FUNDAMENTA
MATHEMATICAE
221 (2013)

On d-finite tuples in random variable structures

by

Shichang Song (Beijing)

Abstract. We prove that the d-finite tuples in models of ARV are precisely the discrete random variables. Then, we apply d-finite tuples to the work by Keisler, Hoover, Fajardo, and Sun concerning saturated probability spaces. In particular, we strengthen a result in Keisler and Sun's recent paper.

1. Introduction. Continuous logic as in [BBHU] and [BU2] shares many properties with classical model theory, such as the compactness theorem, Löwenheim–Skolem theorems, existence of saturated and homogeneous models, Beth's definability theorem, the omitting types theorem, and fundamental results of stability theory.

On the other hand, there exist in continuous logic some phenomena not appearing in classical model theory. For instance, for an $\aleph_0$-categorical theory T, the unique separable model $\mathcal{M}$ might not necessarily be $\aleph_0$-saturated, instead, $\mathcal{M}$ is *approximately* $\aleph_0$*-saturated*, i.e., for every tuple $a \in M$, every $p(x,a) \in S_1(a)$, and every $\epsilon > 0$, there is $a' \in M$ with $d(a,a') \leq \epsilon$ such that $p(x,a')$ is realized in M. In [BU1], Ben Yaacov and Usvyatsov explained this phenomenon by arguing that the notion of finite tuples is not always an appropriate notion in continuous logic. They introduced the notion of d-finite tuples (see Definition 2.1), which are continuous logic analogues of finite tuples in classical model theory. They showed that in every approximately $\aleph_0$-saturated model, every type over a d-finite tuple is realized. Hence, understanding d-finite tuples is a first step toward thorough understanding of the theory T, especially in the case when T is $\aleph_0$-categorical.

Let $(\Omega, \mathcal{F}, \mu)$ be a *probability space*, which means that Ω is a sample space, the σ-algebra $\mathcal{F}$ is the collection of events, and μ is a probability measure on $\mathcal{F}$. We say it is *atomless* if for every $A \in \mathcal{F}$ with $\mu(A) > 0$, there is a $B \in \mathcal{F}$ such that $B \subseteq A$ and $0 < \mu(B) < \mu(A)$. Let

2010 *Mathematics Subject Classification*: Primary 03C90; Secondary 03C98, 60B99.
Key words and phrases: continuous logic, d-finiteness, random variables, saturated probability spaces.

DOI: 10.4064/fm221-3-2

$L^1((\Omega, \mathcal{F}, \mu), [0,1])$, or simply $L^1(\mu, [0,1])$, denote the L^1-space of classes of $[0,1]$-valued $\mathcal{F}$-measurable functions equipped with L^1-metric. Let $\mathcal{ARV}$ denote the class $\{L^1(\mu, [0,1]) \mid (\Omega, \mathcal{F}, \mu)$ is an atomless probability space$\}$. The theory $\mathrm{Th}(\mathcal{ARV})$ of atomless random variable structures, axiomatized by ARV, was first studied via continuous logic by Ben Yaacov [BY], and further studied in [BBH], [S1] and [S2]. Ben Yaacov proved that ARV is a complete theory, has quantifier elimination, and the types (over parameters) in ARV correspond precisely to (conditional) distributions in probability theory (see [BY, Theorem 2.17]). Moreover, ARV is $\aleph_0$-categorical, therefore it is approximately $\aleph_0$-saturated. However, it is not $\aleph_0$-saturated; see Proposition 4.1. For such a theory, as we stated in the preceding paragraph, it is important to know what are the d-finite tuples in models of the theory. We characterize the d-finite tuples in models of ARV as follows:

Main Theorem 1.1. *Let $\mathcal{M} \models$ ARV and $f = (f_1, \ldots, f_n) \in M^n$. Then f is a d-finite tuple in M if and only if f_i is a discrete random variable for every $1 \leq i \leq n$.*

Furthermore, for arbitrary $f \in M^n$, let $\mathsf{ARV}(f) := \mathrm{Th}(\mathcal{M}, f)$. If f is d-finite, then $\mathsf{ARV}(f)$ has only one separable model up to isomorphism. Supposing f is not d-finite, we calculate the number of non-isomorphic separable models of $\mathsf{ARV}(f)$:

Proposition 1.2. *Let $\mathcal{M} \models$ ARV and let f be a tuple in M. If f is not d-finite, then $\mathsf{ARV}(f)$ has continuum many non-isomorphic separable models.*

Starting from 1980s, Keisler, Hoover, Fajardo, and Sun studied model-theoretic properties concerning probability spaces and stochastic processes, e.g., universality, homogeneity, and saturation; see [Ho], [HK], [FK], and [KS]. All those results are closely related to the model theory of ARV, and some of their results are related to d-finiteness.

