm{PGL}(n+1,\mathbf{C})$, $n \ge 1$, *satisfies conditions* 2.48(1) (*for $\pi_*(N)$-determined automorphisms*), 2.51(1), 2.51(2), *and* 2.51(3).

Proof. Condition (1) of 2.48 (for $\pi_*(N)$-determined automorphisms) is concerned with centralizers $C_{\mathrm{PGL}(n+1,\mathbf{C})}(L,\lambda)$ of rank one objects (3.5). The condition is satisfied for all connected rank one centralizers by the induction hypothesis and 2.42, 2.39. The condition is satisfied for the nonconnected rank one centralizer (when $n+1$ is even) by 2.35 since $H(C_2;\mathbf{Z}/2^\infty) = 0$ for the nontrivial action of the cyclic group C_2 of order two on $\mathbf{Z}/2^\infty$.

We use 2.54 to verify conditions (1) and (2) of 2.51. Let (V,ν) be a toral elementary abelian 2-subgroup of $\mathrm{PGL}(n+1,\mathbf{C})$ of rank ≤ 2 and $C(\nu) =$

$C_{\mathrm{PGL}(n+1,\mathbf{C})}(\nu)$ its centralizer. We have seen that $C(\nu)$ is LHS (Chapter 3, §2) and that $\check{Z}(C(\nu)_0) = \check{Z}(N_0(C(\nu)))$ as $C(\nu)_0$ does not contain a direct factor isomorphic to $\mathrm{GL}(2,\mathbf{C})/\mathrm{GL}(1,\mathbf{C}) = \mathrm{SO}(3)$ (2.32, (3.5)). The identity component $C(\nu)_0$ has $\pi_*(N)$-determined automorphisms according to 2.38 and 2.39, and $C(\nu)$ has N-determined automorphisms by 2.35. The identity component $C(\nu)_0$ is N-determined according to 2.42 and 2.43, and $C(\nu)$ is N-determined by 2.40. Thus $C(\nu)$ is LHS and totally N-determined.

The functor $H^1(W/W_0; \check{T}_0^W)$ is zero on $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}_{\leq 2}$ except on the object $(V,\nu) = (i_0, i_0, i_0, i_0)$, when $n+1 = 4i_0$, where it has value $\mathbf{Z}/2$. However, this object has Quillen automorphism group $\mathrm{GL}(V)$ and since the only $\mathrm{GL}(V)$-equivariant homomorphism $\mathrm{St}(V) = V \to \mathbf{Z}/2$ is the trivial homomorphism, $\lim^1(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}_{\leq 2}; H^1(W/W_0; \check{T}_0^W)) = 0$ follows from Oliver's cochain complex [46].

We now turn to condition (3) of 2.51. When $n+1$ is odd there are no nontoral rank two objects (3.3) and so there is nothing to prove. When $n+1 = 2m$ is even, let (H,ν) be the unique nontoral rank two object of $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))$ (3.19(1)). Let X' be a connected 2-compact group with maximal torus normalizer $j' \colon N(\mathrm{PGL}(2m,\mathbf{C})) \to X'$. We must show that ν'_L and $f_{\nu,L} \colon C_{\mathrm{PGL}(2m,\mathbf{C})}(H,\nu) \to C_{X'}(H,\nu'_L)$ as defined in 2.51(3) are independent of the choice of rank one subgroup $L \subset V$. When $m=1$, the claim follows from 2.59, 2.60, (2.62) (where $\overline{\nu}(V)$ and $\overline{\nu}'(V)$ are isomorphisms in this case) since $\mathrm{PGL}(2,\mathbf{C})$ does contain a unique rank one elementary abelian 2-group with nonconnected centralizer (3.11) and a unique nontoral rank two elementary abelian 2-group (3.19(1)). When $m > 1$, we use 2.63 which immediately implies that ν'_L is independent of the choice of $L < V$. There exists a torus $T_\nu \to C_N(V, \nu^N_L)$ as in 2.63(2) because the three preferred lifts ν^N_L, $L < V$, differ by an automorphism of H (the Quillen automorphism group of (H,ν) is the full automorphism group $\mathrm{Aut}(H)$ of H (3.19(1)). Since the identity component of $C_{\mathrm{PGL}(n+1,\mathbf{C})}(H,\nu)$ is uniquely N-determined by induction hypothesis, the restriction $(f_{\nu,L})_0$ of $f_{\nu,L}$ to the identity components is independent of the choice of L (2.14(2)). Also $\pi_0(f_{\nu,L})$ is independent of the choice of $L < V$ by (2.62) (where $\pi_0(\overline{\nu}(V))$ and $\pi_0(\overline{\nu}'(V))$ are isomorphisms). But since $\mathrm{PGL}(m,\mathbf{C})$ is centerfree, $f_{\nu,L}$ is in fact determined (use one half of [33, 5.2]) by $(f_{\nu,L})_0$ and $\pi_0(f_{\nu,L})$. We conclude that $f_{\nu,L}$ is independent of the choice of $L < V$. ∎

Proof of Theorem 1.4. The proof is by induction on $n \geq 1$. The start of the induction is provided by 4.1. The induction step is provided by 4.4 and 3.20 using 2.48 and 2.51.

According to 2.16, the automorphism group

$$\mathrm{Aut}(\mathrm{PGL}(n+1,\mathbf{C})) = W \backslash N_{\mathrm{GL}(L)}(W) = W \backslash \langle \mathbf{Z}_2^\times, W \rangle = Z(W) \backslash \mathbf{Z}_2^\times$$

is isomorphic to $\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ for $n=1$ and to $\mathbf{Z}_2^\times$ for $n>1$. Here we use [30] or the exact sequence (2.8) where we note that $\mathrm{Out}_{\mathrm{tr}}(W)$, is trivial for all $n\geq 1$; $\mathrm{Out}(\Sigma_{n+1})$ is trivial for all $n\neq 5$ [23, II.5.5] and the nontrivial outer automorphism of Σ_6 does not preserve trace. ∎

Proof of Corollary 1.5. Let $X=\mathrm{GL}(n,\mathbf{C})$, $n\geq 1$, and write $\check{T}$, W and L for $\check{T}(X)$, $W(X)$, and $L(X)$. Since the adjoint form $PX=\mathrm{PGL}(n,\mathbf{C})$ of X is uniquely N-determined (1.4), so is X (2.38, 2.42). The extension class $e(X)\in H^2(W;\check{T})$ (2.5) is the zero class since the maximal torus normalizer $N(X)=\mathrm{GL}(1,\mathbf{C})\wr\Sigma_n$ splits. Therefore, $\mathrm{Aut}(X)$ is isomorphic to $W\backslash N_{\mathrm{GL}(L)}(W)$ (2.16). Using the exact sequence (2.8), we conclude, as in the proof of Theorem 1.4, that $\mathrm{Aut}(X)\cong Z(W)\backslash\mathrm{Aut}_{\mathbf{Z}_2W}(L)=Z(W)\backslash\mathrm{Aut}_{\mathbf{Z}_2\Sigma_n}(\mathbf{Z}_2^n)$. ∎

5. MISCELLANEOUS

This chapter contains standard facts used at various places in Parts I and II of this paper.

1. Real representation theory. Real representations are semisimple and determined by their characters [24, 2.11, 3.12(c)]. Any simple real representation arises from a simple complex representation in the following way: Let χ be the character of a simple complex representation of a finite group G. Then [24, 13.1, 13.11, 13.12]:

- $\chi\neq\overline{\chi}$, $\varepsilon_2(\chi)=0$: $\psi=\chi+\overline{\chi}$ is the character of a simple $\mathbf{R}$-module of complex type,
- $\chi=\overline{\chi}$, $\varepsilon_2(\chi)=+1$: χ is the character of a simple $\mathbf{R}$-module of real type,
- $\chi=\overline{\chi}$, $\varepsilon_2(\chi)=-1$: $\psi=2\chi$ is the character of a simple $\mathbf{R}$-module of quaternion type,

where $\varepsilon_2(\chi)=\frac{1}{|G|}\sum_{g\in G}\chi(g^2)$.

5.1. Example. (1) The character table of the cyclic group C_4 of order 4

C_4	ε_2	1	-1	i	$-i$
χ_1	$+$	1	1	1	1
χ_2	$+$	1	1	-1	-1
χ_3	0	1	-1	i	$-i$
χ_4	0	1	-1	$-i$	i

shows that there are two 1-dimensional real representations and one 2-dimensional simple real faithful representation of complex type with character $\psi=\chi_3+\chi_4=(2,-2,0,0)$.

