FUNDAMENTA
MATHEMATICAE
221 (2013)

Relational quotients

by

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Abstract. Let $\mathcal{K}$ be a class of finite relational structures. We define $\mathcal{EK}$ to be the class of finite relational structures $\mathbf{A}$ such that $\mathbf{A}/E \in \mathcal{K}$, where E is an equivalence relation defined on the structure $\mathbf{A}$. Adding arbitrary linear orderings to structures from $\mathcal{EK}$, we get the class $\mathcal{OEK}$. If we add linear orderings to structures from $\mathcal{EK}$ such that each E-equivalence class is an interval then we get the class $\mathcal{CE}[\mathcal{K}^*]$. We provide a list of Fraïssé classes among $\mathcal{EK}$, $\mathcal{OEK}$ and $\mathcal{CE}[\mathcal{K}^*]$. In addition, we classify $\mathcal{OEK}$ and $\mathcal{CE}[\mathcal{K}^*]$ according to the Ramsey property. We also conduct the same analysis after adding additional structure to each equivalence class. As an application, we give a topological interpretation using the technique introduced in Kechris, Pestov and Todorčević. In particular, we extend the lists of known extremely amenable groups and universal minimal flows.

1. Introduction. A standard mathematical approach is to take a structure $\mathbf{A}$ with an equivalence relation E and define its quotient structure $\mathbf{A}/E$. In this paper, we present the opposite approach. We start with a relational structure $\mathbf{B}$ and consider a relational structure $\mathbf{A}$ with relation E such that $\mathbf{A}/E = \mathbf{B}$. Roughly speaking, we replace points by finite equivalence classes. It turns out that we get a class of structures suitable for application of the technique developed by Kechris–Pestov–Todorčević (KPT) (see [11]).

We start with a class $\mathcal{K}$ of finite relational structures in a given signature $L = \{R_i\}_{i\in I}$. We then define the class $\mathcal{EK}$ of finite relational structures in the signature $L \cup \{E\}$, $E \notin L$, such that for all $\mathbf{A} \in \mathcal{EK}$ we have $\mathbf{A}/E \in \mathcal{K}$. The structure $\mathbf{A}/E$ will be defined in Definition 2.1. We continue by adding linear orderings to structures from $\mathcal{EK}$. Adding arbitrary linear orderings to structures from $\mathcal{EK}$, we get the class $\mathcal{OEK}$. We say that a linear ordering $\leq$ defined on the structure $\mathbf{A} \in \mathcal{EK}$, which has the equivalence relation E, is *convex* if each E-equivalence class is an interval with respect to $\leq$. Let $\mathcal{K}^*$ be obtained by adding linear orderings to structures from $\mathcal{K}$ in some way. The class $\mathcal{K}^*$ is a subclass of the class of all finite order expansions of $\mathcal{K}$.

2010 *Mathematics Subject Classification*: Primary 05D10; Secondary 03C15, 54H20.
Key words and phrases: Ramsey property, Fraïssé class, linear ordering.

DOI: 10.4064/fm221-3-1

We define the class $\mathcal{CE}[\mathcal{K}^*] \subseteq \mathcal{OEK}$ to consist of structures $\mathbf{A}$ such that $\mathbf{A}/E \in \mathcal{K}^*$.

Let $\mathcal{L}$ be a class of finite relational structures in a given signature $L' = \{R_j\}_{j\in J}$ such that $L \cap L' = \emptyset$. We define a class $\mathcal{LEK}$ of finite relational structures in the signature $L \cup L'$. Structures from $\mathcal{LEK}$ are obtained by putting a structure from $\mathcal{L}$ on each equivalence class of a structure from $\mathcal{EK}$. On distinct equivalence classes we may put different structures from $\mathcal{L}$. If we add arbitrary linear orderings to structures from $\mathcal{LEK}$, we obtain the class $\mathcal{OLEK}$. Let $\mathcal{K}^*$ and $\mathcal{L}^*$ be the classes obtained by adding linear orderings to structures from $\mathcal{K}$ and $\mathcal{L}$ respectively. Using $\mathcal{K}^*$ and $\mathcal{L}^*$, we define the class $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$, which is obtained from $\mathcal{LEK}$ by adding linear orderings to structures from $\mathcal{LEK}$. If $\mathbf{A} \in \mathcal{OLEK}$ then by dropping relations which interpret symbols from L' we obtain a structure $\mathbf{A}^*$ from $\mathcal{OEK}$. If $\mathbf{A}^* \in \mathcal{CE}[\mathcal{K}^*]$, and the ordered structures defined on each equivalence class with respect to L' belongs to $\mathcal{L}^*$, then $\mathbf{A}^*$ belong to the class $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$.

Fraïssé classes and structures are extensively studied in model theory (see [6] and [10]). Fraïssé classes of graphs have been classified in [12]. The class of hypergraphs of a given type is also a Fraïssé class (see [11]). The list of known Fraïssé classes includes finite vector spaces over a given finite field (see [30]), finite boolean algebras (see [11]), finite metric spaces with rational distances (see [19]), and finite ultrametric spaces (see [18]). In particular, we are interested in extending the list of known Fraïssé classes. In Section 3, we extend this list with the following results.

Theorem 1.1. *If $\mathcal{K}$ and $\mathcal{K}^*$ are Fraïssé classes then $\mathcal{EK}$, $\mathcal{OEK}$ and $\mathcal{CE}[\mathcal{K}^*]$ are Fraïssé classes.*

We denote by $\mathbf{EK}$, $\mathbf{OEK}$ and $\mathbf{CE}[\mathbf{K}^*]$ the Fraïssé limits of $\mathcal{EK}$, $\mathcal{OEK}$ and $\mathcal{CE}[\mathcal{K}^*]$ respectively.

Theorem 1.2. *If $\mathcal{K}$, $\mathcal{L}$, $\mathcal{K}^*$ and $\mathcal{L}^*$ are Fraïssé classes then $\mathcal{LEK}$ and $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ are Fraïssé classes.*

Theorems 1.1 and 1.2 are proved in Section 3 (see discussion after Lemma 3.8). Note that $\mathcal{OLEK}$ is not always a Fraïssé class (see Section 3). If $\mathcal{LEK}$, $\mathcal{OLEK}$ and $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ are Fraïssé classes then we denote their Fraïssé limits by $\mathbf{LEK}$, $\mathbf{OLEK}$ and $\mathbf{C}[\mathbf{L}^*]\mathbf{E}[\mathbf{K}^*]$ respectively.

Let $\mathbf{C}$ and $\mathbf{B}$ be given structures. If $\mathbf{C}$ is isomorphic to $\mathbf{B}$, we write $\mathbf{C} \cong \mathbf{B}$, but if $\mathbf{C}$ is a substructure of $\mathbf{B}$ we write $\mathbf{C} \leq \mathbf{B}$. Also, we use the notation

$$\binom{\mathbf{B}}{\mathbf{A}} = \{\mathbf{C} : \mathbf{C} \leq \mathbf{B}, \mathbf{C} \cong \mathbf{A}\}.$$

A class of structures $\mathcal{K}$ has the *Ramsey property* (RP) if for every natural number r and every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there is $\mathbf{C} \in \mathcal{K}$ such that for every coloring

$$c : \binom{\mathbf{C}}{\mathbf{A}} \to \{1, \dots, r\},$$

there is $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ satisfying

$$c\restriction\binom{\mathbf{B}'}{\mathbf{A}} = \text{const}.$$

We then use the arrow notation

$$\mathbf{C} \to (\mathbf{B})^A_r.$$

Section 4 is devoted to the analysis of the Ramsey property for the classes $\mathcal{OEK}$ and $\mathcal{CE}[\mathcal{K}^*]$. The list of Ramsey classes includes the class of linearly ordered graphs (see [1] and [16]), the class of linearly ordered finite metric spaces (see [13]), and the class of finite convexly ordered ultrametric spaces (see [18]). The following results are generalizations of a result from [14] in the sense that we examine general relational structures instead of partial orderings.

Theorem 1.3. *If $\mathcal{K}^*$ is a Ramsey class, then $\mathcal{CE}[\mathcal{K}^*]$ is a Ramsey class.*

Theorem 1.4. *If $\mathcal{K}^*$ and $\mathcal{L}^*$ are Ramsey classes, then $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ is a Ramsey class.*

We say that a structure $\mathbf{A}$ is *rigid* if its group of automorphisms $\mathrm{Aut}(\mathbf{A})$ is trivial. Ramsey classes of rigid structures are of particular interest.

Theorem 1.5. *If $\mathcal{K}$ is a Ramsey class of rigid structures, then $\mathcal{OEK}$ is a Ramsey class.*

We obtain a similar result for the class $\mathcal{OLEK}$ under additional requirements on the class $\mathcal{L}$ (see Section 4). Theorems 1.3, 1.4 and 1.5 are proved as Theorems 4.5, 4.4 and 4.9 respectively.

In Section 6 we present a topological application of our results. Let G be a topological group. A *G-flow* X is a continuous action $G \times X \to X$ on a compact Hausdorff space X. A G-flow X is called *minimal* if each of its orbits is dense, i.e. $X = \overline{G \cdot x}$ for all $x \in X$. Among all minimal G-flows there is a largest one called the *universal minimal G-flow* (see [2]). If every G-flow X has a fixed point $x \in X$, $gx = x$ for all $g \in G$, then we say that G is an *extremely amenable group.* The list of extremely amenable groups includes the group of increasing homeomorphisms of the unit interval (see [21]), the unitary group of infinite-dimensional separable Hilbert space (see [8]), the group of isometries of Urysohn space (see [22]), the pathological groups (see [9]), and the group of automorphisms of the rationals (see [21]). The technique from [11] helps us to extend this list and to calculate some universal minimal flows. In Section 6 we present the calculation of the universal minimal flow for the group $\mathrm{Aut}(\mathbf{EK})$, as well as the following; in the following theorems we assume that $\mathcal{K}$, $\mathcal{L}$, $\mathcal{K}^*$ and $\mathcal{L}^*$ are Fraïssé classes.

Theorem 1.6. *If $\mathcal{K}^*$ is a Ramsey class obtained by adding linear orderings to structures from $\mathcal{K}$, then $\mathrm{Aut}(\mathbf{CE}[\mathbf{K}^*])$ is an extremely amenable group.*

Theorem 1.7. *If $\mathcal{K}^*$ and $\mathcal{L}^*$ are Ramsey classes, then* $\mathrm{Aut}(\mathbf{C}[\mathbf{L}^*]\mathbf{E}[\mathbf{K}^*])$ *is an extremely amenable group.*

Theorems 1.6 and 1.7 are proved as Theorems 6.4(ii) and 6.5(ii) respectively.

Section 7 is devoted to applications of our results to the case where $\mathcal{K}$ is the class of finite posets. Then $\mathcal{E}\mathcal{K}$ is the class of finite quasi ordered sets. We use the connection between the class of finite topological spaces and the class of finite quasi ordered sets, so our results can be viewed as results about linearly ordered finite topological spaces. Section 8 is dedicated to the application of our results to the case where $\mathcal{K}$ is the class of finite metric spaces with rational distances, so $\mathcal{E}\mathcal{K}$ represents the class of finite pseudometric spaces with rational distances. In Sections 9 and 10 we give applications to the class of ultrametric spaces, the class of graphs and the class of chains respectively.

2. Preliminaries. A *relational signature* is a collection of distinct relational symbols, $L = \{R_i\}_{i\in I}$. Each relational symbol R_i has an assigned non-zero natural number n_i, called its *arity*.

A *structure* in a relational signature $L = \{R_i\}_{i\in I}$ with arity $\{n_i\}_{i\in I}$ is a set A together with relations $\{R_i^A\}_{i\in I}$ on A such that $R_i^A \subseteq A^{n_i}$ for all $i \in I$. R_i^A is called the *interpretation* of the relational symbol R_i in the structure $\mathbf{A} = (A, \{R_i^A\}_{i\in I})$.

Let $\mathbf{B} = (B, \{R_i^B\}_{i\in I})$ be a structure in the signature L. The map $f : A \to B$ is called a *morphism* if for all $i \in I$ and all $x_1, \ldots, x_{n_i} \in A$,

$$R_i^A(x_1, \ldots, x_{n_i}) \;\Leftrightarrow\; R_i^B(f(x_1), \ldots, f(x_{n_i})).$$

In this case we write $f : \mathbf{A} \to \mathbf{B}$. An injective morphism is called an *embedding*, while a bijective morphism is called an *isomorphism*. If there is an embedding of the structure $\mathbf{A}$ into the structure $\mathbf{B}$, we write $\mathbf{A} \hookrightarrow \mathbf{B}$, while isomorphic structures are denoted by $\mathbf{A} \cong \mathbf{B}$. A structure $\mathbf{A}$ is a *substructure* of $\mathbf{B}$, in symbols $\mathbf{A} \le \mathbf{B}$, if $A \subseteq B$, and for all $i \in I$, $R_i^A = R_i^B \cap A^{n_i}$.

In this paper we consider only classes of finite structures closed under isomorphisms.

A class of finite structures $\mathcal{K}$ has the:

(1) *hereditary property* (HP) if for $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B} \le \mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$;
(2) *joint embedding property* (JEP) if for all structures $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there is $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{A} \hookrightarrow \mathbf{C}$ and $\mathbf{B} \hookrightarrow \mathbf{C}$;
(3) *amalgamation property* (AP) if for all structures $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and all embeddings $i_B : \mathbf{A} \to \mathbf{B}$, $i_C : \mathbf{A} \to \mathbf{C}$ there are $\mathbf{D} \in \mathcal{K}$ and embeddings $j_B : \mathbf{B} \to \mathbf{D}$, $j_C : \mathbf{C} \to \mathbf{D}$ such that $j_B \circ i_B = j_C \circ i_C$;

(4) *strong joint embedding property* (SJEP) if for all structures $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ defined on sets A and B respectively, there are $\mathbf{C} \in \mathcal{K}$ and embeddings $i_A : \mathbf{A} \to \mathbf{C}$, $i_B : \mathbf{B} \to \mathbf{C}$ such that $i_A(A) \cap i_B(B) = \emptyset$;
(5) *strong amalgamation property* (SAP) if for all structures $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ defined on sets A, B, C respectively, and all embeddings $i_B : \mathbf{A} \to \mathbf{B}$, $i_C : \mathbf{A} \to \mathbf{C}$ there are $\mathbf{D} \in \mathcal{K}$ and embeddings $j_B : \mathbf{B} \to \mathbf{D}$, $j_C : \mathbf{C} \to \mathbf{D}$ such that $j_B(B) \cap j_C(C) = j_B \circ i_B(A) = j_C \circ i_C(A)$.

It should be clear that SJEP and SAP imply JEP and AP respectively. For our analysis the properties HP, JEP and AP are of main importance, and the properties SJEP and SAP appear naturally.

Consider two relational signatures $L_I = \{R_i\}_{i\in I}$ and $L_J = \{R_i\}_{i\in J}$ such that $I \subseteq J$. Assume that the structures $\mathbf{A}_I = (A, \{R_i^{A_I}\}_{i\in I})$ in the signature L_I, and $\mathbf{A}_J = (A, \{R_i^{A_J}\}_{i\in J})$ in the signature J, are given on the same set A. If $R_i^{A_I} = R_i^{A_J}$ for all $i \in I$ then we say that $\mathbf{A}_J$ is an *expansion* of $\mathbf{A}_I$ or that $\mathbf{A}_I$ is a *reduct* of $\mathbf{A}_J$. We write $\mathbf{A}_I = \mathbf{A}_J | L_I$ and $\mathbf{A}_J = (\mathbf{A}_I, \{R_i^{A_J}\}_{i\in J\setminus I})$. For a class $\mathcal{K}_J$ of structures in the signature L_J, we denote the class of their reducts by $\mathcal{K}_J | L_I = \{\mathbf{A}|L_I : \mathbf{A} \in \mathcal{K}\} = \mathcal{K}_I$, and we say that $\mathcal{K}_J$ is an *expansion of* $\mathcal{K}_I$ or that $\mathcal{K}_I$ is a *reduct of* $\mathcal{K}_J$.

