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A universal planar completely regular continuum

by

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Abstract. We construct a universal planar completely regular continuum. This gives a positive answer to a problem posed by J. Krasinkiewicz (1986).

1. Introduction. We use the term *continuum* for any (nonempty) compact and connected metric space. A continuum K is said to be

- *completely regular* if each subcontinuum (except single points) of K has nonempty interior;
- *regular* if K has a basis consisting of open sets with finite boundaries;
- *hereditarily locally connected* if each subcontinuum of K is locally connected.

Completely regular continua are studied in [4] under the name "continua which contain no nowhere dense subcontinua (except single points)". Every completely regular continuum is regular and every regular continuum is hereditarily locally connected [4, §51, IV]. Simple examples of completely regular continua are connected graphs.

An *arc* is any space A homeomorphic to the segment $I = [0, 1]$. The points a and b of A which correspond to 0 and 1 under the homeomorphism are called the *endpoints* of A and the arc A is written as ab. We denote $(ab) = ab \setminus \{a, b\}$. An arc ab of a space X is called *free* (in X) if (ab) is open in X.

We recall the following characterization of the completely regular continua [2, Lemma 2], [3, Theorem 1.3]:

THEOREM 1.1. *A nondegenerate continuum K is completely regular if and only if there exist a subset F homeomorphic to the Cantor set and a null sequence of free arcs $a_1b_1, a_2b_2, \ldots$ of K such that*

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(i) $K = F \cup \bigcup_{n=1}^{\infty} a_n b_n$;
(ii) $a_n b_n \cap F = \{a_n, b_n\}$ *for any* n;
(iii) $a_n b_n \cap a_m b_m = \emptyset$ *if* $n \neq m$.

A triple $(K, F, \mathcal{A})$, where K is a completely regular continuum, F is a zero-dimensional compact subset of K, and $\mathcal{A}$ is a sequence of arcs of K satisfying the conditions of Theorem 1.1, is called a *completely regular continuum with structure.*

A completely regular continuum with structure $(\tilde{K}, \tilde{F}, \tilde{\mathcal{A}})$ is said to be *universal* for a family $\mathcal{F}$ of completely regular continua with structure if $(\tilde{K}, \tilde{F}, \tilde{\mathcal{A}}) \in \mathcal{F}$ and for every $(K, F, \mathcal{A}) \in \mathcal{F}$ there exists a homeomorphism $h : K \to \tilde{K}$ preserving the structure, that is, $h(F) \subseteq \tilde{F}$ and $h(A) \in \tilde{\mathcal{A}}$ for every $A \in \mathcal{A}$ ([1], [6]).

A continuum X is *universal* for a family $\mathcal{F}$ of continua provided that $X \in \mathcal{F}$ and each member of $\mathcal{F}$ can be homeomorphically embedded in X. It is known that:

- *There exists a universal completely regular continuum* [2].
- *There exists a universal planar completely regular dendrite* [5].
- *There is no universal completely regular continuum with structure* [1], [6].
- *There is no universal element in the class of planar completely regular continua with structure* [6].

In this paper we construct a universal planar completely regular continuum. This gives a positive answer to a problem posed by J. Krasinkiewicz [3].

2. Notations. All spaces considered in the paper are subspaces of the plane $\mathbb{E}^2$ with a system Oxy of orthogonal coordinates. By a *disk* is meant any space homeomorphic to the standard disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$.

For any set X we denote by $|X|$ the cardinality of X.

We denote $\mathbb{N} = \{0, 1, \ldots\}$.

For two points x and y of the plane we denote by $\overline{xy}$ the segment joining x and y. If ab is an arc and $x \in (ab)$, then we write $a < x < b$.

Given a finite family $\mathcal{F}$ of bounded subsets and a subset Q of the plane we denote $\mathcal{F}^* = \bigcup\{F : F \in \mathcal{F}\}$, $\operatorname{st}(Q, \mathcal{F}) = \{F \in \mathcal{F} : F \cap Q \neq \emptyset\}$, and $\operatorname{mesh}(\mathcal{F}) = \max\{\operatorname{diam}(F) : F \in \mathcal{F}\}$.

2.1. The family L_n of ordered n-tuples. Put $L_0 = \{\emptyset\}$ and denote by L_n, $n \in \mathbb{N} \setminus \{0\}$, the set of all ordered n-tuples $\bar{i} = i_1 \ldots i_n$, where $i_t = 0$ or $i_t = 1$ for any $t = 1, \ldots, n$. Also denote $\bar{i}0 = i_1 \ldots i_n 0$ and $\bar{i}1 = i_1 \ldots i_n 1$. For $\bar{i} = \emptyset \in L_0$ we set $\bar{i}0 = 0$ and $\bar{i}1 = 1$. We write $i_1 \ldots i_m \leq j_1 \ldots j_n$ if either $m = 0$, or $1 \leq m \leq n$ and $i_t = j_t$ for every $1 \leq t \leq m$.

For $\bar{i} = i_1 \dots i_n \in L_n$, $n \geq 1$, we denote by $I_{\bar{i}}$ the set of all points of I for which the tth digit of the triadic expansion, $t = 1, \dots, n$, is 0 if $i_t = 0$, and is 2 if $i_t = 1$. For $\bar{i} = \emptyset \in L_0$ we denote $I_{\bar{i}} = I_\emptyset = I$.

For each $\bar{i} \in \bigcup_{n=0}^{\infty} L_n$ we denote

$$a_{\bar{i}} = \min\{x : x \in I_{\bar{i}}\}, \quad b_{\bar{i}} = \max\{x : x \in I_{\bar{i}}\}, \quad a(\bar{i}) = b_{\bar{i}0}, \quad b(\bar{i}) = a_{\bar{i}1}.$$

2.2. The family $\mathcal{W}_n$ of squares. Let C denote the Cantor ternary set. For every $n \in \mathbb{N}$ consider the finite cover $\mathcal{W}_n = \{I_{\bar{i}} \times I_{\bar{j}} \mid \bar{i}, \bar{j} \in L_n\}$ of C^2 by squares. We denote by $V(\mathcal{W}_n)$ the set of all vertices of these squares.

Two elements $F_1 = I_{\bar{i}_1} \times I_{\bar{j}_1}$ and $F_2 = I_{\bar{i}_2} \times I_{\bar{j}_2}$ of $\mathcal{W}_n$ are called *adjacent* if: (α) either $\bar{i}_1 = \bar{i}_2$ or $\bar{j}_1 = \bar{j}_2$, and (β) no segment $\overline{ab}$ with $a \in F_1$ and $b \in F_2$ intersects any other element of $\mathcal{W}_n$.

2.3. Joining family of segments $\mathcal{A}(\bar{i}, \bar{j})$. Let $\bar{i}, \bar{j} \in L_k$, $k \in \mathbb{N}$. By a *joining family of segments for* $\operatorname{st}(I_{\bar{i}} \times I_{\bar{j}}, \mathcal{W}_{k+1})$ is meant any finite collection $\mathcal{A}(\bar{i}, \bar{j})$ of disjoint segments $\overline{xy} \subseteq I_{\bar{i}} \times I_{\bar{j}}$ with the properties:

(α) for any adjacent $F_1, F_2 \in \operatorname{st}(I_{\bar{i}} \times I_{\bar{j}}, \mathcal{W}_{k+1})$ there exists $\overline{xy} \in \mathcal{A}(\bar{i}, \bar{j})$ such that one of the points x, y is in F_1 and the other in F_2,

(β) if $\overline{xy} \in \mathcal{A}(\bar{i}, \bar{j})$, then one of the following four cases holds:

$$\begin{aligned}
x &\in \{a(\bar{i})\} \times I_{\bar{j}0} \quad \text{and} \quad y \in \{b(\bar{i})\} \times I_{\bar{j}0}, \\
x &\in \{a(\bar{i})\} \times I_{\bar{j}1} \quad \text{and} \quad y \in \{b(\bar{i})\} \times I_{\bar{j}1}, \\
x &\in I_{\bar{i}0} \times \{a(\bar{j})\} \quad \text{and} \quad y \in I_{\bar{i}0} \times \{b(\bar{j})\}, \\
x &\in I_{\bar{i}1} \times \{a(\bar{j})\} \quad \text{and} \quad y \in I_{\bar{i}1} \times \{b(\bar{j})\},
\end{aligned}$$

(γ) if $\overline{xy} \in \mathcal{A}(\bar{i}, \bar{j})$, then $x, y \in C^2 \setminus \bigcup_{n=0}^{\infty} V(\mathcal{W}_n)$.

2.4. Primary n-frames of I^2. In what follows, $\mathcal{A}(\bar{i}, \bar{j})$, where $\bar{i}, \bar{j} \in L_k$, $k \in \mathbb{N}$, denotes a (nonempty) joining family of segments for $\operatorname{st}(I_{\bar{i}} \times I_{\bar{j}}, \mathcal{W}_{k+1})$.

By a *primary n-frame of I^2*, $n \in \mathbb{N} \setminus \{0\}$, is meant any continuum $\mathcal{K}_n$ of the form

$$\begin{aligned}
\mathcal{K}_n &= \mathcal{W}_n^* \cup \bigcup \{\mathcal{A}^*(\bar{i}, \bar{j}) : \bar{i}, \bar{j} \in L_k,\ 0 \leq k \leq n-1\} \\
&= \mathcal{W}_n^* \cup \mathcal{A}^*(\mathcal{K}_n),
\end{aligned}$$

where $\mathcal{A}(\mathcal{K}_n) = \bigcup \{\mathcal{A}(\bar{i}, \bar{j}) : \bar{i}, \bar{j} \in L_k,\ 0 \leq k \leq n-1\}$.

