FUNDAMENTA
MATHEMATICAE
221 (2013)

Gibbs states for non-irreducible countable Markov shifts

by

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Abstract. We study Markov shifts over countable (finite or countably infinite) alphabets, i.e. shifts generated by incidence matrices. In particular, we derive necessary and sufficient conditions for the existence of a Gibbs state for a certain class of infinite Markov shifts. We further establish a characterization of the existence, uniqueness and ergodicity of invariant Gibbs states for this class of shifts. Our results generalize the well-known results for finitely irreducible Markov shifts.

1. Introduction. Symbolic dynamics has been a subject of scrutiny for more than a century. Symbolic systems are fundamental as they can be used to encode information and since they are often embedded into more general dynamical systems. Among all symbolic systems, shifts and especially Markov shifts have drawn the interest of a great many mathematicians (see, for instance, [4] and [3]). Given an incidence matrix $A : E^2 \to \{0,1\}$, where E is a finite or countably infinite alphabet, we study the dynamics of the shift map, which acts on the space E_A^∞ of all one-sided infinite sequences of letters so that A equals 1 on all pairs of consecutive letters. The action of the shift map consists in removing the first letter of every word and shifting each of the remaining letters one place to the left. The resulting dynamical system is called a Markov shift. Depending on the cardinality of E, the system is called finite or infinite.

Finite Markov shifts have been studied for a long time. More recently, infinite systems have caught attention. In an overwhelming majority of instances, irreducible shifts have been examined (see, among others, [1] in the finite case and [6] in the infinite case). In this paper, we shall be especially interested in non-irreducible Markov shifts, i.e. Markov shifts that can be split in some sense. To study these systems, we need to pay close attention

2010 *Mathematics Subject Classification*: Primary 37B10; Secondary 37B05, 37C40, 28A80.
Key words and phrases: Gibbs states, countable-state Markov shifts, finitely irreducible, reducible.

DOI: 10.4064/fm221-3-3

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to their irreducible components, as well as their isolated letters. Such systems have been scarcely investigated in the past (see, for instance, [7], [2] and [8]).

Our aim is to provide a characterization of the existence of Gibbs states, as well as to give necessary and sufficient conditions for the existence, uniqueness and ergodicity of invariant Gibbs states, for a certain natural class of infinite Markov shifts when subject to Hölder continuous potentials.

The question about invariant states was solved by Bowen [1] in the topologically mixing (also called irreducible) case for subshifts of finite type. For topologically mixing countable Markov shifts, Mauldin and Urbański [6] and Sarig [9] proved that a necessary and sufficient condition for the existence of an invariant Gibbs state is that the incidence matrix be finitely irreducible (cf. Theorem 2.2.6 in [6]). This Gibbs state is unique, ergodic, and even completely ergodic when the matrix is finitely primitive (cf. Theorems 2.3.3 and 2.2.4 in [6]).

In Section 2 we present some preliminaries on Markov shifts. In Section 3 we define the notion of normalization for boundedly super- and submultiplicative sequences of real numbers. In particular, we prove that the normalization of a boundedly supermultiplicative and boundedly submultiplicative sequence is bounded. We also provide some bounds for the normalization sequence. Furthermore, we define normalized partition functions (see Definition 3.6). In Section 4 we introduce the concepts of connected and strongly connected components of a system, and we establish a partial ordering on the latter. This ordering is a one-way communication between strongly connected components. We further define the notion of isolated letter and finitely linked isolated letters. In addition, we give sufficient conditions for the existence of a component of maximal pressure for some class of Markov shifts (see Proposition 4.10), which includes all finite Markov shifts. Finally, in Section 5 we describe the conditions under which Gibbs states exist (see Theorem 5.1) and under which invariant Gibbs states exist, are unique and are ergodic (see Theorem 5.2) for a natural class of Markov shifts, which includes all finite Markov shifts. To do so, we derive a series of results, most of which revolve around the boundedness (resp. the unboundedness) of the normalized partition functions. Furthermore, we point out conditions under which non-irreducible infinite Markov shifts do not admit any Gibbs state (see Propositions 5.10 and 5.11).

2. Preliminaries on Markov shifts. Let E be a finite or countably infinite set. We shall think of E as an *alphabet*. The elements of E will thereafter be called *letters*. Let $A : E^2 \to \{0,1\}$ be a 0-1 matrix, also called an *incidence* or *transition matrix*. Let

$$E_A^\infty = \{\omega \in E^\infty : A_{\omega_n\omega_{n+1}} = 1,\ \forall n \geq 1\}$$

be the set of all infinite one-sided A-*admissible words*. For each $n \geq 1$, we will denote by E_A^n the set of all blocks of n consecutive letters appearing in words in E_A^∞, also called A-*admissible n-words*. The word ϵ, called the *empty word*, is by convention the only A-admissible word of length 0. Thus, $E_A^0 = \{\epsilon\}$. We shall further denote by E_A^* the set of all finite A-admissible words, i.e. $E_A^* = \bigcup_{n\geq 0} E_A^n$. We will, throughout this paper, assume that the matrix A has at least one positive entry in each of its rows and in each of its columns.

Given $\omega \in E_A^* \cup E_A^\infty$, we denote by $|\omega|$ the length of ω. For $\omega \in E_A^* \cup E_A^\infty$ and $1 \leq n \leq |\omega|$, let

$$\omega|_n = \omega_1 \ldots \omega_n$$

denote the initial n-subword of ω. In particular, note that $\omega|_{|\omega|} = \omega$. Furthermore, we shall denote by

$$[\omega] = \{\tau \in E_A^\infty : \tau|_{|\omega|} = \omega\}$$

the set of all infinite A-admissible words whose initial subword is ω. Such a set is often called a *cylinder*. Given $\omega, \tau \in E_A^* \cup E_A^\infty$, we define $\omega \wedge \tau$ to be the longest initial subword common to both ω and τ. For every $\alpha > 0$ we define a metric d_α on E_A^∞ by setting $d_\alpha(\omega, \tau) = e^{-\alpha|\omega\wedge\tau|}$. These metrics are all Hölder equivalent and therefore induce the same topology and Borel σ-algebra. In fact, the induced topology has for base the cylinders. Moreover, a function is continuous (resp. uniformly continuous) with respect to one of these metrics if and only if it is continuous (resp. uniformly continuous) with respect to all. A function is Hölder continuous with respect to one of these metrics if and only if it is Hölder with respect to all. The Hölder order depends on the metric, though. If no metric is specified, we shall take it to be d_1. The following result is obvious.

Lemma 2.1. *A function $f : E_A^\infty \to \mathbb{R}$ is Hölder continuous with exponent or order $\alpha > 0$ if and only if*

$$V_\alpha(f) := \sup_{n\geq 1} V_{\alpha,n}(f) < \infty,$$

where

$$V_{\alpha,n}(f) := \sup\{|f(\omega) - f(\tau)|e^{\alpha n} : \omega, \tau \in E_A^\infty,\ |\omega \wedge \tau| \geq n\}.$$

The dynamics on E_A^∞ will be governed by the (left) shift map $\sigma : E_A^\infty \to E_A^\infty$ which acts on every word by throwing away its first letter. The couple (σ, E_A^∞) is called a *Markov shift*. It is said to be *finite* if E is finite, and *infinite* if E is infinite. The incidence matrix A is said to be *irreducible* if there is a set $\varOmega \subseteq E_A^*$ such that for every $e_1, e_2 \in E$ there exists a word $\omega = \omega(e_1, e_2) \in \varOmega$ such that $e_1\omega e_2 \in E_A^*$. The matrix A is called *finitely irreducible* if there exists a finite set $\varOmega$ with the above property. Of course, the concept of finite irreducibility coincides with that of irreducibility when

E is finite. Irreducibility of A is equivalent to saying that σ is *topologically mixing*, i.e. for any two non-empty open subsets U, V of E_A^∞ there exists $n \geq 1$ such that $\sigma^n(U) \cap V \neq \emptyset$.

It is also fundamental to recall that the ergodic sums of every Hölder continuous function $f : E_A^\infty \to \mathbb{R}$ exhibit a bounded variation. Remember that the nth *ergodic sum* of f, also called the nth *partial orbit sum* of f with respect to σ, is the function (cf. [10], [6]) defined as

$$S_n f = \sum_{j=0}^{n-1} f \circ \sigma^j.$$

These sums obey the following principle of bounded variation.

LEMMA 2.2. *Let $f : E_A^\infty \to \mathbb{R}$ be a Hölder continuous function. There exists a constant $K = K(f) \geq 1$, called a* constant of bounded variation for f, *such that for every $\omega, \tau \in E_A^\infty$ and $1 \leq n \leq |\omega \wedge \tau|$ we have*

$$K^{-1} \exp(S_n f(\tau)) \leq \exp(S_n f(\omega)) \leq K \exp(S_n f(\tau)).$$

Functions $f : E_A^\infty \to \mathbb{R}$ are sometimes called *potentials*.

DEFINITION 2.3. A potential $f : E_A^\infty \to \mathbb{R}$ is said to be *summable* if

$$\sum_{e \in E} \exp(\sup(f|_{[e]})) < \infty.$$

Notice that if E is finite, then every continuous potential is summable since E_A^∞ is then compact.

Given $F \subseteq E$, we let

$$F_A^\infty = \{\omega \in E_A^\infty : \omega_n \in F, \forall n \geq 1\}$$

be the set of all infinite A-admissible words consisting exclusively of letters from the subalphabet F. Given a potential $f : E_A^\infty \to \mathbb{R}$, we define the nth(-level) *partition function* of $f|_{F_A^\infty}$ by

$$Z_{n,F}(f) = \sum_{\omega \in F_A^n} \exp(\sup(S_n f|_{[\omega \cap F]})),$$

where $[\omega \cap F] := [\omega] \cap F_A^\infty$. To alleviate notation, in the case $F = E$ we simply omit E. Notice that $Z_{n,G}(f) \leq Z_{n,F}(f)$ for all $n \geq 1$ whenever $G \subseteq F$. Note further that f is summable if and only if $Z_1(f) < \infty$. It is also easy to show that the sequence $(Z_{n,F}(f))_{n \geq 1}$ is submultiplicative and hence the following definition makes sense. The *topological pressure* of f with respect to the shift map $\sigma : F_A^\infty \to F_A^\infty$ is defined as

$$P_F(f) = \lim_{n \to \infty} \frac{1}{n} \log Z_{n,F}(f) = \inf_{n \geq 1} \frac{1}{n} \log Z_{n,F}(f).$$

In particular, note that $P_G(f) \leq P_F(f)$ whenever $G \subseteq F$. Observe also that $P(f) < \infty$ whenever f is summable.

The following theorem is a characterization of the topological pressure as a Poincaré exponent. This is Theorem 2.1.3 in [6].

THEOREM 2.4. *For every function* $f : F_A^\infty \to \mathbb{R}$, *we have*

$$P_F(f) = \inf\Big\{t \in \mathbb{R} : \sum_{\omega \in F_A^*} \exp(\sup(S_{|\omega|} f|_{[\omega\cap F]}))e^{-t|\omega|} < \infty\Big\}.$$

Notice that $P_F(f) > -\infty$ if $f|_{F_A^\infty}$ is bounded from below or is Hölder continuous and F_A^∞ has a periodic point. In particular, note that $-\infty < P_F(f) < \infty$ if f is continuous and F is finite. Finally, note that if f is summable, Hölder continuous and F_A^∞ has a periodic point, then $-\infty < P_F(f) \le P(f) < \infty$.

Moreover, the topological pressure of a finitely irreducible shift is equal to the supremum of the topological pressures of its finite subshifts. This is Theorem 2.1.5 in [6].

THEOREM 2.5. *For every Hölder function* $f : E_A^\infty \to \mathbb{R}$ *and finitely irreducible matrix* A, *we have*

$$P(f) = \sup\{P_F(f) : F \text{ is a finite subset of } E\}.$$

We now turn our attention to measures.

DEFINITION 2.6. A Borel probability measure m on E_A^∞ is called a *Gibbs state* for $f : E_A^\infty \to \mathbb{R}$ if there exist constants $Q \ge 1$ and $P_m \in \mathbb{R}$ such that for every $\omega \in E_A^*$ and for every $\tau \in [\omega]$ we have

$$Q^{-1} \le \frac{m([\omega])}{\exp(S_{|\omega|} f(\tau) - P_m|\omega|)} \le Q. \tag{2.1}$$

If, in addition, m is σ-invariant, then it is called an *invariant* Gibbs state.

REMARK 2.7. The sum $S_{|\omega|} f(\tau)$ in (2.1) can be replaced by $\sup(S_{|\omega|} f|_{[\omega]})$ or by $\inf(S_{|\omega|} f|_{[\omega]})$. Note also that if m is a Gibbs state and μ and m are boundedly equivalent measures, i.e. there is some $B \ge 1$ so that $B^{-1} \le \mu([\omega])/m([\omega]) \le B$ for all $\omega \in E_A^*$, then μ is a Gibbs state for the function f too.