Let $L_J = L_I \cup \{<\}$ be a signature such that $<$ is a binary relational symbol satisfying $< \notin L_I$. Assume that $\mathcal{K}_I$ and $\mathcal{K}_J$ are classes of finite structures in the signatures L_I and L_J respectively such that the symbol $<$ is interpreted as a linear ordering in each structure from $\mathcal{K}_J$. If $\mathcal{K}_J | L_I = \mathcal{K}_I$ then:

(1) We say that $\mathcal{K}_J$ is an *ordered class.*
(2) If for all $\mathbf{A}_I, \mathbf{B}_I \in \mathcal{K}_I$, every linear ordering $<^{A_I}$ satisfying $(\mathbf{A}_I, <^{A_I}) \in \mathcal{K}_J$, and every embedding $\pi : \mathbf{A}_I \to \mathbf{B}_I$, there is a linear ordering $<^{B_I}$ such that $(\mathbf{B}_I, <^{B_I}) \in \mathcal{K}_J$ and π is also an embedding from $(\mathbf{A}_I, <^{A_I})$ into $(\mathbf{B}_J, <^{B_J})$, then we say that $\mathcal{K}_J$ is a *reasonable expansion* of $\mathcal{K}_I$.
(3) If for all $\mathbf{A}_I, \mathbf{B}_I \in \mathcal{K}_I$ satisfying $\mathbf{A}_I \hookrightarrow \mathbf{B}_I$, and all linear orderings $<^{A_I}$ and $<^{B_I}$ satisfying $(\mathbf{A}_I, <^{A_I}) \in \mathcal{K}_J$ and $(\mathbf{B}_I, <^{B_I}) \in \mathcal{K}_J$, we have $(\mathbf{A}_I, <^{A_I}) \hookrightarrow (\mathbf{B}_I, <^{B_I})$, then we say that $\mathcal{K}_J$ has the *ordering property* (OP) *with respect to* $\mathcal{K}_I$.

Definition 2.1. Let $\mathcal{K}$ be a class of finite relational structures in the signature $L = \{R_i\}_{i\in I}$, and let E be a binary relational symbol such that $E \notin L$. Then $\mathcal{EK}$ is the class of relational structures of the form $\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A)$ in the signature $L \cup \{E\}$ such that:

(1) E^A is an equivalence relation on the set A.
(2) For all $i \in I$ and all $x_1, \ldots, x_{n_i}, y_1, \ldots, y_{n_i} \in A$ with $x_j E^A y_j$, $1 \leq j \leq n_i$, we have

$$R_i^A(x_1, \ldots, x_{n_i}) \ \Leftrightarrow\ R_i^A(y_1, \ldots, y_{n_i}).$$

(3) $\mathbf{A}/E = (A/E, \{R_i^{A/E}\}_{i\in I}) \in \mathcal{K}$, where $A/E = \{[a]_{E^A} : a \in A\}$ is the set of equivalence classes of the form $[a]_{E^A} = \{x \in A : aE^A x\}$, where the relations $\{R_i^{A/E}\}_{i\in I}$ are well-defined on the set A/E^A according to condition (2) with
$$R_i^{A/E}([a_1]_{E^A}, \ldots, [a_{n_i}]_{E^A}) \;\Leftrightarrow\; R_i^A(x_1, \ldots, x_{n_i}),$$
where $a_j E^A x_j$ for all j.

Note that each element of $\mathcal{K}$ can be treated as an element of $\mathcal{EK}$, where each equivalence class has exactly one element. Let $\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A)$ and $\mathbf{B} = (B, \{R_i^B\}_{i\in I}, E^B)$ be two structures from $\mathcal{EK}$, and let $f : \mathbf{A} \to \mathbf{B}$ be a morphism. Then for all $x, y \in A$ we have $xE^A y \Leftrightarrow f(x)E^B f(y)$. Consequently, each morphism of structures from $\mathcal{EK}$ produces an injective map from the set A/E into B/E, and moreover this induces an embedding of structures from the class $\mathcal{K}$. We denote by $f_E : \mathbf{A}/E \to \mathbf{B}/E$ the map in the class $\mathcal{K}$ induced by f.

The approach from [11] needs examination of ordered classes. Naturally, we add linear orderings to structures from $\mathcal{EK}$.

Definition 2.2. Let $\mathcal{K}$ be a class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let E and $\leq$ be binary relational symbols such that $E \notin L$ and $\leq\, \notin L$. Then $\mathcal{OEK}$ is the class of structures of the form $(A, \{R_i^A\}_{i\in I}, E^A, \leq^A)$ such that $(A, \{R_i^A\}_{i\in I}, E^A) \in \mathcal{EK}$ and $\leq^A$ is a linear ordering on A.

Let $\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A)$ be a structure from $\mathcal{EK}$. A linear ordering $\leq^A$ on the set A which satisfies for all $x, y, z \in A$:
$$x \leq^A y \leq^A z,\, xE^A z \;\Rightarrow\; xE^A yE^A z,$$
is called *convex with respect to* E^A.

Let $\mathcal{K}^*$ be an order expansion of the class $\mathcal{K}$. Structures from $\mathcal{K}^*$ have the form $(A, \{R_i\}_{i\in I}, \prec)$ such that $(A, \{R_i\}_{i\in I}) \in \mathcal{K}$ and $\prec$ is a linear ordering on the set A, and $\prec\, \neq R_i$ for $i \in I$.

Definition 2.3. Let $\mathcal{K}$ be a class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let E, $\leq$ and $\preceq$ be binary relational symbols such that $E \notin L$, $\leq\, \notin L$ and $\preceq\, \notin L$. Let $\mathcal{K}^*$ be a class of finite relational structures in a signature $L \cup \{\preceq\}$ such that $\mathcal{K}^*$ is an ordered expansion of $\mathcal{K}$ and $\mathcal{K}^*|L = \mathcal{K}$. Then $\mathcal{CE}[\mathcal{K}^*]$ is the class of structures $\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A, \leq^A) \in \mathcal{OEK}$ such that:

(1) $\leq^A$ is convex with respect to E^A.

(2) $\mathbf{A}/E = (A/E, \{R_i^{A/E}\}_{i\in I}, \leq^{A/E}) \in \mathcal{K}^*$, where the linear ordering $\leq^{A/E}$ on A/E is well-defined according to (1) with, for all $a, b \in A$,
$$[a]_{E^A} \leq^{A/E} [b]_{E^A} \;\Leftrightarrow\; ([a]_{E^A} = [b]_{E^A} \text{ or } ([a]_{E^A} \neq [b]_{E^A}, a \leq^A b)).$$

For the rest of this section we fix the relational signatures $L_I = \{R_i\}_{i\in I}$ and $L_J = \{R_i\}_{i\in J}$ with arities $\{n_i\}_{i\in I}$ and $\{n_j\}_{j\in J}$ respectively such that $L_I \cap L_J = \emptyset$. We also assume that E and $\leq$ are binary symbols which do not belong to $L_I \cup L_J$. Let $\mathcal{K}$ be a class of finite relational structures in L_I, and let $\mathcal{L}$ be a class of finite relational structures in L_J.

Definition 2.4. $\mathcal{LEK}$ is the class of relational structures of the form $\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A, \{R_j^A\}_{j\in J})$ in the signature $L_I \cup \{E\} \cup L_J$ such that:

(1) $(A, \{R_i^A\}_{i\in I}, E^A) \in \mathcal{EK}$.
(2) For all $j \in J$ and all $x_1, \ldots, x_{n_j} \in A$ we have

$$R_j^A(x_1, \ldots, x_{n_j}) \ \Rightarrow\ [x_1]_{E^A} = \cdots = [x_{n_j}]_{E^A}.$$

(3) For all $a \in A$ we have

$$([a]_{E^A}, \{R_j^A \cap ([a]_{E^A})^{n_j}\}_{i\in J}) \in \mathcal{L}.$$

We will represent the structures from the class $\mathcal{LEK}$ in abbreviated notation. Let $\mathbf{C}$ be the structure in the class $\mathcal{K}$ with underlying set C such that $\mathbf{C} = (A/E, \{R_i^{A/E}\}_{i\in I})$. Let $\mathbf{D}_c$ be the structure from $\mathcal{L}$ such that $\mathbf{D}_c = (c, \{R_j^A \cap (c^{n_j}\}_{j\in J})$. Then we denote the structure $\mathbf{A}$ from the previous definition by $\langle \mathbf{C}, (\mathbf{D}_c)_{c\in C}\rangle$.

Note that each element of $\mathcal{LEK}$ is obtained by transforming the points of some structure from $\mathcal{K}$ into equivalence classes which have their own structure from $\mathcal{L}$. We recall that each element of $\mathcal{K}$ can be treated as an element of $\mathcal{EK}$, where each equivalence class has exactly one element. This is not the case for the class $\mathcal{LEK}$ in general since points may carry additional structure.

Let $\mathbf{A}=(A, \{R_i^A\}_{i\in I}, E^A, \{R_j^A\}_{i\in J})$ and $\mathbf{B} = (B, \{R_i^B\}_{i\in I}, E^B, \{R_j^B\}_{i\in J})$ be two structures from $\mathcal{LEK}$, and let $f : \mathbf{A} \to \mathbf{B}$ be a morphism. Then we have the induced morphism of reducts, which we also denote by f, $f : \mathbf{A}|(L_I \cup \{E\}) \to \mathbf{B}|(L_I \cup \{E\})$, and a sequence of morphisms $\{f_a\}_{a\in A}$ of structures in $\mathcal{L}$. Each f_a is a morphism from the substructure of $\mathbf{A}$ induced on $[a]_{E^A}$ into the substructure of $\mathbf{B}$ induced on $[f(a)]_{E^B}$. As in the case of the class $\mathcal{EK}$, the map f produces an injective map from A/E into the set B/E, and an embedding of structures from $\mathcal{K}$, which we denote by $f_E : \mathbf{A}/E \to \mathbf{B}/E$.

If we add arbitrary linear orderings to structures from $\mathcal{LEK}$, we obtain the following class.

Definition 2.5. $\mathcal{OLEK}$ is the class of structures of the form

$$(A, \{R_i^A\}_{i\in I}, E^A, \{R_j^A\}_{i\in J}, \leq^A)$$

such that $\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A, \{R_j^A\}_{i\in J}) \in \mathcal{LEK}$ and $\leq^A$ is a linear ordering on A.

If in the previous definition we denote the structure $\mathbf{A}$ by $\langle \mathbf{C}, (\mathbf{D}_c)_{c\in C}\rangle$, then the structure obtained by adding the linear ordering $\leq^A$ is denoted by $\langle \mathbf{C}, (\mathbf{D}_c)_{c\in C}, \leq^A\rangle$.

Let $\mathcal{K}^*$ and $\mathcal{L}^*$ be order expansions of $\mathcal{K}$ and $\mathcal{L}$ respectively. Structures from $\mathcal{K}^*$ and $\mathcal{L}^*$ have the form $(A, \{R_i\}_{i\in I}, \prec^A)$ and $(B, \{R_j\}_{j\in J}, \prec^B)$ such that $(A, \{R_i\}_{i\in I}) \in \mathcal{K}$, $(B, \{R_j\}_{j\in J}) \in \mathcal{L}$ and $\prec^A$ and $\prec^B$ are linear orderings on the sets A and B respectively. We assume that $\prec \notin L_I \cup L_J \cup R_i \cup \{E, \leq\}$.

Definition 2.6. $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ is the class of structures

$$\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A, \{R_j^A\}_{i\in J}, \leq^A) \in \mathcal{OLEK}$$

such that:

(1) $(A, \{R_i^A\}_{i\in I}, E^A, \leq^A) \in \mathcal{CE}[\mathcal{K}^*]$.
(2) $([a]_{E^A}, \{R_j^A \cap ([a]_{E^A})^{n_j}\}_{i\in J}, \leq^A \cap ([a]_{E^A})^2) \in \mathcal{L}^*$ for every $a \in A$.

Suppose that the structure $(A, \{R_i^A\}_{i\in I}, E^A, \{R_j^A\}_{i\in J})$ in the previous definition is denoted by $\langle \mathbf{C}, (\mathbf{D}_c)_{c\in C}\rangle$. The linear ordering $\leq^A$ induces a linear ordering $\leq^C$ on C and linear orderings $\leq^{D_c}$ for each structure $\mathbf{D}_c$, $c \in C$, such that $(\mathbf{C}, \leq^C) \in \mathcal{K}^*$ and $(\mathbf{D}_c, \leq^{D_c}) \in \mathcal{L}^*$ for all $c \in C$. Then we denote the structure $\mathbf{A}$ from the previous definition by $\langle (\mathbf{C}, \leq^C), ((\mathbf{D}_c, \leq^{D_c}))_{c\in C}\rangle$.

Note that in the class $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$, the linear ordering restricted to an equivalence class can depend on the structure, but in the class $\mathcal{CE}[\mathcal{K}^*]$ this is not the case.

3. Fraïssé classes. A structure $\mathbf{A}$ is called *ultrahomogeneous* if every isomorphism between finite substructures $\mathbf{S}$ and $\mathbf{T}$ of $\mathbf{A}$ can be extended to an automorphism of $\mathbf{A}$. A countable infinite structure which is ultrahomogeneous and has the property that every finitely generated substructure is finite is called a *Fraïssé structure*. Note that every relational structure has the property that every finitely generated substructure is finite. We say that a class of finite structures is *countable* if the maximal set of its non-isomorphic structures is countable. Given a signature L, a *Fraïssé class in L* is a countable class of finite structures in L which contains structures of arbitrarily large finite cardinality and has HP, JEP, and AP. The *age* of a structure $\mathbf{A}$ is the class $\mathrm{Age}(\mathbf{A})$ of all finitely generated substructures that can be embedded into $\mathbf{A}$.

The following two results establish a connection between Fraïssé classes and Fraïssé structures (see [11, 6, 10]).

Theorem 3.1. *If $\mathbf{A}$ is a Fraïssé structure, then* $\mathrm{Age}(\mathbf{A})$ *is a Fraïssé class.*

Theorem 3.2. *Let $\mathcal{K}$ be a Fraïssé class. Then there is a unique, up to isomorphism, countable structure $\mathbf{A}$ such that $\mathbf{A}$ is a Fraïssé structure and* $\mathcal{K} = \mathrm{Age}(\mathbf{A})$.

The structure from Theorem 3.2 is called the *Fraïssé limit* of $\mathcal{K}$, written $\mathbf{A} = \mathrm{Flim}(\mathcal{K})$.

We examine the classes $\mathcal{EK}$, $\mathcal{CE}[\mathcal{K}^*]$ and $\mathcal{OEK}$ in order to recognize Fraïssé classes among them. We avoid tedious and easy proofs, and we mention only some of the properties.

LEMMA 3.3. *Let $\mathcal{K}$ and $\mathcal{L}$ be classes of finite relational structures in signatures $L = \{R_i\}_{i\in I}$ and $L_J = \{R_j\}_{j\in J}$ respectively. Let $\mathcal{EK}$ and $\mathcal{LEK}$ be as in Definitions* 2.1 *and* 2.4. *Then if $\mathcal{K}$ is a Fraïssé class then so is $\mathcal{EK}$. If $\mathcal{K}$ and $\mathcal{L}$ are Fraïssé classes then so is $\mathcal{LEK}$.*

Note that the class $\mathcal{EK}$ always has SAP if $\mathcal{K}$ has AP, but $\mathcal{LEK}$ will not always have SAP. A similar conclusion holds for SJEP. The following result will help us to examine JEP and AP for the classes $\mathcal{OEK}$ and $\mathcal{OLEK}$.