(2) The character table of the dihedral group $D_8 = 2_+^{1+2}$

D_8	ε_2	1	-1	R_1	R_2	i
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	-1	1	-1
χ_3	$+$	1	1	1	-1	-1
χ_4	$+$	1	1	-1	-1	1
χ_5	$+$	2	-2	0	0	0

shows that there are four 1-dimensional real representations and one 2-dimensional simple real faithful representation of real type with $\chi_5 = (2, -2, 0, 0, 0)$ as its character.

(3) The character table of the quaternion group $Q_8 = 2_-^{1+2}$ (identical to the one for D_8 except for one value of ε_2)

Q_8	ε_2	1	-1	k	j	i
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	-1	1	-1
χ_3	$+$	1	1	1	-1	-1
χ_4	$+$	1	1	-1	-1	1
χ_5	$-$	2	-2	0	0	0

shows that there are four 1-dimensional real representations and one 4-dimensional simple real faithful representation of quaternion type with $\psi = 2\chi_5 = (4, -4, 0, 0, 0)$ as its character.

We are interested in real oriented representations, i.e. homomorphisms of finite groups into the special linear group $\mathrm{SL}(2n, \mathbf{R})$ (as opposed to homomorphisms into the general linear group $\mathrm{GL}(2n, \mathbf{R})$). The outer automorphism of $\mathrm{SL}(2n, \mathbf{R})$ is conjugation by any orientation reversing matrix such as $D = \mathrm{diag}(-1, 1, \ldots, 1)$.

5.2. Lemma. *Let $V \subset \mathrm{PSL}(2n, \mathbf{R})$ be an object of $\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R}))$ and $G = V^* \subset \mathrm{SL}(2n, \mathbf{R})$ its inverse image in $\mathrm{SL}(2n, \mathbf{R})$. Then*

$$V \text{ and } V^D \text{ are nonisomorphic objects of } \mathbf{A}(\mathrm{PSL}(2n, \mathbf{R})) \Leftrightarrow N_{\mathrm{GL}(2n,\mathbf{R})}(G) \subset \mathrm{SL}(2n, \mathbf{R}).$$

Proof. We note that

$$\begin{aligned} V, V^D &\text{ are isomorphic objects of } \mathbf{A}(\mathrm{PSL}(2n, \mathbf{R})) \\ &\Leftrightarrow G, G^D \text{ are conjugate subgroups of } \mathrm{SL}(2n, \mathbf{R}) \\ &\Leftrightarrow G \in G^{D\mathrm{SL}(2n,\mathbf{R})} \\ &\Leftrightarrow N_{\mathrm{GL}(2n,\mathbf{R})}(G) \cap D\mathrm{SL}(2n, \mathbf{R}) \neq \emptyset \\ &\Leftrightarrow N_{\mathrm{GL}(2n,\mathbf{R})}(G) \not\subset \mathrm{SL}(2n, \mathbf{R}) \end{aligned}$$

for any nontrivial elementary abelian 2-group $V \subset \mathrm{PSL}(2n, \mathbf{R})$. ∎

For instance, all representations of elementary abelian p-groups are conjugate in $\mathrm{SL}(2n,\mathbf{R})$ if and only if they are conjugate in $\mathrm{GL}(2n,\mathbf{R})$.

Let $\mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(G)$ be the subgroup of $\mathrm{Out}(G)$ consisting of all outer automorphisms of G induced by conjugation with some element of $\mathrm{GL}(2n,\mathbf{R})$ [38, 5.8] (i.e. $\mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(G)$ is the group $\mathrm{Out}_{\mathrm{tr}}(G)$ of all trace preserving outer automorphisms of G) and $\mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))(G)$ the subgroup of $\mathrm{Out}(G)$ consisting of all outer automorphisms of G induced by conjugation with some element of $\mathrm{SL}(2n,\mathbf{R})$. Since

$$N_{\mathrm{GL}(2n,\mathbf{R})}(G)/GC_{\mathrm{GL}(2n,\mathbf{R})}(G) \xrightarrow{\cong} \mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(G) \tag{5.3}$$

we conclude from 5.2 that

$$\begin{aligned} &G, G^D \text{ are nonconjugate subgroups of } \mathrm{SL}(2n,\mathbf{R}) \\ &\qquad \Leftrightarrow\ N_{\mathrm{GL}(2n,\mathbf{R})}(G) \subset \mathrm{SL}(2n,\mathbf{R}) \\ &\qquad \Leftrightarrow\ \begin{cases} C_{\mathrm{GL}(2n,\mathbf{R})}(G) \subset \mathrm{SL}(2n,\mathbf{R}), \\ \mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))(G) = \mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(G). \end{cases} \end{aligned}$$

Let V and E be objects of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ such that $\dim V + 1 = \dim E$. If there are morphisms $V \to E$ and $V^D \to E$, then (some representative of) $E = \langle V, V^D\rangle$ is generated by the images of (some representatives of) V and V^D so that $E = E^D$. Conversely, if $E = E^D$ and there is a morphism $V \to E$ then there is also a morphism $V^D \to E^D = E$.

We have $2(\phi^D) = 2\phi \in \mathrm{Rep}(G, \mathrm{SL}(4n,\mathbf{R}))$ for any oriented real degree $2n$ representation $\phi \in \mathrm{Rep}(G, \mathrm{SL}(2n,\mathbf{R}))$ as the conjugating matrix $2D$ is orientation preserving.

5.4. Example. (1) Let $G \subset \mathrm{SL}(2d,\mathbf{R})$ be a finite group making $\mathbf{R}^{2d}$ a simple $\mathbf{R}G$-module of complex type. Consider the image of $G \subset \mathrm{SL}(2nd,\mathbf{R})$ of G under the n-fold diagonal $\mathrm{SL}(2d,\mathbf{R}) \xrightarrow{\Delta_n} \mathrm{SL}(2dn,\mathbf{R})$. The centralizer $C_{\mathrm{GL}(2nd,\mathbf{R})}(G) = \mathrm{GL}(n,\mathbf{C})$ is connected, hence contained in $\mathrm{SL}(2dn,\mathbf{R})$. Since $C_{\mathrm{GL}(2d,\mathbf{R})}(G) = \mathrm{GL}(1,\mathbf{C})$, the elements of G commute with $i \in \mathrm{GL}(1,\mathbf{C})$ and we may factor the inclusion of G into $\mathrm{SL}(2dn,\mathbf{R})$ as

$$G \to C_{\mathrm{GL}(2d,\mathbf{R})}(i) = \mathrm{GL}(d,\mathbf{C}) \xrightarrow{\Delta_n} \mathrm{GL}(dn,\mathbf{C}) \to \mathrm{SL}(2dn,\mathbf{R}).$$

Let χ be the character for G in $\mathrm{GL}(d,\mathbf{C})$ so that the character for G in $\mathrm{GL}(2d,\mathbf{R})$ is $\chi + \overline{\chi}$. There are inclusions

$$\begin{tikzcd} \mathbf{A}(\mathrm{GL}(2dn,\mathbf{R}))(G) \arrow[r, equal] & \mathbf{A}(\mathrm{GL}(2d,\mathbf{R}))(G) \arrow[r, equal] & \mathrm{Out}_{\chi+\overline{\chi}}(G) \\ \mathbf{A}(\mathrm{SL}(2dn,\mathbf{R}))(G) \arrow[u, hook] & \mathbf{A}(\mathrm{GL}(d,\mathbf{C}))(G) \arrow[l, hook'] \arrow[u, hook] \arrow[r, equal] & \mathrm{Out}_{\chi}(G) \arrow[u, hook] \end{tikzcd}$$

where $\mathrm{Out}_\phi(G)$ is the group of all outer automorphisms that respect the function ϕ.