The following result is Proposition 2.2.2 in [6].

PROPOSITION 2.8. *For every Gibbs state* m, *we have* $P_m = P(f)$. *Furthermore, any two Gibbs states for the function* f *are boundedly equivalent, with Radon–Nikodym derivatives bounded away from zero and infinity.*

Finally, we remind the reader of a well-known situation in which a Gibbs state is guaranteed to exist. Note that the existence of a Gibbs state implies that f must be summable. The next result is Corollary 2.7.5(c) in [6].

Theorem 2.9. *Suppose that $f : E_A^\infty \to \mathbb{R}$ is a summable, Hölder continuous potential and that the incidence matrix A is finitely irreducible. Then f admits a unique σ-invariant Gibbs state.*

3. Super- and submultiplicative sequences of real numbers. Recall that a sequence $(a_n)_{n\geq 1}$ of positive real numbers is said to be *boundedly submultiplicative with constant* $B > 0$ if

$$a_{m+n} \leq B a_m a_n$$

for all $m, n \geq 1$. If $B = 1$, then the sequence is simply said to be *submultiplicative.*

Similarly, a sequence $(a_n)_{n\geq 1}$ of positive real numbers is said to be *boundedly supermultiplicative with constant* $C > 0$ if

$$C^{-1} a_m a_n \leq a_{m+n}$$

for all $m, n \geq 1$. If $C = 1$, then the sequence is simply said to be *supermultiplicative.*

The following result is due to Fekete (for a proof, see Theorem 4.9 in [10]).

Theorem 3.1. *If $(a_n)_{n\geq 1}$ is a boundedly submultiplicative sequence with constant B, then*

$$\lim_{n\to\infty} \frac{1}{n} \log a_n = \inf_{n\geq 1} \frac{1}{n} \log(B a_n).$$

The limit may be $-\infty$ but if $\inf_{n\geq 1} a_n > 0$, then the limit is non-negative.

The counterpart to this result is the following.

Theorem 3.2. *If $(a_n)_{n\geq 1}$ is a boundedly supermultiplicative sequence with constant C, then*

$$\lim_{n\to\infty} \frac{1}{n} \log a_n = \sup_{n\geq 1} \frac{1}{n} \log(C^{-1} a_n).$$

The limit may be ∞ but if $\sup_{n\geq 1} a_n < \infty$, then the limit is finite.

We now introduce a concept of normalization for boundedly super- or submultiplicative sequences.

Definition 3.3. The *normalization* of a boundedly super- or submultiplicative sequence $(a_n)_{n\geq 1}$ is the sequence $(\tilde{a}_n)_{n\geq 1}$, where

$$\tilde{a}_n = a_n e^{-Pn} \quad \text{and} \quad P = \lim_{n\to\infty} \frac{1}{n} \log a_n.$$

Observe that the normalization of a boundedly submultiplicative sequence $(a_n)_{n\geq 1}$ is a boundedly submultiplicative sequence with the same constant B and that

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{a}_n = \inf_{n\geq 1} \frac{1}{n} \log(B \tilde{a}_n) = 0.$$

Similarly, the normalization of a boundedly supermultiplicative sequence $(a_n)_{n\geq 1}$ is a boundedly supermultiplicative sequence with the same constant C and

$$\lim_{n\to\infty}\frac{1}{n}\log\tilde{a}_n = \sup_{n\geq 1}\frac{1}{n}\log(C^{-1}\tilde{a}_n) = 0.$$

THEOREM 3.4. *The normalization $(\tilde{a}_n)_{n\geq 1}$ of a boundedly supermultiplicative and boundedly submultiplicative sequence $(a_n)_{n\geq 1}$ of positive numbers such that $-\infty < P < \infty$ is a bounded sequence. More precisely, it is bounded from below by B^{-1}, where B is any constant of submultiplicativity, and from above by any constant C of supermultiplicativity.*

Proof. Let $(a_n)_{n\geq 1}$ be a sequence such that

$$C^{-1}a_m a_n \leq a_{m+n} \leq B a_m a_n$$

for all $m, n \geq 1$ and such that $-\infty < P := \lim_{n\to\infty} n^{-1}\log a_n < \infty$. Let $\tilde{a}_n = a_n e^{-Pn}$. Then

$$\lim_{n\to\infty}\frac{1}{n}\log\tilde{a}_n = \inf_{n\geq 1}\frac{1}{n}\log(B\tilde{a}_n) = \sup_{n\geq 1}\frac{1}{n}\log(C^{-1}\tilde{a}_n) = 0.$$

Thus, $C^{-1}\tilde{a}_n \leq 1 \leq B\tilde{a}_n$ for all $n \geq 1$, i.e. $B^{-1} \leq \tilde{a}_n \leq C$ for all $n \geq 1$. ∎

It is well known that the partition functions $(Z_n(f))_{n\geq 1}$ form a submultiplicative sequence of positive real numbers. We shall now prove that the partition functions of a finitely irreducible Markov shift are boundedly supermultiplicative. This is a generalization of Proposition 2.3(v) in [8].

PROPOSITION 3.5. *Let $\sigma : E_A^\infty \to E_A^\infty$ be a finitely irreducible Markov shift under a summable, Hölder continuous potential $f : E_A^\infty \to \mathbb{R}$. Let $\Omega \subseteq E_A^*$ be a finite set which witnesses the irreducibility of A and denote by E_Ω the set of all letters in words of Ω. Then there is a constant $\tilde{K} \geq 1$ such that*

$$\tilde{K}^{-1} Z_{m,F}(f) Z_{n,F}(f) \leq Z_{m+n,F}(f) \leq Z_{m,F}(f) Z_{n,F}(f)$$

for all $m, n \geq 1$ and all $F \supseteq E_\Omega$.

Proof. Let $E_\Omega \subseteq F \subseteq E$. The submultiplicativity of the partition functions $(Z_{n,F}(f))_{n\geq 1}$ is well known and holds for any Markov shift $\sigma : F_A^\infty \to F_A^\infty$. For instance, it has been proved in Lemma 2.1.2 of [6]. To prove the bounded supermultiplicativity, let $\rho = \min\{\exp(\sup(S_{|\omega|}f|_{[\omega\cap E_\Omega]})) : \omega \in \Omega\}$ and $\lambda = \max\{|\omega| : \omega \in \Omega\}$. Then $\rho > 0$ and $\lambda < \infty$. For every couple $(e_1, e_2) \in F^2$ let $\omega(e_1, e_2) \in \Omega$ be such that $e_1\omega(e_1, e_2)e_2 \in F_A^*$. More generally, for every couple $(\tau, \chi) \in (F_A^*)^2$ define $\omega(\tau, \chi) := \omega(\tau_{|\tau|}, \chi_1)$. Fix $m, n \geq 1$. The map

$$l : F_A^m \times F_A^n \to \bigcup_{k=0}^{\lambda} F_A^{m+n+k}, \qquad (\tau, \chi) \mapsto \tau\omega(\tau, \chi)\chi,$$

is clearly injective. Then

$$\begin{aligned}
Z_{m,F}(f)Z_{n,F}(f) &= \sum_{\tau\in F_A^m}\sum_{\chi\in F_A^n} \exp(\sup(S_m f|_{[\tau\cap F]}))\exp(\sup(S_n f|_{[\chi\cap F]}))\\
&\le \sum_{\tau\in F_A^m}\sum_{\chi\in F_A^n} \exp(\sup(S_m f|_{[\tau\cap F]}))\\
&\qquad\qquad \times \rho^{-1}\exp(\sup(S_{|\omega(\tau,\chi)|} f|_{[\omega(\tau,\chi)\cap F]}))\exp(\sup(S_n f|_{[\chi\cap F]}))\\
&\le \rho^{-1}\sum_{\tau\in F_A^m}\sum_{\chi\in F_A^n} K^2\exp(\sup(S_{|l(\tau,\chi)|} f|_{[l(\tau,\chi)\cap F]}))\\
&\le K^2\rho^{-1}\sum_{k=0}^{\lambda}\sum_{\omega\in F_A^{m+n+k}} \exp(\sup(S_{m+n+k} f|_{[\omega\cap F]}))\\
&= K^2\rho^{-1}\sum_{k=0}^{\lambda} Z_{m+n+k,F}(f) \le K^2\rho^{-1}\sum_{k=0}^{\lambda} Z_{k,F}(f)Z_{m+n,F}(f)\\
&\le K^2\rho^{-1}(\lambda+1)\max\{1,(Z_1(f))^{\lambda}\}Z_{m+n,F}(f) =: \tilde{K}Z_{m+n,F}(f),
\end{aligned}$$

where K is a constant of bounded variation for the Hölder potential f. Note that $\tilde{K}<\infty$ since f is summable. This establishes the bounded supermultiplicativity. Finally, observe that $\tilde{K}$ depends only on K, ρ, λ and $Z_1(f)$, and these latter depend solely on f and Ω. Thus, $\tilde{K}$ is independent of F. ∎

The preceding proposition states that all the sequences $(Z_{n,F}(f))_{n\ge1}$, $F\supseteq E_\Omega$, share a common constant of bounded supermultiplicativity and submultiplicativity (i.e. independent of $F\supseteq E_\Omega$). This observation will later play a crucial role.

The normalization of the partition functions will play an important role as well.

Definition 3.6. Let $f : E_A^\infty\to\mathbb{R}$ be a potential. For every $n\ge1$ and $F\subseteq E$, let

$$W_{n,F}(f) = Z_{n,F}(f)e^{-nP_F(f)}.$$

This quantity will be called the nth(-level) *normalized partition function* of $f|_{F_A^\infty}$.

Again, we shall drop the subscript F when $F=E$. This definition and terminology are justified by the fact that replacing the original nth partition function by the corresponding normalized one brings the pressure of the system to 0. Indeed, $\lim_{n\to\infty} n^{-1}\log W_{n,F}(f)=0$.

The following result states that the sequences $(W_{n,F}(f))_{n\ge1}$, $F\supseteq E_\Omega$, share a common constant of bounded supermultiplicativity and submultiplicativity, as well as common lower and upper bounds (i.e. independent of $F\supseteq E_\Omega$).

COROLLARY 3.7. *Let $\sigma : E_A^\infty \to E_A^\infty$ be a finitely irreducible Markov shift under a summable, Hölder continuous potential $f : E_A^\infty \to \mathbb{R}$. Let $\Omega \subseteq E_A^*$ be a finite set which witnesses the irreducibility of A and denote by E_Ω the set of all letters appearing in words of Ω. Then*

$$1 \le W_{n,F}(f) \le \tilde{K}$$

for all $n \ge 1$ and $F \supseteq E_\Omega$, where $\tilde{K}$ is the constant from Proposition 3.5.

Proof. This result follows directly from Proposition 3.5 and Theorem 3.4, and the fact that $-\infty < P_F(f) < \infty$ since f is summable, Hölder continuous, and F_A^∞ has a periodic point due to the irreducibility of $A|_{F\times F}$. ■

4. Connected components of Markov shifts. When studying non-irreducible Markov shifts, we need to pay close attention to their strongly connected components and their isolated letters. Let us define these concepts, which have appeared in a more rudimentary form in [7], [2] and [8].

DEFINITION 4.1. A letter $e_1 \in E$ *leads to* a letter $e_2 \in E$ provided there exists $\omega \in E_A^*$ such that $e_1\omega e_2 \in E_A^*$. Equivalently, we say that e_2 *follows* e_1.

Next, we define the concept of connected component.

DEFINITION 4.2. A subset $C \subseteq E$ of letters is called a *connected component* of E if for any two letters $e_1, e_2 \in C$ there exists $\omega \in C_A^*$ so that $e_1\omega e_2 \in C_A^*$. Equivalently, C is a connected component of E if and only if the matrix $A|_{C^2}$ is irreducible, i.e. $\sigma|_{C_A^\infty}$ is a topologically mixing subshift of σ.

The largest connected components are said to be strongly connected.

DEFINITION 4.3. A connected component C of E is called *strongly connected* if C is maximal in the set-theoretic sense, that is, in the sense of inclusion. In other words, E does not have a component that strictly contains C.

Note that the strongly connected components of E are mutually disjoint.

DEFINITION 4.4. A connected component C is said to *lead to* a letter e if there is some letter in C which leads to e. Equivalently, we say that e *follows* C. A connected component C is said to *follow* a letter e if there is some letter in C which follows e.

In particular, a component C leads to and follows each of its letters from within itself.

DEFINITION 4.5. A connected component C_1 is said to *lead to* a connected component C_2 if some letter in C_1 leads to some letter in C_2. Equivalently, we say that C_2 *follows* C_1.

This defines a partial order on the set of all strongly connected components of E. In particular, note that if a strongly connected component C_1 leads to another strongly connected component C_2, then C_2 cannot lead to C_1.

DEFINITION 4.6. Two connected components are said to *communicate* if one of them leads to the other.

This implies a one-way communication between distinct strongly connected components.

DEFINITION 4.7. A letter is called *isolated* if it does not belong to any (strongly) connected component. The set of isolated letters will be denoted by I.