Let $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{N}$. By a *primary n-frame of* $F = I_{\bar{i}_F} \times I_{\bar{j}_F} \in \mathcal{W}_m$ is meant any continuum $\mathcal{K}_n(F)$ of the form $\operatorname{st}^*(F, \mathcal{W}_{m+n}) \cup \mathcal{A}^*(\mathcal{K}_n(F))$, where

$$\mathcal{A}(\mathcal{K}_n(F)) = \bigcup \{\mathcal{A}(\bar{i}, \bar{j}) : \bar{i}, \bar{j} \in L_k,\ \bar{i}_F \leq \bar{i},\ \bar{j}_F \leq \bar{j},\ m \leq k \leq m+n-1\}.$$

We say that a primary $(m+n)$-frame $\mathcal{K}_{m+n}$ of I^2 is *n-inscribed in* a primary m-frame $\mathcal{K}_m$ of I^2 if $\mathcal{K}_{m+n} = \mathcal{A}^*(\mathcal{K}_m) \cup \bigcup \{\mathcal{K}_n(F) : F \in \mathcal{W}_m\}$, where each $\mathcal{K}_n(F)$ is a primary n-frame of F.

2.5. The family $\mathcal{C}$. Let $\{n_i\}_{i=1}^{\infty}$ be an increasing sequence in $\mathbb{N} \setminus \{0\}$ and $\mathcal{K}_{n_1} \supseteq \mathcal{K}_{n_2} \supseteq \cdots$ a decreasing sequence of inscribed primary n_i-frames of I^2. From Theorem 1.1 it follows that $\mathcal{K} = \bigcap_{i=1}^{\infty} \mathcal{K}_{n_i}$ is a completely regular continuum.

Let $\mathcal{C}$ denote the family of all completely regular continua which are intersections of some decreasing sequence of inscribed primary frames of I^2. Clearly, $\mathcal{K} \in \mathcal{C}$ if and only if $\mathcal{K} = C^2 \cup \bigcup\{\mathcal{A}^*(\bar{i}, \bar{j}) : \bar{i}, \bar{j} \in L_k,\ k = 0, 1, \ldots\}$.

We say that $\mathcal{K} \in \mathcal{C}$ is a *$\mathcal{C}$-representation* of a completely regular continuum X if X is homeomorphic to a subspace of $\mathcal{K}$. The following theorem is proved in [7, Theorem 4.2].

THEOREM 2.1. *For any planar completely regular continuum there exists a $\mathcal{C}$-representation.*

2.6. Generalized frames. *A generalized frame* $\mathcal{G}$ is any planar continuum that can be written in the form $\mathcal{O}^*(\mathcal{G}) \cup \mathcal{A}^*(\mathcal{G})$, where

(i) $\mathcal{O}(\mathcal{G})$ is a finite nonempty family of pairwise disjoint squares,
(ii) $\mathcal{A}(\mathcal{G})$ is a finite nonempty family of arcs,
(iii) $(ab) \cap \mathcal{O}^*(\mathcal{G}) = \emptyset$ for any $ab \in \mathcal{A}(\mathcal{G})$.

A generalized frame $\mathcal{F}$ is *transitively inscribed* in a generalized frame $\mathcal{G}$ if:

(i) $\mathcal{F} \subseteq \mathcal{G}$.
(ii) For any $F \in \mathcal{O}(\mathcal{F})$ there exists $G \in \mathcal{O}(\mathcal{G})$ such that $F \subseteq \mathrm{Int}(G)$.
(iii) If $G \in \mathcal{O}(\mathcal{G})$, $F \in \mathcal{O}(\mathcal{F})$, and $F \subseteq \mathrm{Int}(G)$, then there exists a finite family $\mathcal{A}(F, G) = \{a_i b_i\}_{i=1}^{n}$ of pairwise disjoint arcs of $\mathcal{F}$ such that $a_i \in \mathrm{Bd}(F)$, $\{b_i\}_{i=1}^{n} = \mathrm{Bd}(G) \cap \mathcal{A}^*(\mathcal{G})$, and $(a_i b_i) \subseteq \mathrm{Int}(G) \setminus F$ for $i = 1, \ldots, n$.

The following proposition is an easy consequence of the definition of a completely regular continuum.

PROPOSITION 2.2. *If $\{G_n\}_{n=1}^{\infty}$ is a sequence of generalized frames such that G_{n+1} is transitively inscribed in G_n for any n and* $\lim_{n\to\infty} \mathrm{mesh}(\mathcal{O}(G_n)) = 0$, *then the continuum $\bigcap_{n=1}^{\infty} G_n$ is completely regular.*

2.7. n-frames. For $n \in \mathbb{N} \setminus \{0\}$, by *$n$-frame* is meant any generalized frame that is homeomorphic to some primary n-frame of I^2. If $\mathcal{P}_n$ is an n-frame, then there exist a primary n-frame $\mathcal{K}_n = \mathcal{W}_n^* \cup \mathcal{A}^*(\mathcal{K}_n)$ of I^2 and a homeomorphism $h : \mathcal{K}_n \to \mathcal{P}_n$. We denote

$$\mathcal{O}(\mathcal{P}_n) = \{h(W) : W \in \mathcal{W}_n\}, \qquad \mathcal{A}(\mathcal{P}_n) = \{h(A) : A \in \mathcal{A}(\mathcal{K}_n)\}.$$

Clearly, $\mathcal{P}_n = \mathcal{O}^*(\mathcal{P}_n) \cup \mathcal{A}^*(\mathcal{P}_n)$, where $\mathcal{O}(\mathcal{P}_n)$ is a finite family of pairwise disjoint squares and $\mathcal{A}(\mathcal{P}_n)$ is a finite family of pairwise disjoint arcs. We denote

$$S(\mathcal{O}(\mathcal{P}_n)) = \{s : s \text{ is a side of a square } P \in \mathcal{O}(\mathcal{P}_n)\}.$$

Squares $P, P' \in \mathcal{O}(\mathcal{P}_n)$ are called *adjacent* if the squares $h^{-1}(P)$ and $h^{-1}(P')$ of $\mathcal{W}_n$ are adjacent. Given adjacent squares $P, P' \in \mathcal{O}(\mathcal{P}_n)$ we denote

$$\mathcal{A}_{\mathcal{P}_n}(P, P') = \operatorname{st}(P, \mathcal{A}(\mathcal{P}_n)) \cap \operatorname{st}(P', \mathcal{A}(\mathcal{P}_n)).$$

3. Construction of a universal planar completely regular continuum $\mathcal{Z}$

PROPOSITION 3.1. *Let D be a disk of the plane, $n \geq 2$ be a natural number, and $e_1, \ldots, e_n, b_n, \ldots, b_1, a_n, \ldots, a_1$ be cyclically ordered points on* $\operatorname{Bd}(D)$. *There exist families of disjoint arcs $A = \{e_1a_1, \ldots, e_na_n\}$ and $B = \{e_1b_1, \ldots, e_nb_n\}$ such that:*

(i) $(e_ia_i), (e_ib_i) \subseteq \operatorname{Int}(D)$ *for any i,*
(ii) $e_ia_i \cap e_jb_j = \emptyset$ *for any $i < j$.*

Proof. If D is the standard disk, then the segments $\overline{e_ia_i}$ and $\overline{e_ib_i}$ have properties (i) and (ii). In the other case it suffices to map D homeomorphically onto the standard disk and then take the inverse images of the corresponding segments. ∎

REMARK 3.2. From property (ii) of Proposition 3.1 it follows that for any $k, m \in \mathbb{N}$ such that $k + m \leq n$ and for any strongly increasing subsequence $\{i_1, \ldots, i_{k+m}\}$ of $\{1, \ldots, n\}$ the family $\{e_{i_1}a_{i_1}, \ldots, e_{i_k}a_{i_k}, e_{i_{k+1}}b_{i_{k+1}}, \ldots, e_{i_{k+m}}b_{i_{k+m}}\}$ consists of pairwise disjoint arcs.

We say that a subcontinuum F of a disk D is an *n-frame* of D if there exist a primary n-frame $\mathcal{K}_n$ of I^2 and a homeomorphism h of D onto I^2 such that $F = h^{-1}(\mathcal{K}_n)$.

For any square P we can define a 1-frame $\mathcal{K}(P)$ of P in a way similar to the definition of a primary 1-frame for I^2 (dividing P into nine equal squares, taking only the corner squares and joining any pair of adjacent corner squares by a finite number of disjoint segments).

We say that a frame $\mathcal{K}(P)$ is *n-joined*, $n \in \mathbb{N}\backslash\{0\}$, if any adjacent squares of $\mathcal{K}(P)$ are joined by exactly n disjoint segments.

In what follows, $\mathcal{K}^n(P)$ denotes an n-joined 1-frame of the square P.

For any square $P = [p_1, p_2] \times [q_1, q_2]$ of the plane we denote

$$v_1(P) = (p_1, q_1), \quad v_2(P) = (p_1, q_2), \quad v_3(P) = (p_2, q_2), \quad v_4(P) = (p_2, q_1),$$
$$s_1(P) = \overline{v_1(P)v_2(P)}, \qquad s_2(P) = \overline{v_2(P)v_3(P)},$$
$$s_3(P) = \overline{v_3(P)v_4(P)}, \qquad s_4(P) = \overline{v_4(P)v_1(P)}.$$

Denoting $v_5 \equiv v_1$ we obtain $s_\ell(P) = \overline{v_\ell(P)v_{\ell+1}(P)}$ for any $\ell \in \{1, 2, 3, 4\}$.

