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Stabilizers of closed sets in the Urysohn space

by

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Abstract. Building on earlier work of Katětov, Uspenskij proved in [8] that the group of isometries of Urysohn's universal metric space $\mathbb{U}$, endowed with the pointwise convergence topology, is a universal Polish group (i.e. it contains an isomorphic copy of any Polish group). Answering a question of Gao and Kechris, we prove here the following, more precise result: for any Polish group G, there exists a closed subset F of $\mathbb{U}$ such that G is topologically isomorphic to the group of isometries of $\mathbb{U}$ which map F onto itself.

1. Introduction. In a posthumously published article [7], P. S. Urysohn constructed a complete separable metric space $\mathbb{U}$ that is *universal* (meaning that it contains an isometric copy of every complete separable metric space), and *ω-homogeneous* (i.e. such that its isometry group acts transitively on isometric r-tuples contained in it).

In recent years, interest in the properties of $\mathbb{U}$ has greatly increased, especially since V. V. Uspenskij, building on earlier work of Katětov, proved in [8] that the isometry group of $\mathbb{U}$ (endowed with the product topology) is a universal Polish group, that is, any Polish group is isomorphic to a (necessarily closed) subgroup of it.

In [2], S. Gao and A. S. Kechris used properties of $\mathbb{U}$ to study the complexity of the equivalence relation of isometry between certain classes of Polish metric spaces; as a side-product of their construction, they proved the beautiful fact that any Polish group is (topologically) isomorphic to the isometry group of some Polish space. A consequence of their construction is that, for any Polish group G, there exists a sequence (X_n) of closed subsets of $\mathbb{U}$ such that G is isomorphic to $\mathrm{Iso}(\mathbb{U},(X_n)) = \{\varphi \in \mathrm{Iso}(\mathbb{U})\colon \forall n\ \varphi(X_n) = X_n\}$. This led them to ask the following question (cf. [2]):

Can every Polish group be represented, up to isomorphism, by a group of the form $\mathrm{Iso}(\mathbb{U}, F)$ for a single subset $F \subseteq \mathbb{U}$?

The purpose of this article is to provide a positive answer to this question.

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THEOREM 1.1. *Let G be a Polish group. There exists a closed set $F \subseteq \mathbb{U}$ such that G is* (*topologically*) *isomorphic to* $\mathrm{Iso}(F)$, *and every isometry of F is the restriction of a unique isometry of* $\mathbb{U}$; *in particular*, *G is isomorphic to* $\mathrm{Iso}(\mathbb{U}, F)$.

This gives a somewhat concrete realization of any Polish group as a subgroup of $\mathrm{Iso}(\mathbb{U})$.

The construction, which will be detailed in Section 3, starts with a bounded Polish metric space X such that G is isomorphic to $\mathrm{Iso}(X)$ (the isometry group of X, endowed with the product topology) (Gao and Kechris [2] proved that such an X always exists). Identifying G with $\mathrm{Iso}(X)$, we construct an embedding of X in $\mathbb{U}$ and a discrete, unbounded sequence $(x_n) \subseteq \mathbb{U}$ such that $F = X \cup \{x_n\}$ has the desired properties (here we identify X with its image via the embedding provided by our construction).

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2. Notations and definitions. If (X, d) is a complete separable metric space, we say that it is a *Polish metric space*, and often write it simply X.

To avoid confusion, if (X, d) and (X', d') are two metric spaces, we say that f is an *isometric map* if $d(x, y) = d'(f(x), f(y))$ for all $x, y \in X$; if f is moreover onto, then we say that f is an *isometry*.

A *Polish group* is a topological group whose topology is Polish. If X is a separable metric space, then we denote its isometry group by $\mathrm{Iso}(X)$, and endow it with the product topology, which turns it into a second countable topological group, and into a Polish group if X is Polish (see [1] or [5] for a thorough introduction to the theory of Polish groups).

We say that a metric space X is *finitely injective* if for any finite subsets $K \subseteq L$ and any isometric map $\varphi\colon K \to X$ there exists an isometric map $\widetilde{\varphi}\colon L \to X$ such that $\widetilde{\varphi}|_K = \varphi$. Up to isometry, $\mathbb{U}$ is the only finitely injective Polish metric space (see [7]).

Let (X, d) be a metric space; we say that $f\colon X \to \mathbb{R}$ is a *Katětov map* if

$$\forall x, y \in X \quad |f(x) - f(y)| \le d(x, y) \le f(x) + f(y).$$

These maps correspond to one-point metric extensions of X. We denote by $E(X)$ the set of all Katětov maps on X and endow it with the sup-metric, which turns it into a complete metric space.