DEFINITION 4.8. The alphabet E will be said to have *finitely linked* isolated letters if there is a finite set $\mathcal{L} \subseteq I_A^*$ such that for every $e \in I$ there is a component $C(e)$, a letter $\beta(e) \in C(e)$ and a word $\tau(e) \in \mathcal{L}$ such that $e\tau(e)\beta(e) \in E_A^*$.

The components of maximal pressure will play a crucial role.

DEFINITION 4.9. A connected component C is called a *component of maximal f-pressure* if $P_C(f) = P(f)$.

We now prove that some class of Markov shifts, including all finite Markov shifts, admit at least one component of maximal pressure. Note that under our standing assumption that the matrix A has a 1 in each of its rows, every finite Markov shift has at least one and, of course, at most $|E|$ strongly connected components. Infinite Markov shifts might not have any such component, though. The hypothesis in the following proposition prevents this from happening. It is also worth mentioning that the following result generally does not hold if condition (iii) is not satisfied. This result is inspired from Theorem 3.11 and Corollary 3.15 in [8].

PROPOSITION 4.10. *Let $\sigma : E_A^\infty \to E_A^\infty$ be a Markov shift under a summable, Hölder continuous potential $f : E_A^\infty \to \mathbb{R}$ such that*

(i) *f has finitely many strongly connected components;*
(ii) *all strongly connected components are finitely irreducible;*
(iii) *words consisting solely of isolated letters are uniformly bounded in length.*

Then

$$\begin{aligned} P(f) &= \max\{P_C(f) : C \textit{ is a strongly connected component of } E\} \\ &= \max\{P_C(f) : C \textit{ is a connected component of } E\}. \end{aligned}$$

Proof. We need only prove that $P(f) \le \max_C P_C(f)$ since the converse is clearly true. The fact that words consisting solely of isolated letters are uniformly bounded in length ensures that the system has at least one strongly connected component. Let $\overline{C}$ be a strongly connected component such that

$$P_{\overline{C}}(f) = \max\{P_C(f) : C \text{ is a strongly connected component}\}. \tag{4.1}$$

Since $\overline{C}$ is finitely irreducible, Proposition 3.5 asserts that there exists a constant $\tilde{K}_{\overline{C}} \ge 1$ such that

$$Z_{m,\overline{C}}(f)Z_{n,\overline{C}}(f) \le \tilde{K}_{\overline{C}} Z_{m+n,\overline{C}}(f) \le \tilde{K}_{\overline{C}} Z_{m,\overline{C}}(f)Z_{n,\overline{C}}(f) \tag{4.2}$$

for all $m, n \ge 1$. Let $\varepsilon > 0$. There exists $N \ge 1$ such that

$$Z_{n,\overline{C}}(f)e^{n\varepsilon} \ge Z_{n,C}(f) \tag{4.3}$$

for all $n \ge N$ and all C. Let κ be the number of strongly connected components and B the maximal length of words consisting of isolated letters only. Let $n > (\kappa+1)(B+N)$. In particular, this implies that there are no n-words consisting solely of isolated letters, i.e. every n-word must visit at least one component. Then

$$\begin{aligned} Z_n(f) &= \sum_{\omega\in E_A^n} \exp(\sup(S_{|\omega|}f|_{[\omega]})) \\ &= \sum_{k=1}^{\kappa} \sum_{\substack{\omega\in E_A^n \text{ visits exactly} \\ k \text{ components}}} \exp(\sup(S_{|\omega|}f|_{[\omega]})). \end{aligned} \tag{4.4}$$

Consider the set of all $\omega \in E_A^n$ which visit the k strongly connected components $C_k, \dots, C_1$, and no others. In order for this set to be non-empty, the components $C_k, \dots, C_1$ must form a subchain of a one-way communication chain. Then there is a unique way of writing ω as a concatenation

$$\omega = \beta^{(k+1)}\alpha^{(k)}\beta^{(k)}\alpha^{(k-1)}\beta^{(k-1)}\dots\alpha^{(2)}\beta^{(2)}\alpha^{(1)}\beta^{(1)},$$

where $\beta^{(j)} \in I_A^*$ and $\alpha^{(j)} \in (C_j)_A^*$. Note that some (possibly all) of the $\beta^{(j)}$ may be the empty word ϵ. For (sub)words consisting of isolated letters only, we have the following fact.

Lemma 4.11.

$$\sum_{\beta\in I_A^*} \exp(\sup(S_{|\beta|}f|_{[\beta]})) \le \sum_{i=1}^{B}\Big(\sum_{\beta\in I} \exp(\sup(f|_{[\beta]}))\Big)^i =: L < \infty.$$

Proof. Indeed,

$$\sum_{\beta\in I_A^*}\exp(\sup(S_{|\beta|}f|_{[\beta]})) = \sum_{i=1}^{B}\sum_{\beta\in I_A^i}\exp(\sup(S_{|\beta|}f|_{[\beta]}))$$
$$\le \sum_{i=1}^{B}\sum_{\beta_1\in I}\ldots\sum_{\beta_i\in I}\exp(\sup(f|_{[\beta_1]}))\cdots\exp(\sup(f|_{[\beta_i]}))$$
$$= \sum_{i=1}^{B}\Big(\sum_{\beta\in I}\exp(\sup(f|_{[\beta]}))\Big)^i < \infty$$

since f is summable. ∎

We now continue the proof of Proposition 4.10. Since $|\beta^{(j)}|\le B$ for all j, we have $n \ge \sum_{j=1}^{k}|\alpha^{(j)}| \ge n-(k+1)B$. Therefore

$$(4.5)\quad \sum_{\substack{\omega\in E_A^n \text{ visits}\\ C_k,\ldots,C_1 \text{ only}}}\exp(\sup(S_{|\omega|}f|_{[\omega]}))$$
$$\le \prod_{i=1}^{k+1}\sum_{\beta^{(i)}\in I_A^*}\exp(\sup(S_{|\beta^{(i)}|}f|_{[\beta^{(i)}]}))$$
$$\cdot \sum_{\substack{(\alpha^{(k)},\ldots,\alpha^{(1)})\in(C_k)_A^*\times\cdots\times(C_1)_A^*\\ n\ge\sum_{j=1}^{k}|\alpha^{(j)}|\ge n-(k+1)B}}\prod_{i=1}^{k}\exp(\sup(S_{|\alpha^{(i)}|}f|_{[\alpha^{(i)}]}))$$
$$= \Big(\sum_{\beta\in I_A^*}\exp(\sup(S_{|\beta|}f|_{[\beta]}))\Big)^{k+1}$$
$$\cdot \sum_{\substack{(\alpha^{(k)},\ldots,\alpha^{(1)})\in(C_k)_A^*\times\cdots\times(C_1)_A^*\\ n\ge\sum_{j=1}^{k}|\alpha^{(j)}|\ge n-(k+1)B}}\prod_{i=1}^{k}\exp(\sup(S_{|\alpha^{(i)}|}f|_{[\alpha^{(i)}]}))$$
$$\le L^{k+1}\sum_{l=1}^{k}\sum_{\substack{(\alpha^{(k)},\ldots,\alpha^{(1)})\in(C_k)_A^*\times\cdots\times(C_1)_A^*\\ n\ge\sum_{j=1}^{k}|\alpha^{(j)}|\ge n-(k+1)B\\ \text{with exactly } l\ \alpha^{(j)}\text{'s such that } |\alpha^{(j)}|<N}}\prod_{i=1}^{k}\exp(\sup(S_{|\alpha^{(i)}|}f|_{[\alpha^{(i)}]})).$$

Recall that there are $\binom{k}{l}$ combinations of l components among k. For any such combination, the l chosen components form a unique subchain $C_{j_k},\ldots,C_{j_{k-(l-1)}}$. Similarly, the remaining $k-l$ components constitute a subchain $C_{j_{k-l}},\ldots,C_{j_1}$.

For any component (and in particular for the l chosen components), we have the following estimate.

LEMMA 4.12. *For every component C we have*

$$\sum_{\substack{\alpha\in C_A^* \\ |\alpha|<N}} \exp(\sup(S_{|\alpha|}f|_{[\alpha]})) \le K\sum_{i=1}^{N-1}(Z_1(f))^i =: Z < \infty.$$

Proof. Indeed,

$$\sum_{\substack{\alpha\in C_A^* \\ |\alpha|<N}} \exp(\sup(S_{|\alpha|}f|_{[\alpha]})) \le \sum_{i=1}^{N-1} KZ_{i,C}(f)$$

$$\le K\sum_{i=1}^{N-1} Z_i(f) \le K\sum_{i=1}^{N-1}(Z_1(f))^i < \infty$$

since f is summable. ∎

For the remaining $k-l$ components, we have the following estimate.

LEMMA 4.13. *For any $m \le M$ and $1 \le s \le k-l$, we have*

$$\sum_{\substack{(\alpha^{(j_s)},\dots,\alpha^{(j_1)})\in(C_{j_s})_A^*\times\dots\times(C_{j_1})_A^* \\ M\ge\sum_{i=1}^s|\alpha^{(j_i)}|\ge m \\ |\alpha^{(j_i)}|\ge N,\,\forall i}} \prod_{i=1}^{s} \exp(\sup(S_{|\alpha^{(j_i)}|}f|_{[\alpha^{(j_i)}]}))$$

$$\le K^s\tilde{K}_{\overline{C}}^{s-1}M^{s-1}e^{M\varepsilon}\sum_{r=m}^{M} Z_{r,\overline{C}}(f).$$

Proof. When $s=1$ we deduce, using (4.3), that

$$\sum_{\substack{\alpha^{(j_1)}\in(C_{j_1})_A^* \\ M\ge|\alpha^{(j_1)}|\ge\max\{m,N\}}} \exp(\sup(S_{|\alpha^{(j_1)}|}f|_{[\alpha^{(j_1)}]})) = \sum_{q=\max\{m,N\}}^{M} KZ_{q,C_{j_1}}(f)$$

$$\le \sum_{q=\max\{m,N\}}^{M} Z_{q,\overline{C}}(f)e^{q\varepsilon} \le K\tilde{K}_{\overline{C}}^{0}M^0e^{M\varepsilon}\sum_{r=m}^{M} Z_{r,\overline{C}}(f).$$

Suppose now that the statement holds for some $1 \le s < k-l$. Using (4.3) and (4.2), we obtain

$$\sum_{\substack{(\alpha^{(j_{s+1})},\dots,\alpha^{(j_1)})\in(C_{j_{s+1}})_A^*\times\cdots\times(C_{j_1})_A^* \\ M\geq\sum_{i=1}^{s+1}|\alpha^{(j_i)}|\geq m \\ |\alpha^{(j_i)}|\geq N,\forall i}} \prod_{i=1}^{s+1}\exp(\sup(S_{|\alpha^{(j_i)}|}f|_{[\alpha^{(j_i)}]}))$$

$$= \sum_{p=N}^{M-sN} \sum_{\alpha^{(j_{s+1})}\in(C_{j_{s+1}})_A^p} \exp(\sup(S_{|\alpha^{(j_{s+1})}|}f|_{[\alpha^{(j_{s+1})}]}))$$

$$\cdot \sum_{\substack{(\alpha^{(j_s)},\dots,\alpha^{(j_1)})\in(C_{j_s})_A^*\times\cdots\times(C_{j_1})_A^* \\ M-p\geq\sum_{i=1}^{s}|\alpha^{(j_i)}|\geq\max\{m-p,sN\} \\ |\alpha^{(j_i)}|\geq N,\forall i}} \prod_{i=1}^{s}\exp(\sup(S_{|\alpha^{(j_i)}|}f|_{[\alpha^{(j_i)}]}))$$

$$\leq \sum_{p=N}^{M-sN} KZ_{p,C_{j_{s+1}}}(f)$$

$$\cdot \sum_{\substack{(\alpha^{(j_s)},\dots,\alpha^{(j_1)})\in(C_{j_s})_A^*\times\cdots\times(C_{j_1})_A^* \\ M-p\geq\sum_{i=1}^{s}|\alpha^{(j_i)}|\geq\max\{m-p,sN\} \\ |\alpha^{(j_i)}|\geq N,\forall i}} \prod_{i=1}^{s}\exp(\sup(S_{|\alpha^{(j_i)}|}f|_{[\alpha^{(j_i)}]}))$$

$$\leq \sum_{p=N}^{M-sN} KZ_{p,\overline{C}}(f)e^{p\varepsilon}\cdot K^s\tilde{K}_{\overline{C}}^{s-1}\cdot(M-p)^{s-1}e^{(M-p)\varepsilon}$$

$$\cdot \sum_{q=\max\{m-p,sN\}}^{M-p} Z_{q,\overline{C}}(f)$$

$$\leq K^{s+1}\tilde{K}_{\overline{C}}^{s-1}\cdot M^{s-1}e^{M\varepsilon}\sum_{p=N}^{M-sN}\sum_{q=\max\{m-p,sN\}}^{M-p} Z_{p,\overline{C}}(f)Z_{q,\overline{C}}(f)$$

$$\leq K^{s+1}\tilde{K}_{\overline{C}}^{s-1}\cdot M^{s-1}e^{M\varepsilon}\sum_{p=N}^{M-sN}\sum_{q=\max\{m-p,sN\}}^{M-p} \tilde{K}_{\overline{C}}Z_{p+q,\overline{C}}(f)$$

$$\leq K^{s+1}\tilde{K}_{\overline{C}}^{s-1}\cdot M^{s-1}e^{M\varepsilon}\cdot\tilde{K}_{\overline{C}}(M-(s+1)N+1)\sum_{r=m}^{M} Z_{r,\overline{C}}(f)$$

$$\leq K^{s+1}\tilde{K}_{\overline{C}}^{s}\cdot M^{s}e^{M\varepsilon}\sum_{r=m}^{M} Z_{r,\overline{C}}(f). \ \blacksquare$$