(2) Let $G \subset \mathrm{GL}(d,\mathbf{R})$ be a finite group making $\mathbf{R}^d$ a simple $\mathbf{R}G$-module of real type. Consider the image of $G \subset \mathrm{SL}(2nd,\mathbf{R})$ under the $2n$-fold diagonal $\mathrm{GL}(d,\mathbf{R}) \xrightarrow{\Delta_{2n}} \mathrm{SL}(2dn,\mathbf{R})$. The centralizer $C_{\mathrm{GL}(2nd,\mathbf{R})}(G) = \mathrm{GL}(2n,\mathbf{R})$ is contained in $\mathrm{SL}(2dn,\mathbf{R})$ when d is even. We may factor the inclusion of G into $\mathrm{SL}(2dn,\mathbf{R})$ as

$$G \to \mathrm{GL}(d,\mathbf{R}) \to \mathrm{GL}(d,\mathbf{C}) \xrightarrow{\Delta_n} \mathrm{GL}(nd,\mathbf{C}) \to \mathrm{SL}(2nd,\mathbf{R}),$$

and as the trace functions for G in $\mathrm{GL}(d,\mathbf{C})$ and $\mathrm{GL}(2nd,\mathbf{R})$ are proportional, $\mathbf{A}(\mathrm{GL}(2nd,\mathbf{R}))(G) = \mathbf{A}(\mathrm{GL}(d,\mathbf{C}))(G) \subset \mathbf{A}(\mathrm{SL}(2dn,\mathbf{R}))(G)$. Hence $G \neq G^D$ in $\mathrm{SL}(2nd,\mathbf{R})$.

(3) Let $G \subset \mathrm{SL}(4d,\mathbf{R})$ be a finite group making $\mathbf{R}^{4d}$ a simple $\mathbf{R}G$-module of quaternion type. Consider the image of $G \subset \mathrm{SL}(4nd,\mathbf{R})$ under the n-fold diagonal $\mathrm{SL}(4d,\mathbf{R}) \xrightarrow{\Delta_n} \mathrm{SL}(4dn,\mathbf{R})$. The centralizer $C_{\mathrm{GL}(4dn,\mathbf{R})}(G) = \mathrm{GL}(n,\mathbf{H})$ is connected so it is contained in $\mathrm{SL}(4dn,\mathbf{R})$. Since $C_{\mathrm{GL}(4d,\mathbf{R})}(G) = \mathrm{GL}(1,\mathbf{H}) \subset \mathrm{GL}(2,\mathbf{C})$ the elements of G commute with $i \in \mathrm{GL}(2,\mathbf{C})$ and we may factor the inclusion of G into $\mathrm{SL}(4dn,\mathbf{R})$ as

$$G \to C_{\mathrm{GL}(4d,\mathbf{R})}(i) = \mathrm{GL}(2d,\mathbf{C}) \xrightarrow{\Delta_n} \mathrm{GL}(2nd,\mathbf{C}) \to \mathrm{SL}(4nd,\mathbf{R}),$$

and as the trace functions for G in $\mathrm{GL}(2d,\mathbf{C})$ and $\mathrm{GL}(4nd,\mathbf{R})$ are proportional, $\mathbf{A}(\mathrm{GL}(4nd,\mathbf{R}))(G) = \mathbf{A}(\mathrm{GL}(2d,\mathbf{C}))(G) \subset \mathbf{A}(\mathrm{SL}(4dn,\mathbf{R}))(G)$. Hence $G \neq G^D$ in $\mathrm{SL}(4nd,\mathbf{R})$.

(4) $\mathbf{R}^2$ is a simple $\mathbf{R}C_4$-module of complex type with respect to the group

$$C_4 = \langle I \rangle \subset \mathrm{SL}(2,\mathbf{R}), \qquad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Consider the image of C_4 in $\mathrm{SL}(2n,\mathbf{R})$ under the n-fold diagonal. The Quillen automorphism group $\mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(C_4) = \mathbf{A}(\mathrm{GL}(2,\mathbf{R}))(C_4) = \mathrm{Out}(C_4)$ since the trace lives on $\mho_1(C_4) = \langle -E \rangle$ only. However,

$$\mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))(C_4) = \begin{cases} \mathrm{Out}(C_4), & n \text{ even}, \\ \{1\}, & n \text{ odd}, \end{cases}$$

so that $C_4 \neq C_4^D \Leftrightarrow n$ even.

(5) $\mathbf{R}^4$ is a simple $\mathbf{R}G_{16}$-module of complex type with respect to the group

$$G_{16} = 4 \circ 2_{\pm}^{1+2} = \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \right\rangle \subset \mathrm{SL}(4, \mathbf{R}),$$

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider the image of G_{16} in $\mathrm{SL}(4n, \mathbf{R})$ under the n-fold diagonal. The Quillen automorphism group $\mathbf{A}(\mathrm{GL}(4n, \mathbf{R}))(G_{16}) = \mathbf{A}(\mathrm{GL}(4, \mathbf{R}))(G_{16}) = \mathrm{Out}(G_{16}) \cong \mathrm{Out}(C_4) \times \mathrm{Sp}(2, \mathbf{F}_2)$ since the trace lives on the derived group $[G_{16}, G_{16}] = \langle -E \rangle$ only. In fact, $\mathbf{A}(\mathrm{GL}(2, \mathbf{C}))(G_{16})$ is the factor $\mathrm{Sp}(2, \mathbf{F}_2)$ and since the generator of the factor $\mathrm{Out}(C_4)$ is induced from conjugation with the matrix $\left(\begin{smallmatrix} 0 & E \\ E & 0 \end{smallmatrix}\right)$ of $\mathrm{SL}(4, \mathbf{R})$, we see that also $\mathbf{A}(\mathrm{SL}(4, \mathbf{R}))(G_{16}) \subset \mathbf{A}(\mathrm{SL}(4n, \mathbf{R}))(G_{16})$ is the full outer automorphism group of G_{16}. Hence $G_{16} \neq G_{16}^D$ in $\mathrm{SL}(4n, \mathbf{R})$.

(6) $\mathbf{R}^2$ is a simple $\mathbf{R}G$-module of real type with respect to the group

$$G = 2_{+}^{1+2} = \langle R, T \rangle \subset \mathrm{GL}(2, \mathbf{R}).$$

Consider the image of G in $\mathrm{SL}(4n, \mathbf{R})$ under the $2n$-fold diagonal map. Then $G \neq G^D$ in $\mathrm{SL}(4n, \mathbf{R})$.

(7) $\mathbf{R}^4$ is a simple $\mathbf{R}G$-module of quaternion type with respect to the group

$$G = 2_{-}^{1+2} = \left\langle \begin{pmatrix} 0 & -R \\ R & 0 \end{pmatrix}, \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \right\rangle \subset \mathrm{SL}(4, \mathbf{R}).$$

Consider the image of G in $\mathrm{SL}(4n, \mathbf{R})$ under the n-fold diagonal map. Then $G \neq G^D$ in $\mathrm{SL}(4n, \mathbf{R})$.

5.5. *Representations of (generalized) extraspecial 2-groups.* The extraspecial 2-groups $G = 2_{\pm}^{1+2d}$ have [24, 7.5] 2^d linear characters (that vanish on $\mho_1(G) = G' = Z(G) = C_2$) and one simple complex character

$$\chi(g) = \begin{cases} 0, & g \notin Z(G), \\ 2^d \lambda(g), & g \in Z(G), \end{cases}$$

induced from the nontrivial linear character $\lambda\colon Z(G) \to \{\pm 1\}$ (which is a group isomorphism).

If $G = 2_{+}^{1+2d}$ is of positive type, then $\varepsilon_2(\chi) = +1$ and $\chi\alpha = \chi$ for all $\alpha \in \mathrm{Out}(G)$, isomorphic to $O^+(2d, 2)$ [23, III.13.9.b]. This complex character is also the character of the unique simple real representation $G \to \mathrm{GL}(2^d, \mathbf{R})$ which is of real type; when d is even this representation actually takes values in $\mathrm{SL}(2^d, \mathbf{R})$ but when d is odd this representation is not oriented. The unique faithful real representation $G \to \mathrm{GL}(2 \cdot 2^d, \mathbf{R})$ with central $\mho_1$ has character 2χ and it splits into two distinct oriented real faithful representations $\psi, \psi^D\colon G \to \mathrm{SL}(2 \cdot 2^d, \mathbf{R})$ invariant under the action of $\mathrm{Out}(G)$ (5.4(2)).

If $G = 2_{-}^{1+2d}$ is of negative type, then $\varepsilon_2(\chi) = -1$ and $\chi\alpha = \chi$ for all $\alpha \in \mathrm{Out}(G)$, isomorphic to $O^-(2d, 2)$ [23, III.13.9.b]. The unique simple real representation $G \to \mathrm{GL}(2 \cdot 2^d, \mathbf{R})$ with character 2χ is of quaternion type. It splits into two distinct oriented representations $\psi, \psi^D \colon G \to \mathrm{SL}(2 \cdot 2^d, \mathbf{R})$ invariant under the action of $\mathrm{Out}(G)$ (5.4(3)).