Obviously, $V(P) = \{v_1(P), v_2(P), v_3(P), v_4(P)\}$ is the set of vertices of P, and $S(P) = \{s_1(P), s_2(P), s_3(P), s_4(P)\}$ is the set of sides of P.

Given a 1-frame $\mathcal{K}(P)$ of P, we denote by P_κ, $\kappa \in \{1, 2, 3, 4\}$, the unique element of $\mathcal{O}(\mathcal{K}(P))$ that contains the vertex $v_\kappa(P)$ (see Figure 1).

3.1. Grafting construction. Given a square $P = [p_1, p_2] \times [q_1, q_2]$, a finite set $E_P \subseteq \mathrm{Bd}(P) \setminus V(P)$ that intersects each side of P, and $n \in \mathbb{N} \setminus \{0\}$, we will define a corresponding generalized frame $G_n(P, E_P)$.

Let $\widetilde{P} = [\tilde{p}_1, \tilde{p}_2] \times [\tilde{q}_1, \tilde{q}_2]$ be a square such that $\widetilde{P} \subseteq \mathrm{Int}(P)$ and $\mathcal{K}^n(\widetilde{P})$ be any n-joined 1-frame of $\widetilde{P}$. We denote by D^ℓ, $\ell \in \{1, 2, 3, 4\}$, the disk bounded by the closed curve (see Figure 2)

$$\mathrm{Bd}(D^\ell) = \overline{v_\ell(\widetilde{P})v_\ell(P)} \cup \overline{v_\ell(P)v_{\ell+1}(P)} \cup \overline{v_{\ell+1}(P)v_{\ell+1}(\widetilde{P})} \cup \overline{v_{\ell+1}(\widetilde{P})v_\ell(\widetilde{P})}.$$

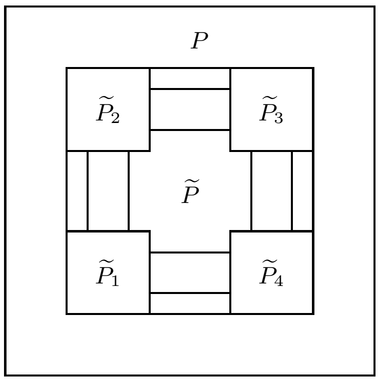

Fig. 1

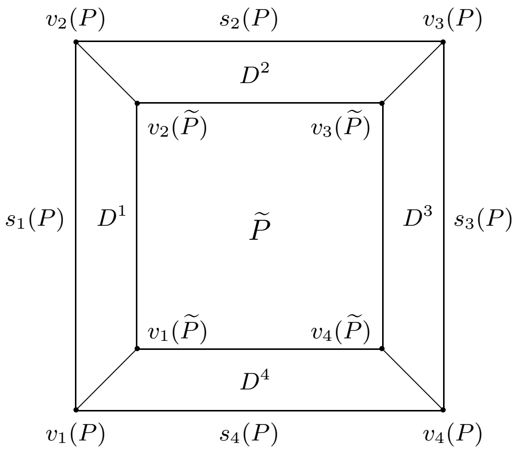

Fig. 2

Construction of families $A^\ell_\kappa(P)$. To each side $s_\ell(P)$ of P we will associate two families $A^\ell_\ell(P)$ and $A^\ell_{\ell+1}(P)$ of pairwise disjoint arcs joining points of $s_\ell(P) \cap E_P$ to points of $s_\ell(\widetilde{P}_\ell)$ and of $s_\ell(\widetilde{P}_{\ell+1})$, respectively (see Figure 3).

Let $s_\ell(P) \cap E_P = \{e^\ell_1, \ldots, e^\ell_{n_\ell}\}$ be cyclically ordered in $\mathrm{Bd}(P)$.

Note that $\mathrm{st}(s_\ell(\widetilde{P}), \mathcal{O}(\mathcal{K}^n(\widetilde{P}))) = \{\widetilde{P}_\ell, \widetilde{P}_{\ell+1}\}$.

Fix cyclically ordered (in $\mathrm{Bd}(\widetilde{P})$) sets $\{a^\ell_1, \ldots, a^\ell_{n_\ell}\} \subseteq s_\ell(\widetilde{P}_\ell) \setminus V(\widetilde{P}_\ell)$ and $\{b^\ell_1, \ldots, b^\ell_{n_\ell}\} \subseteq s_\ell(\widetilde{P}_{\ell+1}) \setminus V(\widetilde{P}_{\ell+1})$. Apply Proposition 3.1 to the disk D^ℓ and the points $e^\ell_1, \ldots, e^\ell_{n_\ell}, b^\ell_{n_\ell}, \ldots, b^\ell_1, a^\ell_{n_\ell}, \ldots, a^\ell_1 \in \mathrm{Bd}(D^\ell)$ to obtain families $A^\ell_\ell(P) = \{e^\ell_1 a^\ell_1, \ldots, e^\ell_{n_\ell} a^\ell_{n_\ell}\}$ and $A^\ell_{\ell+1}(P) = \{e^\ell_1 b^\ell_1, \ldots, e^\ell_{n_\ell} b^\ell_{n_\ell}\}$ of pairwise disjoint arcs that satisfy conditions (i) and (ii) of Proposition 3.1.

Set $A^\ell(P) = A^\ell_\ell(P) \cup A^\ell_{\ell+1}(P)$ for any ℓ. It is easily seen that

(a) $(A^{\ell_1}(P))^* \cap (A^{\ell_2}(P))^* = \emptyset$ whenever $\ell_1 \neq \ell_2$.

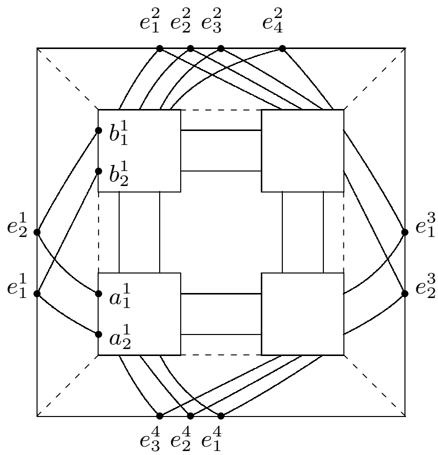

Fig. 3

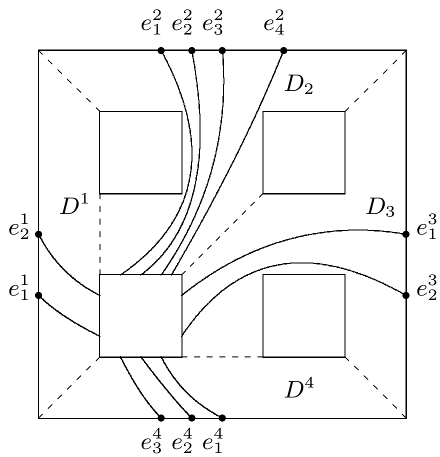

Fig. 4

(b) If $ae \in A^\ell(P)$, then $a \in (s_\ell(\widetilde{P}_\ell) \setminus V(\widetilde{P}_\ell)) \cup (s_\ell(\widetilde{P}_{\ell+1}) \setminus V(\widetilde{P}_{\ell+1}))$, $e \in s_\ell(P) \cap E_P$, and $(ea) \subseteq \mathrm{Int}(P) \setminus \widetilde{P}$.

(c) If $k, m \in \mathbb{N}$, $k+m \le n_\ell$, and $\{i_1, \ldots, i_{k+m}\}$ is a strongly increasing subsequence of $\{1, \ldots, n_\ell\}$, then the families $A^\ell_{\ell,k}(P) = \{e_{i_1}a_{i_1}, \ldots, e_{i_k}a_{i_k}\}$ and $A^\ell_{\ell+1,m}(P) = \{e_{i_{k+1}}b_{i_{k+1}}, \ldots, e_{i_{k+m}}b_{i_{k+m}}\}$ have the following properties: $A^\ell_{k+m}(P) = A^\ell_{\ell,k}(P) \cup A^\ell_{\ell+1,m}(P)$ consists of pairwise disjoint arcs, $|\mathrm{st}(s_\ell(\widetilde{P}_\ell), A^\ell_{k+m}(P))| = k$, and $|\mathrm{st}(s_\ell(\widetilde{P}_{\ell+1}), A^\ell_{k+m}(P))| = m$.

Construction of families $B^\ell(P_\kappa)$. To each side $s_\ell(P)$ of P we will associate a family $B^\ell(\widetilde{P}_1)$ of pairwise disjoint arcs joining points of $E_P \cap s_\ell(P)$ to points of the side $s_\ell(\widetilde{P}_1)$ of $\widetilde{P}_1$ (the choice of $\widetilde{P}_1$ is accidental, in place of $\widetilde{P}_1$ we could take any other element of $\mathcal{O}(\mathcal{K}^n_1(\widetilde{P}))$) in such a way that (see Figure 4):

(d) $(B^{\ell_1}(\widetilde{P}_1))^* \cap (B^{\ell_2}(\widetilde{P}_1))^* = \emptyset$ whenever $\ell_1 \ne \ell_2$.

(e) If $ae \in B^\ell(\widetilde{P}_1)$, then $a \in s_\ell(\widetilde{P}_1) \setminus V(\widetilde{P}_1)$, $e \in s_\ell(P) \cap E_P$, and $(ea) \subseteq \mathrm{Int}(P) \setminus \mathcal{O}^*(\mathcal{K}^n(\widetilde{P}))$.

Set $B^1(\widetilde{P}_1) = A^1_1$ and $B^4(\widetilde{P}_1) = A^4_1$, where the families A^1_1 and A^4_1 of pairwise disjoint arcs in $\mathrm{Int}(D^1)$ and $\mathrm{Int}(D^2)$, respectively, have already been defined.

