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Cardinal sequences of length $< \omega_2$ under GCH

by

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Abstract. Let $\mathcal{C}(\alpha)$ denote the class of all cardinal sequences of length α associated with compact scattered spaces (or equivalently, superatomic Boolean algebras). Also put

$$\mathcal{C}_\lambda(\alpha) = \{s \in \mathcal{C}(\alpha) : s(0) = \lambda = \min[s(\beta) : \beta < \alpha]\}.$$

We show that $f \in \mathcal{C}(\alpha)$ iff for some natural number n there are infinite cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ and ordinals $\alpha_0, \ldots, \alpha_{n-1}$ such that $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ and $f = f_0{}^\frown f_1{}^\frown \ldots {}^\frown f_{n-1}$ where each $f_i \in \mathcal{C}_{\lambda_i}(\alpha_i)$. Under GCH we prove that if $\alpha < \omega_2$ then

(i) $\mathcal{C}_\omega(\alpha) = \{s \in {}^\alpha\{\omega, \omega_1\} : s(0) = \omega\}$;
(ii) if $\lambda > \operatorname{cf}(\lambda) = \omega$,

$$\mathcal{C}_\lambda(\alpha) = \{s \in {}^\alpha\{\lambda, \lambda^+\} : s(0) = \lambda,\ s^{-1}\{\lambda\} \text{ is } \omega_1\text{-closed in } \alpha\};$$

(iii) if $\operatorname{cf}(\lambda) = \omega_1$,

$$\mathcal{C}_\lambda(\alpha) = \{s \in {}^\alpha\{\lambda, \lambda^+\} : s(0) = \lambda,\ s^{-1}\{\lambda\} \text{ is } \omega\text{-closed and successor-closed in } \alpha\};$$

(iv) if $\operatorname{cf}(\lambda) > \omega_1$, $\mathcal{C}_\lambda(\alpha) = {}^\alpha\{\lambda\}$.

This yields a complete characterization of the classes $\mathcal{C}(\alpha)$ for all $\alpha < \omega_2$, under GCH.

1. Introduction. For a scattered space X and an ordinal α we let $\mathrm{I}_\alpha(X)$ denote the αth Cantor–Bendixson level of X. The *height* of X, $\operatorname{ht}(X)$, is the minimal ordinal β with $\mathrm{I}_\beta(X) = \emptyset$. The *reduced height* $\operatorname{ht}^-(X)$ is the smallest ordinal α such that $\mathrm{I}_\alpha(X)$ is finite. Clearly, one has

$$\operatorname{ht}^-(X) \le \operatorname{ht}(X) \le \operatorname{ht}^-(X) + 1.$$

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The sequence of infinite cardinals

$$\langle |\mathrm{I}_\alpha(X)| : \alpha < \mathrm{ht}^-(X)\rangle$$

is called the *cardinal sequence* of X and is denoted by $\mathrm{SEQ}(X)$.

We let $\mathcal{C}(\alpha)$ denote the class of all cardinal sequences of length α of locally compact scattered T_2 (for short: LCS) spaces. We also put, for any fixed infinite cardinal λ,

$$\mathcal{C}_\lambda(\alpha) = \{s \in \mathcal{C}(\alpha) : s(0) = \lambda \wedge \forall \beta < \alpha\ [s(\beta) \geq \lambda]\}.$$

We shall see later that $\mathcal{C}(\alpha)$ is uniquely determined if we know $\mathcal{C}_\lambda(\beta)$ for all cardinals λ and for all ordinals $\beta \leq \alpha$.

As usual, the concatenation of a sequence f of length α and of a sequence g of length β is denoted by $f^\frown g$. So the domain of $h = f^\frown g$ is $\alpha + \beta$, $h(\xi) = f(\xi)$ for $\xi < \alpha$, and $h(\alpha + \eta) = g(\eta)$ for $\eta < \beta$.

We shall use the notation $\langle \kappa \rangle_\alpha$ to denote the constant κ-valued sequence of length α.

In [4], the following simple (to formulate, not to prove!) characterization of $\mathcal{C}(\omega_1)$ was given in ZFC: The sequence $\langle \kappa_\xi : \xi < \omega_1 \rangle \in \mathcal{C}(\omega_1)$ iff $\kappa_\eta \leq \kappa_\xi^\omega$ holds whenever $\xi < \eta < \omega_1$. It follows that cardinal arithmetic (in fact just the operation $\kappa \mapsto \kappa^\omega$) alone decides whether a sequence of cardinals of length ω_1 belongs to $\mathcal{C}(\omega_1)$ or not. The situation changes dramatically for longer sequences, in fact already for sequences of length $\omega_1 + 1$. For example, the question if $\langle \omega \rangle_{\omega_1}{}^\frown \langle \omega_2 \rangle_1 \in \mathcal{C}(\omega_1 + 1)$ is not decided by the following cardinal arithmetic: $2^\omega = \omega_2$ and $2^\kappa = \kappa^+$ for all $\kappa > \omega$ (see [5] and [8]). Moreover, any cardinal arithmetic is consistent with

$$\langle \omega_1 \rangle_{\omega_1}{}^\frown \langle \omega_2 \rangle_1 \in \mathcal{C}(\omega_1 + 1),$$

while one has

$$\langle \omega_1 \rangle_{\omega_1}{}^\frown \langle \omega_2 \rangle_1 \notin \mathcal{C}(\omega_1 + 1)$$

in the Mitchell model (see [1]).

However, as we shall show in this paper, the elements of $\mathcal{C}(\alpha)$ can be characterized for all $\alpha < \omega_2$ if we assume GCH. For example, one has $\langle \omega \rangle_{\omega_1}{}^\frown \langle \omega_2 \rangle_1 \notin \mathcal{C}(\omega_1 + 1)$ and $\langle \omega_1 \rangle_{\omega_1}{}^\frown \langle \omega_2 \rangle_1 \in \mathcal{C}(\omega_1 + 1)$ under GCH.

The following piece of notation is taken from [4]: If X is a scattered space and $x \in X$ then we write $\mathrm{ht}(x, X) = \alpha$ iff $x \in I_\alpha(X)$. Trivially, then

$$\mathrm{ht}(X) = \min\{\beta : \forall x \in X\ [\mathrm{ht}(x, X) < \beta]\}.$$

It is obvious that if $Y \subset X$ then $\mathrm{ht}(x, X) \geq \mathrm{ht}(x, Y)$ whenever $x \in Y$, and if Y is also *open* in X then actually $\mathrm{ht}(x, X) = \mathrm{ht}(x, Y)$. On the other hand, for the points of X outside of Y one can get the following upper bound.

FACT 1.1. *If Y is an open subspace of the scattered space X then for every point $x \in X \setminus Y$ we have* $\mathrm{ht}(x, X) \leq \mathrm{ht}(Y) + \mathrm{ht}(x, X \setminus Y)$. *Consequently,* $\mathrm{ht}(X) \leq \mathrm{ht}(Y) + \mathrm{ht}(X \setminus Y)$.

Indeed, this is proved by a straightforward transfinite induction on $\mathrm{ht}(x, X)$, using $Y \subset \mathrm{I}_{<\mathrm{ht}(Y)}(X)$.

It is well known that any ordinal, as an ordered topological space, is LCS. It is easy to see that if $\alpha < \beta$ are ordinals then $\mathrm{ht}(\alpha, \beta) = \gamma$ iff α can be written in the form $\omega^\gamma \cdot (2\delta + 1)$, or equivalently, γ is minimal such that α can be written as $\alpha = \varepsilon + \omega^\gamma$. Note that in the notation $\mathrm{ht}(\alpha, \beta)$ the ordinals play a double role: α is considered as a "point" in the set β. Using the above characterization of the Cantor–Bendixson levels of ordinal spaces, it is easy to show that for any infinite cardinal λ and for any ordinal $\alpha < \lambda^+$ we have $\langle \lambda \rangle_\alpha \in \mathcal{C}(\alpha)$.

This paper is a natural sequel to [4], so now we recall a few general statements concerning cardinal sequences from [4] that will be needed later.

FACT 1.2 ([4, Lemma 1]). *If $s \in \mathcal{C}(\beta)$ then $|\beta| \le 2^{s(0)}$ and $s(\alpha) \le 2^{s(0)}$ for each $\alpha < \beta$.*

FACT 1.3 ([4, Lemma 2]). *If $s \in \mathcal{C}(\beta)$ and $\alpha + 1 < \beta$ then $s(\alpha + 1) \le s(\alpha)^\omega$.*

FACT 1.4 ([4, Lemma 3]). *If $s \in \mathcal{C}(\beta)$, $\delta < \beta$ is a limit ordinal and C is any cofinal subset of δ, then*

$$s(\delta) \le \prod \{s(\alpha) : \alpha \in C\}.$$

We shall also need the following general construction from [4] that is used to obtain an LCS space by gluing together certain others.