LEMMA 3.4 (see [11]). *Let L be a relational signature and let $< \notin L$ be a binary relational symbol. Let $\mathcal{K}$ be a class of structures in L and let*

$$\mathcal{OK} = \{(\mathbf{A}, <) : \mathbf{A} \in \mathcal{K} \text{ and } < \text{ is a linear ordering on the underlying set of } \mathbf{A}\}.$$

(i) *$\mathcal{OK}$ has SJEP $\Leftrightarrow$ $\mathcal{K}$ has SJEP.*
(ii) *$\mathcal{OK}$ has SAP $\Leftrightarrow$ $\mathcal{OK}$ has AP $\Leftrightarrow$ $\mathcal{K}$ has SAP.*

The previous two lemmas give the following results.

LEMMA 3.5. *Let $\mathcal{K}$ be a class of finite relational structures in a signature L. Let $\mathcal{EK}$ and $\mathcal{OEK}$ be as in Definitions* 2.1 *and* 2.2. *Then if $\mathcal{K}$ has AP then $\mathcal{OEK}$ has SAP.*

LEMMA 3.6. *Let $\mathcal{K}$ and $\mathcal{L}$ be classes of finite relational structures in signatures $L_I = \{R_i\}_{i\in I}$ and $L_J = \{R_j\}_{j\in J}$ respectively. Let $\mathcal{LEK}$ and $\mathcal{OLEK}$ be as in Definitions* 2.4 *and* 2.5. *Then if $\mathcal{K}$ has AP and $\mathcal{L}$ has SAP then $\mathcal{OLEK}$ has SAP.*

LEMMA 3.7. *Let $\mathcal{K}$ be a class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let $\preceq$ be a binary relational symbol such that $\preceq \notin L$ and let $\mathcal{K}^*$ be a class of finite relational structures in the signature $L \cup \{\preceq\}$ such that $\mathcal{K}^*$ is an ordered expansion of $\mathcal{K}$ and $\mathcal{K}^*|L = \mathcal{K}$. Let $\mathcal{CE}[\mathcal{K}^*]$ be as in Definition* 2.3. *Then if $\mathcal{K}^*$ has AP then $\mathcal{CE}[\mathcal{K}^*]$ has SAP.*

The following lemma is proved by arguments similar to Lemma 3.7.

LEMMA 3.8. *Let $\mathcal{K}$ and $\mathcal{L}$ be classes of finite relational structures in signatures $L_I = \{R_i\}_{i\in I}$ and $L_J = \{R_j\}_{j\in J}$ respectively. Let $\mathcal{K}^*$ and $\mathcal{L}^*$ be ordered expansions of $\mathcal{K}$ and $\mathcal{L}$ respectively. Let $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ be as in Definition* 2.6. *Then if $\mathcal{K}^*$ has AP and $\mathcal{L}^*$ has SAP then $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ has SAP.*

Let I be a countable index set, and let $\mathcal{K}$ be a class of finite relational structures in the signature $L = \{R_i\}_{i\in I}$. If $\mathcal{K}^*$ is an order expansion of $\mathcal{K}$, then $\mathcal{K}^*$ is also countable. Moreover the classes $\mathcal{EK}$, $\mathcal{OEK}$, and $\mathcal{CE}[\mathcal{K}^*]$ are also countable. Therefore, for Fraïssé classes $\mathcal{K}$ and $\mathcal{K}^*$ with $\mathcal{K} = \mathcal{K}^*|L$, we have a list of Fraïssé classes

$$\mathcal{EK}, \quad \mathcal{OEK}, \quad \mathcal{CE}[\mathcal{K}^*],$$

and their Fraïssé limits

$$\mathbf{EK} = \mathrm{Flim}(\mathcal{EK}), \quad \mathbf{OEK} = \mathrm{Flim}(\mathcal{OEK}), \quad \mathbf{CE}[\mathbf{K}^*] = \mathrm{Flim}(\mathcal{CE}[\mathcal{K}^*]).$$

It is straightforward to see that $\mathcal{OEK}$ is a reasonable expansion of $\mathcal{EK}$, and that $\mathcal{CE}[\mathcal{K}^*]$ is a reasonable expansion of $\mathcal{EK}$ if $\mathcal{K}^*$ is a reasonable expansion of $\mathcal{K}$.

Let $\mathcal{K}$ and $\mathcal{L}$ be classes of finite relational structures in signatures $L_I = \{R_i\}_{i\in I}$ and $L_J = \{R_j\}_{j\in J}$ respectively. We assume that $L_I \cap L_J = \emptyset$. Let $\mathcal{K}^*$ and $\mathcal{L}^*$ be ordered expansions of $\mathcal{K}$ and $\mathcal{L}$ respectively. If $\mathcal{K}$ and $\mathcal{L}$ are countable then $\mathcal{K}^*$ and $\mathcal{L}^*$ are countable, as also are the classes $\mathcal{LEK}$, $\mathcal{OLEK}$, and $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$. Therefore, for Fraïssé classes $\mathcal{K}$ and $\mathcal{L}$, the class $\mathcal{LEK}$ is also Fraïssé with limit

$$\mathbf{LEK} = \mathrm{Flim}(\mathcal{LEK}).$$

If $\mathcal{K}$ and $\mathcal{L}$ are Fraïssé classes such that one of them has SJEP and $\mathcal{L}$ has SAP then $\mathcal{OLEK}$ is a Fraïssé class with limit

$$\mathbf{OLEK} = \mathrm{Flim}(\mathcal{OLEK}).$$

If $\mathcal{K}^*$ and $\mathcal{L}^*$ are Fraïssé classes then $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ is a Fraïssé class with limit

$$\mathbf{C}[\mathbf{L}^*]\mathbf{E}[\mathbf{K}^*] = \mathrm{Flim}(\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]).$$

It is straightforward to see that $\mathcal{OLEK}$ is a reasonable expansion of $\mathcal{LEK}$ and that $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ is a reasonable expansions of $\mathcal{EK}$ if $\mathcal{K}^*$ and $\mathcal{L}^*$ are reasonable expansions of $\mathcal{K}$ and $\mathcal{L}$ respectively.

4. Ramsey property. A structure is *rigid* if the group of its automorphisms is trivial.

LEMMA 4.1 ([11]). *Let $\mathcal{K}$ be a class of finite rigid structures. If $\mathcal{K}$ has HP, JEP, and RP then $\mathcal{K}$ has AP.*

The following is the well-known product Ramsey theorem (see [7]).

THEOREM 4.2 ([7]). *Let l and r be natural numbers, and let $(a_i)_{i=1}^l$ and $(b_i)_{i=1}^l$ be sequences of natural numbers satisfying $a_i \leq b_i$ for all $i \leq l$. There is a natural number c such that for all sequences $(C_i)_{i=1}^l$ of sets with $|C_i| \geq c$ and any coloring*

$$p : \{A_1 \times \cdots \times A_l : (\forall i \leq l)[A_i \subseteq C_i,\ |A_i| = a_i]\} \to \{1, \ldots, r\},$$

there is a sequence $(B_i)_{i=1}^l$ *of sets satisfying* $B_i \subseteq C_i$, $|B_i| = b_i$ *for all* $i \le l$ *and*

$$p\restriction\{A_1 \times \cdots \times A_l : (\forall i \le l)[A_i \subseteq B_i,\ |A_i| = a_i]\} = \text{const.}$$

Using the Erdős–Rado arrow notation, we then write

$$c \to (b_1, \dots, b_l)_r^{(a_1,\dots,a_l)}.$$

Note that Theorem 4.2 is the special case of Theorem 4.3 below when we take the sequence $(\mathcal{K}_i)_{i=1}^l$ such that the structures from these classes have no relations on themselves.

Theorem 4.3 ([26]). *Let* l *and* r *be natural numbers, and let* $(\mathcal{K}_i)_{i=1}^l$ *be a sequence of Ramsey classes of finite structures. Let* $(\mathbf{A}_i)_{i=1}^l$ *and* $(\mathbf{B}_i)_{i=1}^l$ *be sequences of structures such that* $\mathbf{A}_i \in \mathcal{K}_i$ *and* $\mathbf{B}_i \in \mathcal{K}_i$ *for all* $i \le l$. *Then there is a sequence* $(\mathbf{C}_i)_{i=1}^l$ *of structures such that* $\mathbf{C}_i \in \mathcal{K}_i$ *for all* $i \le l$ *and for any coloring*

$$p : \{(\mathbf{A}'_1, \dots, \mathbf{A}'_l) : (\forall i \le l)\big[\mathbf{A}'_i \in \tbinom{\mathbf{C}_i}{\mathbf{A}}\big]\} \to \{1, \dots, r\},$$

there is a sequence $(\mathbf{B}'_i)_{i=1}^l$ *of structures satisfying* $\mathbf{B}'_i \in \binom{\mathbf{C}_i}{\mathbf{B}_i} \subseteq C_i$ *for all* $i \le l$ *and*

$$p\restriction\{(\mathbf{A}'_1, \dots, \mathbf{A}'_l) : (\forall i \le l)\big[\mathbf{A}'_i \in \tbinom{\mathbf{B}'_i}{\mathbf{A}}\big]\} = \text{const.}$$

Using the Erdős–Rado arrow notation, we write

$$(\mathbf{C}_1, \dots, \mathbf{C}_l) \to (\mathbf{B}_1, \dots, \mathbf{B}_l)_r^{(\mathbf{A}_1,\dots,\mathbf{A}_l)}.$$

We point out that the proof of the following result is a simplified version of the Nešetřil–Rödl partite construction.

Theorem 4.4. *Let* $\mathcal{K}$ *and* $\mathcal{L}$ *be classes of finite relational structures in signature* $L_I = \{R_i\}_{i\in I}$ *and* $L_J = \{R_j\}_{j\in J}$ *respectively. Let* $\mathcal{K}^*$ *and* $\mathcal{L}^*$ *be ordered expansions of* $\mathcal{K}$ *and* $\mathcal{L}$ *respectively such that* $\mathcal{L}^*$ *has JEP. Let* $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ *be as in Definition* 2.6. *Then* $\mathcal{K}^*$ *and* $\mathcal{L}^*$ *have RP iff* $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ *has RP.*

Proof. ($\Rightarrow$) We suppose that $\mathcal{K}^*$ and $\mathcal{L}^*$ have RP, and will verify it for $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$. Let $\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A, \{R_j^A\}_{i\in J}, \le^A)$ and $\mathbf{B} = (B, \{R_i^B\}_{i\in I}, E^B, \{R_j^B\}_{i\in J}, \le^B)$ be two structures from $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ such that $\binom{\mathbf{B}}{\mathbf{A}} \ne \emptyset$, and let r be a fixed natural number. The structures $\mathbf{A}/E = (A/E, \{R_i^{A/E}\}_{i\in I}, \le^{A/E})$ and $\mathbf{B}/E = (B/E, \{R_i^{B/E}\}_{i\in I}, \le^{B/E})$ belong to the Ramsey class $\mathcal{K}^*$, so there is a structure $\mathbf{C} = (C, \{R_i^C\}_{i\in I}, \le^C) \in \mathcal{K}^*$ such that

$$\mathbf{C} \to (\mathbf{B}/E)_r^{\mathbf{A}/E}.$$

We define the structure $\mathbf{D} = (D, \{R_i^D\}_{i\in I}, E^D, \{R_j^D\}_{i\in J}, \le^D) \in \mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ such that $\mathbf{D}/E = (D/E, \{R_i^{D/E}\}_{i\in I}, \le^{D/E}) = \mathbf{C}$. Each point of C will be replaced with one E^D-equivalence class which carries a structure from $\mathcal{L}^*$.

The relations $\{R_i^D\}_{i\in I}$ are defined by setting for all $i \in I$, $j \leq n_i$, $[x_j]_{E^D} = c_j$,

$$R_i^D(x_1, \ldots, x_{n_i}) \Leftrightarrow R_i^C(c_1, \ldots, c_{n_i}).$$

The linear ordering $\leq^D$ is defined so that each E^D-equivalence class is an interval with respect to $\leq^D$, and for all $[y_1]_{E^D} = c_1$, $[y_2]_{E^D} = c_2$, $c_1 \neq c_2$,

$$y_1 \leq^D y_2 \Leftrightarrow c_1 \leq^C c_2.$$

In order to define each E^D-equivalence class and relations $\{R_j^D\}_{i\in J}$ we define a sequence $\{\mathbf{D}_s\}_{s=0}^a$ of structures from $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$. Finally, we take $\mathbf{D} = \mathbf{D}_a$. We list all embeddings $\mathbf{A}/E \hookrightarrow \mathbf{C}$ as $\{\mathbf{A}_l\}_{l=1}^a$, and we list all embeddings $\mathbf{B}/E \hookrightarrow \mathbf{C}$ as $\{\mathbf{B}_l\}_{l=1}^b$. Therefore a and b are the sizes of the sets $\binom{\mathbf{C}}{\mathbf{A}/E}$ and $\binom{\mathbf{C}}{\mathbf{B}/E}$ respectively.

Now we define $\mathbf{D}_0 = (D_0, \{R_i^{D_0}\}_{i\in I}, E^{D_0}, \{R_j^{D_0}\}_{i\in J}, \leq^{D_0}) \in \mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ such that $\mathbf{D}_0/E = (D_0/E, \{R_i^{D_0/E}\}_{i\in I}, \leq^{D_0/E}) = \mathbf{C}$. We consider structures $\mathbf{B}_{0,l} = (B_{0,l}, \{R_i^{B_{0,l}}\}_{i\in I}, E^{B_{0,l}}, \{R_j^{B_{0,l}}\}_{i\in J}, \leq^{B_{0,l}})$ from $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ such that $\mathbf{B}_{0,l}/E = (B_{0,l}/E, \{R_i^{B_{0,l}/E}\}_{i\in I}, \leq^{B_{0,l}/E}) = \mathbf{B}_l$. JEP for $\mathcal{L}^*$ implies that we may take $\mathbf{D}_0$ such that $\mathbf{B}_{0,l} \leq \mathbf{D}_0$ for all $1 \leq l \leq b$.

Suppose that we have $\mathbf{D}_t$ and we want to define $\mathbf{D}_{t+1}$ for $0 \leq t < a$. Then we denote by $\mathbf{D}_t{\restriction}\mathbf{A}_{t+1}$ the restriction of the structure $\mathbf{D}_t$ to E^{D_t}-equivalence classes given by $\mathbf{A}_{t+1}$, and we denote by $\mathbf{D}_t{\restriction}\mathbf{A}_{t+1}^c$ the restriction of $\mathbf{D}_t$ to the other E^{D_t}-equivalence classes. Denote by $\{\mathbf{D}_{t,k}\}_{k=1}^p$ the structures from $\mathcal{L}^*$ defined on E^{D_t}-equivalence classes. Note that in this list we may have structures that occur more than once. Without loss of generality we may assume that the linear ordering $\leq^{D_t}$ induces a linear ordering on $\{\mathbf{D}_{t,k}\}_{k=1}^p$ such that $\mathbf{D}_{t,1} \leq^{D_t} \cdots \leq^{D_t} \mathbf{D}_{t,p}$. We consider the structure $\mathbf{A}$ in a similar way. Let $\{\mathbf{A}^k\}_{k=1}^p$ be the list of structures from $\mathcal{L}^*$ defined by E^A-equivalence classes. Again, this is a list of structures that can appear more than once. Without loss of generality, we may assume that the linear ordering $\leq^A$ induces a linear ordering on $\{\mathbf{A}^k\}_{k=1}^p$ such that $\mathbf{A}^1 \leq^A \cdots \leq^A \mathbf{A}^p$. By Theorem 4.3 there is a sequence $\overrightarrow{\mathbf{E}} = \{\mathbf{E}_{t,k}\}_{k=1}^p$ of structures from $\mathcal{L}^*$ such that

$$(\mathbf{E}_{t,1}, \ldots, \mathbf{E}_{t,p}) \to (\mathbf{D}_{t,1}, \ldots, \mathbf{D}_{t,p})_r^{(\mathbf{A}^1, \ldots, \mathbf{A}^p)}.$$

The sequence $\overrightarrow{\mathbf{E}}$ defines a structure $\mathbf{E}_t$ from $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ such that $\mathbf{E}_t/E = \mathbf{A}_{t+1}$. The structures $\mathbf{E}_t$ and $\mathbf{D}_t{\restriction}\mathbf{A}_{t+1}^c$ define $\mathbf{D}_{t+1}$.