Obviously, there are disks $D_2, D_3 \subseteq P$ such that: (i) the interiors of D_2, D_3, D^1, and D^4 are pairwise disjoint, (ii) $s_2(P), s_2(\widetilde{P}_1) \subseteq \mathrm{Bd}(D_2)$, and (iii) $s_3(P), s_3(\widetilde{P}_1) \subseteq \mathrm{Bd}(D_3)$.

Fix cyclically ordered (in $\mathrm{Bd}(\widetilde{P}_1)$) sets $\{a_1, \ldots, a_{n_2}\} \subseteq s_2(\widetilde{P}_1) \setminus V(\widetilde{P}_1)$ and $\{b_1, \ldots, b_{n_3}\} \subseteq s_3(\widetilde{P}_1) \setminus V(\widetilde{P}_1)$. Apply Proposition 3.1 to the disks

D_2 and D_3 to obtain families $B^2(\widetilde{P}_1) = \{e_1^2 a_1, \ldots, e_{n_2}^2 a_{n_2}\}$ and $B^3(\widetilde{P}_1) = \{e_1^2 b_1, \ldots, e_{n_2}^2 b_{n_3}\}$ of pairwise disjoint arcs that satisfy conditions (d) and (e). Set

$$G_n(P, E_P) = \mathcal{K}^n(\widetilde{P}) \cup \Big(\bigcup_{\ell=1}^{4} (A^\ell(P))^*\Big) \cup \Big(\bigcup_{\ell,\kappa=1}^{4} (B^\ell(P_\kappa))^*\Big),$$

$$\mathcal{A}(G_n(P, E_P)) = \mathcal{A}(\mathcal{K}^n(\widetilde{P})) \cup \Big(\bigcup_{\ell=1}^{4} A^\ell(P)\Big) \cup \Big(\bigcup_{\ell,\kappa=1}^{4} B^\ell(P_\kappa)\Big).$$

Clearly, $\mathrm{mesh}(\mathcal{O}(G_n(P, E_P))) = \mathrm{diam}(\widetilde{P})/9$.

3.2. Construction of $\mathcal{Z}$. We will define a sequence $\{\mathcal{G}_n\}_{n=1}^\infty$ of generalized frames such that $\mathcal{G}_{n+1}$ is transitively inscribed in $\mathcal{G}_n$ for all n.

Let $T = [t_1, t_2]^2$ be any square of the plane. In each side s_ℓ of T take a point $e_\ell \in s_\ell(T) \setminus V(T)$. Select $E_T = \{e_1, e_2, e_3, e_4\} \subseteq \mathrm{Bd}(T)$. We define

$$\mathcal{G}_1 = G_1(T, E_T) \quad \text{and} \quad \mathcal{A}(\mathcal{G}_1) = \mathcal{A}(G_1(T, E_T)).$$

Clearly, $\mathcal{O}(\mathcal{G}_1) = \mathcal{O}(\mathcal{K}^1(\widetilde{P}))$. From the definition of $G_1(T, E_T)$ it follows that for any $P \in \mathcal{O}(\mathcal{G}_1)$ and for any side $s_\ell(P)$ of P the set $\mathcal{A}^*(\mathcal{G}_1) \cap s_\ell(P)$ is a nonempty subset of $s_\ell(P) \setminus V(P)$.

Suppose that a generalized frame $\mathcal{G}_n = \mathcal{O}^*(\mathcal{G}_n) \cup \mathcal{A}^*(\mathcal{G}_n)$, $n \in \mathbb{N} \setminus \{0\}$, is defined and for any $P \in \mathcal{O}(\mathcal{G}_n)$ and any side $s_\ell(P)$ of P the set $\mathcal{A}^*(\mathcal{G}_n) \cap s_\ell(P)$ is a nonempty subset of $s_\ell(P) \setminus V(P)$. Set $E_P = \mathcal{A}^*(\mathcal{G}_n) \cap \mathrm{Bd}(P)$ and define

$$\mathcal{G}_{n+1} = \Big(\mathcal{G}_n \cap \bigcup_{P \in \mathcal{O}(\mathcal{G}_n)} G_{n+1}(P, E_P)\Big) \cup \mathcal{A}(\mathcal{G}_n),$$

$$\mathcal{A}(\mathcal{G}_{n+1}) = \mathcal{A}(\mathcal{G}_n) \cup \bigcup_{P \in \mathcal{O}(\mathcal{G}_n)} \mathcal{A}(G_{n+1}(P, E_P)).$$

3.3. Properties of $\{\mathcal{G}_n\}_{n=1}^\infty$. For any $n \in \mathbb{N} \setminus \{0\}$ the following properties are satisfied:

(1) $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$.
(2) $\mathrm{mesh}(\mathcal{O}(\mathcal{G}_{n+1})) < \mathrm{mesh}(\mathcal{O}(\mathcal{G}_n))/9$.
(3) If $P \in \mathcal{O}(\mathcal{G}_n)$, then there exists $\widetilde{P} \subseteq \mathrm{Int}(P)$ such that

$$P \cap \mathcal{G}_{n+1} = G_n(P, E_P) = \mathcal{K}^{n+1}(\widetilde{P}) \cup \Big(\bigcup_{\ell=1}^{4} (A^\ell(P))^*\Big) \cup \Big(\bigcup_{\ell,\kappa=1}^{4} (B^\ell(P_\kappa))^*\Big).$$

(4) $\mathcal{G}_{n+k}$ is transitively inscribed in $\mathcal{G}_n$ for any $k \in \mathbb{N} \setminus \{0\}$. Moreover, if $\widehat{P} \in \mathcal{O}(\mathcal{G}_n)$ and $P \in \mathrm{st}(\widehat{P}, \mathcal{O}(\mathcal{G}_{n+k}))$, then for each $\ell \in \{1, 2, 3, 4\}$ there exists a finite family $B^\ell(\widehat{P}, P)$ consisting of pairwise disjoint

arcs $ab \in \mathcal{G}_{n+k}$ such that

$$a \in s_\ell(\widehat{P}) \cap \mathcal{A}^*(\mathcal{G}_n), \qquad b \in s_\ell(P) \cap \mathcal{A}^*(\mathcal{G}_{n+k}), \qquad (ab) \subseteq \mathrm{Int}(\widehat{P}) \setminus P.$$

Also, $(B^{\ell_1}(\widehat{P}, P))^* \cap (B^{\ell_2}(\widehat{P}, P))^* = \emptyset$ for $\ell_1 \neq \ell_2$.

We define $\mathcal{Z} = \bigcap_{n=1}^{\infty} \mathcal{G}_n$. By Proposition 2.2, $\mathcal{Z}$ is a planar completely regular continuum.

4. Main theorem

LEMMA 4.1. *Let A, B, C be disks of the plane such that $A \subseteq \mathrm{Int}(B)$ and $B \subseteq \mathrm{Int}(C)$. Let also $\{b_1a_1, \ldots, b_na_n\}$, $\{c_1b_1, \ldots, c_nb_n\}$ be families of pairwise disjoint arcs such that for any $i = 1, \ldots, n$:*

- (i) *$\{a_1, \ldots, a_n\} \subseteq \mathrm{Bd}(A)$, $\{b_1, \ldots, b_n\} \subseteq \mathrm{Bd}(B)$, and $\{c_1, \ldots, c_n\} \subseteq \mathrm{Bd}(C)$,*
- (ii) *$(b_ia_i) \subseteq \mathrm{Int}(B) \setminus A$ and $(c_ib_i) \subseteq \mathrm{Int}(C) \setminus B$.*

Suppose also that for $i = 1, \ldots, n$ there are given homeomorphisms $g_i : c_ib_i \to c_ib_i \cup b_ia_i$ such that $g_i(c_i) = c_i$ and $g_i(b_i) = a_i$. Then for any homeomorphism $h : B \to A$ such that $h(b_i) = a_i$ for any i, there exists a homeomorphism $\overline{h} : C \to C$ such that

- (iii) *$\overline{h}|_B = h$,*
- (iv) *$\overline{h}|_{\mathrm{Bd}(C)}$ is identity, and*
- (v) *$\overline{h}|_{c_ib_i} = g_i$ for any i.*

Proof. We denote $b_{n+1} = b_1$ and $c_{n+1} = c_1$. For any $i = 1, \ldots, n$ we consider the arc c_ic_{i+1} in $\mathrm{Bd}(C)$ for which $(c_ic_{i+1}) \cap \{c_1, \ldots, c_n\} = \emptyset$, the arc b_ib_{i+1} in $\mathrm{Bd}(B)$ for which $(b_ib_{i+1}) \cap \{b_1, \ldots, b_n\} = \emptyset$, and the arc a_ia_{i+1} in $\mathrm{Bd}(B)$ for which $(a_ia_{i+1}) \cap \{a_1, \ldots, a_n\} = \emptyset$. Note that

- (vi) $C \setminus \mathrm{Int}(B)$ is a union of disks D_i^B, $i = 1, \ldots, n$, bounded by the closed curves $\mathrm{Bd}(D_i^B) = c_ic_{i+1} \cup b_ib_{i+1} \cup c_ib_i \cup c_{i+1}b_{i+1}$,
- (vii) $C \setminus \mathrm{Int}(A)$ is a union of disks D_i^A, $i = 1, \ldots, n$, bounded by the closed curves $\mathrm{Bd}(D_i^A) = c_ic_{i+1} \cup a_ia_{i+1} \cup (c_ib_i \cup b_ia_i) \cup (c_{i+1}b_{i+1} \cup b_{i+1}a_{i+1})$.