LEMMA 1.5 ([4, Lemma 7]). *Let X be an LCS space with a closed discrete subset S such that for each $s \in S$ there is given a sequence $\langle U_{s,n} : n \in \omega \rangle$ of pairwise disjoint compact open subsets of $X \setminus S$, converging to the point s. Also, for each $s \in S$ let Y_s be a separable LCS space such that the collection $\{X\} \cup \{Y_s : s \in S\}$ of spaces is disjoint. Then there is an LCS space Z with the following three properties*:

(i) *$Z = (X \setminus S) \cup \bigcup \{Y_s : s \in S\}$ with $X \setminus S$ as an open subspace and each Y_s as a closed subspace. Moreover, $\{Y_s : s \in S\}$ forms a discrete collection in Z.*
(ii) *$\mathrm{ht}(x, Z) = \mathrm{ht}(x, X)$ for $x \in X \setminus S$.*
(iii) *$\mathrm{ht}(y, Z) = \delta_s + \mathrm{ht}(y, Y_s)$ for $y \in Y_s$, where δ_s is the least ordinal δ such that the set $\{n < \omega : U_{s,n} \cap I_\delta(X) \ne \emptyset\}$ is finite. (Clearly, $\delta_s \le \mathrm{ht}(s, X)$.)*

2. A reduction theorem. We have noted in the introduction that, in order to characterize those sequences of length $< \omega_2$ which are cardinal sequences of LCS spaces, it suffices to characterize the classes $\mathcal{C}_\lambda(\alpha)$ for any ordinal $\alpha < \omega_2$ and any infinite cardinal λ. In fact, this follows from the following general reduction theorem that is valid in ZFC.

THEOREM 2.1. *For any ordinal α we have $f \in \mathcal{C}(\alpha)$ iff for some natural number n there is a decreasing sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ of infinite cardinals and there are ordinals $\alpha_0, \ldots, \alpha_{n-1}$ such that $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ and $f = f_0 ^\frown f_1 ^\frown \ldots ^\frown f_{n-1}$ with $f_i \in \mathcal{C}_{\lambda_i}(\alpha_i)$ for each $i < n$.*

Proof. We first prove that the condition is necessary, so fix $f \in \mathcal{C}(\alpha)$. Let us say that $\beta < \alpha$ is a *drop point* in f if for all $\gamma < \beta$ we have $f(\gamma) > f(\beta)$. Clearly f has only finitely many, say n, drop points; let $\{\beta_i : i < n\}$ enumerate all of them in increasing order. (In particular, we then have $\beta_0 = 0$.) For each $i < n$ let us set $\lambda_i = f(\beta_i)$; then we clearly have $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$.

For each $i < n - 1$ let $\alpha_i = \beta_{i+1} - \beta_i$ be the unique ordinal such that $\beta_i + \alpha_i = \beta_{i+1}$, and define the sequence f_i on α_i by setting

$$f_i(\xi) = f(\beta_i + \xi)$$

for all $\xi < \alpha_i$. Similarly, let $\alpha_{n-1} = \alpha - \beta_{n-1}$ be the unique ordinal such that $\beta_{n-1} + \alpha_{n-1} = \alpha$, and define f_{n-1} on α_{n-1} by

$$f_{n-1}(\xi) = f(\beta_{n-1} + \xi)$$

for all $\xi < \alpha_{n-1}$. Now, it is obvious that we have $f = f_0 ^\frown f_1 ^\frown \ldots ^\frown f_{n-1}$ where $f_i \in \mathcal{C}_{\lambda_i}(\alpha_i)$ for each $i < n$.

We shall now prove that the condition is also sufficient. In fact, this will follow from the next lemma.

LEMMA 2.2. *If $f \in \mathcal{C}(\alpha)$, $g \in \mathcal{C}(\beta)$, and $f(\nu) \geq g(0)$ for each $\nu < \alpha$ then $f ^\frown g \in \mathcal{C}(\alpha + \beta)$.*

Indeed, given the sequences f_i for $i < n$, this lemma enables us to inductively define for every $i = 0, \ldots, n-1$ an LCS space Z_i with cardinal sequence $f_0 ^\frown \ldots ^\frown f_i$ because

$$f_i(0) = \lambda_i < \lambda_{i-1} = \min\{f_j(\nu) : j < i,\, \nu < \alpha_j\}.$$

In particular, we then have $\mathrm{SEQ}(Z_{n-1}) = f_0 ^\frown f_1 ^\frown \ldots ^\frown f_{n-1}$. $\blacksquare_{2.1}$

Proof of Lemma 2.2. Let Y be an LCS space with cardinal sequence g and satisfying $\mathrm{I}_\beta(Y) = \emptyset$. Next fix LCS spaces X_y for all $y \in \mathrm{I}_0(Y)$, each having the cardinal sequence f and satisfying $\mathrm{I}_\alpha(X_y) = \emptyset$. Assume also that the family $\{Y\} \cup \bigcup\{X_y : y \in Y\}$ is disjoint.

We then define the space $Z = \langle Z, \tau \rangle$ as follows. Let us first set

$$Z = Y \cup \bigcup\{X_y : y \in \mathrm{I}_0(Y)\}.$$

For any subset $V \subset Y$ we let

$$Z(V) = V \cup \bigcup\{X_y : y \in \mathrm{I}_0(Y) \cap V\};$$

moreover we put

$$\mathcal{T} = \{W : W \text{ is compact open in some } X_y\}.$$

Then the family

$$\mathcal{B} \;=\; \mathcal{T} \cup \left\{ Z(V) \setminus \bigcup \mathcal{T}' \;:\; V \text{ is compact open in } Y \text{ and } \mathcal{T}' \in [\mathcal{T}]^{<\omega} \right\}$$

clearly covers Z and is closed under finite intersections, hence it forms a base for a topology τ on Z.

Since $\mathcal{T} \subset \tau$ and $B \cap X_y$ is open in X_y for all $B \in \mathcal{B}$, each X_y is an open subspace of Z. We also have

$$\{B \cap Y : B \in \mathcal{B}\} = \{V \subset Y : V \text{ is compact open in } Y\},$$

and the latter is a base of Y, hence Y is a closed subspace of Z. It easily follows then that any non-empty subspace $A \subset Z$ has an isolated point, hence Z is scattered. It is also easy to check that Z is Hausdorff because so are Y and all the X_y.

So to see that Z is LCS, it remains to check that it is locally compact. Now, let V be compact open in Y; we claim that then $Z(V)$ is compact in Z. Indeed, this can be proved by a straightforward transfinite induction on

$$\sigma(V) = \max\{\mathrm{ht}(z, Y) : z \in V\}.$$

It clearly follows from this that all members of $\mathcal{B}$ are compact in Z, hence Z is locally compact.

Note that for any isolated point y of Y the space

$$Z(\{y\}) = \{y\} \cup X_y,$$

as a subspace of Z, is the one-point compactification of X_y. This clearly implies that $\mathrm{ht}(y, Z) = \alpha$. From this, with an easy transfinite induction, one can prove that for all points $z \in Y$ we have

$$\mathrm{ht}(z, Z) = \alpha + \mathrm{ht}(z, Y).$$

On the other hand, since each X_y is an open subspace of Z, it follows that for every point $x \in X_y$ we have

$$\mathrm{ht}(x, Z) = \mathrm{ht}(x, X_y) < \alpha.$$

Consequently, for each $\nu < \alpha$ we have

$$\mathrm{I}_\nu(Z) = \bigcup \{\mathrm{I}_\nu(X_y) : y \in \mathrm{I}_0(Y)\}.$$

This implies that $\mathrm{ht}(Z) = \alpha + \beta$, and if $\nu < \alpha$ then

$$|\mathrm{I}_\nu(Z)| = |\mathrm{I}_0(Y)| \cdot f(\nu) = g(0) \cdot f(\nu) = f(\nu).$$

Moreover, if $\eta < \beta$ then $|\mathrm{I}_{\alpha+\eta}(Z)| = |\mathrm{I}_\eta(Y)| = g(\eta)$, and consequently $\mathrm{SEQ}(Z) = f^\frown g$. $\blacksquare_{2.2}$

3. A general existence theorem on "long" cardinal sequences. In order to obtain the promised GCH characterization of the classes $\mathcal{C}(\alpha)$ we need one more result, in addition to the ones from [4] that were collected in the introduction. Unlike those, this result, Theorem 3.9 to be formulated and proved below, is new. In fact, it extends the main result, Theorem 2.19, of [2]. So for the reader who wants to follow its proof, acquaintance with [2] is recommended although not absolutely necessary. (The reader who is mainly interested in the GCH characterization, and is willing to accept Theorem 3.9 without proof, may simply skip this section.) In any case, we start by recalling a few definitions from [2].

Definition 3.1. For any family $\mathcal{A}$ of sets we define the topological space $X(\mathcal{A}) = \langle \mathcal{A}, \tau_{\mathcal{A}} \rangle$ as follows: $\tau_{\mathcal{A}}$ is the coarsest topology on $\mathcal{A}$ such that the sets $\mathrm{U}_{\mathcal{A}}(A) = \mathcal{A} \cap \mathcal{P}(A)$ are clopen for each $A \in \mathcal{A}$. In other words: $\{\mathrm{U}_{\mathcal{A}}(A), \mathcal{A} \setminus \mathrm{U}_{\mathcal{A}}(A) : A \in \mathcal{A}\}$ is a subbase for $\tau_{\mathcal{A}}$.