For a probability space Ω and two random variables x, y on Ω, let $\mathrm{dist}(x)$ denote the distribution of the random variable x and let $\mathrm{dist}(x, y)$ denote the distribution of the pair (x, y) of random variables.

The probability space Ω is said to have the *saturation property for* $\mathrm{dist}(x, y)$ if for all random variables x' on Ω with $\mathrm{dist}(x) = \mathrm{dist}(x')$, there is a random variable y' on Ω such that $\mathrm{dist}(x, y) = \mathrm{dist}(x', y')$. The probability space Ω is said to be *Hoover–Keisler saturated* if it has the saturation property for every such $\mathrm{dist}(x, y)$. In a recent paper [KS], Keisler and Sun gave a local sufficient condition for the saturation property ([KS, Theorem 2.7]). We strengthen their result as follows:

Theorem 1.3. *Let f and g be two random variables valued in Polish spaces X and Y respectively, where f is not a discrete random variable. If* *the probability space* $(\Omega, \mathcal{F}, \mu)$ *has the saturation property for* $\mathrm{dist}(f, g)$ *while the standard Lebesgue unit interval* $([0,1], \mathcal{L}, \lambda)$ *does not, then* $(\Omega, \mathcal{F}, \mu)$ *is Hoover–Keisler saturated.*

Theorem 1.3 strengthens [KS, Theorem 2.7] so the conclusion is true for non-atomic distributions $\mathrm{dist}(f)$ in place of atomless ones.

Outline of the paper. In Section 2, following [BU1] we give the definition of d-finite tuples and some of their properties, and recall some useful results in measure theory. In Section 3, we characterize the d-finite tuples in models of ARV and calculate the number of separable models of $\mathsf{ARV}(f)$. In Section 4, we apply d-finite tuples to the work by Keisler, Hoover, Fajardo, and Sun.

2. Preliminaries

Definition 2.1. Let a and c be tuples, and let $p = \mathrm{tp}(a/c)$. Then we say a is *d-finite* over c, or the type p is d-finite, if for every tuple b and every $\epsilon > 0$, there is $\delta = \delta^{a/c}_{b,\epsilon} > 0$ such that whenever $a' \equiv_c a$ and $d(a, a') \le \delta$, there is b' such that $d(b, b') \le \epsilon$ and $a'b' \equiv_c ab$. If $c = \emptyset$ we omit it.

For a general definition when a and c could be possibly infinite, see [BU1].

Proposition 2.2 ([BU1, Corollary 2.5]). *If* $\mathcal{M}$ *is an approximately* $\aleph_0$*-saturated model of* T *and* $a \in M$ *is d-finite, then every type in at most countably many variables over* a *is realized in* M. □

Therefore, for such a theory, it is of importance to understand the d-finite tuples in models of the theory. The following result characterizes the d-finite tuples in models of an $\aleph_0$-categorical theory.

Proposition 2.3 ([BU1, Proposition 2.9]). *Let* $\mathcal{M}$ *be a structure and let* a *be a tuple in* M. *Let* $T = \mathrm{Th}(\mathcal{M})$ *and* $T(a) = \mathrm{Th}(\mathcal{M}, a)$. *Suppose* T *is* $\aleph_0$*-categorical. Then* a *is d-finite (over* $\emptyset$*) if and only if* $T(a)$ *is* $\aleph_0$*-categorical.* □

We need the following results in measure theory in the next section.

Theorem 2.4 ([Wa, Theorem 2.2]). *Let* X_1 *and* X_2 *be complete separable metric spaces, let* $B(X_1)$ *and* $B(X_2)$ *be their* σ*-algebras of Borel subsets, and let* μ_1 *and* μ_2 *be probability measures on* $B(X_1)$ *and* $B(X_2)$ *respectively. Let* $\Phi : \widehat{B}(X_2) \to \widehat{B}(X_1)$ *be an isomorphism of probability algebras. Then there are* $M_1 \in B(X_1)$ *and* $M_2 \in B(X_2)$ *with* $\mu_1(M_1) = \mu_2(M_2) = 1$, *and an invertible measure-preserving transformation* $\varphi : M_1 \to M_2$ *such that* $\Phi([b]_{\mu_2}) = [\varphi^{-1}(b \cap M_2)]_{\mu_1}$ *for every* $b \in B(X_2)$. *If* ϕ *is any other isomorphism from* $(X_1, B(X_1), \mu_1)$ *to* $(X_2, B(X_2), \mu_2)$ *that induces* Φ, *then*