We now resume the proof of Proposition 4.10. It follows from the previous two lemmas that

$$
\begin{aligned}
(4.6)\qquad &\sum_{\substack{(\alpha^{(j_k)},\dots,\alpha^{(j_1)})\in(C_{j_k})^*_A\times\cdots\times(C_{j_1})^*_A\\ n\geq\sum_{i=1}^k|\alpha^{(j_i)}|\geq n-(k+1)B\\ |\alpha^{(j_i)}|<N,\forall i>k-l,\, |\alpha^{(j_i)}|\geq N,\forall i\leq k-l}} \prod_{i=1}^{k}\exp(\sup(S_{|\alpha^{(j_i)}|}f|_{[\alpha^{(j_i)}]}))\\
&\leq \prod_{i=1}^{l}\Big(\sum_{\substack{\alpha^{(j_{k-l+i})}\in(C_{j_{k-l+i}})^*_A\\ |\alpha^{(j_{k-l+i})}|<N}}\exp(\sup(S_{|\alpha^{(j_{k-l+i})}|}f|_{[\alpha^{(j_{k-l+i})}]}))\Big)\\
&\quad\cdot\sum_{\substack{(\alpha^{(j_{k-l})},\dots,\alpha^{(j_1)})\in(C_{j_{k-l}})^*_A\times\cdots\times(C_{j_1})^*_A\\ n\geq\sum_{i=1}^{k-l}|\alpha^{(j_i)}|\geq n-(k+1)B-l(N-1)\\ |\alpha^{(j_i)}|\geq N,\forall i\leq k-l}}\prod_{i=1}^{k-l}\exp(\sup(S_{|\alpha^{(j_i)}|}f|_{[\alpha^{(j_i)}]}))\\
&\leq Z^l\sum_{\substack{(\alpha^{(j_{k-l})},\dots,\alpha^{(j_1)})\in(C_{j_{k-l}})^*_A\times\cdots\times(C_{j_1})^*_A\\ n\geq\sum_{i=1}^{k-l}|\alpha^{(j_i)}|\geq n-(k+1)B-l(N-1)\\ |\alpha^{(j_i)}|\geq N,\forall i\leq k-l}}\prod_{i=1}^{k-l}\exp(\sup(S_{|\alpha^{(j_i)}|}f|_{[\alpha^{(j_i)}]}))\\
&\leq Z^lK^{k-l}\tilde{K}_{\overline{C}}^{k-l-1}\cdot n^{k-l-1}e^{n\varepsilon}\sum_{r=n-(k+1)(B+N)}^{n}Z_{r,\overline{C}}(f)\\
&\leq \max\{1,Z^\kappa\}(K\tilde{K}_{\overline{C}})^\kappa\cdot n^\kappa e^{n\varepsilon}\sum_{r=n-(\kappa+1)(B+N)}^{n}Z_{r,\overline{C}}(f)\\
&=: \hat{K}\cdot n^\kappa e^{n\varepsilon}\sum_{r=n-(\kappa+1)(B+N)}^{n}Z_{r,\overline{C}}(f).
\end{aligned}
$$

Since there are $\binom{k}{l}\leq k!\leq\kappa!$ combinations of l components among k, we deduce from (4.5) that

$$
\sum_{\substack{\omega\in E_A^n\text{ visits}\\ \text{only }C_k,\dots,C_1}}\exp(\sup(S_{|\omega|}f|_{[\omega]}))
\leq \max\{1,L^{\kappa+1}\}\kappa\kappa!\hat{K}\cdot n^\kappa e^{n\varepsilon}\sum_{r=n-(\kappa+1)(B+N)}^{n}Z_{r,\overline{C}}(f).
$$

As there are at most $\binom{\kappa}{k}\leq\kappa!$ combinations of k components among a grand total of κ, we conclude that

$$
\sum_{\substack{\omega\in E_A^n\text{ visits exactly}\\ k\text{ components}}}\exp(\sup(S_{|\omega|}f|_{[\omega]}))
\leq \kappa!\max\{1,L^{\kappa+1}\}\kappa\kappa!\hat{K}\cdot n^\kappa e^{n\varepsilon}\sum_{r=n-(\kappa+1)(B+N)}^{n}Z_{r,\overline{C}}(f).
$$

It follows from (4.4) that

$$Z_n(f) \le \kappa^2(\kappa!)^2 \max\{1, L^{\kappa+1}\}\hat{K} \cdot n^\kappa e^{n\varepsilon} \sum_{r=n-(\kappa+1)(B+N)}^{n} Z_{r,\overline{C}}(f)$$
$$=: \hat{\hat{K}} \cdot n^\kappa e^{n\varepsilon} \sum_{r=n-(\kappa+1)(B+N)}^{n} Z_{r,\overline{C}}(f).$$

For each n, choose $r_{\max}(n)$ so that $n-(\kappa+1)(B+N) \le r_{\max}(n) \le n$ and

$$Z_{r_{\max}(n),\overline{C}}(f) = \max_{n-(\kappa+1)(B+N)\le r\le n} Z_{r,\overline{C}}(f).$$

Then

$$Z_n(f) \le ((\kappa+1)(B+N)+1)\hat{\hat{K}} \cdot n^\kappa e^{n\varepsilon} Z_{r_{\max}(n),\overline{C}}(f).$$

Consequently,

$$\begin{aligned} P(f) &= \lim_{n\to\infty} \frac{1}{n} \log Z_n(f) \le \lim_{n\to\infty} \frac{1}{n} \log Z_{r_{\max}(n),\overline{C}}(f) + \varepsilon \\ &= \lim_{n\to\infty} \frac{r_{\max}(n)}{n} \lim_{n\to\infty} \frac{1}{r_{\max}(n)} \log Z_{r_{\max}(n),\overline{C}}(f) + \varepsilon \\ &= 1 \cdot P_{\overline{C}}(f) + \varepsilon = \max_C P_C(f) + \varepsilon, \end{aligned}$$

where the last equality follows from (4.1). Since ε is arbitrary, we deduce that $P(f) \le \max_C P_C(f)$. Since it is clear that $P(f) \ge \max_C P_C(f)$, we conclude that $P(f) = \max_C P_C(f)$. The proof of Proposition 4.10 is complete. ∎

We obtain the following immediate corollary.

Corollary 4.14. *For any Markov shift satisfying the conditions of Proposition* 4.10*, the set of strongly connected components of maximal f-pressure is non-empty.*

5. The existence of Gibbs states. In this section, we shall study necessary and sufficient conditions for the existence of Gibbs states and the existence, uniqueness and ergodicity of invariant Gibbs states. Let us state the most important results of this paper. It is worth observing that these results do apply to all finite Markov shifts. The first result is a characterization of the existence of Gibbs states.

Main Theorem 5.1. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable, Hölder continuous potential on a Markov shift which has finitely many strongly connected components each of which is finitely irreducible, has finitely linked isolated letters, and the words consisting solely of isolated letters are uniformly bounded in length. Then this shift admits a Gibbs state if and only if none of its strongly connected components of maximal f-pressure communicate and each of its letters leads to a component of maximal f-pressure.*

The second result is a characterization of the existence, uniqueness and ergodicity of invariant Gibbs states.

MAIN THEOREM 5.2. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable, Hölder continuous potential on a Markov shift which has finitely many strongly connected components each of which is finitely irreducible, has finitely linked isolated letters, and the words consisting solely of isolated letters are uniformly bounded in length. Then this shift admits an invariant Gibbs state if and only if all its strongly connected components are of maximal f-pressure and none of them communicate. In other words, such a shift admits an invariant Gibbs state if and only if it is the disjoint union of finitely irreducible subshifts each generating the same amount of pressure.*

Such a shift has a unique invariant Gibbs state if and only if it is finitely irreducible. Otherwise, it has uncountably many invariant Gibbs states, each of which is a non-trivial convex combination of the componentwise-invariant Gibbs states.

Furthermore, such a shift admits an ergodic Gibbs state if and only if it is finitely irreducible. If the system is finitely primitive, then the unique invariant Gibbs state is completely ergodic.

To prove these characterizations, we will need a series of propositions. Several of those results revolve around the normalized partition functions.

We first obtain a common positive lower bound for the normalized partition functions of any shift. The existence of this lower bound follows directly from the definition of the pressure.

PROPOSITION 5.3. *If f is summable, then $W_n(f) \geq 1$ for every $n \geq 1$.*

Proof. When f is summable, we know that $P(f) < \infty$. For every $n \geq 1$,

$$P(f) \leq \frac{1}{n} \log Z_n(f).$$

Then $e^{nP(f)} \leq Z_n(f)$. Thus, $1 \leq Z_n(f)e^{-nP(f)} = W_n(f)$. ∎

We immediately deduce that any system under a summable potential can only have finitely many (if any) strongly connected components of maximal pressure.

PROPOSITION 5.4. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable potential. Then E has finitely many (if any) strongly connected components of maximal f-pressure.*

Proof. Since f is summable, we know that $P(f) < \infty$. Fix $n \geq 1$. For every strongly connected component C of maximal f-pressure, Proposition 5.3 states that $W_{n,C}(f) \geq 1$. If E had infinitely many strongly connected components of maximal f-pressure, then we would have

$$Z_n(f)e^{-nP(f)} = W_n(f) \ge \sum_C W_{n,C}(f) = \infty,$$

where the summation would run over all strongly connected components of maximal f-pressure. Thus, we would get $Z_n(f) = \infty$ for all $n \ge 1$ and hence $P(f) = \infty$. This would be a contradiction. ∎

We now show that the normalized partition functions have a common finite upper bound for any shift that admits a Gibbs state (cf. Corollary 3.7).

PROPOSITION 5.5. *If a Gibbs state exists under a summable potential f, then there exists $M \ge 1$ so that $1 \le W_n(f) \le M$ for every $n \ge 1$.*

Proof. The lower bound is the one from Proposition 5.3 (recall that the existence of a Gibbs state forces f to be summable). For the upper bound, let m be a Gibbs state. Then there is $Q \ge 1$ so that for every $\omega \in E_A^*$ and for every $\tau \in [\omega]$ we have

$$Q^{-1} \le \frac{m([\omega])}{\exp(S_{|\omega|}f(\tau) - P(f)|\omega|)}.$$

Since $\sum_{\omega \in E_A^n} m([\omega]) = 1$, we get

$$Q^{-1}e^{-nP(f)} \sum_{\omega \in E_A^n} \exp(\sup(S_n f|_{[\omega]})) \le 1$$

for all $n \ge 1$, i.e. $Q^{-1}e^{-nP(f)}Z_n(f) \le 1$. Therefore $W_n(f) \le Q =: M$ for all $n \ge 1$. ∎

Next, we estimate the contribution brought to the system by the complement of the components of maximal pressure.