That definition was introduced by Katětov in [4], and it turns out to be pertinent to the study of finitely injective spaces, since one can easily see by induction that a nonempty metric space X is finitely injective if, and only if,

$$\forall A \subset X \text{ finite } \forall f \in E(A)\ \exists z \in X\ \forall a \in A \quad d(z, a) = f(a).$$

If $Y \subseteq X$ and $f \in E(Y)$, define $\widehat{f}\colon X \to \mathbb{R}$ (the *Katětov extension* of f) by

$$\widehat{f}(x) = \inf\{f(y) + d(x,y)\colon y \in Y\}.$$

Then $\widehat{f}$ is the greatest 1-Lipschitz map on X which is equal to f on Y; one checks easily (see for instance [4]) that $\widehat{f} \in E(X)$ and $f \mapsto \widehat{f}$ is an isometric embedding of $E(Y)$ into $E(X)$.

To simplify future definitions, if f and $S \subseteq X$ are such that

$$\forall x \in X \quad f(x) = \inf\{f(s) + d(x,s)\colon s \in S\},$$

then we say that S is a *support* of f, or that S *controls* f. Notice that if S controls $f \in E(X)$ and $S \subseteq T$, then T controls f.

Also, X isometrically embeds in $E(X)$ via the Kuratowski map $x \mapsto f_x$, where $f_x(y) = d(x,y)$.

A crucial fact for our purposes is that

$$\forall f \in E(X)\ \forall x \in X \quad d(f, f_x) = f(x).$$

Thus, if one identifies X with its image in $E(X)$ via the Kuratowski map, then $E(X)$ is a metric space containing X and such that all one-point metric extensions of X embed isometrically in $E(X)$.

We now go on to sketching Katětov's construction of $\mathbb{U}$; we refer the reader to [2], [3], [7] and [8] for a more detailed presentation and proofs of the results we will use below.

Most important for the construction is the following result:

THEOREM 2.1 (Urysohn). *If X is a finitely injective metric space*, *then the completion of X is also finitely injective.*

Since $\mathbb{U}$ is, up to isometry, the unique finitely injective Polish metric space, this proves that the completion of any separable finitely injective metric space is isometric to $\mathbb{U}$.

The basic idea of Katětov's construction is this: if one lets $X_0 = X$ and $X_{i+1} = E(X_i)$ then, identifying each X_i to a subset of X_{i+1} via the Kuratowski map, we let Y be the inductive limit of the sequence X_i.

The definition of Y makes it clear that Y is finitely injective, since any $\{x_1, \ldots, x_n\} \subseteq Y$ must be contained in some X_m, so that for any $f \in E(\{x_1, \ldots, x_n\})$ there exists $z \in X_{m+1}$ such that $d(z, x_i) = f(x_i)$ for all i.

Thus, if Y were separable, its completion would be isometric to $\mathbb{U}$, and one would have obtained an isometric embedding of X into $\mathbb{U}$. The problem is that $E(X)$ is in general not separable: at each step, we have added too many functions.

Define then $E(X,\omega) = \{f \in E(X)\colon f$ is controlled by some finite $S \subseteq X\}$. Then $E(X,\omega)$ is easily seen to be separable if X is, and the Kuratowski map actually maps X into $E(X,\omega)$, since each f_x is controlled by $\{x\}$. Notice also that, if $\{x_1, \ldots, x_n\} \subseteq X$ and $f \in E(\{x_1, \ldots, x_n\})$, then its Katětov extension $\widehat{f}$ is in $E(X,\omega)$, and $d(\widehat{f}, f_{x_i}) = f(x_i)$ for all i. Thus, if one defines this time $X_0 = X$, $X_{i+1} = E(X_i,\omega)$, then $Y = \bigcup X_i$ is separable and finitely injective, hence its completion Z is isometric to $\mathbb{U}$, and $X \subseteq Z$.

The most interesting property of this construction is that it enables one to keep track of the isometries of X: indeed, any $\varphi \in \mathrm{Iso}(X)$ is the restriction of a unique isometry $\widetilde{\varphi}$ of $E(X,\omega)$, and the mapping $\varphi \mapsto \widetilde{\varphi}$ from $\mathrm{Iso}(X)$ into $\mathrm{Iso}(E(X,\omega))$ is an isomorphic embedding (of topological groups).