By [24, 7.5] the generalized extraspecial 2-group $G = 4 \circ 2_{\pm}^{1+2d}$ has 2^{1+d} linear characters (that vanish on $\mho_1(G) = G' = C_2 \subsetneq Z(G) = C_4$) and two simple complex characters

$$\chi(g) = \begin{cases} 0, & g \notin Z(G), \\ 2^d \lambda(g), & g \in Z(G), \end{cases}$$

induced from the two faithful linear characters $\lambda \colon Z(G) \to \langle i \rangle = C_4$. These two degree 2^d simple characters, χ and $\overline{\chi}$, are interchanged by the action of $\mathrm{Out}(G) = \mathrm{Out}(C_4) \times \mathrm{Sp}(2d, 2)$ [17] (interchanged by the first factor $\mathrm{Out}(C_4)$ and preserved by the second factor $\mathrm{Sp}(2d, 2)$). The unique simple real representation $G \to \mathrm{GL}(2 \cdot 2^d, \mathbf{R})$ has character $\chi + \overline{\chi}$ and is of complex type as $\varepsilon_2(\chi) = 0$. It splits up into two distinct oriented representations $\psi, \psi^D \colon G \to \mathrm{SL}(2 \cdot 2^d, \mathbf{R})$ invariant under the action of $\mathrm{Out}(G)$ (5.4(1)).

These irreducible faithful representations have easy explicit constructions that we now explain.

Let E be a nontrivial elementary abelian 2-group of rank $d \geq 1$ and $\mathbf{R}[E]$ its real group algebra. For $\zeta \in E^\vee = \mathrm{Hom}(E, \mathbf{R}^\times)$ and $u \in E$, let $R_\zeta, T_u \in \mathrm{GL}(\mathbf{R}[E])$ be the linear automorphisms given by $R_\zeta(v) = \zeta(v)v$ and $T_u(v) = u + v$ for all $v \in E$. The computation

$$R_\zeta T_u(v) = R_\zeta(u \cdot v) = \zeta(u)\zeta(v)(u \cdot v) = \zeta(u) T_u(\zeta(v)v) = \zeta(u) T_u R_\zeta(v)$$

shows that $R_\zeta T_u = \zeta(u) T_u R_\zeta$ or, equivalently, $[R_\zeta, T_u] = \zeta(u)$.

The group $2_{+}^{1+2d} = \langle R_\zeta, T_u \rangle \subset \mathrm{GL}(\mathbf{R}[E]) \subset \mathrm{GL}(\mathbf{C}[E]) \overset{\mathcal{T}}{\subset} \mathrm{SL}(2^{d+1}, \mathbf{R})$ is extraspecial and the quadratic form on its abelianization 2^{2d} is given by

$$q(x_1, \ldots, x_d, y_1, \ldots, y_d) = x_1 y_1 + \cdots + x_d y_d,$$

because

$$(R_1^{x_1} \cdots R_d^{x_d} T_1^{y_1} \cdots T_d^{y_d})^2 = \prod_{i=1}^{d} (R_i^{x_i} T_i^{y_i})^2 = \prod_{i=1}^{d} (-E)^{x_i y_i}$$

where $T_1, \ldots, T_d$ correspond to a basis of E, $R_1, \ldots, R_d$ correspond to the dual basis, and $x_i, y_i \in \{0, 1\} = \mathbf{F}_2$. This is the unique faithful complex representation of degree 2^d. It is also the character of a simple real representation $G \to \mathrm{GL}(2^d, \mathbf{R})$, or even $G \to \mathrm{SL}(2^d, \mathbf{R})$ when d is even, of real type.

The group $2_{-}^{1+2d} = \langle R_1, \ldots, R_{d-1}, iR_d, T_1, \ldots, T_{d-1}, iT_d \rangle \subset \mathrm{GL}(\mathbf{C}[E]) \overset{\mathcal{T}}{\subset} \mathrm{SL}(2^{d+1}, \mathbf{R})$ is extraspecial and the quadratic form on its abelianization 2^{2d}

is given by

$$q(x_1,\dots,x_d,y_1,\dots,y_d)=x_1y_1+\dots+x_{d-1}y_{d-1}+x_d^2+x_dy_d+y_d^2,$$

because

$$(R_1^{x_1}\cdots R_{d-1}^{x_{d-1}}(iR_d)^{x_d}T_1^{y_1}\cdots T_{d-1}^{y_{d-1}}(iT_d)^{y_d})^2 = (-E)^{x_d^2+y_d^2}(R_1^{x_1}\cdots R_d^{x_d}T_1^{y_1}\cdots T_d^{y_d})^2$$

where $x_i,y_i\in\{0,1\}=\mathbf{F}_2$. This is the unique faithful complex representation of degree 2^d.

The group $4\circ 2_{\pm}^{1+2d}=4\circ 2_{+}^{1+2d}=\langle i,R_\zeta,T_u\rangle=\langle 2_{+}^{1+2d},2_{-}^{1+2d}\rangle=4\circ 2_{-}^{1+2d}\subset \mathrm{GL}(\mathbf{C}[E])\overset{\tau}{\subset}\mathrm{SL}(2^{d+1},\mathbf{R})$ is generalized extraspecial with derived group $[4\circ 2_{\pm}^{1+2d},4\circ 2_{\pm}^{1+2d}]=\mho_1(4\circ 2_{\pm}^{1+2d})=C_2\subset C_4=Z(4\circ 2_{\pm}^{1+2d})$, and elementary abelian abelianization 2×2^{2d}. The quadratic form on its abelianization is given by

$$q(z,x_1,\dots,x_d,y_1,\dots,y_d)=z^2+\sum_{i=1}^{d}x_iy_i,$$

because

$$(i^zR_1^{x_1}\cdots R_d^{x_d}T_1^{y_1}\cdots T_d^{y_d})^2=(-E)^{z^2}\prod_{i=1}^{d}(R_i^{x_i}T_i^{y_i})^2 = (-E)^{z^2}(R_1^{x_1}\cdots R_d^{x_d}T_1^{y_1}\cdots T_d^{y_d})^2$$

where $z,x_i,y_i\in\{0,1\}$. This representation and its conjugate are the two faithful complex representations of degree 2^d.

In the first two cases the associated symplectic inner product is

$$[(x_1,\dots,x_d,y_1,\dots,y_d),(x_1',\dots,x_d',y_1',\dots,y_d')]=\sum_{i=1}^{d}(x_iy_i'+x_i'y_i)$$

while it is

$$[(z,x_1,\dots,x_d,y_1,\dots,y_d),(z',x_1',\dots,x_d',y_1',\dots,y_d')]=\sum_{i=1}^{d}(x_iy_i'+x_i'y_i)$$

in the last case.

5.6. *Tensor products of real representations.* Suppose that $\mathbf{R}^m$ is an $\mathbf{R}G$-module with trace χ and $\mathbf{R}^n$ an $\mathbf{R}H$ module with trace ϱ. Consider $\mathbf{R}^{mn}=\mathbf{R}^m\otimes\mathbf{R}^n$ as an $\mathbf{R}(G\times H)$-module in the usual way where $(g,h)(u\otimes v)=gu\otimes hv$. The trace of this representation is $\chi\#\varrho(g,h)=\chi(g)\varrho(h)$ and the determinant is $\det(g,h)=(\det g)^n(\det h)^m$. This means that if

- m and n are both even, or if
- m is even and $\mathbf{R}^m$ an oriented G-representation,

then $\mathbf{R}^{mn}$ is a real oriented $G\times H$-representation.

5.7. *Embedding* $\mathrm{GL}(n,\mathbf{C})$ *in* $\mathrm{SL}(2n,\mathbf{R})$. Here are two embeddings $\tau\colon \mathrm{GL}(n,\mathbf{C}) \to \mathrm{GL}^+(2n,\mathbf{R})$ with the property that $\mathrm{tr}(\tau(A)) = \mathrm{tr}(A) + \overline{\mathrm{tr}(A)}$ for all $\mathbf{A} \in \mathrm{GL}(n,\mathbf{C})$.