LEMMA 5.6. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable, Hölder continuous potential on a Markov shift that has finitely many strongly connected components not of maximal f-pressure, each of which is finitely irreducible, and whose words consisting solely of isolated letters are uniformly bounded in length. Let $C_1, \dots, C_j$, where $j \le \infty$, denote the strongly connected components of maximal f-pressure. Let $C_0 = E \setminus \bigcup_{1 \le k \le j} C_k$. Then $P_{C_0}(f) < P(f)$ and therefore there exist constants $K_0 > 0$ and $0 < a < 1$ such that*

$$Z_{n,C_0}(f)e^{-nP(f)} \le K_0 a^n$$

for every $n \ge 1$. In particular, there exists $M_0 > 0$ so that

$$\sum_{\omega \in C_0^*} \exp(\sup(S_{|\omega|} f|_{[\omega]}))e^{-|\omega|P(f)} = \sum_{n \ge 1} Z_{n,C_0}(f)e^{-nP(f)} \le M_0.$$

Proof. First, note that if C_0 does not contain any connected component of E, then $C_0 \subseteq I$ and hence there exists $N \ge 1$ such that $(C_0)_A^n = \emptyset$ for all $n \geq N$. Therefore $Z_{n,C_0}(f) = 0$ for all $n \geq N$ and $P_{C_0}(f) = -\infty$. Since $P(f) > -\infty$, the result is trivial in this case. Secondly, observe that if C_0 contains a connected component of E, then it contains the strongly connected component of E comprising that component, and this strongly connected component is not of maximal f-pressure. Applying Proposition 4.10 with E replaced by C_0, we deduce that

$$\begin{aligned}
P_{C_0}(f) &= \max\{P_C(f) : C \text{ is a connected component of } C_0\} \\
&= \max\{P_C(f) : C \subseteq C_0 \text{ is a connected component of } E\} \\
&= \max\{P_C(f) : C \subseteq C_0 \text{ is a strongly connected component of } E\} \\
&\leq \max\{P_C(f) : C \text{ is a strongly connected component of } E, \\
&\qquad\qquad\qquad\qquad\qquad\qquad C \text{ is not of maximal } f\text{-pressure}\} \\
&< P(f).
\end{aligned}$$

Since

$$P_{C_0}(f) = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,C_0}(f),$$

we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log Z_{n,C_0}(f) < b$$

for any $P_{C_0}(f) < b < P(f)$. Hence for every n large enough,

$$Z_{n,C_0}(f) e^{-nP(f)} < e^{(b-P(f))n}.$$

Since f is summable, all the $Z_n(f)$'s are finite. The preceding inequality thus holds for all $n \geq 1$ up to a multiplicative constant K_0. ■

We now demonstrate that the normalized partition functions are uniformly bounded away from 0 and ∞ when none of the strongly connected components of maximal f-pressure of the system communicate. This feature is already known for all finitely irreducible systems (cf. Corollary 3.7).

PROPOSITION 5.7. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable, Hölder continuous function. Suppose that a shift has finitely many strongly connected components, that all of them are finitely irreducible, and that words consisting solely of isolated letters are uniformly bounded in length. If none of the strongly connected components of maximal f-pressure communicate, then there exists a constant $M \geq 1$ so that $1 \leq W_n(f) \leq M$ for all $n \geq 1$.*

Proof. Of course, the lower bound is the one from Proposition 5.3. For the upper bound, let C be a strongly connected component of maximal f-pressure. Let C_A^{**} be the set of all finite A-admissible words with at least one letter from C, and for each $m \geq 1$, let $C_A^{**,m}$ be the set of words in C_A^{**} which contain exactly m letters from C. Let $n \geq 1$. Let $C_1, \dots, C_j$ be the strongly connected components of maximal f-pressure. Let $C_0 = E \setminus \bigcup_{1\leq k\leq j} C_k$.

Based on our hypothesis, no admissible word contains letters from distinct strongly connected components of maximal f-pressure. Thus, every n-word τ in $C_A^{**,m}$ has the form $\alpha\omega\beta$, where $\omega \in C_A^m$ and $\alpha, \beta \in C_0^*$, with $|\alpha|+|\beta| = n-m$ (α and/or β might be the empty word ϵ). Since the incidence matrix $A|_{C^2}$ is finitely irreducible and f is summable and Hölder, Corollary 3.7 ensures that there exists a constant $\tilde{K}_C \geq 1$ such that

$$1 \leq W_{m,C}(f) \leq \tilde{K}_C$$

for all $m \geq 1$. Since C is of maximal f-pressure, we have $P_C(f) = P(f)$. Using these last two facts and Lemma 5.6, we deduce that

$$\begin{aligned}
&\sum_{\tau \in C_A^{**} \cap E_A^n} \exp(\sup(S_n f|_{[\tau]}))e^{-nP(f)} \\
&\quad = \sum_{m=1}^{n} \sum_{\tau \in C_A^{**,m} \cap E_A^n} \exp(\sup(S_n f|_{[\tau]}))e^{-nP(f)} \\
&\quad \leq \sum_{m=1}^{n} \sum_{\omega \in C_A^m} \exp(\sup(S_m f|_{[\omega]}))e^{-mP(f)} \\
&\qquad \cdot \sum_{\substack{\alpha,\beta \in C_0^* \\ |\alpha|+|\beta|=n-m}} \exp(\sup(S_{|\alpha|} f|_{[\alpha]}))e^{-|\alpha|P(f)} \exp(\sup(S_{|\beta|} f|_{[\beta]}))e^{-|\beta|P(f)} \\
&\quad \leq \sum_{m=1}^{n} K W_{m,C}(f) \sum_{l=0}^{n-m} \Big[\sum_{\alpha \in C_0^l} \exp(\sup(S_l f|_{[\alpha]}))e^{-lP(f)} \\
&\qquad\qquad \cdot \sum_{\beta \in C_0^{n-m-l}} \exp(\sup(S_{n-m-l} f|_{[\beta]}))e^{-(n-m-l)P(f)} \Big] \\
&\quad \leq \sum_{m=1}^{n} K\tilde{K}_C \sum_{l=0}^{n-m} K Z_{l,C_0}(f)e^{-lP(f)} K Z_{n-m-l,C_0}(f)e^{-(n-m-l)P(f)} \\
&\quad \leq K^3 \tilde{K}_C \sum_{m=1}^{n} \sum_{l=0}^{n-m} K_0 a^l K_0 a^{n-m-l} = K^3 K_0^2 \tilde{K}_C \sum_{m=1}^{n} \sum_{l=0}^{n-m} a^{n-m} \\
&\quad = K^3 K_0^2 \tilde{K}_C \sum_{m=1}^{n} (n-m+1)a^{n-m} \\
&\quad \leq K^3 K_0^2 \tilde{K}_C \sum_{k=0}^{\infty} (k+1)a^k = \frac{K^3 K_0^2 \tilde{K}_C}{(1-a)^2} =: \tilde{K}_{C_A^{**}},
\end{aligned}$$

where K is a constant of bounded variation for the ergodic sums of the Hölder potential f. Consequently,

$$W_n(f) = \sum_{\omega \in E_A^n} \exp(\sup(S_n f|_{[\omega]}))e^{-nP(f)}$$

$$= \Big[\sum_{k=1}^{j} \sum_{\omega \in (C_k)_A^{**} \cap E_A^n} \exp(\sup(S_n f|_{[\omega]})) + \sum_{\omega \in C_0^* \cap E_A^n} \exp(\sup(S_n f|_{[\omega]}))\Big] e^{-nP(f)}$$

$$\le \sum_{k=1}^{j} \tilde{K}_{(C_k)_A^{**}} + M_0 =: M. \ \blacksquare$$

We now aim at showing that the normalized partition functions are not bounded from above when some finitely irreducible connected components of maximal f-pressure communicate. But first we need two intermediate results.

LEMMA 5.8. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable, Hölder continuous function. For any finitely irreducible connected component C of E and any $e \in C$, we have*

$$\inf_{n \ge 1} W_{n,C}^{t,e}(f) > 0,$$

where

$$W_{n,C}^{t,e}(f) = \sum_{\omega \in C_A^{n,t,e}} \exp(\sup(S_n f|_{[\omega \cap C]}))e^{-nP_C(f)}$$

and $C_A^{n,t,e}$ is the set of all words in C_A^n whose terminal letter is e.

Proof. Let C be a finitely irreducible connected component of E and let μ_C be a $\sigma|_{C_A^\infty}$-invariant Gibbs state for the component C. Such a state exists according to Corollary 2.7.5(c) in [6]. Let $e \in C$ and $n \ge 1$. Then

$$\begin{aligned}\mu_C([e \cap C]) &= \mu_C(\sigma|_{C_A^\infty}^{-(n-1)}[e \cap C]) = \sum_{\omega \in C_A^{n,t,e}} \mu_C([\omega \cap C]) \\ &\le \sum_{\omega \in C_A^{n,t,e}} Q_C \exp(\sup(S_{|\omega|} f|_{[\omega \cap C]}) - P_C(f)|\omega|) \\ &= Q_C e^{-nP_C(f)} \sum_{\omega \in C_A^{n,t,e}} \exp(\sup(S_n f|_{[\omega \cap C]})) = Q_C W_{n,C}^{t,e}(f).\end{aligned}$$

Hence $\inf_{n \ge 1} W_{n,C}^{t,e}(f) \ge Q_C^{-1} \mu_C([e \cap C]) > 0$. ∎

Observe also that if there is an infinite component C of maximal f-pressure and $n \ge 1$ such that $\inf_{e \in C} W_{n,C}^{t,e}(f) > 0$, then $W_n(f) \ge W_{n,C}(f) = \sum_{e \in C} W_{n,C}^{t,e}(f) = \infty$ and the system does not admit a Gibbs state according to Proposition 5.5. Moreover, since $W_{n,C}(f) = \infty$, the subshift $\sigma|_{C_A^\infty}$ does not have a Gibbs state either, and Theorem 2.3.3 in [6] shows that such a component C cannot be finitely irreducible.

The counterpart of Lemma 5.8 for a fixed initial letter is

LEMMA 5.9. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable, Hölder continuous function. For any finitely irreducible component C of E and any $e \in C$, we have*

$$\inf_{n\geq 1} W_{n,C}^{i,e}(f) > 0,$$

where

$$W_{n,C}^{i,e}(f) = \sum_{\omega\in C_A^{n,i,e}} \exp(\sup(S_n f|_{[\omega\cap C]}))e^{-nP_C(f)}$$

and $C_A^{n,i,e}$ is the set of all words in C_A^n whose initial letter is e.

Proof. Let C be a finitely irreducible connected component of E and let μ_C be a (not necessarily $\sigma|_{C_A^\infty}$-invariant) Gibbs state for the component C. Such a state exists by Corollary 2.7.5(c) in [6]. Let $e \in C$ and $n \geq 1$. Then

$$\begin{aligned}
\mu_C([e\cap C]) &= \sum_{\omega\in C_A^{n,i,e}} \mu_C([\omega\cap C]) \\
&\leq \sum_{\omega\in C_A^{n,i,e}} Q_C \exp(\sup(S_{|\omega|} f|_{[\omega\cap C]}) - P_C(f)|\omega|) \\
&= Q_C e^{-nP_C(f)} \sum_{\omega\in C_A^{n,i,e}} \exp(\sup(S_n f|_{[\omega\cap C]})) = Q_C W_{n,C}^{i,e}(f).
\end{aligned}$$

Thus, $\inf_{n\geq 1} W_{n,C}^{i,e}(f) \geq Q_C^{-1}\mu_C([e\cap C]) > 0$. ∎

From the last two lemmas, we establish the following fact.

PROPOSITION 5.10. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable, Hölder continuous function. If two finitely irreducible, strongly connected components of maximal f-pressure communicate, then $\sup_{n\geq 1} W_n(f) = \infty$. In particular, a Gibbs state does not exist.*

Proof. Let C_1 and C_2 be two finitely irreducible, strongly connected components of maximal f-pressure such that C_1 leads to C_2. Let $e_1 \in C_1$ and $e_2 \in C_2$ be such that there exists $\omega \in [E \setminus (C_1 \cup C_2)]_A^*$ so that $e_1\omega e_2 \in E_A^*$. According to Lemma 5.8, we have

$$\mu_1 := \inf_{n\geq 1} W_{n,C_1}^{t,e_1}(f) > 0.$$

Similarly, according to Lemma 5.9,

$$\mu_2 := \inf_{n\geq 1} W_{n,C_2}^{i,e_2}(f) > 0.$$

Thus, for any $n \geq 2$ we obtain

$$
\begin{aligned}
W_{n+|\omega|}(f) \geq & \sum_{\substack{\tau \in E_A^{n+|\omega|}: \\ e_1\omega e_2 \text{ subword of } \tau}} \exp(\sup(S_{n+|\omega|}f|_{[\tau]}))e^{-(n+|\omega|)P(f)} \\
& \geq K^{-3} \sum_{k=1}^{n-1} \sum_{\alpha \in (C_1)_A^{k,t,e_1}} \sum_{\beta \in (C_2)_A^{n-k,i,e_2}} \exp(\sup(S_k f|_{[\alpha \cap C_1]}))e^{-kP_{C_1}(f)} \\
& \quad \cdot \exp(\sup(S_{|\omega|}f|_{[\omega]}))e^{-|\omega|P(f)} \exp(\sup(S_{n-k}f|_{[\beta\cap C_2]}))e^{-(n-k)P_{C_2}(f)} \\
& = K^{-3}\exp(\sup(S_{|\omega|}f|_{[\omega]}))e^{-|\omega|P(f)} \sum_{k=1}^{n-1} W_{k,C_1}^{t,e_1}(f) W_{n-k,C_2}^{i,e_2}(f) \\
& \geq K^{-3}\exp(\sup(S_{|\omega|}f|_{[\omega]}))e^{-|\omega|P(f)}\mu_1\mu_2(n-1),
\end{aligned}
$$

where K is a constant of bounded variation for the ergodic sums of the Hölder potential f. Since $P(f) < \infty$, we conclude that $\sup_{n\geq 1} W_n(f) = \infty$. It follows from Proposition 5.5 that a Gibbs state cannot exist. ∎

Now, we shall prove that the presence of a letter which does not lead to a component of maximal pressure prevents a system from having a Gibbs measure.