We claim that $\mathbf{D} = \mathbf{D}_a$ is such that

$$\mathbf{D}_a \to (\mathbf{B})_r^{\mathbf{A}}.$$

To see this, consider a coloring

$$p : \binom{\mathbf{D}}{\mathbf{A}} \to \{1, \ldots, r\}.$$

Note that each embedding of the structure $\mathbf{A}$ into the structure $\mathbf{D}$ must be placed on the collection of E^D-equivalence classes given by one of the $(\mathbf{A}_l)_{l=1}^a$. The choice of the sequence $\{\mathbf{D}_s\}_{s=0}^a$ tells us that there is some $\mathbf{F}_{a-1}$ such that:

(1) $\mathbf{F}_{a-1} = (F_{a-1}, \{R_i^{F_{a-1}}\}_{i\in I}, E^{F_{a-1}}, \{R_j^{F_{a-1}}\}_{j\in J}, \leq^{F_{a-1}}) \in \mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$, $\mathbf{F}_{a-1} \leq \mathbf{D}_a$.
(2) $\mathbf{F}_{a-1}/E = \mathbf{C}$ & $\mathbf{F}_{a-1} \cong \mathbf{D}_{a-1}$.
(3) All embeddings of $\mathbf{A}$ into $\mathbf{F}_{a-1}$ placed on the equivalence classes given by $\mathbf{A}_a$ are monochromatic.

In general, there is a sequence $(\mathbf{F}_s)_{s=0}^{a-1}$ of structures from $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ such that $\mathbf{F}_i \leq \mathbf{F}_{i+1}$ for all $0 \leq i < a$, $\mathbf{F}_{a-1}/E = \mathbf{C}$ & $\mathbf{F}_{a-1} \cong \mathbf{D}_{a-1}$, and all embeddings of $\mathbf{A}$ into $\mathbf{F}_{i+1}$ placed on the equivalence classes given with $\mathbf{A}_{i+1}$ are monochromatic. In particular, the color of an embedding of $\mathbf{A}$ into $\mathbf{F}_0$ depends only on the equivalence classes given by the respective $\mathbf{A}_l$, so the coloring p induces a coloring

$$\bar{p} : \binom{\mathbf{C}}{\mathbf{A}/E} \to \{1, \dots, r\}.$$

The choice of the structure $\mathbf{C}$ provides $\mathbf{B}' \leq \mathbf{C}$, $\mathbf{B}' \cong \mathbf{B}/E$ such that $\bar{p}\restriction\binom{\mathbf{B}'}{\mathbf{A}/E} = \text{const}$. Clearly, E^{D_0}-equivalence classes given by $\mathbf{B}'$ contain $\bar{\mathbf{B}} \in \mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ such that $\bar{\mathbf{B}} \leq \mathbf{D}_0 \leq \mathbf{D}$, $\bar{\mathbf{B}} \cong \mathbf{B}$ and $\bar{p}\restriction\binom{\bar{\mathbf{B}}}{\mathbf{A}} = \text{const}$.

($\Leftarrow$) We suppose that $\mathcal{K}^*$ does not have RP in order to show that the class $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ does not satisfy RP. Since $\mathcal{K}^*$ is not a Ramsey class there is a natural number r and structures $\mathbf{A}, \mathbf{B} \in \mathcal{K}^*$ such that $\binom{\mathbf{B}}{\mathbf{A}} \neq \emptyset$ and

($*$) For all $\mathbf{C} \in \mathcal{K}^*$ satisfying $\binom{\mathbf{C}}{\mathbf{B}} \neq \emptyset$, there is a coloring $p : \binom{\mathbf{C}}{\mathbf{A}} \to \{1, \dots, r\}$ such that for every $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ we have $p\restriction\binom{\mathbf{C}}{\mathbf{A}} \neq \text{const}$.

The structures $\mathbf{A}$ and $\mathbf{B}$ can be treated as members of the class $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ in which each equivalence class has exactly one element and each one point equivalence class carries the same structure from $\mathcal{L}^*$. Let $\mathbf{D}$ be a given structure such that $\mathbf{D} = (D, \{R_i^D\}_{i\in I}, E^D, \{R_j^D\}_{j\in J}, \leq^D) \in \mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ and $\binom{\mathbf{D}/E}{\mathbf{B}} \neq \emptyset$. We consider the coloring $p : \binom{\mathbf{D}/E}{\mathbf{A}} \to \{1, \dots, r\}$ that satisfies statement ($*$) for $\mathcal{K}^*$, and we define the coloring

$$\bar{p} : \binom{\mathbf{D}}{\mathbf{A}} \to \{1, \dots, r\}, \quad \bar{p}(\mathbf{A}') = p(\mathbf{A}'/E),$$

where we consider $\mathbf{A}'/E$ as a substructure of $\mathbf{D}/E$ given by the E^D-equivalence classes on which $\mathbf{A}'$ is placed. Since the $\bar{p}$-color of $\mathbf{A}'$ depends only on the E^D-equivalence classes incident with $\mathbf{A}'$, it is straightforward to see that for all $\mathbf{B}' \in \binom{\mathbf{D}}{\mathbf{B}}$ we have $p\restriction\binom{\mathbf{B}'}{\mathbf{A}} \neq \text{const}$.

Since $\mathcal{L}^*$ can be seen as a subclass of $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ when we consider only structures with one equivalence class, it follows that RP for $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ implies RP for $\mathcal{L}^*$. ∎

If we take the class $\mathcal{L}$ in Theorem 4.4 to be the class of finite sets in the empty signature then we obtain the following result.

THEOREM 4.5. *Let $\mathcal{K}$ be a class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let $\preceq$ be a binary relational symbol such that $\preceq \notin L$, and let $\mathcal{K}^*$ be a class of finite relational structures in the signature $L \cup \{\preceq\}$ such that $\mathcal{K}^*$ is an ordered expansion of $\mathcal{K}$ and $\mathcal{K}^*|L = \mathcal{K}$. Let $\mathcal{CE}[\mathcal{K}^*]$ be as in Definition* 2.3. *Then $\mathcal{K}^*$ has RP iff $\mathcal{CE}[\mathcal{K}^*]$ has RP.*

REMARK 4.6. We point out that in Theorems 4.5 and 4.4 we do not assume that the classes $\mathcal{K}$ and $\mathcal{L}$ are Fraïssé classes. Therefore we cannot talk about Fraïssé limits and their automorphism groups. If $\mathcal{K}$ and $\mathcal{L}$ are Fraïssé classes we may obtain the conclusions of the previous two theorems by noticing that the automorphism group of the corresponding limit is a semidirect product of two extremely amenable groups (see [11]).

In contrast with the second part of the proof of Theorem 4.5, we are not able to use a similar argument for the class $\mathcal{OEK}$.

LEMMA 4.7. *Let $\mathcal{K}$ be a class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let $\leq$ be a binary relational symbol such that $\leq \notin L$, and let $\mathcal{OEK}$ be as in Definition* 2.2. *If there is $\mathbf{A} \in \mathcal{K}$ such that* $\mathrm{Aut}(\mathbf{A})$ *is not a one-element group, then $\mathcal{OEK}$ does not satisfy RP.*

In order to examine RP for the class $\mathcal{OEK}$ when the class $\mathcal{K}$ from the statement of Lemma 4.7 has rigid structures, we need to recall the definition of α-colored sets from [1]. Let $\alpha = \{1, \ldots, n\}$ where n is a given natural number. Then the structure $\mathbf{A} = (A, <, f)$, where A is a non-empty set with linear ordering $<$ and a function $f : A \to \alpha$, is called an *α-colored set*. The linear ordering $<$ is called the underlying linear ordering of the α-colored set $\mathbf{A}$. An *embedding of α-colored sets* $(A, <_A, f_A)$ and $(B, <_B, f_B)$ is a map $F : A \to B$ such that for all x, $y \in A$ we have

$$F(x) = f(F(x)), \qquad x <^A y \Leftrightarrow F(x) <^B F(y).$$

An embedding which is a bijection is called an *isomorphism*, and we use the symbol $\cong$.

THEOREM 4.8 ([1]). *Let n be a given natural number. Then the class of finite α-colored sets satisfies RP.*

THEOREM 4.9. *Let $\mathcal{K}$ be a class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let $\leq$ and E be binary relational symbols such that $\leq \notin L$, $E \notin L$. Assume that all $\mathbf{A} \in \mathcal{K}$ are rigid, i.e.* $\mathrm{Aut}(\mathbf{A})$ *is a one-element group. If the class $\mathcal{K}$ satisfies RP, then $\mathcal{OEK}$ satisfies RP.*

Proof. Let $\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A, \leq^A)$ and $\mathbf{B} = (B, \{R_i^B\}_{i\in I}, E^B, \leq^B)$ be from $\mathcal{OEK}$ such that $\binom{\mathbf{B}}{\mathbf{A}} \neq \emptyset$, and let r be a given natural number. Then

$\mathbf{A}/E = (A/E, \{R_i^{A/E}\}_{i\in I})$ and $\mathbf{B}/E = (B/E, \{R_i^{B/E}\}_{i\in I})$ are structures from the Ramsey class $\mathcal{K}$, so there is $\mathbf{C} = (C, \{R_i^C\}_{i\in I}) \in \mathcal{K}$ such that

$$\mathbf{C} \to (\mathbf{B}/E)_r^{\mathbf{A}/E}.$$

We use the structure $\mathbf{C}$ to obtain $\mathbf{D} = (D, \{R_i^D\}_{i\in I}, E^D, \leq^D) \in \mathcal{OEK}$. Each E^D-equivalence class is obtained by replacing each point from C with a set. Placing a linear ordering $\leq^D$ on D is the hardest part of the proof. The rest is motivated by the partite construction of Nešetřil–Rödl (see [15] and [16]).

We list all substructures inside $\mathbf{C}$ isomorphic to $\mathbf{A}/E$ as $(\mathbf{A}_j)_{j=1}^a$ and all substructures inside $\mathbf{C}$ isomorphic to $\mathbf{B}/E$ as $(\mathbf{B}_j)_{j=1}^b$. Recursively, we define a sequence $(\mathbf{C}_j)_{j=0}^a$ of structures of the form $\mathbf{C}_j = (C_j, \{R_i^{C_j}\}_{i\in I}, E^{C_j}, \leq^{C_j}) \in \mathcal{OEK}$ such that for all j, j',

$$(C_j/E, \{R_i^{C_j/E}\}_{i\in I}) = (C_{j'}/E, \{R_i^{C_{j'}/E}\}_{i\in I}).$$

C_0 is a disjoint union: $C_0 = \bigcup_{l=1}^b C_{0,l}$. The set $C_{0,l}$ is placed on the E^D-equivalence classes given by $\mathbf{B}_l$ such that:

(1) $E^{C_{0,l}}$-equivalence classes are given by E^D, i.e. $xE^{C_{0,l}}y \Leftrightarrow xE^Dy$.

(2) The relations $\{R_i^{C_{0,l}}\}_{i\in I}$ are inherited from $\{R_i^C\}_{i\in I}$, i.e. for $i \in I$, $(x_j)_{j=1}^{n_i} \in C_{0,i}$, $[x_j]_{E^{C_{0,l}}} = c_j$,

$$R_i^{C_{0,l}}(x_1, \ldots, x_{n_i}) \;\Leftrightarrow\; R_i^C(c_1, \ldots, c_{n_i}).$$

(3) The linear ordering $\leq^{C_{0,l}}$ on $C_{0,l}$ is such that

$$\mathbf{C}_{0,l} = (C_{0,l}, \{R_i^{C_{0,l}}\}_{i\in I}, E^{C_{0,l}}, \leq^{C_{0,l}}) \cong \mathbf{B}.$$

There is only one way to define relations $\{R_i^{C_0}\}_{i\in I}$ and E^{C_0} on C_0 such that $(C_0/E, \{R_i^{C_0}\}_{i\in I}) = \mathbf{C}$, and it comes from the relations $\{R_i^C\}_{i\in I}$ and from the fact that each point from C produces one E^D-equivalence class. The linear ordering $\leq^{C_0}$ is defined so that

$$\leq^{C_0}|C_{0,l} = \leq^{C_{0,l}} \quad \text{for each } l, \quad C_{0,1} \leq^{C_0} C_{0,2} \leq^{C_0} \cdots \leq^{C_0} C_{0,b}.$$

In this way we have a structure from $\mathcal{OEK}$ that contains copies of $\mathbf{B}$ placed on b disjoint sets.