Let $h_i : \mathrm{Bd}(D_i^B) \to \mathrm{Bd}(D_i^A)$ be a homeomorphism such that $h_i(b_i) = a_i$, $h_i(b_{i+1}) = a_{i+1}$, $h_i(b_ib_{i+1}) = a_ia_{i+1}$, h_i is the identity on c_ic_{i+1}, $h_i|_{c_ib_i} = g_i$, and $h_i|_{c_{i+1}b_{i+1}} = g_{i+1}$. Then there is a homeomorphism $\overline{h}_i : D_i^B \to D_i^A$ such that $\overline{h}_i|_{\mathrm{Bd}(D_i^B)} = h_i$. The required homeomorphism $\overline{h} : C \to C$ is defined by

$$\overline{h}(x) = \begin{cases} h(x) & \text{if } x \in B, \\ \overline{h}_i(x) & \text{if } x \in D_i^B. \end{cases} \blacksquare$$

LEMMA 4.2. *Let rp be an arc and $\{r_i\}_{i=0}^{\infty}$, $\{p_i\}_{i=0}^{\infty}$ be sequences in (rp) such that $\lim_{i\to\infty} p_i = p$ and $r_i < p_i < r_{i+1}$ for any $i \in \mathbb{N}$. Then there is a sequence of homeomorphisms $g_i : rp_{i-1} \to rp_i$, $i = 1, 2, \ldots,$ such that*

(i) $g_i(r) = r$ *and* $g_i(p_{i-1}) = p_i$,
(ii) g_i *is the identity on* rr_{i-1},
(iii) $f = \lim_{i\to\infty}(g_i \circ \cdots \circ g_1)$ *is a homeomorphism of* rp_0 *onto* rp.

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points of (r_0p_0) such that $\lim_{i\to\infty} x_i = p_0$ and $x_i < x_{i+1} < p_0$ for any i.

We have $r < r_0 < x_1 < p_0 < r_1 < p_1$.

Let $g_1 : rp_0 \to rp_1$ be a homeomorphism such that g_1 is the identity on rr_0, $g_1(r_0x_1) = (r_0r_1)$, and $g_1(x_1p_0) = r_1p_1$. Note that $\{g_1(x_i)\}_{i=2}^{\infty} \subseteq (r_1p_1)$.

Assume that for any $1 \le j \le i$ homeomorphisms g_j with properties (i) and (ii) have been defined and that $\{g_i(\ldots g_1(x_k))\}_{k=i+1}^{\infty} \subseteq (r_ip_i)$.

For $x'_{i+1} = g_i(\ldots g_1(x_{i+1}))$ we have $r < r_i < x'_{i+1} < p_i < r_{i+1} < p_{i+1}$.

Let $g_{i+1} : rp_i \to rp_{i+1}$ be a homeomorphism such that g_{i+1} is the identity on rr_i, $g_{i+1}(r_ix'_{i+1}) = (r_ir_{i+1})$, and $g_{i+1}(x'_{i+1}p_i) = r_{i+1}p_{i+1}$. Note that $\{g_{i+1}(\ldots g_1(x_k))\}_{k=i+2}^{\infty} \subseteq (r_{i+1}p_{i+1})$.

Set $f_i = g_i \circ \cdots \circ g_1$. Since $\lim_{i\to\infty} p_i = p$, $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f and since f is defined on the compact set rp_0, we conclude that f is a closed map. Obviously, $f(r) = r$ and $f(p_0) = p$. Hence, $f(rp_0) = rp$.

In order to prove that f is one-to-one assume that $r \le x < y \le p_1$.

If $x, y \in rr_0$, then $f(x) = g_1(x) \ne g_1(y) = f(y)$, because each g_i is the identity on rr_0. In the other case $r \le x \le x_k < y \le p_0$ for some k. Since $f_k(x_k) = r_k$, it follows that $r \le f_k(x) \le r_k < f_k(y) \le f(y) \le p$. Since g_i is the identity on rr_k for each $i \ge k$, it follows that $f(x) = f_k(x) \in rr_k$ and $f(y) \notin rr_k$. Thus $f(x) \ne f(y)$. ■

MAIN THEOREM 4.3. *For any* $\mathcal{K} \in \mathcal{C}$ *there exists a homeomorphism* $H : \mathbb{E}^2 \to \mathbb{E}^2$ *such that* $H(\mathcal{K}) \subseteq \mathcal{Z}$.

Proof. Let $\mathcal{K} \in \mathcal{C}$. Then

$$\mathcal{K} = C^2 \cup \bigcup\{\mathcal{A}^*(\bar{i}, \bar{j}) : \bar{i}, \bar{j} \in L_k,\ k = 0, 1, \ldots\}.$$

For any $i \in \mathbb{N}$ and for any $F \in \mathcal{W}_i$ we denote by $\mathcal{A}(F)$ the joining family of segments for $\mathrm{st}(F, \mathcal{W}_{i+1})$. Then $\mathcal{K}(F) = \mathrm{st}^*(F, \mathcal{W}_{i+1}) \cup \mathcal{A}^*(F)$ is a 1-frame. Note that $\mathcal{K}(F) = (\bigcup_{\ell=1}^{4} F_\ell) \cup \mathcal{A}^*(F)$. We define

$$n_F = \max\{|\mathcal{A}_{\mathcal{K}(F)}(F_\ell, F_{\ell+1})| : F_\ell, F_{\ell+1} \text{ are adjacent in } \mathrm{st}(F, \mathcal{W}_{i+1})\}.$$

Set $\mathcal{A}(\mathcal{K}_i) = \bigcup\{\mathcal{A}(\bar{i}, \bar{j}) : \bar{i}, \bar{j} \in L_k,\ 0 \le k \le i-1\}$ and $\mathcal{K}_i = \mathcal{W}_i^* \cup \mathcal{A}^*(\mathcal{K}_i)$. Note that

$$\mathcal{K}_{i+1} = \mathcal{A}^*(\mathcal{K}_i) \cup \bigcup\{\mathcal{K}(F) : F \in \mathcal{W}_i\}.$$

Clearly each $\mathcal{K}_i$ is a primary i-frame of I^2 which for any $i > 1$ is 1-inscribed in $\mathcal{K}_{i-1}$ and $\mathcal{K} = \bigcap_{i=1}^{\infty} \mathcal{K}_i$.

Let $\{n_i\}_{i=1}^{\infty}$ be a sequence of natural numbers such that $n_{i+1} > n_i + 2$ and $n_i > \max\{n_F : F \in \mathcal{W}_i\}$ for any i. For each $i \ge 1$ we will define an i-frame $\mathcal{M}_i \subseteq \mathcal{G}_{n_i}$ and a homeomorphism $h_i : \mathcal{K}_i \to \mathcal{M}_i$ such that:

(1_i) $\mathcal{M}_i \subseteq \mathcal{M}_{i+1}$.
(2_i) If $F \in \mathcal{O}(\mathcal{K}_i)$ and $F' \in \mathrm{st}(F, \mathcal{O}(\mathcal{K}_{i+1}))$, then $h_{i+1}(F') \subseteq \mathrm{Int}(h_i(F))$.
(3_i) $h_i(A) \subseteq h_{i+1}(A)$ for all $A \in \mathcal{A}(\mathcal{K}_i)$.
(4_i) If x is an endpoint of an arc $A \in \mathcal{A}(\mathcal{K}_i)$, then

$$\mathrm{st}(h_{i+1}(x), \mathcal{O}(\mathcal{M}_{i+1})) = h_{i+1}(\mathrm{st}(x, \mathcal{O}(\mathcal{K}_{i+1}))).$$

Construction of $\mathcal{M}_1$. We begin by taking any $P \in \mathcal{O}(\mathcal{G}_{n_1-1})$. By property (3) of the family $\{\mathcal{G}_n\}_{n=1}^{\infty}$ there are a square $\widetilde{P} \subseteq \mathrm{Int}(P)$ and an n_1-joined 1-frame $\mathcal{K}^{n_1}(\widetilde{P})$ of $\widetilde{P}$ such that $\mathcal{K}^{n_1}(\widetilde{P}) \subseteq \mathcal{G}_{n_1}$.

Since $\mathcal{K}_1$ is an at most n_1-joined 1-frame of I^2, there exists an embedding $h_1 : \mathcal{K}_1 \to \mathcal{K}^{n_1}(\widetilde{P})$ such that

(1_{h_1}) $h_1(I^2_\ell) = \widetilde{P}_\ell$ for all $\ell \in \{1,2,3,4\}$.
(2_{h_1}) $h_1(s_\kappa(I^2_\ell)) = s_\kappa(\widetilde{P}_\ell)$ for all $\ell, \kappa \in \{1,2,3,4\}$.
(3_{h_1}) If $A \in \mathcal{A}_{\mathcal{K}_1}(I^2_{\ell_1}, I^2_{\ell_2})$, then $h_1(A) \in \mathcal{A}_{\mathcal{K}^{n_1}(\widetilde{P})}(h_1(I^2_{\ell_1}), h_1(I^2_{\ell_2}))$.

Let $i \ge 1$ and suppose that for any $1 \le j \le i$ a j-frame $\mathcal{M}_j$ and a homeomorphism $h_j : \mathcal{K}_j \to \mathcal{M}_j$ have been defined.