PROPOSITION 5.11. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable, Hölder continuous function on a Markov shift that has finitely many strongly connected components not of maximal f-pressure, each of which is finitely irreducible, and whose words consisting solely of isolated letters are uniformly bounded in length. If there is a letter which does not lead to a component of maximal f-pressure, then the system cannot have a Gibbs state.*

Proof. Let M be the set of all letters which lead to at least one component of maximal f-pressure. By hypothesis, $\emptyset \neq M \neq E$. Let $e \in E \setminus M$. Assume that the system admits a Gibbs state μ. On the one hand, $\mu([e]) \neq 0$. On the other hand, it is crucial to observe that $e\tau \in E_A^{n+1}$ if and only if $e\tau \in (E \setminus M)_A^{n+1}$. Indeed, $E_A^{n+1} \supset (E \setminus M)_A^{n+1}$. Moreover, observe that if for some k we had $\tau_k \in M$, then τ_k would lead to a component of maximal pressure, and so would e a fortiori. Hence we would conclude $e \in M$, which is not the case. Furthermore, it is fundamental to observe that $P_{E\setminus M}(f) \leq P_{C_0}(f) < P(f)$ by Lemma 5.6. Let $P_{E\setminus M}(f) < b < P(f)$. For all n large enough we have

$$Z_{n,E\setminus M}(f) \leq e^{nb}.$$

It follows that for all n sufficiently large we have, for some constant Q,

$$
\begin{aligned}
\mu([e]) &= \sum_{e\tau\in E_A^{n+1}} \mu([e\tau]) = \sum_{e\tau\in(E\setminus M)_A^{n+1}} \mu([e\tau]) \\
&\le Qe^{-(n+1)P(f)} \sum_{e\tau\in(E\setminus M)_A^{n+1}} \exp(\sup(S_{n+1}f|_{[e\tau]})) \\
&\le Qe^{-(n+1)P(f)} \exp(\sup(f|_{[e]})) \sum_{\tau\in(E\setminus M)_A^{n}} \exp(\sup(S_n f|_{[\tau]})) \\
&\le Q\exp(\sup(f|_{[e]}))e^{-(n+1)P(f)} K Z_{n,E\setminus M}(f) \\
&\le KQ\exp(\sup(f|_{[e]}))e^{-(n+1)P(f)} e^{nb} \\
&\le KQ\exp(\sup(f|_{[e]}))e^{-P(f)} e^{-n(P(f)-b)}.
\end{aligned}
$$

Passing to the limit as $n \to \infty$ gives $\mu([e]) = 0$. This is impossible. So the system does not admit any Gibbs state. ∎

Taken together, Propositions 5.10 and 5.11 establish one of the implications in Theorem 5.1. Moreover, Propositions 5.7 and 5.11 jointly show that the existence of a Gibbs state is generally not equivalent to the normalized partition functions being bounded away from 0 and infinity.

We now prove the converse implication in Theorem 5.1.

PROPOSITION 5.12. *Let $f : E_A^\infty \to \mathbb{R}$ be a summable, Hölder continuous function. Suppose that the system has finitely many strongly connected components each of which is finitely irreducible, that it has finitely linked isolated letters and that the words consisting solely of isolated letters are uniformly bounded in length. If none of the strongly connected components of maximal f-pressure communicate and if every letter leads to a component of maximal f-pressure, then the system has a Gibbs state.*

Proof. First, let us assume that E is finite. For every $j \ge 1$ and every $\omega \in E_A^j$, define the following Borel probability measure on the σ-algebra on E_A^∞ generated by the cylinders of length j:

$$
\mu_j([\omega]) = \frac{\exp(\sup(S_j f|_{[\omega]}))e^{-jP(f)}}{W_j(f)}.
$$

For each letter $e \in E$ there is a strongly connected component of maximal f-pressure $C(e)$, a letter $\beta(e) \in C(e)$ and a word $\tau(e) \in E_A^*$ such that $e\tau(e)\beta(e) \in E_A^*$. For every $\omega \in E_A^j$ and $k \ge \max_{e\in E} |\tau(e)| + 1$, we deduce using Proposition 5.7 that

$$
\mu_{j+k}([\omega]) = \sum_{\omega\alpha\in E_A^{j+k}} \mu_{j+k}([\omega\alpha]) = \sum_{\omega\alpha\in E_A^{j+k}} \frac{\exp(\sup(S_{j+k} f|_{[\omega\alpha]}))e^{-(j+k)P(f)}}{W_{j+k}(f)}
$$

$$
\begin{aligned}
&\geq \sum_{\alpha \in (C(\omega_j))_A^{k-|\tau(\omega_j)|, i, \beta(\omega_j)}} \frac{\exp(\sup(S_{j+k} f|_{[\omega\tau(\omega_j)\alpha]}))e^{-(j+k)P(f)}}{W_{j+k}(f)} \\
&\geq \frac{W^{i,\beta(\omega_j)}_{k-|\tau(\omega_j)|, C(\omega_j)}(f)}{W_{j+k}(f)} K^{-3} \exp(\sup(S_j f|_{[\omega]}))e^{-jP(f)} \\
&\quad \cdot \exp(\sup(S_{|\tau(\omega_j)|} f|_{[\tau(\omega_j)]}))e^{-|\tau(\omega_j)|P(f)} \\
&\quad \cdot \sum_{\alpha \in (C(\omega_j))_A^{k-|\tau(\omega_j)|, i, \beta(\omega_j)}} \frac{\exp(\sup(S_{k-|\tau(\omega_j)|} f|_{[\alpha \cap C(\omega_j)]}))e^{-(k-|\tau(\omega_j)|)P_{C(\omega_j)}(f)}}{W^{i,\beta(\omega_j)}_{k-|\tau(\omega_j)|, C(\omega_j)}(f)} \\
&\geq \frac{\inf_{n\geq 1,\, e \in E} W^{i,\beta(e)}_{n, C(e)}(f)}{M} K^{-3} \exp(\sup(S_j f|_{[\omega]}))e^{-jP(f)} \\
&\quad \cdot \inf_{e \in E} \exp(\sup(S_{|\tau(e)|} f|_{[\tau(e)]}))e^{-|\tau(e)|P(f)} \cdot 1 \\
&= N \exp(\sup(S_j f|_{[\omega]}))e^{-jP(f)},
\end{aligned}
$$

where

$$
N := \frac{1}{MK^3} \inf_{n\geq 1,\, e\in E} W^{i,\beta(e)}_{n,C(e)}(f) \cdot \inf_{e\in E} \exp(\sup(S_{|\tau(e)|} f|_{[\tau(e)]}))e^{-|\tau(e)|P(f)} > 0
$$

using Lemma 5.9. On the other hand, using Proposition 5.7 we obtain

$$
\begin{aligned}
\mu_{j+k}([\omega]) &= \sum_{\omega\alpha \in E_A^{j+k}} \frac{\exp(\sup(S_{j+k} f|_{[\omega\alpha]}))e^{-(j+k)P(f)}}{W_{j+k}(f)} \\
&\leq \frac{W_k(f)}{W_{j+k}(f)} \exp(\sup(S_j f|_{[\omega]}))e^{-jP(f)} \sum_{\alpha\in E_A^k} \frac{\exp(\sup(S_k f|_{[\alpha]}))e^{-kP(f)}}{W_k(f)} \\
&\leq \frac{M}{1} \exp(\sup(S_j f|_{[\omega]}))e^{-jP(f)} \sum_{\alpha \in E_A^k} \mu_k([\alpha]) \\
&= M \exp(\sup(S_j f|_{[\omega]}))e^{-jP(f)}.
\end{aligned}
$$

Now, let μ be a weak* cluster point of the sequence $(\mu_j)_{j\geq 1}$. Such an accumulation point exists since E was assumed to be finite and thus E_A^∞ is compact. The above two inequalities then imply that

$$
N \leq \frac{\mu([\omega])}{\exp(\sup(S_{|\omega|} f|_{[\omega]}))e^{-|\omega|P(f)}} \leq M
$$

for every $\omega \in E_A^*$. This shows that μ is a Gibbs measure with constant $Q := \max\{N^{-1}, M\}$. In particular, note that the constant M is simply an upper bound for the sequence $(W_n(f))_{n\geq 1}$. By Theorem 3.4, it suffices to choose M to be a constant of bounded supermultiplicativity and submultiplicativity for the sequence $(Z_n(f))_{n\geq 1}$. Furthermore, observe that N depends on M, the constant of bounded variation K and the chosen C's, β's and τ's.

Let us now assume that E is infinite. As usual, let I denote the set of isolated letters of E. Denote by $\mathcal{C} = \mathcal{C}(E)$ the set of all strongly connected components of E and by $\mathcal{C}_{\max} = \mathcal{C}_{\max}(E)$ the subset of all those among them that are of maximal f-pressure for E. For each $C \in \mathcal{C}$, there is a finite set $\Omega_C \subseteq C_A^*$ which witnesses the irreducibility of $A|_{C\times C}$. For any $\omega \in E_A^*$ denote by E_ω the set of all letters in the word ω. Let $E_{\Omega_C} = \bigcup_{\omega\in\Omega_C} E_\omega$, the set of all letters in words of Ω_C. Let $\Omega = \bigcup_{C\in\mathcal{C}} \Omega_C$, $\Omega_{\max} = \bigcup_{D\in\mathcal{C}_{\max}} \Omega_D$, $E_\Omega = \bigcup_{C\in\mathcal{C}} E_{\Omega_C}$ and $E_{\Omega_{\max}} = \bigcup_{D\in\mathcal{C}_{\max}} E_{\Omega_D}$. Then Ω, $\Omega_{\max}$, E_Ω and $E_{\Omega_{\max}}$ are all finite sets. By Proposition 4.10, we know that

$$\max_{C\in\mathcal{C}\setminus\mathcal{C}_{\max}} P_C(f) < P(f).$$

Let $0 < \varepsilon < P(f) - \max_{C\notin\mathcal{C}_{\max}} P_C(f)$. By Theorem 2.1.5 in [6], for each $D \in \mathcal{C}_{\max}$ there is a finite set F_D such that $E_{\Omega_D} \subseteq F_D \subseteq D$ and

$$\max_{C\notin\mathcal{C}_{\max}} P_C(f) + \varepsilon < \min_{D\in\mathcal{C}_{\max}} P_{F_D}(f). \tag{5.1}$$

Let $F_{\max} = \bigcup_{D\in\mathcal{C}_{\max}} F_D$ and $F = E_\Omega \cup F_{\max}$. Then F is finite and $E_\Omega \subseteq F \subseteq \bigcup_{C\in\mathcal{C}} C$. Let also $\mathcal{L} \subseteq I_A^*$ be a finite set witnessing the finitely linked feature of the shift. For each letter $e \in E$ there is $D(e) \in \mathcal{C}_{\max}$, a letter $\beta(e) \in E_{\Omega_{D(e)}}$, and a word $\tau(e) \in E_A^*$ such that $e\tau(e)\beta(e) \in E_A^*$. Clearly the sets $\{C(e) : e \in E\} \subseteq \mathcal{C}_{\max}$ and $\{\beta(e) : e \in E\} \subseteq E_\Omega$ are finite and the τ-words can be chosen so that the set $\{\tau(e) : e \in E\}$ is finite (using the words of $\mathcal{L}$ as well as the words of Ω). Let $T = \{E_{\tau(e)} : e \in E\}$.

There exists an increasing sequence $(F_h)_{h\geq 1}$ of finite sets such that $F \cup T \subseteq F_h$ for every $h \geq 1$ and $\bigcup_{h\geq 1} F_h = E$.

Since $F_h \supseteq E_\Omega$, the subalphabet F_h has exactly one strongly connected component in each of the strongly connected components of E, each of them is finitely irreducible, and the set of isolated letters for F_h is a subset of I.

Moreover, since $F_h \supseteq F_{\max}$, by (5.1) the strongly connected components of maximal $f|_{(F_h)_A^\infty}$-pressure for F_h are contained in the strongly connected components of maximal f-pressure for E. Since none of the strongly connected components of maximal f-pressure for E communicate, neither do the strongly connected components of maximal $f|_{(F_h)_A^\infty}$-pressure for F_h.

Furthermore, as $F_h \supseteq T$, every letter of F_h leads to a component of maximal $f|_{(F_h)_A^\infty}$-pressure for F_h via a word in $(F_h)_A^*$.

Then, by the first part (the finite case) of the proof, the subshift $\sigma_h : (F_h)_A^\infty \to (F_h)_A^\infty$ has a Gibbs state μ_h such that

$$N_h \leq \frac{\mu_h([\omega \cap F_h])}{\exp(\sup(S_{|\omega|} f|_{[\omega\cap F_h]}))e^{-|\omega| P_{F_h}(f)}} \leq M_h$$

for every $\omega \in (F_h)_A^*$, where N_h and M_h are determined by any constant of bounded supermultiplicativity and submultiplicativity for the sequence $(Z_{n,F_h}(f))_{n\geq 1}$, the constant of bounded variation K and the C's, β's and τ's.