$$\mu_1(\{x \in X_1 \mid \varphi(x) \neq \phi(x)\}) = 0. \blacksquare$$

Lemma 2.5. *Suppose $(X,\mathscr{B},\mu)$ is an atomless probability space where X is a complete separable metric space and $\mathscr{B}$ is the σ-algebra of Borel subsets of X. For any two measurable partitions P_1 and P_2 of X where $P_1=(A_n)_{n\in\mathbb{N}}$ and $P_2=(B_n)_{n\in\mathbb{N}}$, if $\mu(A_i)=\mu(B_i)$ for every $i\in\mathbb{N}$, then there is an automorphism ϕ of $(X,\mathscr{B},\mu)$ satisfying $\phi(A_i)=B_i$ up to a null set for every $i\in\mathbb{N}$.*

Proof. Since $\mathscr{B}$ is atomless, $\mathscr{B}\restriction A_i$ and $\mathscr{B}\restriction B_i$ are atomless for every $i\in\mathbb{N}$. As $\mu(A_i)=\mu(B_i)$ and using [BH, Corollary 6.2], there is an L_{Pr}-isomorphism $\Phi_i:\widehat{\mathscr{B}\restriction B_i}\to\widehat{\mathscr{B}\restriction A_i}$ for every $i\in\mathbb{N}$.

We define $\Phi:\widehat{\mathscr{B}}\to\widehat{\mathscr{B}}$ as follows:

$$\Phi:\widehat{\mathscr{B}}\to\widehat{\mathscr{B}},\qquad [C]_\mu\mapsto\bigcup_{i\in\mathbb{N}}\Phi_i([C\cap B_i]_\mu)\text{ for every }C\in\mathscr{B}.$$

Then $\Phi([B_i]_\mu)=[A_i]_\mu$ for every $i\in\mathbb{N}$ and Φ is an L_{Pr}-automorphism of $\widehat{\mathscr{B}}$. Then by Theorem 2.4, up to a null set, there is an automorphism ϕ of $(X,\mathscr{B},\mu)$ satisfying $\phi(A_i)=B_i$ for every $i\in\mathbb{N}$. ∎

3. Main theorems

3.1. Characterization of d-finite tuples in ARV. The theory of atomless random variable structures, axiomatized by ARV, was first studied in [BY]. It is an $\aleph_0$-categorical theory, so by the Ryll-Nardzewski theorem for continuous logic due to C. Ward Henson (see Fact 1.14 in [BU1]), it is approximately $\aleph_0$-saturated. But it is not $\aleph_0$-saturated: see Proposition 4.1 below.

Let $\mathcal{M}$ be a model of ARV. Then M is isomorphic to $L^1(\mu,[0,1])$ for some atomless probability space $(\Omega,\mathcal{F},\mu)$. For the type spaces of ARV, there is the following theorem:

Theorem 3.1 ([BY, Theorem 2.17]). *Let $\mathcal{M}=L^1((\Omega,\mathcal{F},\mu),[0,1])$ be a model of* ARV. *Then two tuples f and g in M^n have the same type over a set $A\subseteq M$ if and only if they have the same joint conditional distribution over $\sigma(A)$, the σ-algebra of measurable sets generated by the random variables in A. Moreover,* $\mathrm{dcl}(A)=\mathrm{acl}(A)=L^1((\Omega,\sigma(A),\mu),[0,1])$. ∎

Let $\mathcal{M}\models$ ARV and let $f=(f_1,\dots,f_n)\in M^n$ be an n-tuple. Let ARV(f) denote $\mathrm{Th}(\mathcal{M},f)$. By Proposition 2.3, f is d-finite (over $\emptyset$) if and only if ARV(f) is $\aleph_0$-categorical. We now use this result to prove our Main Theorem:

Proof of Main Theorem 1.1. $\Leftarrow$: Let $\mathcal{N}=L^1(\lambda,[0,1])\models$ ARV, where $([0,1],\mathcal{L},\lambda)$ is the standard Lebesgue space. To show that ARV(f) is $\aleph_0$-categorical, we need only show that for all $g,h\in N^n$ with $\mathrm{tp}(f)=\mathrm{tp}(g)=\mathrm{tp}(h)$, there exists $\varphi\in\mathrm{Aut}(\mathcal{N})$ such that $\varphi(g)=h$.