Clearly $X(\mathcal{A})$ is a 0-dimensional T_2-space.

A family $\mathcal{A}$ is called *well-founded* if the partial order $\langle \mathcal{A}, \subset \rangle$ is well-founded. $\mathcal{A}$ is said to be $\cap$-*closed* iff $A \cap B \in \mathcal{A} \cup \{\emptyset\}$ whenever $A, B \in \mathcal{A}$.

It is easy to see that if $\mathcal{A}$ is $\cap$-closed, then a neighbourhood base of $A \in \mathcal{A}$ in the space $X(\mathcal{A})$ is formed by the clopen sets

$$\mathrm{W}_{\mathcal{A}}(A; B_1, \ldots, B_n) = \mathrm{U}_{\mathcal{A}}(A) \setminus \bigcup_{i=1}^{n} \mathrm{U}_{\mathcal{A}}(B_i),$$

where $n \in \omega$ and $B_i \subsetneq A$ for $i = 1, \ldots, n$. (For $n = 0$ we have $\mathrm{W}_{\mathcal{A}}(A) = \mathrm{U}_{\mathcal{A}}(A)$.) We shall write $\mathrm{U}(A)$ instead of $\mathrm{U}_{\mathcal{A}}(A)$ if $\mathcal{A}$ is clear from the context, and similarly for the W's.

The following statement, proved in [2, Lemma 2.2], shows the relevance of the above concepts to the subject matter of this paper.

Fact 3.2. *Assume that $\mathcal{A}$ is both $\cap$-closed and well-founded. Then $X(\mathcal{A})$ is an LCS space.*

To simplify notation, if $X(\mathcal{A})$ is scattered then we write $\mathrm{I}_\alpha(\mathcal{A})$ instead of $\mathrm{I}_\alpha(X(\mathcal{A}))$, and $\mathrm{I}_{<\alpha}(\mathcal{A})$ instead of $\bigcup\{\mathrm{I}_\zeta(\mathcal{A}) : \zeta < \alpha\}$. In the same spirit, for $A \in \mathcal{A}$ we sometimes write $\mathrm{ht}(A, \mathcal{A})$ instead of $\mathrm{ht}(A, X(\mathcal{A}))$.

We shall say that $\mathcal{A}$ is an *ordinal family* if all members of $\mathcal{A}$ are sets of ordinal numbers, and $\mathcal{A}$ is both $\cap$-closed and well-founded. (As usual, we shall denote by On the class of all ordinals.) The following definition makes sense for any ordinal family $\mathcal{A}$ and will play an important role in our construction.

If $\mathcal{A}$ is an ordinal family and ξ is any ordinal then we let

$$\mathcal{A}\restriction\xi = \{A \cap \xi : A \in \mathcal{A}\}, \quad \mathcal{A}^* = \bigcup\{\mathcal{A}\restriction\xi : \xi \in \mathrm{On}\}.$$

So $\mathcal{A}^*$ is simply the family consisting of all initial segments of all members of $\mathcal{A}$. Clearly,

$$\mathcal{A}^* = \mathcal{A} \cup \{A \cap \xi : A \in \mathcal{A} \wedge \xi \in A\}.$$

It is easy to see that if $\mathcal{A}$ is an ordinal family then so is $\mathcal{A}^*$, hence both $X(\mathcal{A})$ and $X(\mathcal{A}^*)$ are LCS spaces. A key ingredient in our construction is, just as in [2], the clarification of the relationship between these two spaces. The following technical lemma will play a significant role in this. As indicated above, we shall write $\mathrm{U}(A)$ instead of $\mathrm{U}_{\mathcal{A}}(A)$, $\mathrm{U}_*(A)$ instead of $\mathrm{U}_{\mathcal{A}^*}(A)$, and similarly for the W's.

LEMMA 3.3. *Let $\mathcal{A}$ be an ordinal family. Then for any $A \in \mathcal{A}$ we have*

$$\mathrm{ht}(\mathrm{U}_*(A)) \le \sup\{\mathrm{ht}(\mathrm{U}_*(A')) : A' \in \mathrm{U}(A) \setminus \{A\}\} + \mathrm{ht}(\mathrm{tp}\, A + 1).$$

Proof. Set $\mathcal{V} = \bigcup\{\mathrm{U}_*(A') : A' \in \mathrm{U}(A) \setminus \{A\}\}$ and $\mathcal{B} = \mathrm{U}_*(A) \setminus \mathcal{V}$. Then $\mathcal{V}$ is open in $X(\mathcal{A}^*)$, and covered by the family $\{\mathrm{U}_*(A') : A' \in \mathrm{U}(A) \setminus \{A\}\}$ of open sets, hence

$$(\diamond) \qquad \mathrm{ht}(\mathcal{V}) = \sup\{\mathrm{ht}(\mathrm{U}_*(A')) : A' \in \mathrm{U}(A) \setminus \{A\}\}.$$

Let ζ be the smallest ordinal such that $A \cap \zeta \in \mathcal{B}$. Clearly then either $\zeta \in A$ or $\zeta = \bigcup A$; moreover it is easy to see that

$$\mathcal{B} = \{A \cap \xi : \xi \in A \setminus \zeta\} \cup \{A\},$$

hence $\mathcal{B}$ is well-ordered by inclusion in some order type $\beta \le \mathrm{tp}\, A + 1$. It then follows that $\mathcal{B}$ as a subspace of $X(\mathcal{A}^*)$ is homeomorphic to $X(\mathcal{B})$, that is, to the ordinal β (see Example 2.3 in [2]). Hence

$$(\bullet) \qquad \mathrm{ht}(\mathcal{B}) = \mathrm{ht}(\beta) \le \mathrm{ht}(\mathrm{tp}\, A + 1).$$

Finally, by Fact 1.1 we have

$$(\star) \qquad \mathrm{ht}(\mathrm{U}_*(A)) \le \mathrm{ht}(\mathcal{V}) + \mathrm{ht}(\mathcal{B}).$$

Formulas $(\diamond)$, $(\bullet)$, and $(\star)$ together yield what we have to prove. $\blacksquare_{3.3}$

What we shall really need in our construction is the following corollary of Lemma 3.3.

LEMMA 3.4. *Let $\mathcal{A}$ be an ordinal family such that*, *for a fixed indecomposable* $\alpha \in \mathrm{On}$, *we have* $|\mathcal{A}| < \mathrm{cf}(\alpha)$ *and* $\mathrm{ht}(\mathrm{tp}\, A) < \alpha$ *for all* $A \in \mathcal{A}$. *Then* $\mathrm{ht}(X(\mathcal{A}^*)) < \alpha$.

Proof. Clearly, $\{\mathrm{U}_{\mathcal{A}^*}(A) : A \in \mathcal{A}\}$ forms a cover of $X(\mathcal{A}^*)$ by open sets, hence, by $|\mathcal{A}| < \mathrm{cf}(\alpha)$, it suffices to prove that $\mathrm{ht}(\mathrm{U}_{\mathcal{A}^*}(A)) < \alpha$ for all $A \in \mathcal{A}$. This, in turn, is easily proved by well-founded induction on $A \in \mathcal{A}$, using our assumptions and the fact that

$$\mathrm{ht}(\mathrm{U}_{\mathcal{A}^*}(A)) \le \sup\{\mathrm{ht}(\mathrm{U}_*(A')) : A' \in \mathrm{U}(A) \setminus \{A\}\} + \mathrm{ht}(\mathrm{tp}\, A + 1)$$

by Lemma 3.3. $\blacksquare_{3.4}$

Now we shall prove a result showing that, for certain ordinal families $\mathcal{H}$, the space $X(\mathcal{H})$ is a very special subspace of $X(\mathcal{H}^*)$. This result naturally corresponds to [2, Lemma 2.14].