We now aim at showing that the constants N_h and M_h can be chosen so that they are independent of h. To do this, it suffices to prove that the sequences $(Z_{n,F_h}(f))_{n\geq 1}$, $h \geq 1$, share a common constant of bounded supermultiplicativity (since these sequences are all submultiplicative).

By (5.1), there exists $N \geq 1$ such that

$$\max_{C\notin\mathcal{C}_{\max}} Z_{n,C}(f)\cdot e^{n\varepsilon} \leq \min_{D\in\mathcal{C}_{\max}} Z_{n,F_D}(f) \tag{5.2}$$

for all $n \geq N$. For each $h \geq 1$, let $D_h \in \mathcal{C}_{\max}$ be such that $F_h \cap D_h$ is a component of maximal $f|_{(F_h)_A^\infty}$-pressure for F_h. By passing to a subsequence, we may assume that there is $D_{\max} \in \mathcal{C}_{\max}$ such that $F_h \cap D_{\max}$ is a component of maximal $f|_{(F_h)_A^\infty}$-pressure for F_h for all $h \geq 1$. Moreover, Proposition 3.5 and the note following it assert that, with $\tilde{K} := \max_{C\in\mathcal{C}} \tilde{K}_C$, we have

$$\begin{aligned} Z_{m,F_h\cap C}(f)Z_{n,F_h\cap C}(f) &\leq \tilde{K} Z_{m+n,F_h\cap C}(f) \\ &\leq \tilde{K} Z_{m,F_h\cap C}(f)Z_{n,F_h\cap C}(f) \end{aligned} \tag{5.3}$$

for all $m, n \geq 1$, $h \geq 1$ and $C \in \mathcal{C}$.

Let κ be the number of strongly connected components and let B be the maximal length of words consisting of isolated letters only. Let $n > (\kappa+1)(B+N) + \kappa N$ and $h \geq 1$. In particular, this implies that there are no n-words consisting solely of isolated letters, i.e. every n-word must visit at least one component. Then

$$\begin{aligned} Z_{n,F_h}(f) &\leq \sum_{\omega\in(F_h)_A^n} \exp(\sup(S_{|\omega|}f|_{[\omega]})) \\ &= \sum_{k=1}^{\kappa} \sum_{\substack{\omega\in(F_h)_A^n \text{ visits exactly} \\ k \text{ components}}} \exp(\sup(S_{|\omega|}f|_{[\omega]})). \end{aligned} \tag{5.4}$$

Consider the set of all $\omega \in (F_h)_A^n$ that visit the k strongly connected components $C_k, \ldots, C_1$ of F_h, and no others, in this specific order. In order for this set to be non-empty, the components $C_k, \ldots, C_1$ must form a subchain of a chain. Then there is a unique way of writing ω as a concatenation

$$\omega = \beta^{(k+1)}\alpha^{(k)}\beta^{(k)}\alpha^{(k-1)}\beta^{(k-1)}\ldots\alpha^{(2)}\beta^{(2)}\alpha^{(1)}\beta^{(1)}, \tag{5.5}$$

where $\beta^{(j)} \in I_A^*$ and $\alpha^{(j)} \in (C_j)_A^*$. Note that some (possibly all) of the $\beta^{(j)}$ may be the empty word ϵ. For (sub)words consisting of isolated letters only, Lemma 4.11 states that

$$\sum_{\beta\in I_A^*}\exp(\sup(S_{|\beta|}f|_{[\beta]}))\le\sum_{i=1}^{B}\Big(\sum_{\beta\in I}\exp(\sup(f|_{[\beta]}))\Big)^i=:L<\infty.$$

Therefore

$$\sum_{\substack{\omega\in(F_h)_A^n\text{ visits}\\ C_k,\dots,C_1\text{ only}}}\exp(\sup(S_{|\omega|}f|_{[\omega]})) \tag{5.6}$$
$$\le L^{k+1}\sum_{l=1}^{k}\sum_{\substack{(\alpha^{(k)},\dots,\alpha^{(1)})\in(C_k)_A^*\times\dots\times(C_1)_A^*\\ n\ge\sum_{j=1}^{k}|\alpha^{(j)}|\ge n-(k+1)B\\ \text{with exactly } l\ \alpha^{(j)}\text{'s such that } |\alpha^{(j)}|<N}}\prod_{i=1}^{k}\exp(\sup(S_{|\alpha^{(i)}|}f|_{[\alpha^{(i)}]})).$$

Recall that there are $\binom{k}{l}$ combinations of l components among k. For any such combination, the l components C_j such that $|\alpha^{(j)}|<N$ form a unique subchain $C_{j_k},\dots,C_{j_{k-l+1}}$. Similarly, the remaining $k-l$ components constitute a subchain $C_{j_{k-l}},\dots,C_{j_1}$.

For the l components C_j such that $|\alpha^{(j)}|<N$, Lemma 4.12 asserts that

$$\sum_{\substack{\alpha^{(j)}\in(C_j)_A^*\\ |\alpha^{(j)}|<N}}\exp(\sup(S_{|\alpha^{(j)}|}f|_{[\alpha^{(j)}]}))\le\sum_{i=1}^{N-1}(Z_1(f))^i=:Z<\infty.$$

For the remaining $k-l$ components, we have the following estimate when $C_{j_{k-l}}\in\mathcal{C}(F_h)\setminus\mathcal{C}_{\max}(F_h)$.

LEMMA 5.13. *Suppose that $C_{j_s},\dots,C_{j_1}$ are strongly connected components of F_h constituting a chain and that $C_{j_1}\in\mathcal{C}(F_h)\setminus\mathcal{C}_{\max}(F_h)$. Then for any $m\le M$, we have*

$$\sum_{\substack{(\alpha^{(j_s)},\dots,\alpha^{(j_1)})\in(C_{j_s})_A^*\times\dots\times(C_{j_1})_A^*\\ M\ge\sum_{i=1}^{s}|\alpha^{(j_i)}|\ge m\\ |\alpha^{(j_i)}|\ge N,\ \forall i}}\prod_{i=1}^{s}\exp(\sup(S_{|\alpha^{(j_i)}|}f|_{[\alpha^{(j_i)}]}))$$
$$\le K^s\tilde{K}^{s-1}\cdot M^{s-1}e^{-\varepsilon\max\{m,sN\}}\sum_{r=m}^{M}Z_{r,F_h\cap D_{\max}}(f).$$

Proof. The proof is similar to that of Lemma 4.13; it uses (5.2) and (5.3) and the fact that by construction $F_h\cap D_{\max}\supseteq F_{D_{\max}}$. ∎

We resume the proof of Proposition 5.12. When $C_{j_{k-l}}\in\mathcal{C}(F_h)\setminus\mathcal{C}_{\max}(F_h)$, it follows from the previous two lemmas, the choice $n>(\kappa+1)(B+N)+\kappa N$

and (5.3) that (cf. (4.6))

$$
\begin{aligned}
&\sum_{\substack{(\alpha^{(j_k)},\dots,\alpha^{(j_1)})\in(C_{j_k})^*_A\times\dots\times(C_{j_1})^*_A\\ n\geq\sum_{i=1}^k|\alpha^{(j_i)}|\geq n-(k+1)B\\ |\alpha^{(j_i)}|<N,\forall i>k-l,\,|\alpha^{(j_i)}|\geq N,\forall i\leq k-l}} \prod_{i=1}^{k}\exp(\sup(S_{|\alpha^{(j_i)}|}f|_{[\alpha^{(j_i)}]}))\\
&\quad\leq Z^l K^{k-l}\tilde{K}^{k-l-1}\cdot n^{k-l-1}e^{-\varepsilon\max\{n-(k+1)(B+N),kN\}}\\
&\qquad\cdot\sum_{r=n-(k+1)(B+N)}^{n} Z_{r,F_h\cap D_{\max}}(f)\\
&\quad\leq Z^l(K\tilde{K})^k\cdot n^k e^{-\varepsilon(n-(k+1)(B+N))}\frac{[(k+1)(B+N)+1]\tilde{K}}{\min_{0\leq j\leq(k+1)(B+N)} Z_{j,F_h\cap D_{\max}}(f)}\\
&\qquad\cdot Z_{n,F_h\cap D_{\max}}(f)\\
&\quad\leq\frac{2(\kappa+1)(B+N)\max\{1,Z^\kappa\}(K\tilde{K})^{\kappa+1}e^{\varepsilon(\kappa+1)(B+N)}}{\min_{0\leq j\leq(\kappa+1)(B+N)} Z_{j,E_{\Omega_{D_{\max}}}}(f)}\\
&\qquad\cdot n^\kappa e^{-\varepsilon n}Z_{n,F_h\cap D_{\max}}(f)\\
&\quad=:\hat{K}\cdot n^\kappa e^{-\varepsilon n}Z_{n,F_h\cap D_{\max}}(f).
\end{aligned}
$$

Since there are $\binom{k}{l}\leq k!\leq\kappa!$ combinations of l components among k, we deduce from (5.6) that

$$
\begin{aligned}
&\sum_{\substack{\omega\in(F_h)^n_A\text{ visits}\\ \text{only }C_k,\dots,C_1}}\exp(\sup(S_{|\omega|}f|_{[\omega]}))\\
&\qquad\qquad\leq\max\{1,L^{\kappa+1}\}\kappa\kappa!\hat{K}\cdot n^\kappa e^{-\varepsilon n}Z_{n,F_h\cap D_{\max}}(f)\\
&\qquad\qquad=:\hat{\hat{K}}\cdot n^\kappa e^{-\varepsilon n}Z_{n,F_h\cap D_{\max}}(f)
\end{aligned}
$$

whenever $C_1\in\mathcal{C}(F_h)\setminus\mathcal{C}_{\max}(F_h)$.

However, when $C_1\in\mathcal{C}_{\max}(F_h)$, we know that the other C_j's are not in $\mathcal{C}_{\max}(F_h)$. Since $C_1\in\mathcal{C}_{\max}(F_h)$ and $F_h\cap D_{\max}\in\mathcal{C}_{\max}(F_h)$, Corollary 3.7 guarantees that $1\leq W_{n,C_1}(f)\leq\tilde{K}$ and $1\leq W_{n,F_h\cap D_{\max}}(f)\leq\tilde{K}$ for all $n\geq 1$. Then

$$
\begin{aligned}
\frac{Z_{n,C_1}(f)}{Z_{n,F_h\cap D_{\max}}(f)}&=\frac{Z_{n,C_1}(f)e^{-nP_{F_h}(f)}}{Z_{n,F_h\cap D_{\max}}(f)e^{-nP_{F_h}(f)}}=\frac{Z_{n,C_1}(f)e^{-nP_{C_1}(f)}}{Z_{n,F_h\cap D_{\max}}(f)e^{-nP_{F_h\cap D_{\max}}(f)}}\\
&=\frac{W_{n,C_1}(f)}{W_{n,F_h\cap D_{\max}}(f)}\leq\frac{\tilde{K}}{1}
\end{aligned}
$$

for all $n\geq 1$. Then

$$\begin{aligned}
&\sum_{\substack{\omega\in(F_h)^n_A \text{ visits} \\ \text{only } C_k,\dots,C_1}} \exp(\sup(S_{|\omega|}f|_{[\omega]})) \\
&\quad \le \sum_{r=1}^{n}\sum_{\tau\in(C_1)^r_A}\exp(\sup(S_{|\tau|}f|_{[\tau]}))\sum_{\substack{\rho\in(F_h)^{n-r}_A \text{ visits} \\ \text{only } C_k,\dots,C_2}}\exp(\sup(S_{|\rho|}f|_{[\rho]})) \\
&\quad \le \sum_{r=1}^{n} Z_{r,C_1}(f)\cdot\hat{\hat{K}}\cdot(n-r)^{\kappa}e^{-\varepsilon(n-r)}Z_{n-r,F_h\cap D_{\max}}(f) \\
&\quad \le \sum_{r=1}^{n} \tilde{K}Z_{r,F_h\cap D_{\max}}(f)\cdot\hat{\hat{K}}\cdot(n-r)^{\kappa}e^{-\varepsilon(n-r)}Z_{n-r,F_h\cap D_{\max}}(f) \\
&\quad = \tilde{K}\hat{\hat{K}}\sum_{r=1}^{n}(n-r)^{\kappa}e^{-\varepsilon(n-r)}Z_{r,F_h\cap D_{\max}}(f)Z_{n-r,F_h\cap D_{\max}}(f) \\
&\quad \le \tilde{K}\hat{\hat{K}}\sum_{r=1}^{n}(n-r)^{\kappa}e^{-\varepsilon(n-r)}\cdot\tilde{K}Z_{n,F_h\cap D_{\max}}(f) \\
&\quad \le \tilde{K}^2\hat{\hat{K}}\sum_{s=0}^{\infty}s^{\kappa}e^{-\varepsilon s}\cdot Z_{n,F_h\cap D_{\max}}(f)
\end{aligned}$$

whenever $C_1\in\mathcal{C}_{\max}(F_h)$.