Let $g = (g_1, \ldots, g_n) \in N^n$ be such that $\mathrm{tp}(f) = \mathrm{tp}(g)$, so $\mathrm{dist}(f) = \mathrm{dist}(g)$. Since all f_i's are discrete random variables, all g_i's are also discrete. Therefore, we can write $g_i = \sum_{j=1}^{\infty} r_{ij}\chi_{A_{ij}}$ for all $i = 1, \ldots, n$, where $r_{ij} \in [0,1]$ for all j and $(A_{i1}, \ldots, A_{in}, \ldots)$ is a measurable partition of $[0,1]$. Then $\{A_{1i_1} \cap \cdots \cap A_{ni_n} \mid \forall i_1, \ldots, i_n \in \mathbb{N}\}$ also forms a partition of $[0,1]$, although some of its elements might be null sets. Take any other $h = (h_1, \ldots, h_n) \in N^n$ such that $\mathrm{tp}(f) = \mathrm{tp}(g) = \mathrm{tp}(h)$, so $\mathrm{dist}(g) = \mathrm{dist}(h)$. Then for every $i = 1, \ldots, n$, every h_i can be written as $\sum_{j=1}^{\infty} r_{ij}\chi_{B_{ij}}$ such that $(B_{i1}, \ldots, B_{in}, \ldots)$ is a measurable partition of $[0,1]$ with $\mu(A_{ij}) = \mu(B_{ij})$ for all $j \in \mathbb{N}$ and $\mu(A_{1i_1} \cap \cdots \cap A_{ni_n}) = \mu(B_{1i_1} \cap \cdots \cap B_{ni_n})$ for all $i_1, \ldots, i_n \in \mathbb{N}$. Then by Lemma 2.5, there is an automorphism φ of $([0,1], \mathcal{L}, \lambda)$ such that $\varphi(A_{1i_1} \cap \cdots \cap A_{ni_n}) = B_{1i_1} \cap \cdots \cap B_{ni_n}$ (up to null sets) for all $i_1, \ldots, i_n \in \mathbb{N}$. We then extend φ to an automorphism $\widehat{\varphi}$ of $\mathcal{N}$ such that $\widehat{\varphi}(g) = h$. Hence f is d-finite.

$\Rightarrow$: Assume that f is a d-finite element in $\mathcal{N} = L^1(\lambda, [0,1]) \models \mathsf{ARV}$, where $([0,1], \mathcal{L}, \lambda)$ is the standard Lebesgue space. Then for all $g \in N$ such that $\mathrm{tp}(f) = \mathrm{tp}(g)$, there is a $\varphi \in \mathrm{Aut}(\mathcal{N})$ satisfying $\varphi(f) = g$. Suppose f is not a discrete random variable; then f has a decomposition $f = f_d + f_c$, where f_d is a discrete random variable, and f_c is a nonzero continuous random variable. Note that $f_d \cdot f_c = 0$. By Theorem 3.1, $f_c \in \mathrm{dcl}(f)$ and $\mathsf{ARV}(f)$ is $\aleph_0$-categorical. Therefore, $\mathsf{ARV}(f, f_c)$ is also $\aleph_0$-categorical, so that $\mathsf{ARV}(f_c)$ is $\aleph_0$-categorical by [BBHU, Corollary 12.13]. Hence f_c is d-finite, by Theorem 2.3.

Define $F(t) = \lambda(f_c \le t)$ for all $t \in [0,1]$. Because f_c is nonzero, $F(t)$ is continuous and strictly increasing. Thus, the inverse function $g(t) = F^{-1}(t)$ exists for all $t \in [0,1]$. Because $F : [0,1] \to [0,1]$ is continuous and strictly increasing, $\sigma(g)$ is the set $\mathscr{B}$ of all Borel sets in $[0,1]$. Also, note that $\mathrm{dist}(g) = \mathrm{dist}(f_c)$, so $\mathrm{tp}(g) = \mathrm{tp}(f_c)$ by Theorem 3.1. We know that $\mathcal{N} = L^1(([0,1], \mathcal{L}, \lambda), [0,1])$ is isomorphic to $\mathcal{N}' = L^1(([0,1]^2, \mathcal{L} \times \mathcal{L}, \lambda \times \lambda), [0,1])$, since ARV is $\aleph_0$-categorical. Let α be such an isomorphism. Let $g_1 = \alpha(g)$. Because $\sigma(g)$ is the set of all Borel sets in $[0,1]$, we know that $\mathrm{dcl}(g) = L^1(([0,1], \mathcal{L}, \lambda), [0,1])$. Since α is an isomorphism and $\alpha(g) = g_1$, we know that

$$\mathrm{dcl}(g_1) = L^1(([0,1]^2, \sigma(g_1), \lambda \times \lambda), [0,1]) = \mathcal{N}'.$$