Construction of an i-frame $\mathcal{N}_i$ *that is transitively inscribed in* $\mathcal{M}_i$. For any $\widehat{P} \in \mathcal{O}(\mathcal{M}_i)$ we fix any $P \in \mathrm{st}(\widehat{P}, \mathcal{O}(\mathcal{G}_{n_{i+1}-1}))$ and denote it by $\hat{\omega}(\widehat{P})$. Since $\widehat{P} \in \mathcal{O}(\mathcal{G}_{n_i})$, from property (4) of $\{\mathcal{G}_n\}_{n=1}^{\infty}$ it follows that for any $\ell \in \{1,2,3,4\}$ there is a finite family $B^\ell(\widehat{P}, P)$ of pairwise disjoint arcs $\hat{p}p \subseteq \mathcal{G}_{n_{i+1}-1}$ such that $\hat{p} \in s_\ell(\widehat{P}) \cap \mathcal{A}^*(\mathcal{M}_i)$, $p \in s_\ell(P) \cap \mathcal{A}^*(\mathcal{G}_{n_{i+1}-1})$, and $(\hat{p}p) \subseteq \mathrm{Int}(\widehat{P}) \setminus P$.

Let $\hat{p}\hat{q} \in \mathcal{A}(\mathcal{M}_i)$. Then there are adjacent elements $\widehat{P}, \widehat{Q}$ of $\mathcal{O}(\mathcal{M}_i)$ and $\ell_{\hat{p}}, \ell_{\hat{q}} \in \{1,2,3,4\}$ such that $\hat{p} \in s_{\ell_{\hat{p}}}(\widehat{P})$ and $\hat{q} \in s_{\ell_{\hat{q}}}(\widehat{Q})$. Let $\hat{\omega}(\widehat{P}) = P$ and $\hat{\omega}(\widehat{Q}) = Q$.

Consider the points $p \in s_{\ell_{\hat{p}}}(P) \cap \mathcal{A}^*(\mathcal{G}_{n_{i+1}-1})$ and $q \in s_{\ell_{\hat{q}}}(Q) \cap \mathcal{A}^*(\mathcal{G}_{n_{i+1}-1})$ such that $\hat{p}p, \hat{q}q \subseteq \mathcal{G}_{n_{i+1}-1}$, $(\hat{p}p) \subseteq \mathrm{Int}(\widehat{P}) \setminus P$, and $(\hat{q}q) \subseteq \mathrm{Int}(\widehat{Q}) \setminus Q$. We denote $\widehat{\tau}_i(\hat{p}\hat{q}) = \hat{p}\hat{q} \cup \hat{p}p \cup \hat{q}q$.

Set $\mathcal{A}(\mathcal{N}_i) = \{\widehat{\tau}_i(A) : A \in \mathcal{A}(\mathcal{M}_i)\}$ and $\mathcal{O}(\mathcal{N}_i) = \{\hat{\omega}(\widehat{P}) : \widehat{P} \in \mathcal{O}(\mathcal{M}_i)\}$. Clearly, $\widehat{\tau}_i : \mathcal{A}(\mathcal{M}_i) \to \mathcal{A}(\mathcal{N}_i)$ and $\widehat{\omega}_i : \mathcal{O}(\mathcal{M}_i) \to \mathcal{O}(\mathcal{N}_i)$ are bijections.

Set $\mathcal{N}_i = \mathcal{O}^*(\mathcal{N}_i) \cup \mathcal{A}^*(\mathcal{N}_i)$.

Construction of $\mathcal{M}_{i+1}$. Let $\widehat{P} \in \mathcal{M}_i$. Then $\widehat{\omega}_i(\widehat{P}) = P \in \mathcal{O}(\mathcal{N}_i)$. Since $P \in \mathcal{O}(\mathcal{G}_{n_{i+1}-1})$, by property (3) of $\{\mathcal{G}_n\}_{n=1}^{\infty}$ there exist a square $\widetilde{P} \subseteq \mathrm{Int}(P)$ and an n_{i+1}-joined 1-frame $\mathcal{K}^{n_{i+1}}(\widetilde{P})$ of $\widetilde{P}$ such that $P \cap \mathcal{G}_{n_{i+1}} = \mathcal{K}^{n_{i+1}}(\widetilde{P}) \cup \bigcup_{\ell=1}^4 (A^\ell(\widetilde{P}))^*$, where the families of arcs $A^\ell(P)$ have properties (a)–(c) of Subsection 3.1. Clearly, to each $\widehat{P} \in \mathcal{M}_i$ corresponds a unique $\widetilde{P}$. We denote $\widetilde{P} = \widetilde{\omega}_i(\widehat{P})$.

On the other hand $\widehat{P} = h_i(F)$, where $F \in \mathcal{O}(\mathcal{K}_i)$. Since $\mathcal{K}(F) = F \cap \mathcal{K}_{i+1}$ is an at most n_{i+1}-joined 1-frame of F and $\mathcal{K}^{n_{i+1}}(\widetilde{P})$ is an n_{i+1}-joined 1-frame of P, there is an embedding $h_F : \mathcal{K}(F) \to \mathcal{K}^{n_{i+1}}(\widetilde{P})$ such that:

(1_{h_F}) $h_F(F_\ell) = \widetilde{P}_\ell$ for all $\ell \in \{1, 2, 3, 4\}$.
(2_{h_F}) $h_F(s_\kappa(F_\ell)) = s_\kappa(\widetilde{P}_\ell)$ for all $\ell, \kappa \in \{1, 2, 3, 4\}$.
(3_{h_F}) If $A \in \mathcal{A}_{\mathcal{K}(F)}(F_{\ell_1}, F_{\ell_2})$, then $h_F(A) \in \mathcal{A}_{\mathcal{K}^{n_{i+1}}(\widetilde{P})}(h_F(F_{\ell_1}), h_F(F_{\ell_2}))$.

Let $\ell \in \{1, 2, 3, 4\}$ be such that $s_\ell(F) \cap \mathcal{A}^*(\mathcal{K}_i) \neq \emptyset$.

Note that $\mathrm{st}(s_\ell(F), \mathcal{O}(\mathcal{K}_{i+1})) = \{F_\ell, F_{\ell+1}\}$. We denote

$$k = |F_\ell \cap \mathcal{A}^*(\mathcal{O}(\mathcal{K}_i))| \quad \text{and} \quad m = |F_{\ell+1} \cap \mathcal{A}^*(\mathcal{O}(\mathcal{K}_i))|.$$

Then $|s_\ell(P) \cap \mathcal{A}^*(\mathcal{N}_i)| = k + m \leq |s_\ell(P) \cap \mathcal{A}^*(\mathcal{G}_{n_{i+1}})|$.

From property (c) of $A^\ell(P)$ it follows that there are families of pairwise disjoint arcs $A^\ell_{\ell,k}(P)$ and $A^\ell_{\ell+1,m}(P)$ of $\mathcal{G}_{n_{i+1}}$ such that:

(i) $A^\ell_{\ell,k}(P) \cup A^\ell_{\ell+1,m}(P)$ consists of pairwise disjoint arcs.
(ii) $|A^\ell_{\ell,k}(P)| = k$ and $|A^\ell_{\ell+1,m}(P)| = m$.
(iii) If $p\tilde{p} \in A^\ell_{\ell,k}(P)$, then $p \in s_\ell(P) \cap \mathcal{A}^*(\mathcal{N}_1)$, $\tilde{p} \in s_\ell(\widetilde{P}_\ell)$, and $(p\tilde{p}) \subseteq \mathrm{Int}(P) \setminus \widetilde{P}$.
(iv) If $p\tilde{p} \in A^\ell_{\ell+1,m}(P)$, then $p \in s_\ell(P) \cap \mathcal{A}^*(\mathcal{N}_1)$, $\tilde{p} \in s_\ell(\widetilde{P}_{\ell+1})$, and $(p\tilde{p}) \subseteq \mathrm{Int}(P) \setminus \widetilde{P}$.

For any $p \in s_\ell(P) \cap \mathcal{A}^*(\mathcal{N}_1)$ we denote $\widetilde{\tau}(p) = \mathrm{st}(p, A^\ell_{\ell,k}(P) \cup A^\ell_{\ell+1,m}(P))$.

Let $A = p_A q_A \in \mathcal{A}(\mathcal{K}_i)$, $\hat{p} = h_i(p_A)$, and $\hat{q} = h_i(q_A)$. Then $\hat{p}\hat{q} \in \mathcal{A}(\mathcal{M}_i)$ and $\widehat{\tau}_i(\hat{p}\hat{q}) = pq \in \mathcal{A}(\mathcal{N}_i)$. There are adjacent elements P, Q of $\mathcal{O}(\mathcal{N}_i)$ and $\ell_p, \ell_q \in \{1, 2, 3, 4\}$ such that $p \in s_{\ell_p}(P) \cap \mathcal{A}^*(\mathcal{N}_i)$ and $q \in s_{\ell_q}(Q) \cap \mathcal{A}^*(\mathcal{N}_i)$. Let $\widetilde{\tau}(p) = p\tilde{p}$ and $\widetilde{\tau}(q) = p\tilde{q}$. Set $\tilde{p}\tilde{q} = \widetilde{\tau}(p) \cup pq \cup \widetilde{\tau}(q)$.

Let $h_A : p_A q_A \to \tilde{p}\tilde{q}$ be a homeomorphism such that $h_A(p_A) = \tilde{p}$ and $h_A(q_A) = \tilde{q}$. Set $\mathcal{M}_{i+1} = (\bigcup_{F \in \mathcal{O}(\mathcal{K}_i)} h_F(\mathcal{K}(F))) \cup (\bigcup_{A \in \mathcal{A}(\mathcal{K}_i)} h_A(A))$.