If we write $\mathbf{C}^n = (\mathbf{R}+i\mathbf{R})^n$, then

$$\mathrm{GL}(n,\mathbf{C}) \ni A+iB \overset{\tau}{\mapsto} \left(\begin{pmatrix} a_{ij} & -b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix}\right)_{1\le i,j\le n} \in \mathrm{GL}^+(2n,\mathbf{R}).$$

In particular, $i \in \mathrm{GL}(n,\mathbf{C})$ is sent to $\mathrm{diag}(I,\ldots,I) \in \mathrm{SL}(2n,\mathbf{R})$ and $C_{\mathrm{SL}(2n,\mathbf{R})}(i)$ consists of matrices with 2×2 blocks as above. For (2×2)-matrices this embedding has the form

$$\mathrm{GL}(2,\mathbf{C}) \ni \begin{pmatrix} a_1+ia_2 & b_1+ib_2 \\ c_1+ic_2 & d_1+id_2 \end{pmatrix}$$
$$\overset{\tau}{\mapsto} \begin{pmatrix} \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} & \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix} \\ \begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix} & \begin{pmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{pmatrix} \end{pmatrix} \in \mathrm{SL}(4,\mathbf{R})$$

and with this convention the six subgroups of $\mathrm{SL}(4,\mathbf{R})$ isomorphic to D_8, Q_8, $G_{16} = 4\circ 2_{\pm}^{1+2}$ are

$$D_8 = \left\langle \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \right\rangle,$$

$$D_8^D = \left\langle \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \begin{pmatrix} 0 & -R \\ -R & 0 \end{pmatrix} \right\rangle,$$

$$Q_8 = \left\langle \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\rangle,$$

$$Q_8^D = \left\langle \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} \right\rangle,$$

$$G_{16} = \left\langle \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \begin{pmatrix} 0 & E \\ E & 0, \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right\rangle,$$

$$G_{16}^D = \left\langle \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \begin{pmatrix} 0 & -R \\ -R & 0 \end{pmatrix}, \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \right\rangle.$$

If we write $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$ then

$$\mathrm{GL}(n,\mathbf{C}) \ni A+iB \overset{\tau}{\mapsto} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{GL}^+(2n,\mathbf{R}).$$

In particular, $i \in \mathrm{GL}(n, \mathbf{C})$ is sent to $\left(\begin{smallmatrix} 0 & -E \\ E & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2n, \mathbf{R})$ and $C_{\mathrm{SL}(2n,\mathbf{R})}(i)$ consists of block matrices of the form as above. For (2×2)-matrices this embedding has the form

$$\mathrm{GL}(2,\mathbf{C}) \ni \begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ c_1 + ic_2 & d_1 + id_2 \end{pmatrix} \stackrel{\tau}{\mapsto} \begin{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} & -\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} & \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \end{pmatrix} \in \mathrm{SL}(4, \mathbf{R})$$

and with this convention the six subgroups of $\mathrm{SL}(4, \mathbf{R})$ isomorphic to D_8, Q_8, $G_{16} = 4 \circ 2^{1+2}_{\pm}$ are

$$\begin{aligned}
D_8 &= \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right\rangle, \\
D_8^D &= \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} -T & 0 \\ 0 & T \end{pmatrix} \right\rangle, \\
Q_8 &= \left\langle \begin{pmatrix} 0 & -R \\ R & 0 \end{pmatrix}, \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \right\rangle, \\
Q_8^D &= \left\langle \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \right\rangle, \\
G_{16} &= \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \right\rangle, \\
G_{16}^D &= \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} -T & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} 0 & R \\ -R & 0 \end{pmatrix} \right\rangle.
\end{aligned}$$

2. Lie group theory. The facts from Lie theory that are used in this paper are collected here.

5.8. *Centerings.* There are centerings [47]

$$L(\mathrm{pin}(2n)) \xrightarrow{P} L(\mathrm{GL}(2n, \mathbf{R})) \xrightarrow{Q} L(\mathrm{PGL}(2n, \mathbf{R}))$$

where

$$\begin{aligned}
P(x_1, x_2, \dots, x_n) &= (2x_1 - x_2 - \dots - x_n, x_2, \dots, x_n), \\
Q(x_1, x_2, \dots, x_n) &= (2x_1, x_1 - x_2, \dots, x_1 - x_n).
\end{aligned}$$

The expression for P is worked out in [4, pp. 174–175]. The expression for Q follows from the commutative diagram

$$\begin{tikzcd} T(\mathrm{GL}(2n,\mathbf{R})) = \mathrm{U}(1)^n \arrow[r] \arrow[dr, "\varphi"'] & \mathrm{U}(1)^n/\langle(-1,\dots,-1)\rangle = T(\mathrm{PGL}(2n,\mathbf{R})) \arrow[d, "\cong"'] \\ & \mathrm{U}(1)^n \end{tikzcd}$$

where $\varphi(z_1, z_2, \dots, z_n) = (z_1^2, z_1 z_2^{-1}, \dots, z_1 z_n^{-1})$ is surjective with kernel $C_2 = \langle(-1,\dots,-1)\rangle$. Since the action of the Weyl group $C_2 \wr \Sigma_n$ is known in $L(\mathrm{GL}(2n,\mathbf{R}))$, the two other actions can be worked out as well. The action in $L(\mathrm{pin}(2n))$ is $P^{-1}(C_2 \wr \Sigma_n)P$ and the action in $L(\mathrm{PGL}(2n,\mathbf{R}))$ is $Q(C_2 \wr \Sigma_n)Q^{-1}$. Here,

$$P^{-1}(u_1, u_2, \dots, u_n) = \left(\frac{1}{2}(u_1 + \cdots + u_n), u_2, \dots, u_n\right),$$
$$Q^{-1}(u_1, u_2, \dots, u_n) = \left(\frac{1}{2}u_1, \frac{1}{2}u_1 - u_2, \dots, \frac{1}{2}u_1 - u_n\right)$$

are the inverses.

5.9. *Centralizers in semidirect products.* Let $G \rtimes W$ be the semidirect product for a group action of W on G. The following lemma is elementary.

5.10. Lemma. *For any $g \in G$ and $w \in W$,*

$$C_{G\rtimes W}(g, w) = \{(h, v) \mid \exists w \in C_W(w)\colon g(wh) = h(vg)\},$$
$$C_{G\rtimes W}(g) = \{(h, v) \in G \rtimes W \mid vg = g^h\}, \qquad C_{G\rtimes W}(w) = G^w \rtimes C_W(w)$$

where G^w is the fixed point group for the action of w on G. If G is abelian then

$$C_{G\rtimes W}(g) = G \rtimes W(g)$$

where $W(g) = \{w \in W \mid wg = g\}$ is the isotropy subgroup at g.

Let $\mu\colon V \to \check{T} \rtimes W$ be a group homomorphism of an elementary abelian 2-group V into the semidirect product of a discrete 2-compact torus $\check{T}$ and a group W. Write $\mu = (\check{T}(\mu), W(\mu))$ for the two coordinates of μ. Then $W(\mu)\colon V \to W$ is a group homomorphism and $\check{T}(\mu)\colon V \to \check{T}$ a crossed homomorphism into the V-module $V \xrightarrow{W(\mu)} W \to \mathrm{Aut}(\check{T})$. Let $H^1(V; \check{T})$ be the first cohomology group for this V-module and $[\check{T}(\mu)] \in H^1(V; \check{T})$ the cohomology class represented by the crossed homomorphism $\check{T}(\mu)$.

5.11. Lemma. *There is a short exact sequence*

$$0 \to H^0(V; \check{T}) \to C_{\check{T}\rtimes W}(\mu) \to C_W(W(\mu))_{[\check{T}(\mu)]} \to 1$$

where $C_W(W(\mu))_{[\check{T}(\mu)]}$ is the isotropy subgroup at $[\check{T}(\mu)]$ for the action of $C_W(W(\mu))$ on $H^1(V; \check{T})$.