We define $g_2 : [0,1] \times [0,1] \to [0,1]$ by $g_2(t,s) := g(t)$ for all $t, s \in [0,1]$. Then $\mathrm{dist}(g_2) = \mathrm{dist}(g)$, which implies $\mathrm{tp}(g_2) = \mathrm{tp}(g)$ by Theorem 3.1. But clearly, the completion of $\sigma(g_2)$ is not $\mathcal{L} \times \mathcal{L}$. Thus, $\mathrm{dcl}(g_2) \subsetneq \mathrm{dcl}(g_1)$. If there is an automorphism Φ of $\mathcal{N}'$ sending g_1 to g_2, then $\Phi(\mathrm{dcl}(g_1)) = \mathrm{dcl}(g_2)$. Since $\mathrm{dcl}(g_1) = \mathcal{N}'$, we have $\Phi(\mathrm{dcl}(g_1)) = \mathrm{dcl}(g_1)$, and thus $\mathrm{dcl}(g_2) = \mathrm{dcl}(g_1)$, which contradicts $\mathrm{dcl}(g_2) \subsetneq \mathrm{dcl}(g_1)$.

Therefore $\mathsf{ARV}(f_c)$ is not $\aleph_0$-categorical, which contradicts the fact that f_c is d-finite. Hence, f is a discrete random variable.

If $f = (f_1, \dots, f_n)$ is a d-finite tuple, then by the definition of d-finiteness, f_i is d-finite for every $i = 1, \dots, n$. Thus f_i is a discrete random variable for every $i = 1, \dots, n$. ∎

3.2. Number of separable models of $\mathsf{ARV}(f)$**.** Let $\mathcal{M} \models \mathsf{ARV}$. If f is not a d-finite tuple in M, then $\mathsf{ARV}(f)$ is not $\aleph_0$-categorical, but how many nonisomorphic separable models would it have?

PROPOSITION 3.2. *Let* $\mathcal{M} \models \mathsf{ARV}$. *Let* f *a tuple in* M. *Then:*

(1) *If* f *is* d*-finite, then* $\mathsf{ARV}(f)$ *has a unique separable model up to isomorphism.*
(2) *If* f *is not* d*-finite, then* $\mathsf{ARV}(f)$ *has continuum many nonisomorphic separable models.*

Proof. We need only show (2).

Without loss of generality, we may assume that $\mathcal{M} = L^1(\lambda, [0,1])$, where $([0,1], \mathcal{L}, \lambda)$ is the standard Lebesgue space, and f is a finite tuple in M which is not d-finite. By Theorem 1.1, $f = (f_1, \dots, f_n)$ is not a discrete random variable on $[0,1]^n$. Since $\sigma(f)$ is countably σ-generated, there exists $g \in M$ such that $\sigma(g) = \sigma(f)$. Next, we introduce some notations. For all $r, s \in [0,1]$, let $\mathscr{B}([r,s])$ denote the set of all Borel subsets of $[r,s]$. In $\mathscr{B}([r,s])$, we interpret $\mathbf{1}$ as $[r,s]$; then $\mathscr{B}([r,s])$ also forms a σ-algebra. Let $\mathscr{B}$ denote $\mathscr{B}([0,1])$, the set of all Borel subsets in $[0,1]$. In $[0,1] \times [0,1]$, define $\mathscr{B}_r$ as $\sigma(\{A_1 \times [0,1], A_2 \times A_3 : A_1 \in \mathscr{B}([0,r]), A_2 \in \mathscr{B}([r,1]), A_3 \in \mathscr{B}\})$. Then $\mathscr{B}_0$ is just the set of all Borel sets in $[0,1] \times [0,1]$. Since g is not discrete, it has a decomposition $g = g_c + g_d$, where g_c is a nonzero continuous random variable, g_d is a discrete random variable, and $g_c \cdot g_d = 0$. After rearranging the values of g (but keeping the distribution of g), we may further assume that the support of g_c is $[0,t]$ and $\sigma(g_c) = \sigma(\mathscr{B}([0,t]))$ for some $t \in [0,1]$. Let $\mathscr{B}_g$ denote $\sigma(g) = \sigma(g_c) \vee \sigma(g_d)$, the smallest σ-algebra containing $\sigma(g_c)$ and $\sigma(g_d)$.