Assume that we already have the structure $\mathbf{C}_k$. Then we define $\mathbf{C}_{k+1}$ as follows. The restriction of $\mathbf{C}_k$ to E^{C_k}-equivalence classes given by $\mathbf{A}_{k+1}$ is denoted by

$$\mathbf{C}_k|A_{k+1} = (C_k|A_{k+1}, \{R_i^{C_k|A_{k+1}}\}_{i\in I}, E^{C_k|A_{k+1}}, \leq^{C_k|A_{k+1}}).$$

We denote $E^{C_k|A_{k+1}}$-equivalence classes by $(e_s)_{s=1}^g$ and E^A-equivalence classes by $(e'_s)_{s=1}^g$. Without loss of generality, we assume that all embeddings of $\mathbf{A}$ into $\mathbf{C}_k|A_{k+1}$ send e'_s into e_s. This follows from the fact that $(A/E, \{R_i^{A/E}\}_{i\in I})$ is a rigid structure, and $\mathcal{K}$ is the class of rigid structures. In the following we consider α-colored sets with $\alpha = \{1, \dots, g\}$. We code $\mathbf{C}_k|A_{k+1}$ using the α-colored set $(C_k|A_{k+1}, \leq^{C_k|A_{k+1}}, f_{C_k|A_{k+1}})$, so that for $x \in C_k|A_{k+1}$ we have

$$f_{C_k|A_{k+1}}(x) = s \;\Leftrightarrow\; x \in e_s.$$

Similarly we code $\mathbf{A}$ with the α-colored set $(A, \leq^A, f_A)$ so that for $y \in A$ we have

$$f_A(y) = s \;\Leftrightarrow\; y \in e'_s.$$

By Theorem 4.8, there is an α-colored set $(F_k, \leq^{F_k}, f_{F_k})$ such that

$$(F_k, \leq^{F_k}, f_{F_k}) \to (C_k|A_{k+1}, \leq^{C_k|A_{k+1}}, f_{C_k|A_{k+1}})_r^{(A, \leq^A, f_A)}.$$

The structure $(F_k, \leq^{F_k}, f_{F_k})$ gives $\mathbf{F}_k = (F_k, \{R_i^{F_k}\}_{i \in I}, E^{F_k}, \leq^{F_k}) \in \mathcal{OEK}$ in the following way. If $f_{F_k}(x) = s$ for $x \in F_k$, then x is placed on the E^D-equivalence class given by e_s. The relations $\{R_i^{F_k}\}_{i \in I}$ are defined so that for $x_1, \dots, x_{n_i} \in F_k$ we have

$$R_i^{F_k}(x_1, \dots, x_{n_i}) \;\Leftrightarrow\; R_i^{C_k|A_{k+1}}(y_1, \dots, y_{n_i}),$$

where $y_1, \dots, y_{n_i} \in C_k|A_{k+1}$ and

$$f_{F_k}(y_1) = f_{C_k|A_{k+1}}(y_1), \;\dots,\; f_{F_k}(y_{n_i}) = f_{C_k|A_{k+1}}(y_{n_i}).$$

List all embeddings $\mathbf{C}_k|A_{k+1} \hookrightarrow \mathbf{F}_k$ as $(\mathbf{C}_k^l)_{l=1}^{n_k}$. Over each $\mathbf{C}_k^l$ we can amalgamate the structure $\bar{\mathbf{C}}_k^l$ isomorphic to $\mathbf{C}_k$ by preserving E^D-equivalence classes so that $\bar{\mathbb{C}}_k^l$ intersects $\mathbf{F}_k$ exactly over $\mathbf{C}_k^l$. For a detailed explanation of this amalgamation see [25]. Each structure $\bar{\mathbf{C}}_k^l$ has the underlying set $\bar{C}_k^l$. For $l \neq l'$ the structures $\bar{\mathbf{C}}_k^l$ and $\bar{\mathbf{C}}_k^{l'}$ can intersect only over points inside the set F_k. The set C_{k+1} is the union

$$C_{k+1} = F_k \cup \bigcup_{l=1}^{n_k} \bar{C}_k^l$$

and the relations $\{R_i^{C_{k+1}}\}_{i \in I}$ and $E^{C_{k+1}}$ are defined in the obvious way. The linear orderings $\{\leq^{F_k}\} \cup \{\leq^{\bar{C}_k^l}\}_{l=1}^{n_k}$ generate a partial ordering $\prec^k$ on the set C_{k+1} and for $\leq^{C_{k+1}}$ we can take an arbitrary linear extension of $\prec^k$, so the structure $\mathbf{C}_{k+1}$ is defined.

In the end we have $\mathbf{C}_a$, and we claim that

$$\mathbf{C}_a \to (\mathbf{B})_r^{\mathbf{A}}.$$

Now, we take $\mathbf{D} = \mathbf{C}_a$. So let $p : \binom{\mathbf{C}_a}{\mathbf{A}} \to \{1, \dots, r\}$ be a given coloring. The construction of $\mathbf{C}_a$ implies that there is $\hat{\mathbf{C}}_{a-1} \in \mathcal{OEK}$ such that

$$\hat{\mathbf{C}}_{a-1} \cong \mathbf{C}_{a-1}, \qquad \hat{\mathbf{C}}_{a-1} \leq \mathbf{C}_a$$

and the coloring of embeddings $\mathbf{A} \hookrightarrow \hat{\mathbf{C}}_{a-1}$ placed on the E^D-equivalence classes given by $\mathbf{A}_a$ is constant. Recursively, we define a sequence $(\hat{\mathbf{C}}_j)_{j=0}^{a-1}$

of structures such that

$$\hat{\mathbf{C}}_j \in \mathcal{OEK}, \quad \hat{\mathbf{C}}_j \cong \mathbf{C}_j, \quad \hat{\mathbf{C}}_j \le \hat{\mathbf{C}}_{j+1}$$

and the coloring of embeddings $\mathbf{A} \hookrightarrow \hat{\mathbf{C}}_j$ placed on the E^D-equivalence classes given by $\mathbf{A}_{j+1}$ is constant. In the end,

$$\hat{\mathbf{C}}_0 \in \mathcal{OEK}, \quad \hat{\mathbf{C}}_0 \cong \mathbf{C}_0, \quad \hat{\mathbf{C}}_0 \le \mathbf{C}_a,$$

and the coloring of an embedding $\mathbf{A} \hookrightarrow \hat{\mathbf{C}}_0$ depends only on the E^D-equivalence classes on which the embedding is placed. Therefore, we obtain an induced coloring $\hat{p} : \binom{\mathbf{C}}{\mathbf{A}} \to \{1, \dots, r\}$. The choice of the structure $\mathbf{C}$ implies the existence of a structure $\mathbf{B}' \cong \mathbf{B}/E$ such that

$$\hat{p}{\restriction}\binom{\mathbf{B}'}{\mathbf{A}'} = \text{const}.$$

The equivalence classes given by $\mathbf{B}'$ provide a structure $\mathbf{B}^* \in \mathcal{OEK}$, $\mathbf{B}^* \cong \mathbf{B}$, $\mathbf{B}^* \le \hat{\mathbf{C}}_0 \le \mathbf{C}_a$ and $p{\restriction}\binom{\mathbf{B}^*}{\mathbf{A}} = \text{const}$, so RP is verified for the class $\mathcal{OEK}$. ∎

LEMMA 4.10. *Let $\mathcal{K}$ be a class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let $\le$ and E be binary relational symbols such that $\preceq \notin L$, $E \notin L$. Assume that all $\mathbf{A} \in \mathcal{K}$ are rigid, i.e.* $\mathrm{Aut}(\mathbf{A})$ *is a one-element group. If $\mathcal{K}$ does not have RP then neither does $\mathcal{OEK}$.*

Proof. Let r be a natural number, and let $\mathbf{A} = (A, \{R_i^A\}_{i\in I})$ and $\mathbf{B} = (B, \{R_i^B\}_{i\in I})$ be structures from $\mathcal{K}$ such that $\binom{\mathbf{B}}{\mathbf{A}} \ne \emptyset$. We assume that for every $\mathbf{C} \in \mathcal{K}$ with $\binom{\mathbf{C}}{\mathbf{B}} \ne \emptyset$ there is a coloring $p : \binom{\mathbf{C}}{\mathbf{A}} \to \{1, \dots, r\}$ such that for every $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$, we have $p{\restriction}\binom{\mathbf{B}'}{\mathbf{A}} \ne \text{const}$. The structures $\mathbf{A}$ and $\mathbf{B}$ may be viewed as members of $\mathcal{EK}$, in which each E^A-equivalence class and each E^B-equivalence class has exactly one element. Let $(\mathbf{A}_j)_{j=1}^a$ be a list of all structures $\mathbf{A}_j = (A_j, \{R_i^A\}_{i\in I}) \in \mathcal{K}$ such that $\mathbf{A}_j \le \mathbf{B}$ and $\mathbf{A}_j \cong \mathbf{A}$. By adding an arbitrary linear ordering $\le^A$ to the set A, we get a structure $\bar{\mathbf{A}} = (A, \{R_i^A\}_{i\in I}, E^A, \le^A) \in \mathcal{OEK}$. We consider a structure $\bar{\mathbf{B}} = (B, \{R_i^B\}_{i\in I}, E^B, \le^B) \in \mathcal{OEK}$ such that:

(1) $B = \bigcup_{j=1}^a B_j$.
(2) $j \ne j' \Rightarrow B_j \cap B_{j'} = \emptyset$, for all $j, j' \in \{1, \dots, a\}$.
(3) Each B_j is placed on the E^B-equivalence classes given by A_j such that bE^Bb' for no $b \ne b'$ from B_j.
(4) If $\bar{\mathbf{B}}|B_j$ is a substructure of $\bar{\mathbf{B}}$ given by the set B_j then $\bar{\mathbf{B}}|B_j \cong \bar{\mathbf{A}}$.

Now, let $\bar{\mathbf{C}} = (C, \{R_i^C\}_{i\in I}, E^C, \le^C) \in \mathcal{OEK}$ be such that $\binom{\bar{\mathbf{C}}}{\bar{\mathbf{B}}} \ne \emptyset$. Then $\mathbf{C}/E = (C/E, \{R_i^{C/E}\}_{i\in I}) \in \mathcal{K}$, and there is a coloring $p : \binom{\mathbf{C}/E}{\mathbf{A}} \to \{1, \dots, r\}$ such that for every $\mathbf{B}' \in \binom{\mathbf{C}/E}{\mathbf{B}}$, we have $p{\restriction}\binom{\mathbf{B}'}{\mathbf{A}} \ne \text{const}$. Using p, we define the coloring

$$\bar{p} : \binom{\bar{\mathbf{C}}}{\bar{\mathbf{A}}} \to \{1, \dots, r\}, \quad \bar{p}(\mathbf{A}') = p(\mathbf{A}_j),$$

where $\mathbf{A}_j$ gives the E^C-classes on which the structure $\mathbf{A}'$ is placed. Now, it is straightforward to check that for all $\mathbf{B}' \in \binom{\bar{\mathbf{C}}}{\mathbf{B}}$, we have $\bar{p}\restriction\binom{\mathbf{B}'}{\mathbf{A}} \neq \text{const}$, so $\mathcal{OEK}$ does not have RP. ∎

In order to examine the Ramsey property for the class $\mathcal{OLEK}$, we need to introduce new concepts. Let $\mathcal{L}$ be a class of finite relational structures in a signature $L_J = \{R_j\}_{j\in J}$ with arity $\{n_j\}_{j\in J}$. Let n be a natural number and let $\{I_i\}_{i=1}^n$ be a list of unary relational symbols which do not belong to L_J. Then we denote by $\mathcal{L}_n$ the class of structures, in the signature $L_J \cup \{I_i\}_{i=1}^n$, of the form $(A, \{I_i^A\}_{i=1}^n, \{R_j^A\}_{j\in J})$ such that for all $1 \le i, i' \le n$ we have:

(1) $A = \bigcup_{i=1}^n \{a \in A : I_i^A(a)\}$.
(2) $i \neq i' \Rightarrow \{a \in A : I_i^A(a)\} \cap \{a \in A : I_{i'}^A(a)\} = \emptyset$.
(3) If $A_i = \{a \in A : I_i^A(a)\}$ then $(A_i, \{R_j^A \cap A_i^{n_j}\}_{j\in J}) \in \mathcal{L}$.
(4) For all $a_1, \dots, a_{n_j} \in A$ with $R_j^A(a_1, \dots, a_{n_j})$ there is i such that $I_i^A(a_1), \dots, I_i^A(a_{n_j})$.

If we add arbitrary linear orderings to structures from $\mathcal{L}_n$ we obtain the class $\mathcal{OL}_n$. It contains structures in the signature $L_J \cup \{I_i\}_{i=1}^n \cup \{\le\}$ where $\le$ is a binary symbol not in L_J. Structures from $\mathcal{OL}_n$ are of the form $(A, \{I_i^A\}_{i=1}^n, \{R_j^A\}_{j\in J}, \le^A)$ where $(A, \{I_i^A\}_{i=1}^n, \{R_j^A\}_{j\in J})$ is in $\mathcal{L}_n$ and $\le^A$ is a linear ordering on A. Then we say that the class $\mathcal{L}$ is *n-adaptable* if $\mathcal{OL}_n$ has RP. If $\mathcal{L}$ is n-adaptable for all natural n then we say that $\mathcal{L}$ is *adaptable*. If we take $\mathcal{L}$ to be the class of finite graphs we obtain an adaptable class. One can also show that the class of finite metric spaces is adaptable. We introduce adaptable classes in order to transfer the proof of Theorem 4.9 into a corresponding proof for the class $\mathcal{OLEK}$. More precisely, we make the recursive step in the proof of Theorem 4.9 possible for the class $\mathcal{OLEK}$ under the assumption of adaptability. Using the idea of the proof of Theorem 4.9 we obtain

THEOREM 4.11. *Let $\mathcal{K}$ be a class of finite rigid relational structures in a signature $L_I = \{R_i\}_{i\in I}$. Let $\mathcal{L}$ be a class of finite relational structures in a signature $L_J = \{R_j\}_{j\in J}$. Let $\mathcal{OLEK}$ be as in Definition* 2.5. *If $\mathcal{K}$ has RP and $\mathcal{L}$ is adaptable then $\mathcal{OLEK}$ also has RP.*

Note that Theorem 4.9 can be obtained as a corollary of Theorem 4.11 if we take $\mathcal{L}$ to be the class of finite sets in the empty signature.

5. Ordering property

PROPOSITION 5.1. *Let $\mathcal{K}$ be a class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let $\preceq$ be a binary relational symbol such that $\preceq \notin L$, and let $\mathcal{K}^*$ be a class of finite relational structures in the signature $L \cup \{\preceq\}$ such that $\mathcal{K}^*$ is an ordered expansion of $\mathcal{K}$ and $\mathcal{K}^*|L = \mathcal{K}$. Let $\mathcal{CE}[\mathcal{K}^*]$ be as* *in Definition* 2.3. *Then* $\mathcal{K}^*$ *has OP with respect to* $\mathcal{K}$ *iff* $\mathcal{CE}[\mathcal{K}^*]$ *has OP with respect to* $\mathcal{EK}$.

Proof. ($\Rightarrow$) We assume that $\mathcal{K}^*$ has OP with respect to $\mathcal{K}$. For a given $\mathbf{A} = (A, \{R_i^A\}_{i\in I}, E^A) \in \mathcal{EK}$, we have $\mathbf{A}/E = (A/E, \{R_i^{A/E}\}_{i\in I}) \in \mathcal{K}$. OP of $\mathcal{K}^*$ with respect to $\mathcal{K}$ implies the existence of $\mathbf{B} = (B, \{R_i^B\}_{i\in I}) \in \mathcal{K}$ such that for all linear orderings $\leq^{A/E}$ and $\leq^B$ on A/E and B respectively with $(A/E, \{R_i^{A/E}\}_{i\in I}, \leq^{A/E}), (B, \{R_i^B\}_{i\in I}, \leq^B) \in \mathcal{K}^*$ we have

$$(A/E, \{R_i^{A/E}\}_{i\in I}, \leq^{A/E}) \hookrightarrow (B, \{R_i^B\}_{i\in I}, \leq^B).$$

We consider $\mathbf{C} = (C, \{R_i^C\}_{i\in I}, E^C) \in \mathcal{EK}$ such that $\mathbf{C}/E = \mathbf{B}$ and for all $c \in C$,

$$|[c]_{E^C}| = \max\{|[a]_{E^A}| : a \in A\}.$$

Let $\leq^C$ be a given linear ordering on C, and let $\leq^A$ be some linear ordering on A. By OP for $\mathcal{K}^*$, there is an embedding

$$e : (A/E, \{R_i^{A/E}\}_{i\in I}, \leq^{A/E}) \to (B, \{R_i^B\}_{i\in I}, \leq^B) = (C/E, \{R_i^{C/E}\}_{i\in I}, \leq^C),$$

and according to the size of the E^C-equivalence classes in $\mathbf{C}$, there is an embedding $f : \mathbf{A} \to \mathbf{C}$ such that $f_E = e$, so OP is verified for $\mathcal{CE}[\mathcal{K}^*]$ with respect to $\mathcal{EK}$.