Proof. We first determine the kernel of the homomorphism $C_{\check{T}\rtimes W}(\mu) \to C_W(W(\mu))$. Let $t \in \check{T}$. Then

$$\begin{aligned}
&(t,1) \text{ commutes with } V \\
&\qquad \Leftrightarrow\ \forall v \in V\colon (t,1)(\check{T}(\mu)(v), W(\mu)(v)) = (\check{T}(\mu)(v), W(\mu)(v))(t,1) \\
&\qquad \Leftrightarrow\ \forall v \in V\colon t + \check{T}(\mu)(v) = \check{T}(\mu)(v) + W(\mu)(v)(t) \\
&\qquad \Leftrightarrow\ \forall v \in V\colon W(\mu)(v)t = t \\
&\qquad \Leftrightarrow\ t \in H^0(V;\check{T}).
\end{aligned}$$

More generally, for any element $(t,w) \in \check{T} \rtimes W$ we have

$$\begin{aligned}
&(t,w) \text{ commutes with } V \\
&\Leftrightarrow \forall v \in V\colon (t,w)(\check{T}(\mu)(v), W(\mu)(v)) = (\check{T}(\mu)(v), W(\mu)(v))(t,w) \\
&\Leftrightarrow \forall v \in V\colon s + w\check{T}(\mu)(v) = \check{T}(\mu)(v) + W(\mu)(v)(t),\ wW(\mu)(v) = W(\mu)(v)w \\
&\Leftrightarrow w \in C_W(W(\mu)) \text{ and } \forall v \in V\colon (1-w)\check{T}(\mu)(v) = (1 - W(\mu)(v))t \\
&\Leftrightarrow w \in C_W(W(\mu)) \text{ and } \forall v \in V\colon w\check{T}(\mu)(v) = \check{T}(\mu)(v) - (1 - W(\mu)(v))t.
\end{aligned}$$

It follows that for $w \in C_W(W(\mu))$ we have

$$\begin{aligned}
w \in \operatorname{im}(C_{\check{T}\rtimes W}(\mu) \to C_W(W(\mu))) \ &\Leftrightarrow\ \exists t \in \check{T}\colon (t,w) \in C_{\check{T}\rtimes W}(\mu) \\
&\Leftrightarrow\ w[\check{T}(\mu)] = [\check{T}(\mu)],
\end{aligned}$$

that is, w fixes the crossed homomorphism $\check{T}(\mu)$ up to a principal crossed homomorphism. ∎

5.12. *Centers of semidirect products.* Let $G \rtimes \Sigma$ be the semidirect product for the action $\Sigma \to \operatorname{Aut}(G)$ of the group Σ on the group G. Let $G^\Sigma = \{g \in G \mid \Sigma g = g\}$ and $\Sigma_G = \{\sigma \in \Sigma \mid \sigma(g) = g \text{ for all } g \in G\}$.

5.13. Lemma. *The center $Z(G \rtimes \Sigma) = G^\Sigma \times_{\operatorname{Aut}(G)} Z(\Sigma)$ of $G \rtimes \Sigma$ is the pull-back*

$$\begin{tikzcd}
Z(G \rtimes \Sigma) \arrow[r] \arrow[d] & Z(\Sigma) \arrow[d] \\
G^\Sigma \arrow[r] & \operatorname{Aut}(G)
\end{tikzcd}$$

of the action map restricted to the center of Σ along the map $G^\Sigma \to \operatorname{Aut}(G)$ given by inner automorphisms.

Proof. Suppose that $(g,\sigma) \in G \times \Sigma$ is in the center of $G \rtimes \Sigma$. Since

$$(g,\sigma)\cdot(1,\tau) = (g,\sigma\tau) = (1,\tau)\cdot(g,\sigma) = (\tau(g),\tau\sigma)$$

for all $\tau \in \Sigma$, g is fixed by Σ and σ is central in Σ. Moreover, from

$$(g,\sigma)\cdot(h,1) = (g\cdot\sigma(h),\sigma) = (h,1)\cdot(g,\sigma) = (hg,\sigma)$$

we see that $\sigma(h) = h^g$ for all $h \in G$. ∎

5.14. Corollary. *If the center of Σ acts faithfully on G through automorphisms that are not inner, then $Z(G \rtimes \Sigma) = Z(G)^\Sigma$. If G is abelian, then $Z(G \rtimes \Sigma) = G^\Sigma \times Z(\Sigma)_G$ is a direct product.*

Proof. In the first case, the vertical map $Z(\Sigma) \to \mathrm{Aut}(G)$ is injective and its image intersects trivially with the image of the horizontal map $G^\Sigma \to \mathrm{Aut}(G)$. So the pull-back is $G^\Sigma \cap Z(G) = Z(G)^\Sigma$. In the second case, the bottom horizontal homomorphism $G^\Sigma \to \mathrm{Aut}(G)$ is trivial. ∎

5.15. Corollary. *Let G be a group and $Z \neq G$ a central subgroup. Let the cyclic group C_p of prime order p act on G^p/Z by cyclic permutation. Then*

$$Z(G)/Z \times \{z \in Z \mid z^p = 1\} \cong Z(G^p/Z \rtimes C_p)$$

via the isomorphism that takes the element $z \in Z$ of order p to the element $(1,z,\dots,z^{p-1})Z \in G^p/Z$ and is the diagonal on $Z(G)/Z$.

Proof. Observe that

$$G/Z \times \{z \in Z \mid z^p = 1\} \xrightarrow{\cong} \left(G^p/Z\right)^{C_p}$$

via the isomorphism that takes (gZ, z) to $g(1,z,\dots,z^{p-1})Z$. To see this, consider an element $(g_1,\dots,g_p)Z$ which is fixed by C_p. Then

$$(g_1,g_2,\dots,g_p)Z = (g_p,g_1,\dots,g_{p-1})Z$$

so there exists an element $z \in Z$ such that $g_2 = g_1z, g_3 = g_2z = g_1z^2, \dots, g_p = g_1z^{p-1}, g_1 = g_1z^p$. Therefore, $z^p = 1$ and $(g_1,g_2,\dots,g_p) = g_1(1,z,\dots,z^{p-1})$.

Thus $Z(G^p/Z \rtimes C_p)$ is the pull-back of the group homomorphisms

$$G/Z \times \{z \in Z \mid z^p = 1\} \xrightarrow{\varphi} \mathrm{Aut}(G^p/Z) \leftarrow C_p$$

where $\varphi(gZ,z)((g_1,\dots,g_p)Z) = (g_1^g,\dots,g_p^g)Z$. Let $((gZ,z),\sigma)$ be an element of the pull-back. Assume that σ is nontrivial. Since p is a prime number, σ has no fixed points. The formula

$$\forall g_1,\dots,g_p \in G\colon \quad (g_1^g,\dots,g_p^g)Z = (g_{\sigma(1)},\dots,g_{\sigma(p)})Z$$

shows that $g_1^gZ = g_{\sigma(1)}Z$. This is impossible unless σ is the identity since otherwise we can find a $g_1 \in Z$ and a $g_{\sigma(1)} \notin Z$. Thus the permutation σ must be the identity. The requirement for $((gZ,z),1)$ to be in the pull-back is that

$$\forall (g_1,\dots,g_p) \in G^p \ \exists u \in Z\colon \quad (g_1^g,g_2^g,\dots,g_p^g) = (g_1u,g_2u,\dots,g_pu),$$

which implies that $[g_1,g] = u = [g_2,g]$ for all $g_1,g_2 \in G$. If we take $g_1 = 1$, we see that g must be central. ∎

5.16. *Centers of Lie groups and p-compact groups.* Let Y be a compact connected Lie group and ZY its center. Let BY denote the p-completed classifying space of Y, i.e. the p-compact group associated to Y. Lie group multiplication $ZY \times Y \to Y$ induces a homotopy equivalence $BZY \to \mathrm{map}(BY, BY)_{B1}$ [12, 1.4] of the p-completed classifying space BZY to the mapping space component containing the identity map. We need a version that holds for nonconnected Lie groups as well.

Let G be a possibly nonconnected Lie group and ZG its center. Let BZG and BG denote the $\mathbf{F}_p$-localized classifying spaces of ZG and G, respectively. The space $\mathrm{map}(BG, BG)_{B1}$ is the center of the p-compact group BG [12, 1.3].

5.17. Lemma. *The map*

$$BZG \to \mathrm{map}(BG, BG)_{B1}$$

induced by Lie group multiplication $ZG \times G \to G$ is a weak homotopy equivalence.