If ϱ is an ordinal and $L \subset \mathrm{On}$ then we write

$$(L)^\varrho = \{K \subset L : \mathrm{tp}\, K = \varrho\}, \qquad (L)^{<\varrho} = \bigcup \{(L)^\alpha : \alpha < \varrho\}.$$

Lemma 3.5. *Let ϱ be an indecomposable ordinal such that $\mathrm{ht}(\varrho) = \alpha$ is also indecomposable, $\mathrm{cf}(\alpha) = \mathrm{cf}(\varrho)$, and $\mathrm{ht}(\xi) < \alpha$ for all $\xi < \varrho$. Let $\mathcal{H} \subset (\varrho)^\varrho$ be an ordinal family such that $\left|\mathcal{H}\restriction\xi\right| < \mathrm{cf}(\alpha)$ for all $\xi < \varrho$. Then $X(\mathcal{H})$ forms a "tail" of $X(\mathcal{H}^*)$ in the following sense:*

(a) *$X(\mathcal{H})$ is a closed subspace of $X(\mathcal{H}^*)$,*
(b) *$\mathrm{I}_\beta(\mathcal{H}) = \mathrm{I}_{\alpha+\beta}(\mathcal{H}^*)$ for all $\beta < \mathrm{ht}(X(\mathcal{H}))$,*
(c) *$\mathrm{I}_{<\alpha}(\mathcal{H}^*) = \mathcal{H}^* \setminus \mathcal{H}$.*

Proof. As before, we shall write $\mathrm{U}(A)$ for $\mathrm{U}_{\mathcal{H}}(A)$ and $\mathrm{U}_*(A)$ for $\mathrm{U}_{\mathcal{H}^*}(A)$. Since $\mathcal{H} = \{A \in \mathcal{H}^* : \mathrm{tp}\, A = \varrho\}$, we clearly have

$$\mathrm{U}_*(A) \cap \mathcal{H} = \begin{cases} \mathrm{U}(A) & \text{if } A \in \mathcal{H}, \\ \emptyset & \text{if } A \in \mathcal{H}^* \setminus \mathcal{H}, \end{cases}$$

hence (a) holds.

Next we prove "half" of (b) in the case $\beta = 0$, namely

(b′) $\mathrm{I}_0(\mathcal{H}) \subset \mathrm{I}_\alpha(\mathcal{H}^*)$.

Indeed, if $A \in \mathrm{I}_0(\mathcal{H})$ then there are $B_1, \dots, B_n \in \mathrm{U}(A) \setminus \{A\}$ such that

$$(\dagger) \qquad \{A\} = \mathrm{W}(A; B_1, \dots, B_n) = \mathrm{U}(A) \setminus \bigcup_{i=1}^{n} \mathrm{U}(B_i).$$

Since here each $B_i \subsetneq A$ and the order-type of A is a limit ordinal, we can fix $\eta \in A$ such that $A \cap \eta \not\subset B_i$ for every $i = 1, \dots, n$. Now write

$$W = W_*(A; A \cap \eta, B_1, \dots, B_n) = \mathrm{U}_*(A) \setminus \mathrm{U}_*(A \cap \eta) \setminus \bigcup_{i=1}^{n} \mathrm{U}_*(B_i).$$

We claim that the clopen subspace W of $X(\mathcal{H}^*)$ is homeomorphic to the ordinal $\varrho + 1$. Indeed, if $A \not\subset C$ for $C \in \mathcal{H}$ then $A \cap C \subset B_i$ for some $i \le n$ by $(\dagger)$, so clearly we have $C \cap \xi \notin W$ for any $\xi \le \varrho$. It then clearly follows that

$$W = \{A \cap \zeta : \zeta \in A \setminus \eta\} \cup \{A\}.$$

But then $W = X(W)$ is homeomorphic to $\varrho + 1$ because $\mathrm{tp}(A \setminus \eta) = \varrho$ as ϱ is indecomposable. Consequently, we have

$$\mathrm{ht}(A, \mathcal{H}^*) = \mathrm{ht}(A, W) = \mathrm{ht}(\varrho, \varrho + 1) = \mathrm{ht}(\varrho) = \alpha,$$

proving (b′).

As the next step in the proof, we shall prove "half" of (c), namely

(c′) $\mathrm{I}_{<\alpha}(\mathcal{H}^*) \supset \mathcal{H}^* \setminus \mathcal{H} = \bigcup\{\mathcal{H}\restriction\xi : \xi < \varrho\}$.

Indeed, for any $\xi < \varrho$ we have, by assumption, $|\mathcal{H}\restriction\xi| < \mathrm{cf}(\alpha)$, and moreover

$$\mathrm{ht}(\mathrm{tp}(H \cap \xi)) \leq \mathrm{ht}(\xi) < \alpha$$

for any member $H \cap \xi$ of $\mathcal{H}\restriction\xi$. Therefore we can apply Lemma 3.4 to $\mathcal{H}\restriction\xi$ to conclude that

$$\mathrm{ht}[(\mathcal{H}\restriction\xi)^*] = \mathrm{ht}(\mathcal{H}^*\restriction\xi) < \alpha.$$

But $\mathcal{H}^*\restriction\xi$ is clearly an open subset of $X(\mathcal{H})^*$, hence any $K \in \mathcal{H}^*\restriction\xi$ satisfies $\mathrm{ht}(K, \mathcal{H}^*) = \mathrm{ht}(K, \mathcal{H}^*\restriction\xi) < \alpha$, which proves (c′).

Now, (b′) and (c′) together clearly imply

$$\mathrm{I}_{<\alpha}(\mathcal{H}^*) = \mathcal{H}^* \setminus \mathcal{H},$$

and so (c) holds. On the other hand, (b′) and (c) immediately imply the other "half" of (b) for $\beta = 0$, so we have $\mathrm{I}_0(\mathcal{H}) = \mathrm{I}_\alpha(\mathcal{H}^*)$. From this then, with a straightforward induction on $\beta = \mathrm{ht}(H, \mathcal{H})$, the full clause (b) is proved easily. $\blacksquare_{3.5}$

Remark. It is obvious that Lemma 3.5 remains valid in the more general formulation where, instead of $\mathcal{H} \subset (\varrho)^\varrho$, we have $\mathcal{H} \subset (L)^\varrho$ for some set of ordinals $L \subset \mathrm{On}$ with $\mathrm{tp}\, L = \varrho$. (Of course, in this case we also have to assume $|\mathcal{H}\restriction\xi| < \mathrm{cf}(\alpha)$ for all $\xi \in L$ instead of $\xi < \varrho$.) In fact, Lemma 3.5 will be used later in this more general form; we only decided to prove the restricted version to have a clearer presentation.

Lemma 3.5 yields a method to build LCS spaces of the form $X(\mathcal{A})$ such that the sets appearing on levels of height less than α are "small" (of order type less than ϱ), while the sets on all higher levels are "large" (of order type equal to ϱ). The key step of our construction will be an amalgamation procedure aimed at "gluing together" an appropriate family of spaces obtained in this manner. The following concept serves this purpose.

Definition 3.6. Two families of sets $\mathcal{A}_0$ and $\mathcal{A}_1$ are said to be *coherent* if $A_0 \cap A_1 \in (\mathcal{A}_0 \cap \mathcal{A}_1) \cup \{\emptyset\}$ whenever $A_i \in \mathcal{A}_i$ for $i < 2$. A system of families $\{\mathcal{A}_i : i \in I\}$ is *coherent* iff $\mathcal{A}_i$ and $\mathcal{A}_j$ are coherent for each pair $\{i, j\} \in [I]^2$.

If $\{\mathcal{A}_i : i \in I\}$ is a coherent system of well-founded and $\cap$-closed families then we can "amalgamate" the spaces $\{X(\mathcal{A}_i) : i \in I\}$ as follows: According to [2, Lemma 2.7], then $\mathcal{A} = \bigcup\{\mathcal{A}_i : i \in I\}$ is also well-founded and $\cap$-closed, hence $X(\mathcal{A})$ is also an LCS space; moreover $\{X(\mathcal{A}_i) : i \in I\}$ forms an open cover of $X(\mathcal{A})$.

Next we introduce a method that transforms an ordinal family $\mathcal{A}$ into another, isomorphic family $\widehat{\mathcal{A}}$. We do this because, for certain systems of ordinal families $\{\mathcal{A}_i : i \in I\}$, the new system $\{\widehat{\mathcal{A}}_i : i \in I\}$ turns out to be coherent. Since $X(\mathcal{A}_i)$ and $X(\widehat{\mathcal{A}}_i)$ are clearly homeomorphic, in this way the spaces $\{X(\mathcal{A}_i) : i \in I\}$ may be amalgamated in two steps. The definition of the operation $\mathcal{A} \mapsto \widehat{\mathcal{A}}$ given below is a slight generalization of the one given in Definition 2.8 from [2].