On $([0,1] \times [0,1], \mathscr{B}_r)$, there is a natural probability measure λ_r satisfying:

- $\lambda_r(A_1 \times [0,1]) = \lambda(A_1)$ for all $A_1 \in \mathscr{B}([0,r])$,
- $\lambda_r(A_2 \times A_3) = \lambda(A_2)\lambda(A_3)$ for all $A_2 \in \mathscr{B}([r,1])$ and all $A_3 \in \mathscr{B}$.

Let $\mathcal{L}_r$ denote the completion of $\mathscr{B}_r$ under λ_r. Define $M_r := L^1(([0,1] \times [0,1], \mathcal{L}_r, \lambda_r), [0,1])$. Then $M_r \models \mathsf{ARV}$ and it is separable. Define $g_r : [0,1] \times [0,1] \to [0,1]$ by $g_r(s,t) = g(s)$ for all $s, t \in [0,1]$. Note that $\operatorname{dist}(g_r) = \operatorname{dist}(g) = \operatorname{dist}(f)$, which implies $\operatorname{tp}(g_r) = \operatorname{tp}(g) = \operatorname{tp}(f)$ by Theorem 3.1. Therefore, $(M_r, g_r) \models \mathsf{ARV}(f)$ for all $r \in [0,1]$. Since $\sigma(g) = \mathscr{B}_g$, we have $\sigma(g_r) = \mathscr{B}_g \times [0,1] := \{B \times [0,1] : B \in \mathscr{B}_g\}$.

Now we show that for distinct $r \le t$, those models are not isomorphic to each other. For $r_1, r_2 \in [0,t]$ with $r_1 > r_2$, suppose $(M_{r_1}, g_{r_1}) \cong (M_{r_2}, g_{r_2})$;

let α denote such an isomorphism. Then $\alpha(g_{r_1}) = g_{r_2}$. Note that α is induced by an isomorphism β between $(\mathscr{B}_{r_1}, \sigma(g_{r_1}), \lambda_{r_1})$ and $(\mathscr{B}_{r_2}, \sigma(g_{r_2}), \lambda_{r_2})$ such that $\beta(\mathscr{B}_{r_1}) = \mathscr{B}_{r_2}$, $\beta(\sigma(g_{r_1})) = \sigma(g_{r_2})$, and β preserves the measure. Since $\sigma(g_{r_1}) = \sigma(g_{r_2}) = \mathscr{B}_g \times [0,1]$, we have $\beta(\mathscr{B}_g \times [0,1]) = \mathscr{B}_g \times [0,1]$.

For any two measure algebras $\mathscr{A}_1 \subseteq \mathscr{A}_2$, an element $A \in \mathscr{A}_2$ of positive measure is called an *atom over* $\mathscr{A}_1$ if for every $B \in \mathscr{A}_2$ there exists $C \in \mathscr{A}_1$ such that $A \cap B = A \cap C$. If there is no element in $\mathscr{A}_2$ that is an atom over $\mathscr{A}_1$, then we say that $\mathscr{A}_2$ is *atomless over* $\mathscr{A}_1$.

Note that $[0,r] \times [0,1] \in \mathscr{B}_r$ is an atom over $\mathscr{B}_g \times [0,1]$ and $\lambda_r([0,r] \times [0,1]) = r$. Suppose there is an atom $C \in \mathscr{B}_r$ over $\mathscr{B}_g \times [0,1]$ with $\lambda_r(C) > r$. Then $C' := ([r,1] \times [0,1]) \cap C$ is a subset of positive measure. Note that $\sigma(\{A_1 \times A_2 : A_1 \in \mathscr{B}([r,1]), A_2 \in \mathscr{B}\})$ is atomless over $\mathscr{B}([r,1]) \times [0,1]$ and $\mathscr{B}([r,1]) \times [0,1] \supseteq (\mathscr{B}_g \times [0,1]) \cap (\mathscr{B}([r,1]) \times [0,1])$. By the definition, $\sigma(\{A_1 \times A_2 : A_1 \in \mathscr{B}([r,1]),\ A_2 \in \mathscr{B}\})$ is atomless over $(\mathscr{B}_g \times [0,1]) \cap (\mathscr{B}([r,1]) \times [0,1])$. Hence C' is not an atom over $(\mathscr{B}_g \times [0,1]) \cap (\mathscr{B}([r,1]) \times [0,1])$. Therefore, C is not an atom over $\mathscr{B}_g \times [0,1]$. Thus, each atom in $\mathscr{B}_r$ over $\mathscr{B}_g \times [0,1]$ has measure at most r. Because β is an isomorphism, $\beta([0,r_1] \times [0,1])$ is also an atom over $\beta(\mathscr{B}_g \times [0,1]) = \mathscr{B}_g \times [0,1]$. But $\lambda_{r_2}(\beta([0,r_1] \times [0,1])) = \lambda_{r_1}([0,r_1] \times [0,1]) = r_1 > r_2$. This contradicts the fact that each atom in $\mathscr{B}_{r_2}$ over $\mathscr{B}_g \times [0,1]$ has measure at most r_2.