($\Leftarrow$) Assume that $\mathcal{CE}[\mathcal{K}^*]$ has OP with respect to $\mathcal{EK}$, but $\mathcal{K}^*$ does not have OP with respect to $\mathcal{EK}$. Note that the embedding $f : \mathbf{A} \to \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are from $\mathcal{CE}[\mathcal{K}^*]$, induces an embedding $f_E : \mathbf{A}/E \to \mathbf{B}/E$ of structures from $\mathcal{EK}$. Therefore, if $\mathcal{CE}[\mathcal{K}^*]$ has OP with respect to $\mathcal{EK}$, then $\mathcal{K}^*$ would have OP with respect to $\mathcal{K}$. This contradicts our assumption. ∎

PROPOSITION 5.2. *Let* $\mathcal{K}$ *and* $\mathcal{L}$ *be classes of finite relational structures in signatures* $L_I = \{R_i\}_{i\in I}$ *and* $L_J = \{R_j\}_{j\in J}$ *respectively such that* $\mathcal{L}$ *has JEP. Let* $\mathcal{K}^*$ *and* $\mathcal{L}^*$ *be ordered expansions of* $\mathcal{K}$ *and* $\mathcal{L}$ *respectively. Let* $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ *and* $\mathcal{LEK}$ *be as in Definitions* 2.6 *and* 2.4. *Then* $\mathcal{K}^*$ *and* $\mathcal{L}^*$ *have OP with respect to* $\mathcal{K}$ *and* $\mathcal{L}$ *respectively iff* $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ *has OP with respect to* $\mathcal{LEK}$.

Proof. ($\Rightarrow$) Let $\mathbf{A}$ be a given structure from $\mathcal{LEK}$. We assume that $\mathbf{A}$ is represented in the form $\langle \mathbf{C}, (\mathbf{D}_c)_{c\in C}\rangle$ where $\mathbf{C}$ is a structure from $\mathcal{K}$ with underlying set C and $\mathbf{D}_c \in \mathcal{L}$ for all $c \in C$ (see the paragraph after Definition 2.4). Since $\mathcal{L}^*$ has OP with respect to $\mathcal{L}$, for each $c \in C$ there is $\bar{\mathbf{D}}_c \in \mathcal{L}$ which witnesses OP for $\mathbf{D}_c$. JEP for $\mathcal{L}^*$ implies that there is $\mathbf{D} \in \mathcal{L}$ such that $\bar{\mathbf{D}}_c \hookrightarrow \mathbf{D}$. Since $\mathcal{K}^*$ has OP with respect to $\mathcal{K}$, there is $\mathbf{E} \in \mathcal{L}$, with underlying set E, which witnesses OP for $\mathbf{C}$.

We claim that the structure $\langle \mathbf{E}, (\mathbf{F}_e)_{e\in E}\rangle$, where $\mathbf{F}_e = \mathbf{D}$ for all $e \in E$, witnesses OP for $\mathbf{A}$. Let $\leq^A$ be a linear ordering such that $(\mathbf{A}, \leq^A) \in \mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$. Then $(\mathbf{A}, \leq^A)$ is of the form $\langle (\mathbf{C}, \leq^C), ((\mathbf{D}_c, \leq^{D_c}))_{c\in C}\rangle$ where

$(\mathbf{C}, \leq^C) \in \mathcal{K}^*$ and $(\mathbf{D}_c, \leq^{D_c}) \in \mathcal{L}^*$ for $c \in C$ (see the paragraph after Definition 2.6). We consider $\langle(\mathbf{E}, \leq^E), ((\mathbf{F}_e, \leq^{F_e}))_{e\in E}\rangle$ from $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$. By the choice of $\mathbf{E}$ there is an embedding from $(\mathbf{C}, \leq^C)$ into $(\mathbf{E}, \leq^E)$. By the choice of the structure $\mathbf{D}$ there is an embedding of $(\mathbf{D}_c, \leq^{D_c})$ into $(\mathbf{F}_e, \leq^{F_e})$ for all $c \in C$ and all $e \in E$. Therefore we have an embedding from $(\mathbf{A}, \leq^A)$ into $\langle(\mathbf{E}, \leq^E), ((\mathbf{F}_e, \leq^{F_e}))_{e\in E}\rangle$.

($\Leftarrow$) Suppose that $\mathcal{L}^*$ does not have OP with respect to $\mathcal{L}$. Since $\mathcal{L}$ has JEP, there is $\mathbf{D} \in \mathcal{L}$ and a linear ordering $\leq^C$ such that $(\mathbf{D}, \leq^D) \in \mathcal{L}^*$ but for all $\mathbf{E} \in \mathcal{L}$ and all linear orderings $\leq^E$ with $(\mathbf{E}, \leq^E) \in \mathcal{L}^*$ there is no embedding from $(\mathbf{D}, \leq^D)$ into $(\mathbf{E}, \leq^E)$. Then we consider an arbitrary $\mathbf{C} \in \mathcal{K}$, with underlying set C, and $\mathbf{A} = \langle \mathbf{C}, (\mathbf{D}_c)_{c\in C}\rangle$ from $\mathcal{LEK}$ where $\mathbf{D}_c = \mathbf{D}$ for all $c \in C$. We consider $\langle(\mathbf{C}, \leq^C), ((\mathbf{D}_c, \leq^{D_c}))_{c\in C}\rangle$ from $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ such that $(\mathbf{D}_c, \leq^{D_c}) = (\mathbf{D}, \leq^D)$ for all $c \in C$. Let

$$\langle(\mathbf{E}, \leq^E), ((\mathbf{F}_e, \leq^{F_e}))_{e\in E}\rangle \in \mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$$

be such that linear orderings $\leq^{F_e}$ are such that there is no embedding from $(\mathbf{D}, \leq^D)$ into $(\mathbf{F}_e, \leq^{F_e})$. Then there is no embedding

$$\langle(\mathbf{C}, \leq^C), ((\mathbf{D}_c, \leq^{D_c}))_{c\in C}\rangle \hookrightarrow \langle(\mathbf{E}, \leq^E), ((\mathbf{F}_e, \leq^{F_e}))_{e\in E}\rangle,$$

so $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ does not have OP with respect to $\mathcal{LEK}$.

Suppose that $\mathcal{K}^*$ does not have OP with respect to $\mathcal{K}$. Let $\mathbf{A} = \langle(\mathbf{C}, \leq^C), ((\mathbf{D}_c, \leq^{D_c}))_{c\in C}\rangle$ and $\mathbf{B} = \langle(\mathbf{E}, \leq^E), ((\mathbf{F}_e, \leq^{F_e}))_{e\in E}\rangle$ be two structures from $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$. Then any embedding from $\mathbf{A}$ into $\mathbf{B}$ induces an embedding from $(\mathbf{C}, \leq^C)$ into $(\mathbf{E}, \leq^E)$. So if $(\mathbf{C}, \leq^C)$ witnesses that $\mathcal{K}^*$ does not have OP with respect to $\mathcal{K}$, then $\mathbf{A}$ witnesses that $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ does not have OP with respect to $\mathcal{LEK}$. ∎

Lemma 5.1. *Let $\mathcal{K}$ be a class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let $\leq$ and E be binary relational symbols such that $\preceq \notin L$, $E \notin L$, and let $\mathcal{OEK}$ be as in Definition 2.2. Then $\mathcal{OEK}$ does not have OP with respect to $\mathcal{EK}$.*

Proof. Note that for $\mathbf{A} \in \mathcal{OEK} \setminus \mathcal{CE}[\mathcal{K}^*]$ and $\mathbf{B} \in \mathcal{CE}[\mathcal{K}^*]$ there is no embedding $\mathbf{A} \hookrightarrow \mathbf{B}$, so $\mathcal{OEK}$ does not have OP with respect to $\mathcal{EK}$. ∎

Lemma 5.2. *Let $\mathcal{K}$ and $\mathcal{L}$ be classes of finite relational structures in signatures $L_I = \{R_i\}_{i\in I}$ and $L_J = \{R_j\}_{j\in J}$ respectively. Let $\mathcal{K}^*$ and $\mathcal{L}^*$ be ordered expansions of $\mathcal{K}$ and $\mathcal{L}$ respectively. Let $\mathcal{OLEK}$ and $\mathcal{LEK}$ be as in Definitions 2.4 and 2.5. Then $\mathcal{OLEK}$ does not have OP with respect to $\mathcal{LEK}$.*

Proof. This follows from the fact that for $\mathbf{A} \in \mathcal{OLEK} \setminus \mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ and $\mathbf{B} \in \mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ there is no embedding $\mathbf{A} \hookrightarrow \mathbf{B}$. ∎

6. Dynamics. In this section we present a few topological implications of the results from Sections 4 and 5.

Let G be a topological group. A *G-flow* X is a continuous action $G \times X \to X$ on a compact Hausdorff space X. A G-flow X is *minimal* if each of its orbits is dense, i.e. $X = \overline{G \cdot x}$ for all $x \in X$. If every minimal G-flow X is a point then we say that G is an *extremely amenable group*. Equivalently, G is extremely amenable if every G-flow X contains a fixed point x, i.e. we have $gx = x$ for all $g \in G$. The following result shows that among all minimal G-flows there is a largest one.

THEOREM 6.1 ([2]). *Given a topological group G, there is a minimal G-flow X with the following property: For any minimal G-flow Y there is a surjective continuous map $\phi : X \to Y$ such that $g \cdot \phi(x) = \phi(g \cdot x)$ for all $g \in G$, $x \in X$. Moreover the G-flow X is uniquely determined up to isomorphism, i.e. for any other G-flow $\bar{X}$ with the same property there is a continuous bijection $\phi : X \to \bar{X}$ with $g \cdot \phi(x) = \phi(g \cdot x)$ for all $g \in G$, $x \in X$.*

The G-flow given by the previous theorem is called the *universal minimal G-flow*.

A way to recognize extremely amenable groups among groups of automorphisms of Fraïssé structures is given in [11]. In addition, the authors of [11] provide the technique for the calculation of universal minimal flows of certain automorphism groups of Fraïssé structures. In the following, we present material that we need for topological interpretation of our combinatorial results from the previous sections.

If $\mathbf{L}$ is a countable structure, then we consider the group $\mathrm{Aut}(\mathbf{L})$ of its automorphisms as a topological group with pointwise topology (see [3]).

THEOREM 6.2 ([11]). *Let $\mathcal{L}$ be a Fraïssé ordered class and let* $\mathbf{L} = \mathrm{Flim}(\mathcal{L})$. *Then the topological group* $\mathrm{Aut}(\mathbf{L})$ *with the pointwise topology is extremely amenable iff $\mathcal{L}$ has the Ramsey property.*

If the topological group is not extremely amenable, we will try to calculate its universal minimal flow. Let L_0 and L_1 be countable signatures such that $L_0 \subseteq L_1 = L_0 \cup \{\leq\}$, $\leq \notin L_0$, where $\leq$ is a binary relational symbol. We assume that a Fraïssé class $\mathcal{L}_1$ in the signature L_1 is an order expansion of a Fraïssé class $\mathcal{L}_0$ in the signature L_0. Also, we assume that $\leq$ is interpreted as a linear ordering in each structure from $\mathcal{L}_1$. We consider the Fraïssé limits $\mathbf{L}_0 = \mathrm{Flim}(\mathcal{L}_0)$ and $\mathbf{L}_1 = \mathrm{Flim}(\mathcal{L}_1)$. In terms of reducts we have $\mathbf{L}_0 = \mathbf{L}_1|V_0$. The interpretation of the symbol $\leq$ in $\mathbf{L}_1$ is as a linear ordering $\leq^{L_1}$. Without loss of generality we may assume that the structures $\mathbf{L}_0$ and $\mathbf{L}_1$ are defined on the set of natural numbers $\mathbb{N}$. The set of all linear orderings of natural numbers, $\mathrm{LO} \subseteq 2^{\mathbb{N}\times\mathbb{N}}$, is a compact Hausdorff space (see [3]). In particular $\leq^{L_1} \in \mathrm{LO}$. Let $G_0 = \mathrm{Aut}(\mathbf{L}_0)$ and consider the logic action of G_0 on LO (see [3]). The topological closure of the orbit of $\leq^{L_1}$

under the logic action,

$$X_{\mathcal{L}_1} = \overline{G_0 \cdot \leq^{L_1}} \subseteq \mathrm{LO},$$

is important for calculation of universal minimal flows. Linear orderings from $X_{\mathcal{S}}$ will be called *S-admissible.*

THEOREM 6.3 ([11]). *Let L_1 and L_0 be countable signatures such that $L_0 = L_1 \setminus \{\leq\}$ where $\leq$ is a binary relational symbol. Let $\mathcal{L}_1$ be a reasonable Fraïssé order class in L_1 and let $\mathcal{L}_0$ be a Fraïssé class in L_1. Assume that $\mathcal{L}_0 = \mathcal{L}_1|L_0$ and the Fraïssé limits $\mathbf{L}_1 = \mathrm{Flim}(\mathcal{L}_1)$ and $\mathbf{L}_0 = \mathrm{Flim}(\mathcal{L}_0)$ are defined on the set of natural numbers. Let $G_0 = \mathrm{Aut}(\mathbf{L}_0)$ with pointwise topology, and let $X_{\mathcal{L}_1}$ be the set of $\mathcal{L}_1$-admissible linear orderings. Then:*

- (i) *$X_{\mathcal{L}_1}$ is the universal minimal G_0-flow iff $\mathcal{L}_1$ is a Ramsey class with the ordering property with respect to $\mathcal{L}_0$.*
- (ii) *$X_{\mathcal{L}_1}$ is a minimal G_0-flow iff $\mathcal{L}_1$ has the ordering property with respect to $\mathcal{L}_0$.*

If $\mathcal{K}$ is a Fraïssé class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$, then I has to be countable.

THEOREM 6.4. *Let $\mathcal{K}$ be a Fraïssé class of finite relational structures in a signature $L = \{R_i\}_{i\in I}$. Let $\preceq$ and E be binary relational symbols such that $\preceq \notin L$, $E \notin L$. Let $\mathcal{K}^*$ be a Fraïssé class of finite relational structures in the signature $L \cup \{\preceq\}$ such that $\mathcal{K}^*$ is an ordered expansion of $\mathcal{K}$ and $\mathcal{K}^*|L = \mathcal{K}$. Let $\mathcal{EK}$, $\mathcal{OEK}$ and $\mathcal{CE}[\mathcal{K}^*]$ be as in Definitions 2.1–2.3. Then:*

- (i) *The Fraïssé limits $\mathbf{EK} = \mathrm{Flim}(\mathcal{EK})$, $\mathbf{OEK} = \mathrm{Flim}(\mathcal{OEK})$ and $\mathbf{CE}[\mathbf{K}^*] = \mathrm{Flim}(\mathcal{CE}[\mathcal{K}^*])$ are well-defined.*
- (ii) *The topological group $\mathrm{Aut}(\mathbf{CE}[\mathbf{K}^*])$ is extremely amenable iff $\mathcal{K}^*$ has RP.*
- (iii) *The topological group $\mathrm{Aut}(\mathbf{OEK})$ is extremely amenable iff $\mathcal{K}$ is a Ramsey class of finite rigid structures.*
- (iv) *The $\mathrm{Aut}(\mathbf{EK})$-flow $X_{\mathcal{CE}[\mathcal{K}^*]}$ is minimal iff $\mathcal{K}^*$ has OP with respect to $\mathcal{K}$.*
- (v) *$X_{\mathcal{CE}[\mathcal{K}^*]}$ is the universal minimal $\mathrm{Aut}(\mathbf{EK})$-flow iff $\mathcal{K}^*$ has RP and OP with respect to $\mathcal{K}$.*

Proof. (i) This follows from Lemmas 3.3, 3.4, 3.5 and 3.7.

(ii) This follows from Theorems 6.2 and 4.5.

(iii) This follows from Theorem 6.2, Lemma 4.7, Lemma 4.10 and Theorem 4.9.

(iv) This follows from Theorem 6.3 and Proposition 5.1.