We define $h_{i+1} : \mathcal{K}_{i+1} \to \mathcal{M}_{i+1}$ as follows:

$$h_{i+1}(x) = \begin{cases} h_F(x) & \text{if } x \in F \in \mathcal{O}(\mathcal{K}_i), \\ h_A(x) & \text{if } x \in A \in \mathcal{A}(\mathcal{K}_i). \end{cases}$$

Construction of a homeomorphism $H : \mathbb{E}^2 \to \mathbb{E}^2$ that carries $\mathcal{K}$ into $\mathcal{Z}$. Given a square P we denote by $U[P, \delta]$ the square consisting of points that are at distance $\leq \delta$ from P.

For each $i = 1, 2, \ldots,$ we choose $\delta_i > 0$ such that

(i) $U[\widehat{P}, \delta_i] \cap U[\widehat{Q}, \delta_i] = \emptyset$ for any distinct $\widehat{P}, \widehat{Q} \in \mathcal{O}(\mathcal{M}_i)$.
(ii) If $\widehat{Q} \in \mathcal{O}(\mathcal{M}_i)$ and $\widehat{P} \in \mathrm{st}(Q, \mathcal{O}(\mathcal{M}_{i+1}))$, then $U[\widehat{P}, \delta_{i+1}] \subseteq \mathrm{Int}(\widehat{Q})$.

Obviously, $\lim_{i \to \infty} \delta_i = 0$.

To each i-frame $\mathcal{M}_i$ we associate the i-frame $\mathcal{U}_i = \mathcal{O}^*(\mathcal{U}_i) \cup \mathcal{A}^*(\mathcal{U}_i)$, where

$$\mathcal{O}(\mathcal{U}_i) = \{U[\widehat{P}, \delta_i] : \widehat{P} \in \mathcal{O}(\mathcal{M}_i)\},$$
$$\mathcal{A}(\mathcal{U}_i) = \{\mathrm{Cl}(A \setminus \mathcal{O}^*(\mathcal{U}_i)) : A \in \mathcal{A}(\mathcal{M}_i)\}.$$

For each $A \in \bigcup_{i=1}^{\infty} \mathcal{A}(\mathcal{K}_i)$ we will define an embedding $H^A : A \to \mathcal{Z}$. The final homeomorphism H will be such that $H|_A = H^A$.

Let $A = p_A q_A \in \bigcup_{i=1}^{\infty} \mathcal{A}(\mathcal{K}_i)$. Since $\mathcal{A}(\mathcal{K}_1) \subsetneq \mathcal{A}(\mathcal{K}_2) \subsetneq \cdots$, there is a least i_A such that $A \in \mathcal{A}(\mathcal{K}_{i_A})$. Consider adjacent $F^{p_A}, F^{q_A} \in \mathcal{O}(\mathcal{K}_{i_A})$ such that $p_A \in F^{p_A}$ and $q_A \in F^{q_A}$. Set $h_{i_A+i}(p_A) = p_i$, $h_{i_A+i}(q_A) = q_i$, $h_{i_A+i}(F^{p_A}) = P_i$, and $h_{i_A+i}(F^{q_A}) = Q_i$ for any $i \in \mathbb{N}$. Then $p_i \in P_i$, $q_i \in Q_i$, and P_i, Q_i are adjacent in $\mathcal{O}(\mathcal{M}_{i_A+i})$ for any $i \in \mathbb{N}$.

Since the sets P_i and Q_i are compact and since, from (2_i), we have $P_{i+1} \subseteq P_i$ and $Q_{i+1} \subseteq Q_i$, it follows that $\bigcap_{i=1}^{\infty} P_i = \{p\}$ and $\bigcap_{i=1}^{\infty} Q_i = \{q\}$.

Let $p_i q_i = h_{i_A+i}(p_A q_A)$. Then $p_i q_i \subseteq p_{i+1} q_{i+1}$ from (3_i). It is easy to see that $\bigcup_{i=1}^{\infty} p_i q_i = pq$ and pq is an arc of $\mathcal{Z}$.

Note that $p \in \mathrm{Int}(P_i) \subseteq U[P_i, \delta_{i_A+i}]$ and $q \in \mathrm{Int}(Q_i) \subseteq U[Q_i, \delta_{i_A+i}]$ for all $i \in \mathbb{N}$. Denote $r_i = pq \cap \mathrm{Bd}(U[P_i, \delta_{i_A+i}])$ and $s_i = pq \cap \mathrm{Bd}(U[Q_i, \delta_{i_A+i}])$.

Fix any $r \in (r_0 s_0)$. Note that the sequences $\{r_i\}_{i=0}^{\infty}$ and $\{p_i\}_{i=0}^{\infty}$ of (rp) as well as the sequences $\{s_i\}_{i=0}^{\infty}$ and $\{q_i\}_{i=0}^{\infty}$ of (rq) satisfy the conditions of Lemma 4.2. Since $rp_i \cup rq_i = p_i q_i$, there is a sequence of homeomorphisms $g_i^A : p_{i-1}q_{i-1} \to p_i q_i$, $i = 1, 2, \ldots$, such that:

(i) $g_i^A(r) = r$, $g_i^A(p_{i-1}) = p_i$, and $g_i^A(q_{i-1}) = q_i$.
(ii) g_i^A is the identity on $r_{i-1}s_{i-1}$.
(iii) $f^A = \lim_{i\to\infty}(g_i^A \circ \cdots \circ g_1^A)$ is a homeomorphism of $p_0 q_0$ onto pq.

Obviously, $H^A = f^A \circ h_{i_A}$ is a homeomorphism of A onto pq.

Since $\mathcal{K}_1$ is a union of finitely many pairwise disjoint disks joined by finitely many pairwise disjoint arcs and since $h_1 : \mathcal{K}_1 \to \mathcal{M}_1$ is a homeomorphism, there exists a homeomorphism $H_1 : \mathbb{E}^2 \to \mathbb{E}^2$ such that $H_1|_{\mathcal{K}_1} = h_1$.

Let $\widehat{P} \in \mathcal{O}(\mathcal{M}_1)$ and $\mathrm{st}(\widehat{P}, \mathcal{A}(\mathcal{M}_1)) = \{A_1^{\widehat{P}}, \ldots, A_n^{\widehat{P}}\}$. Since $\mathcal{M}_1 = h_1(\mathcal{K}_1)$, there exist pairwise disjoint arcs $A_1, \ldots, A_n \in \mathcal{A}(\mathcal{K}_1)$ such that $A_i^{\widehat{P}} = h_1(A_i)$ for $i = 1, \ldots, n$. Clearly,

$$\mathrm{st}(U[\widehat{P}, \delta_1], \mathcal{A}(\mathcal{U}_1)) = \{\mathrm{Cl}(A_i^{\widehat{P}} \setminus \mathcal{O}^*(\mathcal{U}_1))\}_{i=1}^{n}.$$

Also, for $\widetilde{P} = \widetilde{\omega}_1(\widehat{P})$ we have $\mathrm{st}(\widetilde{P}, \mathcal{A}(\mathcal{M}_2)) = \{h_2(A_i)\}_{i=1}^{n}$.

Obviously, we have $\mathrm{Cl}(A_i^{\widehat{P}} \setminus \mathcal{O}^*(\mathcal{U}_1)) \subseteq A_i^{\widehat{P}} \subseteq h_2(A_i)$ for $i = 1, \ldots, n$. We denote $r_0^i = \mathrm{Bd}(U[\widehat{P}, \delta_1]) \cap \mathrm{Cl}(A_i^{\widehat{P}} \setminus \mathcal{O}^*(\mathcal{U}_1))$, $p_0^i = \mathrm{Bd}(\widehat{P}) \cap A_i^{\widehat{P}}$, and

$p_1^i = \mathrm{Bd}(\widetilde{P}) \cap h_2(A_i)$. Then

$$\begin{aligned}\{r_0^1, \ldots, r_0^n\} &= \mathrm{Bd}(U[\widehat{P}, \delta_1]) \cap \mathcal{A}^*(\mathcal{M}_1),\\ \{p_0^1, \ldots, p_0^n\} &= \mathrm{Bd}(\widehat{P}) \cap \mathcal{A}^*(\mathcal{M}_1),\\ \{p_1^1, \ldots, p_1^n\} &= \mathrm{Bd}(\widetilde{P}) \cap \mathcal{A}^*(\mathcal{M}_2).\end{aligned}$$

Observe that $\widetilde{P}, \widehat{P}$, and $U[\widehat{P}, \delta_1]$ are disks such that $\widetilde{P} \subseteq \mathrm{Int}(\widehat{P})$ and $\widehat{P} \subseteq \mathrm{Int}(U[\widehat{P}, \delta_1])$.

Since $p_0^i \in \mathrm{Bd}(\widehat{P}) \cap H_1(\mathcal{K}_2)$ and $p_1^i \in \mathrm{Bd}(\widetilde{P}) \cap \mathcal{M}_2$ for all i, there exists a homeomorphism $g_{\widehat{P}} : \widehat{P} \to \widetilde{P}$ such that $g_{\widehat{P}}(H_1(\mathcal{K}_2) \cap \widehat{P}) = \mathcal{M}_2 \cap \widetilde{P}$ and $g_{\widehat{P}}(p_0^i) = p_1^i$.

By Lemma 4.1 there is a homeomorphism $\overline{g}_{\widehat{P}} : U[\widehat{P}, \delta_1] \to U[\widehat{P}, \delta_1]$ such that $\overline{g}_{\widehat{P}}|_{\mathrm{Bd}(U[\widehat{P}, \delta_1])}$ is the identity, $\overline{g}|_{\widehat{P}} = g_{\widehat{P}}$, and $\overline{g}_{\widehat{P}}|_{r_0^i p_0^i} = g_1^{A_i}|_{r_0^i p_0^i}$ for any i.