Definition 3.7. Given a set of ordinals $L \subset \mathrm{On}$ of limit order type and a family $\mathcal{A}$ with $\{L\}^* \subset \mathcal{A} \subset \mathcal{P}(L)$, we first define the map $\mathrm{k}_\mathcal{A}$ on L by the formula $\mathrm{k}_\mathcal{A}(\eta) = \mathcal{A}\restriction\eta + 1$ for $\eta \in L$; then we define the map $\chi_\mathcal{A}$ on $\mathcal{A}$ by putting $\chi_\mathcal{A}(A) = \mathrm{k}''_\mathcal{A}(A)$, i.e. $\chi_\mathcal{A}(A)$ is the $\mathrm{k}_\mathcal{A}$-image of $A \in \mathcal{A}$. Finally, we define the family $\widehat{\mathcal{A}}$ by

$$\widehat{\mathcal{A}} = \{\chi_\mathcal{A}(A) : A \in \mathcal{A}\}.$$

Let us remark that $L \cap (\eta + 1) \in \{L\}^* \subset \mathcal{A}$ for each $\eta \in L$, hence we have

$$\mathrm{k}_\mathcal{A}(\eta) = \mathcal{A}\restriction\eta + 1 = \mathrm{U}_\mathcal{A}(L \cap (\eta + 1)).$$

By the same token, for any $\eta \in L$ we also have

$$\max\Big[\bigcup \mathrm{k}_\mathcal{A}(\eta)\Big] = \max[L \cap (\eta + 1)] = \eta,$$

and consequently $\mathrm{k}_\mathcal{A}$ is a bijection between the sets L and $\chi_\mathcal{A}(L)$. Therefore, $\chi_\mathcal{A}$ is indeed an isomorphism between the partial orders $\langle \mathcal{A}, \subset\rangle$ and $\langle \widehat{\mathcal{A}}, \subset\rangle$, and so the spaces $X(\mathcal{A})$ and $X(\widehat{\mathcal{A}})$ are homeomorphic. Note, however, that the sets $\chi_\mathcal{A}(A)$ from $\widehat{\mathcal{A}}$ are not sets of ordinals any more. This is the price we have to pay for transforming the ordinal family $\mathcal{A}$ into the coherent family $\widehat{\mathcal{A}}$.

Next, if $\mathcal{A}_0 \neq \mathcal{A}_1$ are two families of sets of ordinals then we let

$$\Delta(\mathcal{A}_0, \mathcal{A}_1) = \min\{\delta : \mathcal{A}_0\restriction\delta \neq \mathcal{A}_1\restriction\delta\}.$$

Since $\mathcal{A}_i^*\restriction\xi = \bigcup\{\mathcal{A}_i\restriction\eta : \eta \le \xi\}$, we clearly have

$$\Delta(\mathcal{A}_0, \mathcal{A}_1) = \Delta(\mathcal{A}_0^*, \mathcal{A}_1^*),$$

provided that $\mathcal{A}_0^* \neq \mathcal{A}_1^*$ as well.

The following lemma is a strengthening of [2, Lemma 2.9]. It gives us a condition under which the transforms $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ of two families $\mathcal{A}$ and $\mathcal{B}$, respectively, turn out to be coherent.

Lemma 3.8. *Assume that L and M are two sets of ordinals of limit order type such that $L \cap M$ is a proper initial segment of both L and M, and moreover $\{L\}^* \subset \mathcal{A} \subset \mathcal{P}(L)$ and $\{M\}^* \subset \mathcal{B} \subset \mathcal{P}(M)$ are $\cap$-closed families. If $\Delta(\mathcal{A}, \mathcal{B}) = \delta + 1$ is a successor ordinal then the families $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ are coherent.*

Proof. By symmetry, it is enough to show that if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $\chi_\mathcal{A}(A) \cap \chi_\mathcal{B}(B) \in \widehat{\mathcal{A}}$. Assume that $\mathrm{k}_\mathcal{A}(\alpha) = \mathrm{k}_\mathcal{B}(\beta)$ for some $\alpha \in A$ and $\beta \in B$. Since $\max[\bigcup \mathrm{k}_\mathcal{A}(\alpha)] = \alpha$ and $\max[\bigcup \mathrm{k}_\mathcal{B}(\beta)] = \beta$, we then have

$\alpha = \beta \in L \cap M$ and so $L \cap \alpha = M \cap \beta$ because $L \cap M$ is an initial segment of both L and M.

We also have $B \cap \delta \in \mathcal{B}\restriction\delta = \mathcal{A}\restriction\delta \subset \mathcal{A}$ by the choice of δ, and consequently $A \cap B \cap \delta \in \mathcal{A}$ because $\mathcal{A}$ is $\cap$-closed and $\emptyset \in \mathcal{A}$. So we have

$$\chi_{\mathcal{A}}(A) \cap \chi_{\mathcal{B}}(B) = \big\{\mathcal{A}\restriction\eta + 1 : \eta \in A \cap B \wedge \mathcal{A}\restriction\eta + 1 = \mathcal{B}\restriction\eta + 1\big\}$$
$$= \big\{\mathcal{A}\restriction\eta + 1 : \eta \in A \cap B \ \wedge\ \eta < \delta\big\} = \chi_{\mathcal{A}}(A \cap B \cap \delta) \in \widehat{\mathcal{A}},$$

proving our claim. $\blacksquare_{3.8}$

We can now formulate and prove the result that was promised at the beginning of this section.

THEOREM 3.9. *Let λ be a cardinal with $\mu = \mathrm{cf}(\lambda) > \omega$ and $\lambda = \lambda^{<\mu}$. Then for any cardinal κ with $\lambda < \kappa \le \lambda^\mu$ and for every ordinal $\alpha < \mu^+$ with* $\mathrm{cf}(\alpha) = \mu$ *we have $\langle\lambda\rangle_\alpha {}^\frown \langle\kappa\rangle_{\mu^+} \in \mathcal{C}(\mu^+)$.*

Proof. First we show that it suffices to prove the theorem in the case when α is indecomposable. Indeed, let $\alpha < \mu^+$ be arbitrary with $\mathrm{cf}(\alpha) = \mu$. We may then write $\alpha = \alpha' + \alpha''$ where α'' is indecomposable and, of course, $\mathrm{cf}(\alpha) = \mathrm{cf}(\alpha'') = \mu$. We may apply Lemma 2.2 to the sequences $\langle\lambda\rangle_{\alpha'} \in \mathcal{C}(\alpha')$ and $\langle\lambda\rangle_{\alpha''} {}^\frown \langle\kappa\rangle_{\mu^+} \in \mathcal{C}(\mu^+)$ to conclude that

$$\langle\lambda\rangle_\alpha {}^\frown \langle\kappa\rangle_{\mu^+} = \langle\lambda\rangle_{\alpha'} {}^\frown \langle\lambda\rangle_{\alpha''} {}^\frown \langle\kappa\rangle_{\mu^+} \in \mathcal{C}(\mu^+).$$

So assume now that α is indecomposable and let $\varrho = \omega^\alpha$. Clearly then ϱ is also indecomposable, $\mathrm{cf}(\varrho) = \mathrm{cf}(\alpha) = \mu$, moreover $\mathrm{ht}(\varrho) = \alpha$ and $\mathrm{ht}(\xi) < \alpha$ for all $\xi < \varrho$. Let $\langle\nu_\zeta : \zeta < \mu\rangle$ be a strictly increasing sequence of limit ordinals cofinal in ϱ. As ϱ is indecomposable and $\mathrm{cf}(\varrho) = \mu$, for every set $a \in [\mu]^\mu$ we have $\sum\{\nu_\zeta : \zeta \in a\} = \varrho$.

Next we fix a *disjoint* family $\{K_t : t \in {}^{<\mu}\lambda\}$ of intervals of ordinals such that for any $t, s \in {}^{<\mu}\lambda$ we have

(a) $\mathrm{tp}\, K_t = \nu_{\mathrm{dom}\, t}$,
(b) if s is a proper initial segment of t then $\sup K_s < \min K_t$.

We also choose a family $\mathcal{G} \subset {}^\mu\lambda$ of functions with $|\mathcal{G}| = \kappa$ and for every $g \in \mathcal{G}$ put

$$L_g = \bigcup\{K_{g\restriction\xi} : \xi \in \mu\}.$$

Then we have

(I) $\mathrm{tp}\, L_g = \varrho$ for each $g \in \mathcal{G}$,
(II) $L_g \cap L_h$ is a proper initial segment of both L_g and L_h whenever $\{g, h\} \in [\mathcal{G}]^2$.

In the proof of the main result of [2], namely Theorem 2.19, we constructed, for any fixed cardinal μ and for all ordinals $\gamma < \mu^+$, ordinal families $\mathcal{F}_\gamma$ such that the following five conditions were satisfied:

(i) $\mu \in \mathcal{F}_\gamma \subset [\mu]^\mu$,
(ii) $\mathcal{F}_\gamma$ is well-founded and $\cap$-closed,
(iii) $\mathrm{ht}(X(\mathcal{F}_\gamma)) = \gamma + 1$,
(iv) $\Delta(\mathcal{F}_\gamma, \mathcal{F}_\delta) = \Delta(\mathcal{F}^*_\gamma, \mathcal{F}^*_\delta)$ is a successor ordinal if $\gamma \neq \delta$,
(v) $|\mathcal{F}_\gamma \restriction \xi| \le |\xi| + \omega < \mu$ for each $\xi < \mu$.

We shall also make use of these families $\{\mathcal{F}_\gamma : \gamma < \mu^+\}$, more precisely some transformed versions of them, in the present proof. To this end, we first fix a function $\Gamma : \mathcal{G} \to \mu^+$ such that $|\Gamma^{-1}\{\gamma\}| = \kappa$ for each $\gamma \in \mu^+$. This is possible because $|\mathcal{G}| = \kappa \ge \mu^+$.