Therefore, $\mathsf{ARV}(f)$ has continuum many separable models. ∎

4. Some applications.

4. Some applications. Keisler, Hoover, Fajardo, and Sun have published several papers around model-theoretic results for probability spaces; for example, see [HK], [Ho], [FK], and [KS]. Some of their results are closely related to the d-finiteness in ARV.

PROPOSITION 4.1. *No separable model of* ARV *is* $\aleph_0$*-saturated.*

Proof. Let $\mathcal{M}$ be a separable model of ARV and let f be a non-d-finite tuple of M. Then the theory $\mathsf{ARV}(f)$ is not $\aleph_0$-categorical. By the omitting types theorem [BU1, Theorem 1.11], M as a model of $\mathsf{ARV}(f)$ omits some types in $S_1^{\mathsf{ARV}}(f)$, so it is not $\aleph_0$-saturated. ∎

FACT 4.2 ([S2, Theorem 5.5.4]). *A probability space* $(\Omega, \mathcal{F}, \mu)$ *is Hoover–Keisler saturated if and only if* $L^1(\mu, [0,1])$ *as a model of* ARV *is* $\aleph_0$*-saturated.*

COROLLARY 4.3 ([FK, Theorem 3B.1]). *No ordinary probability space is Hoover–Keisler saturated.* ∎

PROPOSITION 4.4. *Let* $\mathcal{M} \models \mathsf{ARV}$. *Then for every* d*-finite tuple* f *in* M *and every type* $p(f,y)$ *in* $S_1(f)$, *there is a* $g \in M$ *realizing* $p(f,y)$.

Proof. This follows from Proposition 2.2. ∎

Corollary 4.5. *Let Ω be an atomless probability space, and let Γ be another probability space. Let $\mathcal{X}$ and $\mathcal{Y}$ be Polish spaces. Then for every discrete $\mathcal{X}$-valued random variable x on Ω and every pair of random variables (x', y') on Γ with values in $\mathcal{X}$ and $\mathcal{Y}$ such that* $\operatorname{dist}(x') = \operatorname{dist}(x)$, *there exists a random variable y on Ω such that* $\operatorname{dist}(x, y) = \operatorname{dist}(x', y')$.

Proof. By the Borel Isomorphism Theorem [K, Theorem 15.6], we may assume $\mathcal{X} = \mathcal{Y} = [0, 1]$. The rest follows from Theorem 1.1, Theorem 3.1, and Proposition 4.4. ∎

Remark 4.6. Corollary 4.5 is a generalization of [FK, Proposition 3B.3], where x is a simple $\mathcal{X}$-valued random variable instead of being discrete.

Let $\mathcal{M}$ be a model of a theory T. Let $p(x, y)$ and $q(x)$ be two complete types in T such that $q(x) \subseteq p(x, y)$. We say $\mathcal{M}$ has the *saturation property for* $p(x, y)$ if for every $a \in M$ with $a \models q(x)$ there is $b \in M$ such that $(a, b) \models p(x, y)$.

Theorem 4.7. *Let M be $L^1(\lambda, [0, 1])$, where $([0, 1], \mathcal{L}, \lambda)$ is the standard Lebesgue space. For every non-d-finite type $p(x)$ in* ARV, *there is a complete type $q(x, y) \supseteq p(x)$ in* ARV *such that M does not have the saturation property for $q(x, y)$. For every model $\mathcal{N}$ of* ARV, *if $\mathcal{N}$ has the saturation property for $q(x, y)$, then $\mathcal{N}$ is $\aleph_0$-saturated.*

Proof. If $p(x)$ is not d-finite, then for every $a \in M$ with $a \models p(x)$ we find that $\mathsf{ARV}(a)$ is not $\aleph_0$-categorical, and thus there is a type $r \in S(\mathsf{ARV}(a))$ such that r is not realized in (M, a). Hence, there is a complete type $q(x, y) \supseteq p(x)$ in ARV such that M does not have the saturation property for $q(x, y)$.