Proof. Let Y be the identity component of G and $\pi = G/Y$ the group of components. Note that the group π acts on the center ZY of Y and that there is an exact sequence of abelian groups

$$1 \to H^0(\pi; ZY) \to ZG \to Z\pi \to H^1(\pi; ZY)$$

relating the centers ZY, ZG, and $Z\pi$, of Y, G, and π. The abelian Lie group ZG, a product of a torus and a finite abelian group, is described by the data

$$\begin{aligned}
&\pi_1(ZG) \otimes \mathbf{Q} \cong H^0(\pi; \pi_1(ZY) \otimes \mathbf{Q}),\\
&1 \to H^0(\pi; \pi_0(ZY)) \times H^1(\pi; \pi_1(ZY)) \to \pi_0(ZG) \to Z\pi\\
&\qquad\qquad \to H^1(\pi; \pi_0(ZY)) \times H^2(\pi; \pi_1(ZY)),
\end{aligned}$$

where the last two lines are an exact sequence.

Similarly, there is a fibration of mapping spaces

$$\mathrm{map}(BY, BY)^{h\pi} \to \mathrm{map}(BG, BG) \to \mathrm{map}(BG, B\pi)$$

where the fiber over $BG \xrightarrow{B\pi_0} B\pi$ is the space $\mathrm{map}(BY, BY)^{h\pi}$ of self-maps of BG over $B\pi$. If we restrict to a single component of the total space, we obtain a fibration

$$\mathrm{map}(BG, BG)_{B1} \to \mathrm{map}(BG, B\pi)_{B\pi_0} \simeq \mathrm{map}(B\pi, B\pi)_{B1}$$

between path-connected spaces. The base space is $BZ\pi$. The fiber consists of some path-components of $\mathrm{map}(BY, BY)^{h\pi}_{B1} = (BZY)^{h\pi}$, the space of self-maps of BG over $B\pi$ with restriction to BY homotopic to the identity map. We have

$$\pi_i((BZY)^{h\pi}) = H^{1-i}(\pi; \pi_0(ZY)) \times H^{2-i}(\pi; \pi_1(ZY))$$

because $BZY = K(\pi_0 Y, 1) \times K(\pi_1 Y, 2)$ is a product of Eilenberg–MacLane spaces [39, 3.1], [12, 1.1]. It follows that $\mathrm{map}(BG, BG)_{B1}$ is an abelian [11, 3.5, 8.6] p-compact toral group described by the data

$$\pi_2((BZY)^{h\pi}) \otimes \mathbf{Q} \cong \pi_2(\mathrm{map}(BG, BG), B1) \otimes \mathbf{Q},$$
$$1 \to \pi_1((BZY)^{h\pi}) \to \pi_1(\mathrm{map}(BG, BG), B1) \to Z\pi \to \pi_0((BZY)^{h\pi}),$$

where the second line is an exact sequence.

Finally, the left commutative diagram of Lie groups

$$\begin{tikzcd}
(ZY)^\pi \times G \arrow[r] \arrow[d] & G \arrow[d, equal] & B(ZY)^\pi \arrow[r] \arrow[d] & (BZY)^{h\pi} \arrow[d] \\
ZG \times G \arrow[r] \arrow[d] & G \arrow[r, rightsquigarrow] \arrow[d] & BZG \arrow[r] \arrow[d] & \mathrm{map}(BG, BG)_{B1} \arrow[d] \\
Z\pi \times \pi \arrow[r] & \pi & BZ\pi \arrow[r] & \mathrm{map}(B\pi, B\pi)_{B1}
\end{tikzcd}$$

induces the right commutative diagram of mapping spaces. Comparing the homotopy groups, we see that $BZ(G) \to \mathrm{map}(BG, BG)_{B1}$ is a weak homotopy equivalence. ∎

5.18. Lemma. *We have*

$$Z\left(\frac{\mathrm{GL}(i_0, \mathbf{R}) \times \cdots \times \mathrm{GL}(i_t, \mathbf{R})}{\langle -E \rangle}\right) = \frac{\langle -E \rangle \times \cdots \times \langle -E \rangle}{\langle -E \rangle} \cong C_2^t,$$
$$Z\left(\frac{\mathrm{SL}(n, \mathbf{R}) \cap (\mathrm{GL}(i_0, \mathbf{R}) \times \cdots \times \mathrm{GL}(i_t, \mathbf{R}))}{\langle -E \rangle}\right) = \frac{\mathrm{SL}(n, \mathbf{R}) \cap (\langle -E \rangle \times \cdots \times \langle -E \rangle)}{\langle -E \rangle},$$
$$Z\left(\frac{\mathrm{GL}(i_0, \mathbf{H}) \times \cdots \times \mathrm{GL}(i_t, \mathbf{H})}{\langle -E \rangle}\right) = \frac{\langle -E \rangle \times \cdots \times \langle -E \rangle}{\langle -E \rangle} \cong C_2^t$$

for all natural numbers $i_0, \ldots, i_t > 0$ *with sum* n (*which is even in the second formula*).

Proof. Put $G = \mathrm{GL}(i_0, \mathbf{R}) \times \cdots \times \mathrm{GL}(i_t, \mathbf{R})$. There is [38, 5.11] a short exact sequence

$$1 \to Z(G)/\langle -E \rangle \to Z(G/\langle -E \rangle) \to \mathrm{Hom}(G, \langle -E \rangle)_{\mathrm{id}} \to 1$$

where the group on the right consists of all homomorphisms $\phi\colon G \to \langle -E \rangle$ such that the map $g \mapsto \phi(g)g$ is conjugate to the identity of G. Let $B \in \mathrm{GL}(i_j, \mathbf{R})$ be any matrix of positive trace. Then $\phi(E, \ldots, E, B, E, \ldots, E) = E$ since the map $g \mapsto \phi(g)g$ preserves trace. It follows that $\phi(g) = E$ for all $g \in G$ since ϕ is constant on the 2^t components of G. Thus the group on the right in the above short exact sequence, $\mathrm{Hom}(G, \langle -E \rangle)_{\mathrm{id}}$, is trivial.

Suppose that n is even. Since

$$Z\Big(\mathrm{SL}(n,\mathbf{R})\cap\prod\mathrm{GL}(i_j,\mathbf{R})\Big)\subset C_{\prod\mathrm{GL}(i_j,\mathbf{R})}\Big(\prod\mathrm{SL}(i_j,\mathbf{R})\Big)=\prod Z\mathrm{GL}(i_j,\mathbf{R})$$

we see that $Z\big(\mathrm{SL}(n,\mathbf{R})\cap\prod\mathrm{GL}(i_j,\mathbf{R})\big)=\mathrm{SL}(n,\mathbf{R})\cap\prod Z\mathrm{GL}(i_j,\mathbf{R})$. Suppose that the homomorphism $\phi\colon \mathrm{SL}(n,\mathbf{R})\cap\prod\mathrm{GL}(i_j,\mathbf{R})\to\langle -E\rangle$ is such that the map $g\mapsto\phi(g)g$ is conjugate to the identity. Let $B_1\in\mathrm{GL}(i_{j_1},\mathbf{R})$ and $B_2\in\mathrm{GL}(i_{j_2},\mathbf{R})$ be any pair of matrices such that $\mathrm{tr}(B_1)+\mathrm{tr}(B_2)>0$. Then $\phi(E,\ldots,B_{i_1},\ldots,B_{i_2},\ldots,E)=E$ by trace considerations. The short exact sequence from [38, 5.11], similar to that in 5.16, now yields the second formula.

The third formula has a similar proof. ■

It is not true in general that $Z(G)/Z$ is the center of the quotient G/Z of the (nonconnected) Lie group G by the central subgroup Z.

5.19. *Centralizers in quotients.* Let G be a Lie group and $Z\subset G$ a central subgroup. Write $N(G)$ for the normalizer of the maximal torus, $T(G)$, and $W=W(G)=N(G)/T(G)$ for the Weyl group. Suppose that $V\subset T(G)/Z$ is a toral subgroup of the quotient Lie group G/Z and let $V^*\subset T(G)\subset G$ be the preimage of V in G.

There is an exact sequence

$$1\to W(V^*)\to W(V)\to\mathrm{Hom}(V^*,Z)$$

relating the pointwise stabilizer subgroups for the action of the Weyl group W on V^* and V. The image of the rightmost homomorphism consists of all $\zeta\in\mathrm{Hom}(V^*,Z)$ for which the automorphism of V^* given by $v^*\mapsto\zeta(v^*)v^*$, $v^*\in V^*$, is of the form $v^*\mapsto wv^*$ for some Weyl group element $w\in W$.