Fix $g \in \mathcal{G}$ and for all $F \subset \mu$ put

$$\varphi_g(F) = \bigcup\{K_{g\restriction\xi} : \xi \in F\};$$

then we define

$$\mathcal{H}_g = \{\varphi_g(F) : F \in \mathcal{F}_{\Gamma(g)}\}.$$

Note that, by (i), for each $g \in \mathcal{G}$ we have $\mathcal{H}_g \subset (L_g)^\varrho$.

It is obvious that the map φ_g induces an inclusion-preserving isomorphism between the families $\mathcal{F}_{\Gamma(g)}$ and $\mathcal{H}_g$ (i.e. between the partial orders $\langle \mathcal{F}_{\Gamma(g)}, \subset\rangle$ and $\langle \mathcal{H}_g, \subset\rangle$); consequently, the spaces $X(\mathcal{F}_{\Gamma(g)})$ and $X(\mathcal{H}_g)$ are homeomorphic. It is also easy to check that, for each $g \in \mathcal{G}$, the ordinal family $\mathcal{H}_g$ satisfies all the requirements of Lemma 3.5, or actually of its more general version formulated in the Remark following its proof. In view of this and property (iii), we may sum up the relevant properties of the spaces $X(\mathcal{H}^*_g)$ as follows.

Fact 3.9.0. *$X(\mathcal{H}_g)$ is a closed subspace of $X(\mathcal{H}^*_g)$ and we have*

(a) $\mathrm{ht}(X(\mathcal{H}^*_g)) = \alpha + \Gamma(g) + 1$,
(b) $\mathrm{I}_{\alpha+\beta}(\mathcal{H}^*_g) = \mathrm{I}_\beta(\mathcal{H}_g)$ *for all* $\beta < \mathrm{ht}(X(\mathcal{H}_g)) = \Gamma(g) + 1$,
(c) $\mathrm{I}_{<\alpha}(\mathcal{H}^*_g) = \mathcal{H}^*_g \setminus \mathcal{H}_g$.

Our aim is to amalgamate the spaces $\{X(\mathcal{H}^*_g) : g \in \mathcal{G}\}$, but to do that we shall have to transform the families $\mathcal{H}^*_g$ by means of the"hat" operation described in Lemma 3.8.

Since $\mu \in \mathcal{F}_{\Gamma(g)}$, we have $\{L_g\}^* \subset \mathcal{H}^*_g \subset \mathcal{P}(L_g)$ for each $g \in \mathcal{G}$. Also, if $\{g, h\} \in [\mathcal{G}]^2$ then $L_g \cap L_h$ is a proper initial segment of both L_g and L_h by (II). Consequently, by Lemma 3.8, the system $\{\widehat{\mathcal{H}^*_g} : g \in \mathcal{G}\}$ will be proven to be coherent once the following claim is established.

Claim 3.9.1. *$\Delta(\mathcal{H}^*_g, \mathcal{H}^*_h)$ is a successor ordinal for each $\{g, h\} \in [\mathcal{G}]^2$.*

Proof. Let $\xi < \mu$ be minimal such that $g\restriction\xi \neq h\restriction\xi$. Then we clearly have

$$\eta = \min(L_g \bigtriangleup L_h) = \min(K_{g\restriction\xi} \cup K_{h\restriction\xi}).$$

(As usual, we are using $\bigtriangleup$ to denote symmetric difference.) Now, if $\mathcal{F}_{\Gamma(g)}\restriction\xi = \mathcal{F}_{\Gamma(h)}\restriction\xi$ (this happens for instance if $\Gamma(g) = \Gamma(h)$) then $\mathcal{H}_g\restriction\eta = \mathcal{H}_h\restriction\eta$, and so we also have $\mathcal{H}_g^*\restriction\eta = \mathcal{H}_h^*\restriction\eta$. On the other hand, $K_{g\restriction\xi} \cap K_{h\restriction\xi} = \emptyset$ implies that

$$(L_g \cap L_h) \cup \{\eta\} \in (\mathcal{H}_g^*\restriction\eta + 1) \triangle (\mathcal{H}_h^*\restriction\eta + 1),$$

hence $\varDelta(\mathcal{H}_g^*, \mathcal{H}_h^*) = \eta + 1$, and we are done.

Thus we can assume that $\mathcal{F}_{\varGamma(g)}\restriction\xi \neq \mathcal{F}_{\varGamma(h)}\restriction\xi$. In this case, by property (iv), we know that $\varDelta(\mathcal{F}_{\varGamma(g)}, \mathcal{F}_{\varGamma(h)}) \leq \xi$ is a successor ordinal, say $\delta+1$. But then there is a set $A \in \mathcal{F}_{\varGamma(g)}\restriction\delta = \mathcal{F}_{\varGamma(h)}\restriction\delta$ such that

$$A \cup \{\delta\} \in (\mathcal{F}_{\varGamma(g)}\restriction\delta + 1) \triangle (\mathcal{F}_{\varGamma(h)}\restriction\delta + 1).$$

Write $\sigma = \min K_{g\restriction\delta}$ $(= \min K_{h\restriction\delta})$ and put

$$D = \bigcup\{K_{g\restriction\zeta} : \zeta \in A\} \cup \{\sigma\};$$

then clearly

$$D \in (\mathcal{H}_g^*\restriction\sigma + 1) \triangle (\mathcal{H}_h^*\restriction\sigma + 1).$$

On the other hand, $\mathcal{F}_{\varGamma(g)}\restriction\delta = \mathcal{F}_{\varGamma(h)}\restriction\delta$ and $g\restriction\delta = h\restriction\delta$ together imply $\mathcal{H}_g\restriction\sigma = \mathcal{H}_h\restriction\sigma$ and so $\mathcal{H}_g^*\restriction\sigma = \mathcal{H}_h^*\restriction\sigma$. Thus we have $\varDelta(\mathcal{H}_g^*, \mathcal{H}_h^*) = \sigma + 1$, completing the proof. $\blacksquare_{3.9.1}$

Consequently, we may now apply Lemma 3.8 to conclude that the family $\{\widehat{\mathcal{H}_g^*} : g \in \mathcal{G}\}$ is coherent. Therefore, by [2, Lemma 2.7], the family $\mathcal{H} = \bigcup\{\widehat{\mathcal{H}_g^*} : g \in \mathcal{G}\}$ is well-founded and $\cap$-closed, and so the amalgamation $X(\mathcal{H})$ is an LCS space that is covered by its open subspaces $\{X(\widehat{\mathcal{H}_g^*}) : g \in \mathcal{G}\}$. As a consequence, we have

$$(\ddagger) \qquad \mathrm{I}_\beta(\mathcal{H}) = \bigcup\{\mathrm{I}_\beta(\widehat{\mathcal{H}_g^*}) : g \in \mathcal{G}\}$$

for any ordinal β. Since $X(\widehat{\mathcal{H}_g^*})$ is homeomorphic to $X(\mathcal{H}_g^*)$ we easily deduce from Fact 3.9.0 that $\mathrm{ht}(X(\mathcal{H})) = \mu^+$.

Our aim now is to determine the sizes of the levels $\mathrm{I}_\beta(\mathcal{H})$ of the LCS space $X(\mathcal{H})$. To simplify notation, for each $g \in \mathcal{G}$ we shall denote the maps $\mathrm{k}_{\mathcal{H}_g^*}$ and $\chi_{\mathcal{H}_g^*}$, both defined in 3.7, by k_g and χ_g, respectively.

CLAIM 3.9.2. *$|\mathrm{I}_\beta(\mathcal{H})| = \kappa$ whenever $\alpha \leq \beta < \mu^+$.*

Proof. Let γ be the ordinal such that $\alpha + \gamma = \beta$. Then for every $g \in \mathcal{G}$ with $\varGamma(g) \geq \gamma$ we can apply Lemma 3.5 to the family $\mathcal{H}_g$ to conclude that $\mathrm{I}_\beta(\mathcal{H}_g^*) = \mathrm{I}_\gamma(\mathcal{H}_g) \neq \emptyset$.

On the other hand, for any $G \in \mathcal{H}_g$ we have $G \in (L_g)^\varrho$ and so G is cofinal in L_g, while $L_g \cap L_h$ is bounded in both L_g and in L_h whenever $\{g, h\} \in [\mathcal{G}]^2$. So if $G \in \mathcal{H}_g$ and $H \in \mathcal{H}_h$ are arbitrary then

$$\chi_g(G) \cap \chi_h(H) \subset \chi_g(L_g \cap L_h) = \mathrm{k}_g''(L_g \cap L_h)$$

implies $\chi_g(G) \neq \chi_h(H)$. In other words, the families

$$\{\chi_g(G) : G \in \mathcal{H}_g\} = \mathrm{I}_{\geq\alpha}(\widehat{\mathcal{H}_g^*})$$

are pairwise disjoint as g ranges over $\mathcal{G}$. Since we have $\mathrm{I}_\beta(\mathcal{H}) \supset \mathrm{I}_\beta(\widehat{\mathcal{H}_g^*})$ for $g \in \mathcal{G}$ by (‡), we conclude that

$$|\mathrm{I}_\beta(\mathcal{H})| \geq |\{g \in \mathcal{G} : \Gamma(g) \geq \gamma\}| = \kappa.$$

But by $|\mathcal{H}| = \kappa$ we must have equality here. $\blacksquare_{3.9.2}$

Next we show, in a single step, that $|\mathrm{I}_\beta(\mathcal{H})| \leq \lambda$ for each $\beta < \alpha$.