(vi) This follows from Theorem 6.3, Theorem 4.5 and Proposition 5.1. ∎

THEOREM 6.5. *Let $\mathcal{K}$ and $\mathcal{L}$ be Fraïssé classes of finite relational structures in signatures $L_I = \{R_i\}_{i\in I}$ and $L_J = \{R_j\}_{j\in J}$ respectively such that*

$L_I \cap L_J = \emptyset$. Assume that $\mathcal{L}$ has SAP and that at least one of $\mathcal{L}$ or $\mathcal{K}$ has SJEP. Let $\mathcal{K}^*$ and $\mathcal{L}^*$ be reasonable Fraïssé ordered expansions of $\mathcal{K}$ and $\mathcal{L}$ respectively. Let $\mathcal{LEK}$ and $\mathcal{OLEK}$ and $\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]$ be as in Definitions 2.4–2.6. Then:

(i) *The Fraïssé limits* $\mathbf{LEK} = \mathrm{Flim}(\mathcal{LEK})$, $\mathbf{OLEK} = \mathrm{Flim}(\mathcal{OLEK})$ *and* $\mathbf{C}[\mathbf{L}^*]\mathbf{E}[\mathbf{K}^*] = \mathrm{Flim}(\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*])$ *are well-defined.*

(ii) *The topological group* $\mathrm{Aut}(\mathbf{C}[\mathbf{L}^*]\mathbf{E}[\mathbf{K}^*])$ *is extremely amenable iff* $\mathcal{K}^*$ *and* $\mathcal{L}^*$ *have RP.*

(iii) *If* $\mathcal{K}$ *is a Ramsey class and* $\mathcal{L}$ *is adaptable then the topological group* $\mathrm{Aut}(\mathbf{OLEK})$ *is extremely amenable.*

(iv) *The* $\mathrm{Aut}(\mathbf{LEK})$*-flow* $X_{\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]}$ *is minimal iff* $\mathcal{K}^*$ *and* $\mathcal{L}^*$ *have OP with respect to* $\mathcal{K}$ *and* $\mathcal{L}$ *respectively.*

(v) $X_{\mathcal{C}[\mathcal{L}^*]\mathcal{E}[\mathcal{K}^*]}$ *is the universal minimal* $\mathrm{Aut}(\mathbf{LEK})$*-flow iff* $\mathcal{K}^*$ *and* $\mathcal{L}^*$ *have RP and OP with respect to* $\mathcal{K}$ *and* $\mathcal{L}$ *respectively.*

Proof. Argue as in the proof of Theorem 6.4. ∎

7. Finite topological spaces. We denote by $\mathcal{T}$ the class of finite topological spaces, and its elements by (X, τ) where τ is a topology on X. A topology τ on a finite set X has a base $\{U_x : x \in X\}$ where

$$U_x = \bigcap\{U \in \tau : x \in U\},$$

and there is a binary relation $\leq$ on X such that for $x, y \in X$ we have

$$x \leq y \ \Leftrightarrow\ x \in U_y.$$

Clearly, the relation $\leq$ is transitive and reflexive, but it is not antisymmetric. Consequently, $\leq$ is a quasi ordering on X and the pair $(X, \leq)$ is a qoset. In this way we assign to each finite topological space a qoset. It is also possible to make an assignment in the opposite direction. Let $(Y, \leq)$ be a qoset. Then we have a collection $\mathcal{U} = \{U_y : y \in Y\}$ of sets such that

$$U_y = \{z \in Y : z \leq y\}.$$

Clearly, $\mathcal{U}$ is a base of a topology on X. Note that this is a 1-1 correspondence between the classes of finite topological spaces and of finite qosets. Instead of examining the class of finite topological spaces with linear ordering we may examine the class of finite qosets with linear ordering. Also, there is no countable signature describing finite topological spaces, but we have a countable signature describing finite qosets. We refer the reader to [14] and [29] for more details about finite topological spaces.

Let $\mathcal{Q}$ be the class of finite qosets and $\mathcal{P}$ the class of finite posets. According to Definition 2.1 we have $\mathcal{Q} = \mathcal{EP}$.

For a given set A, the collection of all linear orderings on A is denoted by $\mathrm{lo}(A)$. If $\leq$ is a given partial ordering on a set A, then the collection of all linear extensions of the partial ordering $\leq$ is denoted by $\mathrm{le}(\leq)$. Then we have the class of finite posets with linear orderings,

$$\mathcal{P}_1 = \{(A, \preceq, \leq) : (A, \preceq) \in \mathcal{P},\ \leq \in \mathrm{lo}(A)\},$$

and the class of finite posets with linear extensions,

$$\mathcal{P}_2 = \{(A, \preceq, \leq) \in \mathcal{P}_1 : \leq \in \mathrm{le}(\preceq)\}.$$

$\mathcal{P}$ is a Fraïssé class (see [23]), as also are $\mathcal{P}_1$ and $\mathcal{P}_2$ (see [25]), so by Lemmas 3.3 and 3.5. $\mathcal{Q} = \mathcal{EP}$ and $\mathcal{Q}_0 = \mathcal{OEP}$ are Fraïssé classes. If we take $\mathcal{K}^* = \mathcal{P}_1$ or $\mathcal{K}^* = \mathcal{P}_2$ then by Definition 2.3 we get classes $\mathcal{Q}_1 = \mathcal{CE}[\mathcal{P}_1]$ and $\mathcal{Q}_2 = \mathcal{CE}[\mathcal{P}_2]$ which are also Fraïssé by Lemma 3.7. Note that we have three reasonable Fraïssé ordered expansions of the class $\mathcal{Q}$: $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$. We also have the corresponding Fraïssé limits: $\mathbf{Q}$, $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$ and their respective automorphism groups $\mathbf{G}$, $\mathbf{G}_0$, $\mathbf{G}_1$, $\mathbf{G}_2$. The class $\mathcal{P}_1$ does not have OP with respect to $\mathcal{P}$, but $\mathcal{P}_2$ does (see [25]).

THEOREM 7.1.

(i) ([24], [25]) $\mathcal{P}_1$ *is not a Ramsey class.*
(ii) ([17], [20], [5], [24], [25]) $\mathcal{P}_2$ *is a Ramsey class.*

Applying the results of Sections 4 and 5 to the classes $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$, we obtain

COROLLARY 7.2.

(i) $\mathcal{Q}_0$ *and* $\mathcal{Q}_1$ *are not Ramsey classes, but* $\mathcal{Q}_2$ *is a Ramsey class* (see [14]).
(ii) $\mathcal{Q}_0$ *and* $\mathcal{Q}_1$ *do not have OP with respect to* $\mathcal{Q}$, *but* $\mathcal{Q}_2$ *does.*

From Theorem 6.4 we get topological implications:

COROLLARY 7.3.

(i) $\mathbf{G}_2$ *is an extremely amenable group.*
(ii) $X_{\mathcal{Q}_2}$ *is the universal minimal* $\mathbf{G}$*-flow.*

REMARK 7.4. Without loss of generality, we may assume that the qoset $\mathbf{Q} = (\mathbb{N}, \leq)$ has the set of natural numbers as underlying set. On $\mathbb{N}$, we may define the collection $\mathcal{B} = \{U_x : x \in \mathbb{N}\}$ where $U_x = \{y \in \mathbb{N} : y \leq x\}$. Clearly, $\mathcal{B}$ is a base of a topology $\mathcal{T}$ on $\mathbb{N}$. The topological space $(\mathbb{N}, \mathcal{T})$ contains any finite topological space as subspace, but it does not contain all countable topological spaces as subspaces because its weight is countable. The qosets $\mathbf{Q}_0$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ may be viewed as the topological space $(\mathbb{N}, \mathcal{T})$ with an additional linear ordering, so Corollaries 7.2 and 7.3 can be stated in terms of topological spaces. In this case $\mathbf{G}$ is actually the group of homeomorphisms of the topological space $(\mathbb{N}, \mathcal{T})$ with the pointwise topology, and $\mathbf{G}_0$, $\mathbf{G}_1$, $\mathbf{G}_2$ are its subgroups.

8. Finite metric spaces. We denote by $\mathcal{M}$ the class of finite metric spaces with all distances in the set $\mathbb{Q}$ of rational numbers. Then $\mathcal{M}$ may be viewed as a class of finite structures in the signature $L = \{R_i\}_{i\in\mathbb{Q}}$, where each R_i is a binary relational symbol. To a metric space (A, d) we assign a structure $(A, \{R_i^A\}_{i\in\mathbb{Q}})$ such that for all $x, y \in A$ and all $i \in \mathbb{Q}$ we have

$$R_i^A(x, y) \Leftrightarrow d(x, y) = i.$$

Adding linear orderings to finite metric spaces we obtain the class of finite linearly ordered metric spaces:

$$\mathcal{M}_1 = \{(A, \{R_i^A\}_{i\in\mathbb{Q}}, \leq) : (A, \{R_i^A\}_{i\in\mathbb{Q}}) \in \mathcal{M}, \leq \in \mathrm{lo}(A)\}.$$

By Definition 2.1 the class $\mathcal{R} = \mathcal{E}\mathcal{M}$ corresponds to the class of finite pseudometric spaces with rational distances. Definition 2.2 gives us the class $\mathcal{R}_0 = \mathcal{O}\mathcal{E}\mathcal{M}$ of linearly ordered finite pseudometric spaces. If we take $\mathcal{K}^* = \mathcal{M}_1$ then by Definition 2.3 we get the class $\mathcal{R}_1 = \mathcal{C}\mathcal{E}[\mathcal{M}_1]$ of convex linearly ordered pseudometric spaces. It is known that $\mathcal{M}$ and $\mathcal{M}_1$ are Fraïssé classes (see [11]) such that $\mathcal{M}_1$ is a Ramsey class with OP with respect to $\mathcal{M}$ (see [13]). Lemmas 3.3, 3.5 and 3.7 imply that $\mathcal{R}$, $\mathcal{R}_0$ and $\mathcal{R}_1$ are Fraïssé classes with limits $\mathbf{R}$, $\mathbf{R}_0$ and $\mathbf{R}_1$ respectively. The automorphism groups of $\mathbf{R}$, $\mathbf{R}_0$ and $\mathbf{R}_1$ are $\mathbf{H}$, $\mathbf{H}_0$ and $\mathbf{H}_1$ respectively, and the results of Sections 4 and 5 give us the following.

Corollary 8.1.

(i) *$\mathcal{R}_1$ is a Ramsey class, but $\mathcal{R}_0$ is not.*
(ii) *$\mathcal{R}_1$ has OP with respect to $\mathcal{R}$, but $\mathcal{R}_0$ does not.*

We also have a topological implication of Theorem 6.4.

Corollary 8.2.

(i) *$\mathbf{H}_1$ is an extremely amenable group.*
(ii) *$X_{\mathcal{R}_1}$ is the universal minimal $\mathbf{H}$-flow.*

In the rest of this section we will show that $\mathcal{M}$ is an adaptable class. For this we need to consider more classes. Let n be a natural number and let $\{I_i\}_{i=1}^n$ be a list of unary relational symbols. We denote by $\mathcal{I}_n$ the class of structures of the form $(A, \{I_i^A\}_{i=1}^n)$ such that for all $1 \leq i, i' \leq n$ we have:

- $A = \bigcup_{i=1}^n \{a \in A : I_i^A(a)\}$.
- $i \neq i' \Rightarrow \{a \in A : I_i^A(a)\} \cap \{a \in A : I_{i'}^A(a)\} = \emptyset$.

We consider:

- $\mathcal{O}\mathcal{I}_n$, the class of structures of the form $(A, \{I_i^A\}_{i=1}^n, \leq^A)$ such that $(A, \{I_i^A\}_{i=1}^n) \in \mathcal{I}_n$ and $\leq \in \mathrm{lo}(A)$.
- $\mathcal{M}^n$, the class of structures of the form $(A, \{R_i^A\}_{i\in\mathbb{Q}}, \{I_i^A\}_{i=1}^n, \leq^A)$ such that $(A, \{R_i^A\}_{i\in\mathbb{Q}}, \leq^A) \in \mathcal{M}_1$ and $(A, \{I_i^A\}_{i=1}^n, \leq^A) \in \mathcal{O}\mathcal{I}_n$.

Loosely speaking, $\mathcal{I}_n$ is the class of finite sets partitioned into at most n labeled parts, $\mathcal{OI}_n$ is the class of linearly ordered structures from $\mathcal{I}_n$, and $\mathcal{M}^n$ is the class of finite linearly ordered metric spaces partitioned into at most n labeled parts. We use the following result in order to prove adaptability for $\mathcal{M}$.

LEMMA 8.3. *For any natural number $n \geq 1$, $\mathcal{M}^n$ is a Ramsey class.*

Proof. Our proof is based on the technique developed in Section 3 of [27]. This technique gives a method of obtaining a new Ramsey class from two Ramsey classes. We present the main steps of the proof and leave the easy verifications to the reader.

Fix a natural number $r > 1$, and let $\mathbf{A} = (A, \{R_i^A\}_{i\in Q}, \{I_i^A\}_{i=1}^n, \leq^A)$ and $\mathbf{B} = (B, \{R_i^B\}_{i\in Q}, \{I_i^B\}_{i=1}^n, \leq^B)$ be structures from $\mathcal{M}^n$ such that $\binom{\mathbf{B}}{\mathbf{A}} \neq \emptyset$. We consider the structures $\mathbf{A}_1 = (A, \{R_i^A\}_{i\in\mathbb{Q}}, \leq^A)$, $\mathbf{A}_2 = (A, \{I_i^A\}_{i=1}^n, \leq^A)$, $\mathbf{B}_1 = (B, \{R_i^B\}_{i\in Q}, \leq^B)$ and $\mathbf{B}_2 = (B, \{I_i^B\}_{i=1}^n, \leq^B)$. We noticed earlier in this section that $\mathcal{M}_1$ is a Ramsey class; now we observe that $\mathcal{OI}_n$ is also a Ramsey class by Theorem 4.8. By Theorem 4.3 there is a pair $(\mathbf{C}_1, \mathbf{C}_2)$ of structures such that $\mathbf{C}_1 = (C_1, \{R_i^{C_1}\}_{i\in\mathbb{Q}}, \leq^{C_1}) \in \mathcal{OI}_n$, $\mathbf{C}_2 = (C_2, \{R_i^{C_2}\}_{i\in\mathbb{Q}}, \leq^{C_2}) \in \mathcal{M}_1$ and

$$(\mathbf{C}_1, \mathbf{C}_2) \to (\mathbf{B}_1, \mathbf{B}_2)_r^{(\mathbf{A}_1, \mathbf{A}_2)}.$$

In the following we define a structure $\mathbf{C} = (C, \{R_i^C\}_{i\in\mathbb{Q}}, \{I_i^C\}_{i=1}^n, \leq^C) \in \mathcal{M}^n$ on $C = C_1 \times C_2$. Let (c_1, c_2) and (c_1', c_2') be in C. Let q be a positive rational number and let i be an integer such that $1 \leq i \leq n$. Then we define:

- $I_i^C((c_1, c_2)) \Leftrightarrow I_i^{C_1}(c_1)$.
- $(c_1, c_2) \leq^C (c_1', c_2') \Leftrightarrow (c_1 <^{C_1} c_1'$ or $(c_1 = c_1'$ and $c_2 \leq^{C_2} c_2'))$.
- $R_q^C((c_1, c_2), (c_1', c_2')) \Leftrightarrow ((c_2 \neq c_2'$ and $R_q^{C_2}(c_2, c_2'))$ or $(c_2 = c_2'$ and $c_1 \neq c_1'$ and $q = m)$ or $(c_1 = c_1'$ and $c_2 = c_2'$ and $q = 0)$, where m is the minimal nonzero distance between points in $\mathbf{C}_2$.