Similarly, there is an exact sequence [38, 5.11]

$$1\to C_G(V^*)/Z\to C_{G/Z}(V)\to\mathrm{Hom}(V^*,Z)$$

relating the centralizers of $V^*\subset G$ and $G\subset G/Z$. The image of the rightmost homomorphism consists of all $\zeta\in\mathrm{Hom}(V^*,Z)$ for which the automorphism of V^* given by $v^*\mapsto\zeta(v^*)v^*$, $v^*\in V^*$, is of the form $v^*\mapsto g^{-1}v^*g$ for some $g\in G$.

5.20. Lemma. *$W(V)/W(V^*)=C_{G/Z}(V)/C_G(V)$.*

Proof. Any automorphism of the toral subgroup V^* that is induced by conjugation with an element of G is in fact induced by conjugation with an element of $N(G)$ [4, IV.2.5] and hence agrees with the action of a Weyl group element. ■

5.21. *Action on centralizers in the Lie case.* Let $\nu\colon V\to G$ be a monomorphism of a nontrivial elementary abelian p-group to a compact Lie group G. There is a canonical map $BC_G(\nu(V)) \to \mathrm{map}(BV, BG)_{B\nu}$ from the classifying space of the Lie-theoretic centralizer of $\nu(V)$ to the mapping space component containing $B\nu$. Write c_g for conjugation with $g \in G$.

5.22. Lemma. *Suppose that $\nu\alpha = c_g\nu$ for some element $g \in G$ and some automorphism $\alpha \in \mathrm{GL}(V)$. Then conjugation by g takes $C_G(\nu(V))$ to $C_G(c_g\nu(V)) = C_G(\nu\alpha(V)) = C_G(\nu(V))$ and the diagram*

$$\begin{tikzcd} BC_G(\nu(V)) \arrow[r] & \mathrm{map}(BV, BG)_{B\nu} \arrow[d, "\cong"', "(B\alpha)^*"] \\ BC_G(\nu(V)) \arrow[r] \arrow[u, "Bc_g", "\cong"'] & \mathrm{map}(BV, BG)_{B\nu} \end{tikzcd}$$

is homotopy commutative.

Proof. The commutative diagram of Lie group morphisms

$$\begin{tikzcd} V \times C_G(\nu(V)) \arrow[r, "\nu\times 1"] \arrow[d, "\alpha\times c_g"'] & \nu(V) \times C_G(\nu(V)) \arrow[r, "\mathrm{mult}"] & G \arrow[d, equal] \\ V \times C_G(\nu(V)) \arrow[r, "\nu\times 1"] & \nu(V) \times C_G(\nu(V)) \arrow[r, "\mathrm{mult}"] & G \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} BV \times BC_G(\nu(V)) \arrow[r, "B(\mathrm{mult}\circ(\nu\times 1))"] \arrow[d, "B\alpha\times Bc_g"'] & BG \arrow[d, equal] \\ BV \times BC_G(\nu(V)) \arrow[r, "B(\mathrm{mult}\circ(\nu\times 1))"] & BG \end{tikzcd}$$

of classifying spaces. Taking adjoints, we obtain the homotopy commutative diagram

$$\begin{tikzcd} BC_G(\nu(V)) \arrow[r] & \mathrm{map}(BV, BG)_{B\nu} \arrow[d, "(B\alpha)^*"] \\ BC_G(\nu(V)) \arrow[r] \arrow[u, "Bc_g"] & \mathrm{map}(BV, BG)_{B\nu} \end{tikzcd}$$

as claimed. ■

5.23. Corollary. *Suppose that $\mu\colon V \to N(G)$ is a monomorphism and that $\mu\alpha = c_n\mu$ for some $\alpha \in \mathrm{GL}(V)$ and $n \in N(G)$. Then*

$$w^{-1} = \pi_2((B\alpha)^*)\colon \pi_2(BT(G))^{\pi_0(\mu)(V)} \to \pi_2(BT(G))^{\pi_0(\mu)(V)}$$

where $w \in W(G)$ is the image of $n \in N(G)$.

Proof. There is a commutative diagram

$$\begin{tikzcd}
\pi_2(BT) \arrow[r, equal] & \pi_2(BN(G)) & \pi_2(BC_{N(G)}(V,\mu)) \arrow[l, hook'] \arrow[r, "\cong"] & \pi_2(\mathrm{map}(BV,BN),B\mu) \arrow[d, "\pi_2((B\alpha)^*)"] \\
\pi_2(BT) \arrow[u, "w"] \arrow[r, equal] & \pi_2(BN(G)) \arrow[u, "\pi_2(Bc_n)"'] & \pi_2(BC_{N(G)}(V,\mu)) \arrow[l, hook'] \arrow[u, "\pi_2(Bc_n)"'] \arrow[r, "\cong"] & \pi_2(\mathrm{map}(BV,BN),B\mu)
\end{tikzcd}$$

where $\pi_2(BC_{N(G)}(V,\mu)) = \pi_2(BT(G))^{\pi_0(\mu)(V)}$ denotes the fixed point group for the group action $\pi_0(\mu)\colon V \to W(G) \subseteq \mathrm{Aut}(\pi_2(BT(G)))$. Since the induced map $Bc_n\colon BN \to BN$ is freely homotopic to the identity along the loop $w \in \pi_1(BN)$, its effect on the $\mathbf{Z}_p[\pi_1(BN)]$-module $\pi_2(BN)$ is multiplication by w. ∎

5.24. *Low degree identifications.* There are the following low degree identifications [4, pp. 61, 292], [28, above Def. 3.3]:

$$\begin{aligned}
&\mathrm{Spin}(3) = \mathrm{Sp}(1) = \mathrm{SU}(2), && \mathrm{SO}(3) = \mathrm{PSp}(1) = \mathrm{PSU}(2),\\
&\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2), && \mathrm{PSO}(4) = \mathrm{SO}(3) \times \mathrm{SO}(3),\\
&\mathrm{Spin}(5) = \mathrm{Sp}(2), && \mathrm{SO}(5) = \mathrm{PSp}(2),\\
&\mathrm{Spin}(6) = \mathrm{SU}(4), && \mathrm{PSO}(6) = \mathrm{PSU}(4).
\end{aligned} \tag{5.25}$$

5.26. *Reflection group data.* For some computations it is convenient to have information

W	A_n	B_n	C_n	D_n
dim	$n(n+2)$	$n(2n+1)$	$n(2n+1)$	$n(2n-1)$
refl	$\frac{1}{2}n(n+1)$	n^2	n^2	$n(n-1)$
$\lvert W\rvert$	$(n+1)!$	$2^n n!$	$2^n n!$	$2^{n-1}n!$

W	G_2	F_4	E_6	E_7	E_8
dim	14	52	78	133	248
refl	6	24	36	63	120
$\lvert W\rvert$	$2^2 3^1$	$2^7 3^2$	$2^7 3^4 5^1$	$2^{10} 3^4 5^1 7$	$2^{14} 3^5 5^2 7^1$

about the cohomological dimension of the associated 2-compact group, the number of reflections, and the order of the irreducible reflection groups. The product $\prod d_i$ of the degrees d_i is the order of the Weyl group, and $\sum d_i$ is the number of reflections plus the rank.

5.27. *The* Spin-*groups.* Consider the defining universal covering space $\mathrm{Spin}(2m) \to \mathrm{SO}(2m)$ and the maximal elementary abelian 2-group $S\Delta_{2m}$ in $\mathrm{SO}(2m)$ consisting of diagonal matrices. Let $[\ ,\ ]\colon S\Delta_{2m} \times S\Delta_{2m} \to \mathbf{F}_2$ be the quadratic form given by $[D_1, D_2] = 0$ if and only if the pre-images of $D_1, D_2 \in S\Delta_{2m}$ commute in $\mathrm{Spin}(2m)$, and let $q\colon S\Delta_{2m} \to \mathbf{F}_2$ be the quadratic function given by $q(D) = 0$ if and only if the pre-images $D \in S\Delta_{2m}$ have order two in $\mathrm{Spin}(2m)$.

5.28. Lemma. *For diagonal matrices* $D, D_1, D_2 \in S\Delta_{2m}$ *we have*:
$[D_1, D_2] = 0$
$\Leftrightarrow$ *the number of entries that are negative in both matrices is even*,
$q(D) = 0 \ \Leftrightarrow$ *the number of negative entries in* D *is divisible by* 4.

This is well-known or an exercise in covering space theory.

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