Claim 3.9.3. $|\mathrm{I}_{<\alpha}(\mathcal{H})| \leq \lambda$.

Proof. To each $S \in \mathrm{I}_{<\alpha}(\mathcal{H})$ we may assign a quadruple $F(S)$ as follows. First pick $g \in \mathcal{G}$ with $S \in \mathrm{I}_{<\alpha}(\widehat{\mathcal{H}_g^*})$. Then we have $S = \chi_g(T)$ for some $T \in \mathrm{I}_{<\alpha}(\mathcal{H}_g^*)$. By Lemma 3.5(c) this set T must be bounded in L_g, so we can fix $\xi < \mu$ such that $T \subset \bigcup\{K_{g\restriction\zeta} : \zeta < \xi\}$. Then put

$$F(S) = \langle \xi, g\restriction\xi, \mathcal{F}_{\Gamma(g)}\restriction\xi, T\rangle.$$

We shall now show that F is injective, i.e. S can be recovered from the quadruple $F(S)$.

Indeed, both the sequence $\langle K_{g\restriction\zeta} : \zeta < \xi\rangle$ and the ordinal $\eta = \min K_{g\restriction\xi}$ are obviously determined by the map $g\restriction\xi$. Next, the family $\mathcal{H}_g^*\restriction\eta$ is determined by the sequence $\langle K_{g\restriction\zeta} : \zeta < \xi\rangle$ and the family $\mathcal{F}_{\Gamma(g)}\restriction\xi$ because we clearly have

$$\mathcal{H}_g^*\restriction\eta = \Big\{\bigcup\{K_{g\restriction\zeta} : \zeta \in A\} : A \in \mathcal{F}_{\Gamma(g)}\restriction\xi\Big\}^*.$$

It is easy to see that the family $\mathcal{H}_g^*\restriction\eta$ determines the map $\mathrm{k}_g\restriction\eta$ and consequently $\chi_g\restriction(\mathcal{H}_g^*\restriction\eta)$ as well. But $S = \chi_g(T)$ where $T \in \mathcal{H}_g^*\restriction\eta$, and so we are done.

Therefore, to conclude, it suffices to prove that there are at most λ quadruples of the form $F(S)$. To see this, first note that we have μ choices for ξ. Next, since $\lambda^{<\mu} = \lambda$ we have λ choices for $g\restriction\xi$. By (v) we have

$$\mathcal{F}_{\Gamma(g)}\restriction\xi \in [\mathcal{P}(\xi)]^{\leq|\xi+\omega|},$$

and consequently there are at most $2^{|\xi+\omega|} \leq \lambda^{|\xi+\omega|} = \lambda$ choices for $\mathcal{F}_{\Gamma(g)}\restriction\xi$. Finally, it is easy to see that

$$|\mathcal{H}_g^*\restriction\eta| \leq \Big|\bigcup\{K_{g\restriction\zeta} : \zeta < \xi\}\Big| \cdot |\mathcal{F}_{\Gamma(g)}\restriction\xi| \leq \mu,$$

hence, for fixed ξ, $g\restriction\xi$, and $\mathcal{F}_\Gamma(g)\restriction\xi$, there are at most μ choices for T. All this together clearly gives us

$$|\{F(S) : S \in \mathrm{I}_{<\alpha}(\mathcal{H})\}| = |\mathrm{I}_{<\alpha}(\mathcal{H})\}| \leq \lambda. \ \blacksquare_{3.9.3}$$

We are now almost finished with the proof of Theorem 3.9: the LCS space $X = X(\mathcal{H})$ satisfies $|\mathrm{I}_\beta(X)| = \kappa$ for all $\alpha \leq \beta < \mu^+$ and $|\mathrm{I}_\beta(X)| \leq \lambda$ for all $\beta < \alpha$. Thus if Y is the disjoint topological sum of λ copies of X then Y is an LCS space with

$$\mathrm{SEQ}(Y) = \langle\lambda\rangle_\alpha{}^\frown\langle\kappa\rangle_{\mu^+}. \ \blacksquare_{3.9}$$

4. The GCH characterization. From now on we assume GCH. Our aim is to characterize the classes $\mathcal{C}_\lambda(\alpha)$ with $\alpha < \omega_2$. It follows immediately from 1.2 and GCH that

$$\mathcal{C}_\lambda(\alpha) \subset {}^{\alpha}\{\lambda, \lambda^+\}.$$

(For an ordinal α and a set B, as usual, we let ${}^{\alpha}B$ denote the set of all sequences of length α taking values in B.) Now, for any $s \in {}^{\alpha}\{\lambda, \lambda^+\}$ we write

$$\mathrm{A}_\lambda(s) = \{\beta \in \alpha : s(\beta) = \lambda\} = s^{-1}\{\lambda\}.$$

If α is any ordinal, a subset $L \subset \alpha$ is called *κ-closed in α*, where κ is an infinite cardinal, if $\sup\langle \alpha_i : i < \kappa\rangle \in L \cup \{\alpha\}$ for each increasing sequence $\langle \alpha_i : i < \kappa\rangle \in {}^{\kappa}L$. Similarly, L is said to be *successor closed in α* if $\beta + 1 \in L \cup \{\alpha\}$ for all $\beta \in L$. We are now ready to present the promised GCH characterization of the classes $\mathcal{C}_\lambda(\alpha)$ and consequently, in view of 2.1, the characterization of $\mathcal{C}(\alpha)$ for all $\alpha < \omega_2$.

THEOREM 4.1. *Assume GCH and fix $\alpha < \omega_2$.*

(i) $\mathcal{C}_\omega(\alpha) = \{s \in {}^{\alpha}\{\omega, \omega_1\} : s(0) = \omega\}$.

(ii) *If $\lambda > \mathrm{cf}(\lambda) = \omega$, then*

$$\mathcal{C}_\lambda(\alpha) = \{s \in {}^{\alpha}\{\lambda, \lambda^+\} : s(0) = \lambda \text{ and } \mathrm{A}_\lambda(s) \text{ is } \omega_1\text{-closed in } \alpha\}.$$

(iii) *If $\mathrm{cf}(\lambda) = \omega_1$, then*

$$\mathcal{C}_\lambda(\alpha) = \{s \in {}^{\alpha}\{\lambda, \lambda^+\} : s(0) = \lambda \text{ and } \mathrm{A}_\lambda(s) \text{ is both } \omega\text{-closed and successor closed in } \alpha\}.$$

(iv) *If $\mathrm{cf}(\lambda) > \omega_1$, then*

$$\mathcal{C}_\lambda(\alpha) = \{\langle\lambda\rangle_\alpha\}.$$

Proof. The first case, $\lambda = \omega$, follows immediately from [4, Theorem 9], which actually implies

$${}^{\alpha}\{\omega, \omega_1\} \subset \mathcal{C}(\alpha)$$

in ZFC.

Now consider the second case: $\lambda > \mathrm{cf}(\lambda) = \omega$. Let $\langle \lambda_n : n < \omega\rangle$ be an increasing sequence of cardinals cofinal in λ.

Necessity. Assume $s \in \mathcal{C}_\lambda(\alpha)$ and fix an LCS space X with cardinal sequence s. Suppose $\beta < \alpha$, $\mathrm{cf}(\beta) = \omega_1$, and $\mathrm{A}_\lambda(s) \cap \beta$ is cofinal in β. We have to show that $\beta \in \mathrm{A}_\lambda(s)$. Let $\{\beta_\eta : \eta < \omega_1\} \subset \mathrm{A}_\lambda(s)$ be an increasing sequence cofinal in β. For each $x \in \mathrm{I}_\beta(X)$ let U_x be a compact open neighbourhood of x such that

$$U_x \setminus \{x\} \subset \bigcup\{\mathrm{I}_\xi(X) : \xi < \beta\}.$$

For each $\eta < \omega_1$ pick a point $p(x, \eta) \in U_x \cap \mathrm{I}_{\beta_\eta}(X)$. Then the sequence $\langle p(x, \eta) : \eta < \omega_1\rangle$ converges to x. Now since $|\mathrm{I}_{\beta_\eta}(X)| = s(\beta_\eta) = \lambda$ for all

$\eta < \omega_1$, the set

$$S = \bigcup\{\mathrm{I}_{\beta_\eta}(X) : \eta < \omega_1\}$$

has size λ. Let $S = \bigcup\{S_n : n < \omega\}$ where $|S_n| = \lambda_n$ for each $n < \omega$. For each $x \in \mathrm{I}_\beta(X)$ there must be some $n < \omega$ such that

$$S_n \cap \{p(x,\eta) : \eta < \omega_1\}$$

is uncountable and so $x \in \overline{S}_n$. However, by GCH, we have $|\overline{S}_n| \le \lambda_n^+ < \lambda$ for each $n < \omega$, and consequently

$$s(\beta) = |\mathrm{I}_\beta(X)| \le \left|\bigcup\{\overline{S}_n : n \in \omega\}\right| \le \sup_{n<\omega} \lambda_n^+ = \lambda,$$

i.e. $\beta \in \mathrm{A}_\lambda(s)$. This completes the necessity part of the second case.