We leave it to the reader to verify that $\mathbf{C}$ is a structure in $\mathcal{M}^n$. Clearly $\mathcal{OI}_n$ and $\mathcal{M}_1$ are linearly ordered structures. Moreover $(\mathbf{B}_1, \mathbf{B}_2)$ and $(\mathbf{A}_1, \mathbf{A}_2)$ are balanced sequences of structures (see the beginning of Section 3 in [27]). It is easy to see that $\mathbf{C}$ with $(\mathbf{C}_1, \mathbf{C}_2)$ has the diagonal property (see Section 3 in [27]). Finally, all requirements of Lemma 3 in [27] are satisfied. In the conclusion of Lemma 3 in [27] we have the structures $\mathbf{C}\restriction\Delta((\mathbf{B}_1, \mathbf{B}_2))$ and $\mathbf{C}\restriction\Delta((\mathbf{A}_1, \mathbf{A}_2))$ which are isomorphic to $\mathbf{B}$ and $\mathbf{A}$ respectively (see Section 3 in [27] for the definition of $\Delta(\ldots)$). This shows

$$\mathbf{C} \to (\mathbf{B})_r^{\mathbf{A}},$$

which completes the proof of the Ramsey property for $\mathcal{M}^n$. ■

PROPOSITION 8.1. *$\mathcal{M}$ is an adaptable class.*

Proof. We fix natural numbers $n, r \geq 1$ and show that $\mathcal{OM}_n$ has RP. Let $\mathbf{A} = (A, \{I_i^A\}_{i=1}^n, \{R_i^A\}_{i\in\mathbb{Q}}, \leq^A)$ and $\mathbf{B} = (B, \{I_i^B\}_{i=1}^n, \{R_i^B\}_{i\in\mathbb{Q}}, \leq^B)$ be structures from $\mathcal{OM}_n$ such that $\binom{\mathbf{B}}{\mathbf{A}} \neq \emptyset$. We consider structures $\mathbf{A}_1 = (A_1, \{R_i^{A_1}\}_{i\in\mathbb{Q}}, \{I_i^{A_1}\}_{i=1}^n, \leq^{A_1})$ and $\mathbf{B}_1 = (B_1, \{R_i^{B_1}\}_{i\in\mathbb{Q}}, \{I_i^{B_1}\}_{i=1}^n, \leq^{B_1})$ in $\mathcal{M}^n$ such that for all $1 \leq i, i' \leq n$, $a, a' \in A_1$, and $b, b' \in B_1$ we have:

- $A_1 = A$, $B_1 = B$.
- $I_i^{A_1} = I_i^A$, $I_i^{B_1} = I_i^B$.
- $\leq^{A_1} = \leq^A$, $\leq^{B_1} = \leq^B$.
- If $I_i^{A_1}(a)$, $I_i^{B_1}(b)$, $I_{i'}^{A_1}(a')$, $I_{i'}^{B_1}(b')$ then
$$R_q^{A_1}(a,a') \Leftrightarrow ((i = i' \text{ and } R_q^A(a,a')) \text{ or } (i \neq i' \text{ and } q = m)),$$
$$R_q^{B_1}(b,b') \Leftrightarrow ((i = i' \text{ and } R_q^B(b,b')) \text{ or } (i \neq i' \text{ and } q = m)),$$
where m is a rational number strictly greater than any defined non-zero distance between points in $\mathbf{A}$ and between points in $\mathbf{B}$.

Since $\mathbf{A}_1$ and $\mathbf{A}$ have common underlying set, as also do $\mathbf{B}_1$ and $\mathbf{B}$, we have the following observation:

(▲) An embedding of $\mathbf{A}_1$ into $\mathbf{B}_1$ is an embedding of $\mathbf{A}$ into $\mathbf{B}$ and vice versa.

By Lemma 8.3 there is $\mathbf{C}_1 = (C_1, \{R_i^{C_1}\}_{i\in\mathbb{Q}}, \{I_i^{C_1}\}_{i=1}^n, \leq^{C_1}) \in \mathcal{M}^n$ such that $\mathbf{C}_1 \to (\mathbf{B}_1)_r^{\mathbf{A}_1}$. We consider a structure $\mathbf{C} = (C, \{I_i^C\}_{i=1}^n, \{R_i^C\}_{i\in\mathbb{Q}}, \leq^C) \in \mathcal{OM}_n$ such that for all $1 \leq i, i' \leq n$, $a, a' \in C$, and $q \in \mathbb{Q}$ we have:

- $C = C_1$.
- $I_i^C = I_i^{C_1}$.
- $\leq^C = \leq^{C_1}$.
- If $I_i^C(a)$ and $I_i^C(a')$ then $R_q^C(a,a') \Leftrightarrow R_q^{C_1}(a,a')$.

We claim that
$$\mathbf{C} \to (\mathbf{B})_r^{\mathbf{A}}.$$
In order to check this let
$$\chi : \binom{\mathbf{C}}{\mathbf{A}} \to \{1, \ldots, r\}$$
be a given coloring. Then we have an induced coloring
$$\chi_1 : \binom{\mathbf{C}_1}{\mathbf{A}_1} \to \{1, \ldots, r\}$$
such that $\chi_1(\mathbf{A}_1') = \chi(\mathbf{A}')$ where $\mathbf{A}'$ is a substructure of $\mathbf{C}$ which has the same underlying set as $\mathbf{A}_1'$. Then there is $\mathbf{B}_1' \in \binom{\mathbf{C}_1}{\mathbf{B}_1}$ such that
$$\chi_1 \restriction \binom{\mathbf{B}_1'}{\mathbf{A}_1} = \text{const}.$$

If $\mathbf{B}'$ is substructure of $\mathbf{C}$ which has the same underlying set as $\mathbf{B}'_1$, then by the property (▲) we have

$$\chi\restriction\binom{\mathbf{B}'}{\mathbf{A}} = \text{const}.$$

This completes the verification that for a given n, $\mathcal{OM}_n$ is a Ramsey class. Since this is satisfied for all natural n we conclude that $\mathcal{M}$ is an adaptable class. ∎

Now we consider the class $\mathcal{G}$ of finite graphs. It is a class in the signature $\{E\}$ where E is a binary relational symbol. We denote by $\mathcal{G}^n$ the class of structures of the form $(A, E^A, \{I_i^A\}_{i=1}^n, \leq^A)$ where $(A, E^A) \in \mathcal{G}$ and $(A, \{I_i^A\}_{i=1}^n, \leq^A) \in \mathcal{OI}_n$. As a special case of Theorem 4.8 we have the following.

Lemma 8.4 ([1]). *For any natural number $n \geq 1$, $\mathcal{G}^n$ is a Ramsey class.*

Using the same arguments as in the proof of Proposition 8.1 we prove the following.

Proposition 8.2. *$\mathcal{G}$ is an adaptable class.*

In the following we apply adaptability.

Corollary 8.5.

(i) *$\mathcal{OP}_2\mathcal{EM}$, $\mathcal{OP}_2\mathcal{EG}$, $\mathcal{OM}_1\mathcal{EM}$, $\mathcal{OM}_1\mathcal{EG}$ are Fraïssé classes with limits* $\mathbf{OP_2EM}$, $\mathbf{OP_2EG}$, $\mathbf{OM_1EM}$, $\mathbf{OM_1EG}$ *respectively.*
(ii) *All the classes from* (i) *have RP.*
(iii) *The topological groups* $\mathrm{Aut}(\mathbf{OP_2EM})$, $\mathrm{Aut}(\mathbf{OP_2EG})$, *as well as* $\mathrm{Aut}(\mathbf{OM_1EM})$, $\mathrm{Aut}(\mathbf{OM_1EG})$, *are extremely amenable.*

Proof. (i) This follows from the results in Section 3.
(ii) This follows from Theorem 4.11, Proposition 8.1 and Proposition 8.2.
(iii) This follows from Theorem 6.5(iii) and from (i) and (ii). ∎

9. Finite ultrametric spaces. Let $S \neq \emptyset$ be a subset of $(0, \infty)$. We denote by $\mathcal{U}_S$ the class of finite ultrametric spaces with all distances in S. As in the case of metric spaces, we may view $\mathcal{U}_S$ as a class of finite structures in the signature $L = \{R_s\}_{s\in S}$, where each R_s is a binary relational symbol. To each ultrametric space (A, d) we assign a structure $(A, \{R_s^A\}_{s\in S})$ in the same way as we assign such a structure to each metric space. For each countable set S, $\mathcal{U}_S$ is a Fraïssé class with limit $\mathbf{U}_S$ (see [18] and [19]). Note that $\mathcal{M}_S$ is not necessarily a Fraïssé class. Let $\mathbf{A} = (A, d) \in \mathcal{U}_S$ and $s \in S$. Then open balls of radius s form a partition of A. We say that a linear ordering $\leq$ on A is *convex* if for every $a \in A$ and every $s \in S$ the open ball with center a and radius s is an interval with respect to $\leq$. We denote by $\mathrm{co}(\mathbf{A})$ the collection of all convex linear orderings on the ultrametric space $\mathbf{A}$. Adding convex linear orderings to finite ultrametric spaces with all distances in S we obtain the class

$$\mathcal{CU}_S = \{(A, \{R_s^A\}_{s\in S}, \le) : (A, \{R_s^A\}_{s\in S}) \in \mathcal{U}_S,\ \le\ \in \mathrm{co}((A, \{R_s^A\}_{s\in S}))\}.$$

By Definition 2.1 we obtain the class $\mathcal{N}_S = \mathcal{EU}_S$ of finite pseudoultametric spaces with distances in S. Definition 2.2 give us the class $\mathcal{N}_{S_0} = \mathcal{OEU}_S$ of linearly ordered finite pseudoultrametric spaces with distances in S. If we take $\mathcal{K}^* = \mathcal{CU}_S$ then by Definition 2.3 we obtain the class $\mathcal{N}_{S_1} = \mathcal{CE}[\mathcal{CU}_S]$ of convex linearly ordered pseudoultrametric spaces. It is known that $\mathcal{U}_S$ and $\mathcal{CU}_S$ are Fraïssé classes (see [18] and [19]) such that $\mathcal{CU}_S$ is a Ramsey class with OP with respect to $\mathcal{U}_S$. Lemmas 3.3, 3.5 and 3.7 imply that $\mathcal{N}_S$, $\mathcal{N}_{S_0}$ and $\mathcal{N}_{S_1}$ are Fraïssé classes with limits $\mathbf{N}_S$, $\mathbf{N}_{S_0}$ and $\mathbf{N}_{S_1}$ respectively. The results of Sections 4 and 5 give us the following.

Corollary 9.1.

(i) *$\mathcal{N}_{S_1}$ is a Ramsey class.*
(ii) *$\mathcal{N}_{S_1}$ has OP with respect to $\mathcal{N}_S$.*

We also have topological consequences of Theorem 6.4:

Corollary 9.2.

(i) $\mathrm{Aut}(\mathbf{N}_{S_1})$ *is an extremely amenable group.*
(ii) $X_{\mathcal{N}_{S_1}}$ *is the universal minimal* $\mathrm{Aut}(\mathbf{N})$*-flow.*

We now give a new proof that $\mathcal{CU}_S$ is a Ramsey class using our approach. First, we note that it is enough to show that $\mathcal{CU}_S$ is a Ramsey class for S finite. Suppose $S = \{s_0 < s_1 < \cdots < s_n\}$ and $S' = S \setminus \{s_0\}$. It is sufficient to show that the Ramsey property for $\mathcal{CU}_{S'}$ implies the Ramsey property for $\mathcal{CU}_S$. Note that structures from $\mathcal{EU}_{S'}$ may also be viewed as structures from $\mathcal{U}_S$, every equivalence class being an open ball of radius s_0. If we take $\mathcal{K}^* = \mathcal{CU}_{S'}$ then by Definition 2.3 we obtain the class $\mathcal{O}^*\mathcal{EU}_{S'}$ of convex linearly ordered pseudoultrametric spaces. Note that we may view $\mathcal{O}^*\mathcal{EU}_{S'}$ as the class $\mathcal{CU}_S$. Then by the result of Section 4 we see that $\mathcal{CU}_S$ is a Ramsey class. It remains to note that for S with $|S| = 1$, the Ramsey property of $\mathcal{CU}_S$ follows from the classical Ramsey theorem.

Remark 9.3. If there is q such that $0 < q \le \inf S$, then taking $T = S \cup \{q\}$, we can view $\mathcal{EU}_S$ as the class $\mathcal{U}_T$ of ultrametric spaces.

10. Chains. Let $D = \bigcup_{i=1}^{\infty} \{i\} \times \mathbb{Q}$ and let $\mathbf{D} = (D, \le)$ be a poset such that for all (i, p) and (j, q) from D we have

$$(i, p) \le (j, q) \ \Leftrightarrow\ (i = j \text{ and } p \le q).$$

We denote by $\mathcal{D}$ the class of all finite posets isomorphic to some subposet of $\mathbf{D}$. Let $\mathbf{A} = (A, \le)$ be a poset in $\mathcal{D}$, and let $\preceq\ \in \mathrm{lo}(A)$. We say that $\preceq$ is *convex* on $\mathbf{A}$ if for all $x, y, z \in A$ we have

$$(x \preceq y \preceq z \text{ and } x \leq z) \;\Rightarrow\; x \leq y \leq z.$$

This means that $\preceq$ agrees with $\leq$ on every maximal chain in $\mathbf{A}$ and makes every maximal chain into an interval. We denote by $\mathrm{co}(\mathbf{A})$ the collection of convex linear orderings on $\mathbf{A}$. Adding convex linear orderings to structures from $\mathcal{D}$ we obtain the class

$$\mathcal{OD} = \{(A, \leq, \preceq) : (A, \leq) \in \mathcal{D},\ \preceq \in \mathrm{co}((A, \leq))\}.$$

It is shown in [23] that $\mathcal{D}$ is a Fraïssé class, and in [26] that $\mathcal{OD}$ is a Fraïssé class with the Ramsey property.

By Definition 2.1 we obtain the class $\mathcal{J} = \mathcal{ED}$. Definition 2.2 gives us the class $\mathcal{J}_0 = \mathcal{OED}$ of linearly ordered structures from $\mathcal{J}$. If we take $\mathcal{K}^* = \mathcal{OD}$ then by Definition 2.3 we obtain the class $\mathcal{J}_1 = \mathcal{CE}[\mathcal{OD}]$ of convex linearly ordered structures from $\mathcal{J}$. Lemmas 3.3, 3.5 and 3.7 imply that $\mathcal{J}$, $\mathcal{J}_0$ and $\mathcal{J}_1$ are Fraïssé classes with limits $\mathbf{J}$, $\mathbf{J}_0$ and $\mathbf{J}_1$ respectively. The results of Sections 4 and 5 give us the following.

Corollary 10.1.

(i) *$\mathcal{J}_1$ is a Ramsey class.*
(ii) *$\mathcal{J}_1$ has OP with respect to $\mathcal{J}$.*

We also have the following topological consequences of Theorem 6.4.

Corollary 10.2.

(i) $\mathrm{Aut}(\mathbf{J}_1)$ *is an extremely amenable group.*
(ii) $X_{\mathcal{J}_1}$ *is the universal minimal* $\mathrm{Aut}(\mathbf{J})$*-flow.*

Remark 10.3. We have the topological group isomorphism $\mathrm{Aut}(\mathbf{J}) \cong S_\infty^{\mathbb{N}} \rtimes \mathrm{Aut}(\mathbf{D})$ where S_∞ is the group of permutations of the natural numbers. Note that if we drop the equivalence relation from the structure $\mathbf{J}$ then the resulting structure is not isomorphic to $\mathbf{D}$ because it does not have the extension property of the Fraïssé structure $\mathbf{D}$.

Acknowledgements. The author is grateful to the referee for pointing out many errors and suggesting improvements.

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Received 8 February 2011;
in revised form 26 November 2012