Let $g_1 : \mathbb{E}^2 \to \mathbb{E}^2$ be a homeomorphism such that

$$g_1|_{\mathbb{E}^2 \setminus \mathcal{O}^*(\mathcal{U}_1)} = H_1|_{\mathbb{E}^2 \setminus \mathcal{O}^*(\mathcal{U}_1)} \quad \text{and} \quad g_1|_{\widehat{P}} = \overline{g}_{\widehat{P}}$$

for all $\widehat{P} \in \mathcal{O}(\mathcal{M}_1)$. We set $H_2 = g_1 \circ H_1$. Clearly, H_2 sends $\mathcal{K}_2$ onto $\mathcal{M}_2$.

By induction the homeomorphisms $g_i : \mathbb{E}^2 \to \mathbb{E}^2$ and $H_i : \mathbb{E}^2 \to \mathbb{E}^2$, $i \in \mathbb{N} \setminus \{0\}$, can be defined so that the following conditions are satisfied:

(1) $H_i(\mathcal{K}_i) = h_i(\mathcal{K}_i) = \mathcal{M}_i$.
(2) $g_i|_{\mathbb{E}^2 \setminus \mathcal{O}^*(\mathcal{U}_i)} = H_i|_{\mathbb{E}^2 \setminus \mathcal{O}^*(\mathcal{U}_i)}$.
(3) $g_i|_{\mathrm{Bd}(U[\widehat{P}, \delta_i])} = H_i|_{\mathrm{Bd}(U[\widehat{P}, \delta_i])}$ for all $\widehat{P} \in O(\mathcal{M}_i)$.
(4) If $\widehat{P} \in \mathcal{O}(\mathcal{M}_i)$, then $g_i(U[\widehat{P}, \delta_i]) = U[\widehat{P}, \delta_i]$ and $g_i|_{\widehat{P}}$ maps $\widehat{P}$ onto $\widetilde{P} = \widetilde{\omega}_i(\widehat{P})$ in such a way that $g_i(H_i(\mathcal{K}_{i+1}) \cap \widehat{P}) = \mathcal{M}_{i+1} \cap \widetilde{P}$.
(5) If $A \in \mathcal{A}(\mathcal{K}_i)$ and $h_{i_A + j}(A) = p_j q_j \in \mathcal{A}(\mathcal{M}_i)$, then $g_i|_{p_j q_j} = g_{j+1}^A$.
(6) $H_{i+1} = g_i \circ H_i$.

Let $H : \mathbb{E}^2 \to \mathbb{E}^2$ be the limit of the sequence $\{H_i\}_{i=1}^\infty$ of homeomorphisms.

We will prove that H is a homeomorphism and $H(\mathcal{K}) \subseteq \bigcap_{i=1}^\infty \mathcal{M}_i$.

Note that $H_i(\mathcal{K}) \subseteq H_i(\mathcal{K}_i)$ and $H_{i+1}(\mathcal{K}_{i+1}) \subseteq H_i(\mathcal{K}_i)$ for all i. Since $H_i(\mathcal{K}_i) = \mathcal{M}_i$ for all i, we obtain

$$H(\mathcal{K}) = \lim_{i \to \infty} H_i(\mathcal{K}) \subseteq \bigcap_{i=1}^\infty H_i(\mathcal{K}_i) = \bigcap_{i=1}^\infty \mathcal{M}_i.$$

Let $\widehat{H} : \mathbb{E}^2 \to \mathbb{E}^2$ be the limit of the sequence $\{H_i\}_{i=2}^\infty$. Since $H = H_1 \circ \widehat{H}$ and H_1 is a homeomorphism, it suffices to show that $\widehat{H}$ is a homeomorphism.

From properties (2) and (6) it follows that $\widehat{H}_{i+1} \equiv \widehat{H}_i$ on $\mathbb{E}^2 \setminus \mathcal{O}^*(\mathcal{U}_i)$. Since in addition $\lim_{i \to \infty} \mathrm{mesh}(\mathcal{O}^*(\mathcal{U}_i)) = 0$, the homeomorphisms $\widehat{H}_i$ converge uniformly to $\widehat{H}$. Thus $\widehat{H}$ is continuous.

Since $\widehat{H}|_{\mathbb{E}^2\setminus\mathcal{U}_1} = H_2|_{\mathbb{E}^2\setminus\mathcal{U}_1}$, it remains to prove that $\widehat{H}$ is one-to-one on the compact set $\mathcal{U}_1$. From $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$, it follows that $\mathcal{U}_1 = (\bigcup_{i=1}^{\infty}(\mathcal{U}_i \setminus \mathcal{U}_{i+1})) \cup (\bigcap_{i=1}^{\infty}\mathcal{U}_i)$. Since $\widehat{H}|_{\mathcal{U}_i\setminus\mathcal{U}_{i+1}} = H_{i+1}|_{\mathcal{U}_i\setminus\mathcal{U}_{i+1}}$ is a homeomorphism and the family $\{\mathcal{U}_i \setminus \mathcal{U}_{i+1}\}_{i=1}^{\infty}$ consists of pairwise disjoint sets, it suffices to show that $\widehat{H}$ is one-to-one on $\bigcap_{i=1}^{\infty}\mathcal{U}_i$. It is easy to verify that $\bigcap_{i=1}^{\infty}\mathcal{U}_i = \bigcap_{i=1}^{\infty}\mathcal{M}_i = (\bigcap_{i=1}^{\infty}\mathcal{O}^*(\mathcal{M}_i)) \cup (\bigcup_{i=1}^{\infty}\mathcal{A}^*(\mathcal{M}_i))$.

By (4) for any i and for any $\widehat{P} \in \mathcal{O}(\mathcal{M}_i)$ it follows that $H_i(\widehat{P}) = \widetilde{P} \subseteq \mathrm{Int}(\widehat{P})$. Since $\lim_{i\to\infty}\mathrm{mesh}(\mathcal{O}(\mathcal{M}_i)) = 0$, we conclude that $\widehat{H}$ is one-to-one on $\bigcap_{i=1}^{\infty}\mathcal{O}(\mathcal{M}_i)$.

Let $x, y \in \bigcup_{i=1}^{\infty}\mathcal{A}^*(\mathcal{M}_i)$ and $x \neq y$. Then $H_1(x) \neq H_1(y)$.

If there exist $i \in \mathbb{N}\setminus\{0\}$ and $A \in \mathcal{A}(\mathcal{K}_i)$ such that $x, y \in h_i(A) \in \mathcal{A}(\mathcal{M}_i)$, then (5) yields $H|_A = H^A = \widehat{H}|_{H_1(A)} \circ H_1|_A$. Thus $\widehat{H}(x) \neq \widehat{H}(y)$.

In the other case there exist $i_x, i_y \in \mathbb{N}\setminus\{0\}$, $A_x \in \mathcal{A}(\mathcal{K}_{i_x})$, and $A_y \in \mathcal{A}(\mathcal{K}_{i_y})$ with $A_x \cap A_y = \emptyset$, $x \in h_{i_x}(A_x) \in \mathcal{A}(\mathcal{M}_{i_x})$, and $y \in h_{i_y}(A_y) \in \mathcal{A}(\mathcal{M}_{i_y})$.

Without loss of generality we can assume $i_{A_x} \leq i_{A_y}$. Then $A_x, A_y \in \mathcal{A}(\mathcal{K}_i)$ for any $i \geq i_{A_y}$. Thus $h_{i_A+i}(A_x) \cap h_{i_A+i}(A_y) = \emptyset$ for each $i \geq i_{A_y}$.

Since the endpoints of the arcs A_x and A_y are in $\mathcal{O}^*(\mathcal{K}_i)$ for each $i \geq i_{A_y}$ and $\lim_{i\to\infty}\mathrm{mesh}(\mathcal{O}^*(\mathcal{K}_i)) = 0$, there is $i_0 \geq i_{A_y}$ such that the endpoints of arcs A_x and A_y are separated in $\mathcal{O}(\mathcal{K}_{i_A+i_0})$. From (4_i) it follows that the endpoints of arcs $h_{i_A+i}(A_x)$ and $h_{i_A+i}(A_y)$ are separated in $\mathcal{O}(\mathcal{M}_{i_A+i_0})$ for each $i \geq i_0$.

Since $\widehat{H}(H_1(A_x)) = \bigcup_{i=1}^{\infty} h_{i_A+i}(A_x)$ and $\widehat{H}(H_1(A_y)) = \bigcup_{i=1}^{\infty} h_{i_A+i}(A_y)$, it follows that $\widehat{H}(A_x) \cap \widehat{H}(A_y) = \emptyset$. Hence, $\widehat{H}(x) \neq \widehat{H}(y)$. ∎

Theorems 2.1 and 4.3 imply the following corollary.

COROLLARY 4.4. *$\mathcal{Z}$ is a universal planar completely regular continuum.*

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References

[1] S. D. Iliadis, *Rim-finite spaces and the property of universality*, Houston J. Math. 12 (1986), 55–78.

[2] S. D. Iliadis, *Universal continuum for the class of completely regular continua*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 28 (1980), 603–607.

[3] J. Krasinkiewicz, *On two theorems of Dyer*, Colloq. Math. 50 (1986), 201–208.

[4] K. Kuratowski, *Topology, Vol. II*, New York, 1968.
[5] K. Omiljanowski and S. Zafiridou, *Universal completely regular dendrites*, Colloq. Math. 103 (2005), 149–154.
[6] S. Zafiridou, *A note about planar completely regular continua with structure*, Topology Appl. 123 (2002), 199–203.
[7] S. Zafiridou, *Planar completely regular continua and the problem of universality*, Questions Answers Gen. Topology 22 (2004), 61–72.

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