Sufficiency. We first handle some specific sequences $s \in {}^\alpha\{\lambda, \lambda^+\}$. In particular, as was noted in the introduction, the constant sequence $\langle\lambda\rangle_\alpha$ is a member of $\mathcal{C}_\lambda(\alpha)$.

CLAIM 4.1.1. *If* $0 < \beta$, $\gamma < \omega_2$ *and* $\mathrm{cf}(\beta) < \omega_1$ *then*

$$\langle\lambda\rangle_\beta{}^\frown\langle\lambda^+\rangle_\gamma \in \mathcal{C}_\lambda(\beta+\gamma).$$

Proof. First we do the case $\gamma = 1$, that is, we construct an LCS space Z with $\mathrm{SEQ}(Z) = \langle\lambda\rangle_\beta{}^\frown\langle\lambda^+\rangle_1$.

If $\mathrm{cf}(\beta) = \omega$ let $\langle\beta_n : n < \omega\rangle$ be an increasing sequence converging to β. If $\beta = \varrho + 1$ let $\beta_n = \varrho$ for each $n < \omega$. For each $n < \omega$ and each ordinal μ with $\lambda_n \le \mu < \lambda_{n+1}$ let Z_μ be a copy of the one-point compactification of an LCS space of height β_n such that $\mathrm{SEQ}(Z_\mu) = \langle\lambda\rangle_{\beta_n}$ and $\{Z_\mu : \mu < \lambda\}$ is disjoint. Let $\{a_\eta : \eta < \lambda^+\}$ be a collection of almost disjoint subsets of λ such that $|a_\eta \cap (\lambda_{n+1} \setminus \lambda_n)| = 1$ for all $\eta < \lambda^+$ and $n < \omega$. (The existence of such an almost disjoint family of size λ^+ is well known.) Let $\{p_\eta : \eta < \lambda^+\}$ be a set of new points and set

$$Z = \{p_\eta : \eta < \lambda^+\} \cup \bigcup\{Z_\mu : \mu < \lambda\}.$$

Let τ be the topology on Z generated by all sets which are open in any Z_μ along with all sets of the form

$$\{p_\eta\} \cup \bigcup\{Z_\mu : \mu \in a_\eta \text{ and } \mu > \lambda_m\}$$

for $\eta < \lambda^+$ and $m < \omega$. It is straightforward to show that $\langle Z, \tau\rangle$ is an LCS space of height $\beta + 1$ with $\mathrm{SEQ}(Z,\tau) = \langle\lambda\rangle_\beta{}^\frown\langle\lambda^+\rangle_1$.

Now, if $\gamma > 1$, then we extend this space Z using Lemma 1.5 with the choices $S = \{p_\eta : \eta < \lambda^+\}$, $U_{p_\eta,n} = Z_\zeta$ where $\{\zeta\} = a_\eta \cap [\lambda_n, \lambda_{n+1})$, and each Y_{p_η} being an LCS space of height γ satisfying $\mathrm{SEQ}(Y_{p_\eta}) = \langle\omega\rangle_\gamma$, i.e. $|\mathrm{I}_\xi(Y_{p_\eta})| = \omega$ for all $\xi < \gamma$. Note that for each $\eta < \lambda^+$ we have $\delta_{p_\eta} = \beta$ here, and consequently we thus obtain an LCS space with cardinal sequence $\langle\lambda\rangle_\beta{}^\frown\langle\lambda^+\rangle_\gamma$. $\blacksquare_{4.1.1}$

Now let $s \in {}^{\alpha}\{\lambda, \lambda^+\}$ be an arbitrary sequence such that $s(0) = \lambda$ and $\mathrm{A}_\lambda(s)$ is ω_1-closed in α. Our aim is to construct an LCS space X with s as its cardinal sequence.

Let $C = \alpha \setminus \mathrm{A}_\lambda(s) = \{\gamma < \alpha : s(\gamma) = \lambda^+\}$. For each $\gamma \in C$ let $\beta_\gamma = \min\{\beta \in C : [\beta, \gamma] \subset C\}$. By the choice of s we have $\beta_\gamma > 0$ and $\mathrm{cf}(\beta_\gamma) < \omega_1$.

By Claim 4.1.1 above, for each $\gamma \in C$ there is an LCS space X_γ with $\mathrm{SEQ}(X_\gamma) = \langle\lambda\rangle_{\beta_\gamma} {}^\frown \langle\lambda^+\rangle_{(\gamma+1) \dot{-} \beta_\gamma}$. The final X that we require is simply the disjoint topological sum of $\{X_\gamma : \gamma \in C\} \cup \{Y\}$, where $\mathrm{SEQ}(Y) = \langle\lambda\rangle_\alpha$.

Now consider the third case: $\mathrm{cf}(\lambda) = \omega_1$.

Necessity. Let $s \in \mathcal{C}_\lambda(\alpha)$. By 1.3 and GCH we see that if $\beta + 1 < \alpha$ and $s(\beta) = \lambda$ then $s(\beta+1) \le s(\beta)^{\aleph_0} = \lambda$ as well, i.e. $\mathrm{A}_\lambda(s)$ is successor closed in α. Similarly, 1.4 and GCH together guarantee that if $\beta < \alpha$, $\mathrm{cf}(\beta) = \omega$, and $\mathrm{A}_\lambda(s) \cap \beta$ is cofinal in β, then $s(\beta) \le \lambda^\omega = \lambda$, that is, $\mathrm{A}_\lambda(s)$ is indeed ω-closed in α. So we have completed the necessity part of this case.

Sufficiency. We make essential use of the following proposition, which is an immediate corollary to Theorem 3.9.

Proposition 4.2. *Let λ be a cardinal with* $\mathrm{cf}\,(\lambda) = \omega_1$ *and* $\lambda^\omega = \lambda$. *Then for every ordinal $\gamma < \omega_2$ with* $\mathrm{cf}(\gamma) = \omega_1$ *we have*

$$\langle\lambda\rangle_\gamma {}^\frown \langle\lambda^+\rangle_{\omega_2} \in \mathcal{C}_\lambda(\omega_2).$$

Now let $s \in {}^{\alpha}\{\lambda, \lambda^+\}$ be such that $s(0) = \lambda$ and $\mathrm{A}_\lambda(s)$ is both ω-closed and successor closed in α. Let $B = \alpha \setminus \mathrm{A}_\lambda(s) = \{\beta < \alpha : s(\beta) = \lambda^+\}$. For $\beta \in B$ let $\gamma_\beta = \min\{\gamma \in B : [\gamma, \beta] \subset B\}$. Since $s(0) = \lambda$ and $\mathrm{A}_\lambda(s)$ is both ω-closed and successor closed, we have

$$\mathrm{cf}(\gamma_\beta) = \omega_1 = \mathrm{cf}(\lambda)$$

for each $\beta \in B$.

Thus, using Proposition 4.2, we may fix for each $\beta \in B$ an LCS space X_β with $\mathrm{SEQ}(X_\beta) = \langle\lambda\rangle_{\gamma_\beta} {}^\frown \langle\lambda^+\rangle_{\nu_\beta}$, where ν_β is chosen so as to satisfy $\beta + 1 = \gamma_\beta + \nu_\beta$. Now, let X be the disjoint topological sum of the family of spaces

$$\{Y\} \cup \{X_\beta : \beta \in B\},$$

where Y is an LCS space with $\mathrm{SEQ}(Y) = \langle\lambda\rangle_\alpha$. Since $|B| \le \omega_1 \le \lambda$, it is clear that $\mathrm{SEQ}(X) = s$.

Finally, consider the case when $\mathrm{cf}(\lambda) > \omega_1$. For any $s \in \mathcal{C}_\lambda(\alpha)$ we may then use Facts 1.3 and 1.4 along with GCH to inductively show that $s(\xi) = \lambda$ for all $\xi < \alpha$, hence $s = \langle\lambda\rangle_\alpha$. Since $\langle\lambda\rangle_\alpha \in \mathcal{C}_\lambda(\alpha)$ is clear, we are done. $\blacksquare_{4.1}$

Having given a full characterization, under GCH, of the classes $\mathcal{C}(\alpha)$ for all $\alpha < \omega_2$, it is now natural to raise the following question: Can this GCH characterization be extended to longer sequences, i.e. to $\mathcal{C}(\omega_2)$ and beyond? This question, however, remains open. In fact, we do not even know if GCH implies that the constant sequence $\langle\omega_1\rangle_{\omega_2}$ belongs to $\mathcal{C}(\omega_2)$, although this is known to be consistent with GCH.

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