Let $N = L^1((\Omega, \mathcal{F}, \mu), [0, 1])$. Suppose $\mathcal{N}$ is not $\aleph_0$-saturated. By Fact 4.2, $(\Omega, \mathcal{F}, \mu)$ is Hoover–Keisler saturated. By [FK, Theorem 3B.7] and Maharam's Theorem, $(\Omega, \mathcal{F}, \mu)$ is a convex combination of $(\Omega_c, \mathcal{F}_c, \mu_c)$ and $(\Omega_u, \mathcal{F}_u, \mu_u)$, where the Maharam spectrum of Ω_u is a set of uncountable cardinals, while $(\Omega_c, \mathcal{F}_c, \mu_c)$ is isomorphic to $([0, 1], \mathcal{L}, \lambda)$. We assume that $\Omega = r\Omega_c + (1 - r)\Omega_u$, where $0 < r < 1$.

Take $(f', g') \in M^2$ such that $\operatorname{tp}(f', g') = q(x, y)$, and thus $\operatorname{tp}(f') = p(x)$. By Theorem 1.1, $f' = f'_c + f'_d$, where f'_d is discrete, $\sigma(f'_c)$ is atomless, and $f'_d \cdot f'_c = 0$. Since M does not have the saturation property for $q(x, y)$, we infer that $g' \notin \operatorname{acl}(f')$. Thus, $g' \notin \operatorname{acl}(f'_c)$. By Theorem 3.1, g' is not $\sigma(f'_c)$-measurable. Hence there is a $\sigma(f'_c)$-measurable set A with $\mu(A) = r$ such that $g' \restriction_A$ is not $\sigma(f'_c \restriction_A)$-measurable. By Theorem 3.1, we have $\mathbf{1}_A \in \operatorname{acl}(f')$ with $\mu(A) = r$ such that $g' \cdot \mathbf{1}_A \notin \operatorname{acl}(f' \cdot \mathbf{1}_A)$. Since ARV is $\aleph_0$-categorical, there is $f_c : (\Omega_c, r\mu_c) \to [0, 1]$ such that $\operatorname{dist}(f_c) = \operatorname{dist}(f'_c \restriction_A)$ and $\sigma(f_c) = \mathcal{F}_c$, and there is $f_u : (\Omega_u, (1 - r)\mu_u) \to [0, 1]$ such that $\operatorname{dist}(f_u) = \operatorname{dist}(f' \restriction_{A^\complement})$. Define $f : \Omega \to [0, 1]$ as $f_c \sqcup f_u$. Then $\operatorname{dist}(f, \mathbf{1}_{\Omega_c}) = \operatorname{dist}(f', \mathbf{1}_A)$, and thus $\operatorname{tp}(f, \mathbf{1}_{\Omega_c}) = \operatorname{tp}(f', \mathbf{1}_A)$. Since $\mathcal{N}$ has the saturation property for $q(x, y)$, there is $g : \Omega \to [0,1]$ such that $\mathrm{tp}(f,g) = q(x,y) = \mathrm{tp}(f',g')$. Then as $\mathrm{tp}(f,\mathbf{1}_{\Omega_c}) = \mathrm{tp}(f',\mathbf{1}_A)$, $\mathbf{1}_A \in \mathrm{acl}(f')$, and $g' \cdot \mathbf{1}_A \notin \mathrm{acl}(f' \cdot \mathbf{1}_A)$, we have $g \cdot \mathbf{1}_{\Omega_c} \notin \mathrm{acl}(f \cdot \mathbf{1}_{\Omega_c})$. But $\sigma(f_c) = \mathcal{F}_c$ implies that $g\restriction_{\Omega_c}$ is $\sigma(f_c)$-measurable, which contradicts the fact that $g \cdot \mathbf{1}_{\Omega_c} \notin \mathrm{acl}(f \cdot \mathbf{1}_{\Omega_c})$. Hence, $\mathcal{N}$ is $\aleph_0$-saturated. ∎

Now Theorem 1.3 follows from the Borel Isomorphism Theorem, Theorem 1.1, Theorem 4.7, and Fact 4.2.

REMARK 4.8. Theorem 1.3 is an extension of [KS, Theorem 2.7], where $\mathrm{dist}(f)$ is atomless instead of non-atomic.

Acknowledgements. This research is part of the author's thesis [S2] at the University of Illinois at Urbana-Champaign. The author is grateful to his thesis advisor Professor C. Ward Henson for his kind supervision and support, to Professor Alexander Berenstein for inspiring discussions, and to Patrick Reynolds for his revision of this paper. Also, this paper is in memory of Chen Lin (1981–2009), who was the first person the author discussed this research with just a couple of days before his tragedy in Puerto Rico.

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Shichang Song
Department of Mathematics
Beijing Jiaotong University
Beijing, P.R. China
E-mail: song@amss.ac.cn

Received 31 October 2011;
in revised form 31 December 2012