As there are at most $\binom{\kappa}{k}\le\kappa!$ combinations of k components among a grand total of κ, we conclude that

$$\sum_{\substack{\omega\in(F_h)^n_A \text{ visits exactly} \\ k \text{ components}}} \exp(\sup(S_{|\omega|}f|_{[\omega]})) \le \kappa!\hat{\hat{K}}\Big[n^{\kappa}e^{-\varepsilon n}+\tilde{K}^2\sum_{s=0}^{\infty}s^{\kappa}e^{-\varepsilon s}\Big]Z_{n,F_h\cap D_{\max}}(f).$$

It follows from (5.4) and the choice of n that

$$Z_{n,F_h}(f)\le\kappa\kappa!\hat{\hat{K}}\Big[n^{\kappa}e^{-\varepsilon n}+\tilde{K}^2\sum_{s=0}^{\infty}s^{\kappa}e^{-\varepsilon s}\Big]Z_{n,F_h\cap D_{\max}}(f).$$

Let

$$\hat{M}:=\kappa\kappa!\hat{\hat{K}}\Big[\max_{n\ge1}(n^{\kappa}e^{-\varepsilon n})+\tilde{K}^2\sum_{s=0}^{\infty}s^{\kappa}e^{-\varepsilon s}\Big]<\infty.$$

Then

$$Z_{n,F_h}(f)\le\hat{M}\cdot Z_{n,F_h\cap D_{\max}}(f)$$

for all $n>(\kappa+1)(B+N)+\kappa N$ and all $h\ge1$. Furthermore,

$$\begin{aligned} Z_{n,F_h\cap D_{\max}}(f) \geq Z_{n,E_{\Omega_{D_{\max}}}}(f) &= \frac{Z_{n,E_{\Omega_{D_{\max}}}}(f)}{Z_{n,F_h}(f)} \cdot Z_{n,F_h}(f) \\ &\geq \left[\min_{1\leq m\leq(\kappa+1)(B+N)+\kappa N} \frac{Z_{m,E_{\Omega_{D_{\max}}}}(f)}{Z_m(f)} \right] Z_{n,F_h}(f) \end{aligned}$$

for all $n \leq (\kappa+1)(B+N)+\kappa N$ and all $h \geq 1$. Consequently, there exists a constant $\tilde{M} > 0$ such that

$$Z_{n,F_h}(f) \leq \tilde{M} \cdot Z_{n,F_h\cap D_{\max}}(f)$$

for all $n \geq 1$ and all $h \geq 1$. It follows that

$$\begin{aligned} Z_{m,F_h}(f) Z_{n,F_h}(f) \geq Z_{m+n,F_h}(f) &\geq Z_{m+n,F_h\cap D_{\max}}(f) \\ &\geq \tilde{K}^{-1} Z_{m,F_h\cap D_{\max}}(f) Z_{n,F_h\cap D_{\max}}(f) \\ &\geq \tilde{K}^{-1} \tilde{M}^{-2} Z_{m,F_h}(f) Z_{n,F_h}(f) \end{aligned}$$

for all $m, n \geq 1$ and all $h \geq 1$. This proves that the sequence $(Z_{n,F_h}(f))_{n\geq 1}$ is boundedly supermultiplicative with a constant independent of h.

Finally, we prove that the sequence $(\mu_h)_{h\geq 1}$ of Gibbs measures, which have a common Gibbs constant Q, is tight. The proof is inspired from that of Theorem 2.7.3 in [6]. Obviously, $P_{F_1}(f) \leq P_{F_h}(f)$ for all $h \geq 1$. For every $k \geq 1$ let $\pi_k : E_A^\infty \to E$ be the projection onto the kth coordinate, i.e. $\pi_k((e_i)_{i\geq 1}) = e_k$. Then for every $h \geq 1$, each $k \geq 1$ and all $e \in E$ we have

$$\begin{aligned} \mu_h(\pi_k^{-1}(e)) &= \sum_{\omega\in(F_h)_A^k:\,\omega_k=e} \mu_h([\omega]) \\ &\leq Q \sum_{\omega\in(F_h)_A^k:\,\omega_k=e} \exp(\sup(S_k f|_{[\omega]}) - kP_{F_h}(f)) \\ &\leq Q e^{-kP_{F_1}(f)} \sum_{\omega\in E_A^{k-1}} \exp(\sup(S_{k-1} f|_{[\omega]})) \cdot \exp(\sup(f|_{[e]})) \\ &\leq Q e^{-kP_{F_1}(f)} Z_{k-1}(f) \cdot \exp(\sup(f|_{[e]})) \\ &\leq Q e^{-kP_{F_1}(f)} (Z_1(f))^{k-1} \cdot \exp(\sup(f|_{[e]})). \end{aligned}$$

Therefore (for convenience, we set $E = \mathbb{N}$)

$$\mu_h(\pi_k^{-1}([e+1,\infty))) \leq Q e^{-kP_{F_1}(f)} (Z_1(f))^{k-1} \cdot \sum_{i>e} \exp(\sup(f|_{[i]})).$$

Fix $\delta > 0$ and for every $k \geq 1$ choose $n_k \geq 1$ such that

$$Q e^{-kP_{F_1}(f)} (Z_1(f))^{k-1} \cdot \sum_{i>n_k} \exp(\sup(f|_{[i]})) \leq \frac{\delta}{2^k}.$$

Then $\mu_h(\pi_k^{-1}([n_k+1,\infty))) \le \delta/2^k$ for every $h \ge 1$ and every $k \ge 1$. Hence

$$\begin{aligned}\mu_h\Big(E_A^\infty \cap \prod_{k\ge 1}[1,n_k]\Big) &\ge 1-\sum_{k\ge 1}\mu_h(\pi_k^{-1}([n_k+1,\infty))) \\ &\ge 1-\sum_{k\ge 1}\frac{\delta}{2^k} = 1-\delta.\end{aligned}$$

Since $E_A^\infty \cap \prod_{k\ge 1}[1,n_k]$ is a compact subset of E_A^∞, the tightness of the sequence $(\mu_h)_{h\ge 1}$ is proved. Therefore, in view of Prokhorov's Theorem, this sequence has a weak accumulation point μ and this accumulation point is clearly a Gibbs measure with Gibbs constant Q. ■

We shall now give a proof of our two main theorems, namely, Theorems 5.1 and 5.2.

Proof of Theorem 5.1. Taken together, Propositions 5.10 and 5.11 establish one of the implications in Theorem 5.1. The other implication is precisely the object of Proposition 5.12. ■

Proof of Theorem 5.2. Suppose that the system admits an invariant Gibbs state μ. By Theorem 5.1, we know that none of the strongly connected components of maximal f-pressure communicate and each letter leads to a component of maximal f-pressure. Now, assume that the system has a strongly connected component C which is not of maximal f-pressure. Since every letter leads to a component of maximal f-pressure, component C leads to a strongly connected component $\tilde{C}$ of maximal f-pressure. Component C further belongs to a chain of one-way communication between strongly connected components. Let D be the minimal element of that chain. Then D is not of maximal f-pressure since D leads to C and thereafter to $\tilde{C}$, and since strongly connected components of maximal f-pressure do not communicate. Let $0<\varepsilon<P(f)-P_D(f)$. Fix $\omega \in D_A^*$. The minimality of D implies that $\tau\omega \in E_A^*$ if and only if $\tau\omega \in D_A^*$. Since μ is an invariant Gibbs state, for some constant $Q \ge 1$ that for every $n \ge 1$,

$$\begin{aligned}\mu([\omega]) = \mu(\sigma^{-n}([\omega])) &= \sum_{\tau\in E_A^n:\, A_{\tau_n\omega_1}=1}\mu([\tau\omega]) = \sum_{\tau\in D_A^n:\, A_{\tau_n\omega_1}=1}\mu([\tau\omega]) \\ &\le \sum_{\tau\in D_A^n:\, A_{\tau_n\omega_1}=1} Q\exp(\sup(S_{n+|\omega|}f|_{[\tau\omega]}) - P(f)(n+|\omega|)) \\ &\le Q\exp(\sup(S_{|\omega|}f|_{[\omega]})-P(f)|\omega|)\sum_{\tau\in D_A^n}\exp(\sup(S_nf|_{[\tau]})-P(f)n) \\ &\le Q^2\mu([\omega])\sum_{\tau\in D_A^n}\exp(\sup(S_nf|_{[\tau]})-P(f)n)\end{aligned}$$

$$\leq Q^2\mu([\omega])\sum_{\tau\in D_A^n}\exp(\sup(S_nf|_{[\tau]})-(P_D(f)+\varepsilon)n)$$
$$\leq Q^2\mu([\omega])\sum_{\tau\in D_A^n}K\exp(\sup(S_nf|_{[\tau\cap D]})-P_D(f)n)\cdot e^{-n\varepsilon}$$
$$\leq Q^2K\mu([\omega])W_{n,D}(f)\cdot e^{-n\varepsilon}\leq Q^2K\mu([\omega])\tilde{K}_D\cdot e^{-n\varepsilon},$$

where the constant $\tilde{K}_D$ exists according to Corollary 3.7. Letting $n\to\infty$, we deduce that $\mu([\omega])=0$. This is impossible. We hence conclude that the existence of an invariant Gibbs state prevents the existence of strongly connected components not of maximal f-pressure. This proves one implication.

To prove the converse, suppose that a Markov shift is such that all its strongly connected components are of maximal f-pressure and that none of them communicate. This implies that the system is a union of disjoint, non-communicating strongly connected components of maximal f-pressure. Let $C_1,\ldots,C_j$ be the strongly connected components of the system. Then $E_A^\infty=\bigcup_{1\leq k\leq j}(C_k)_A^\infty$ and $P_{C_k}(f)=P(f)$ for all $1\leq k\leq j$. According to Theorems 2.3.3 and 2.2.4 in [6], each subshift $\sigma:(C_k)_A^\infty\to(C_k)_A^\infty$ has a unique invariant Gibbs state μ_k. These measures can be extended to E_A^∞ by setting $\overline{\mu}_k(B):=\mu_k(B\cap(C_k)_A^\infty)$, where B is a Borel subset of E_A^∞. Then any invariant Gibbs state μ for the full system is a non-trivial convex combination of the extensions $\overline{\mu}_k$ of the component invariant Gibbs states, i.e. $\mu=\sum_{1\leq k\leq j}\alpha_k\overline{\mu}_k$ for some $\alpha_k\neq 0$, $1\leq k\leq j$, such that $\sum_{1\leq k\leq j}\alpha_k=1$. This is because the restriction of a Gibbs measure for the full system to any $(C_k)_A^\infty$ is an invariant measure satisfying the Gibbs condition (2.1), albeit not a probability measure. Normalizing this restriction produces an invariant Gibbs measure for the subshift $\sigma:(C_k)_A^\infty\to(C_k)_A^\infty$. But by Theorem 2.2.4 in [6] there is only one such measure.

Hence, if the system has more than one component, i.e. A is not irreducible, then there are uncountably many invariant Gibbs states, albeit all of the above form. None of these invariant Gibbs states is ergodic since every subset $(C_k)_A^\infty$ is invariant and $0\neq\mu((C_k)_A^\infty)=\alpha_k\mu_k((C_k)_A^\infty)=\alpha_k\neq 1$. If the system consists of a single component, i.e. A is finitely irreducible, then the system has a unique invariant Gibbs measure. This measure is ergodic according to Theorem 2.2.4 in [6]. If A is finitely primitive, then Theorem 2.2.4 in [6] asserts that the invariant Gibbs state is completely ergodic. ∎

It is apropos to recall that for irreducible Markov shifts, Mauldin and Urbański [6] and Sarig [9] proved that a necessary and sufficient condition for the existence of an invariant Gibbs state is that the incidence matrix be finitely irreducible (cf. Theorem 2.2.6 in [6]). Thus, Main Theorem 5.2 does not apply to a system consisting of a unique strongly connected component which is not finitely irreducible. This means that the assumption of finite irreducibility on the strongly connected components cannot be relaxed in general.

Moreover, note that under a summable potential, a system with infinitely many strongly connected components possesses (infinitely many) strongly connected components that are not of maximal pressure (because there are finitely many strongly connected components of maximal pressure according to Proposition 5.4). Thus, if Main Theorem 5.2 holds for such systems, then no such system admits an invariant Gibbs state. By the argument in the proof of Theorem 5.2, this is certainly the case for all systems that have a chain with a minimal strongly connected component not of maximal pressure. But we do not know whether this is the case more generally.

Finally, it is unclear whether one can relax the hypothesis that isolated letters are finitely linked and that words consisting only of isolated letters are uniformly bounded in length.

Acknowledgements. The first author would like to thank York University's Glendon College for its hospitality during his visits. Both authors would like to thank the University of Alaska–Fairbanks for its hospitality during their stay while this paper was written.

The second author's research was supported by NSERC (Natural Sciences and Engineering Research Council of Canada).

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Received 25 May 2012;
in revised form 27 March 2013