That way, we obtain for all i isomorphic embeddings $\Psi^i\colon \mathrm{Iso}(X) \to \mathrm{Iso}(X_i)$ such that $\Psi^{i+1}(\varphi)_{|X_i} = \Psi^i(\varphi)$ for all i and all $\varphi \in \mathrm{Iso}(X)$. This in turns defines an isomorphic embedding from $\mathrm{Iso}(X)$ into $\mathrm{Iso}(Y)$, and since extension of isometries defines an isomorphic embedding from the isometry group of any metric space into that of its completion (see [9]), we actually have an isomorphic embedding of $\mathrm{Iso}(X)$ into the isometry group of Z, that is, $\mathrm{Iso}(\mathbb{U})$ (and the image of any $\varphi \in \mathrm{Iso}(X)$ is actually an extension of φ to Z).

3. Proof of the main theorem. To prove Theorem 1.1, we will use ideas very similar to those used in [2]; all the notations are the same as in Section 2.

We will need an additional definition, which was introduced in [2]. If X is a metric space and $i \geq 1$, let

$$E(X,i) = \{f \in E(X)\colon f \text{ has a support of cardinality } \leq i\}.$$

We endow $E(X,i)$ with the sup-metric.

Gao and Kechris proved the following result, of which we will give a new, slightly simpler proof:

THEOREM 3.1 (Gao–Kechris). *If X is a Polish metric space and $i \geq 1$ then $E(X,i)$ is a Polish metric space.*

Proof. Notice first that the separability of $E(X,i)$ is easy to prove; we will prove its completeness by induction on i.

The proof for $i = 1$ is the same as in [2]; we include it for completeness.

First, let (f_n) be a Cauchy sequence in $E(X,1)$. It has to converge uniformly to some Katětov map f, and it is enough to prove that $f \in E(X,1)$. By definition of $E(X,1)$, there exists a sequence (y_n) such that

$$(*) \qquad \forall x \in X \quad f_n(x) = f_n(y_n) + d(y_n,x).$$

Let then $\varepsilon > 0$, and let M be large enough that $m,n \geq M \Rightarrow d(f_n,f_m) \leq \varepsilon$. Then, for $m,n \geq M$, one has

$$2d(y_n,y_m) = (f_n(y_m) - f_m(y_m)) + (f_m(y_n) - f_n(y_n)) \leq 2\varepsilon.$$

This proves that (y_n) is Cauchy, hence has a limit y. One easily checks that

$f(y) = \lim f_n(y_n)$, so that letting $n \to \infty$ in $(*)$ gives

$$\forall x \in X \quad f(x) = f(y) + d(y, x).$$

That does the trick for $i = 1$.

Suppose now we have proved the result for $1, \ldots, i-1$, and let (f_n) be a Cauchy sequence in $E(X, i)$. By definition, there are $y_1^n, \ldots, y_i^n$ such that

$$(**) \qquad \forall x \in X \quad f_n(x) = \min_{1 \le j \le i} \{f_n(y_j^n) + d(y_j^n, x)\}.$$

Once again, (f_n) converges uniformly to some Katětov map f, and we want to prove that $f \in E(X, i)$.

By the induction hypothesis, we can assume that there is $\delta > 0$ such that for all n and all $k \neq j \le i$ one has $d(y_j^n, y_k^n) \ge 2\delta$ (if not, a subsequence of (f_n) can be approximated by a Cauchy sequence in $E(X, i-1)$, and the induction hypothesis applies).

Let $d_n = \min\{f_n(x) \colon x \in X\}$. Then (d_n) is Cauchy, so it has a limit $d \ge 0$; up to extracting a subspace, and some rearrangement of the sequence, we can assume that there are $p \ge 1$ and $\delta' > 0$ such that:

- $\forall j \le p \ f_n(y_j^n) \to d$,
- $\forall j > p \ \forall n \ f_n(y_j^n) > d + \delta'$.

Let $\varepsilon > 0$, $\alpha = \min(\delta, \delta', \varepsilon)$ and choose M large enough that $n, m \ge M \Rightarrow d(f_n, f_m) < \alpha/4$ and $|f_n(y_j^n) - d| < \alpha/4$ for all $j \le p$. Then, for $n, m \ge M$ and $j \le p$ one has $f_n(y_j^m) < d + \alpha/2$, so there exists $k \le p$ such that

$$f_n(y_j^m) = f_n(y_k^n) + d(y_j^m, y_k^n).$$

Such a y_k^n has to be at a distance strictly smaller than δ from y_j^m: there is at most one y_k^n that can work, and there is necessarily one. Thus, one sees, as in the case $i = 1$, that $d(y_k^n, y_j^m) \le \varepsilon$. This means that one can assume, choosing an appropriate rearrargement, that for $k \le p$ each sequence $(y_i^n)_n$ is Cauchy, hence has a limit y_k.

Define

$$\widetilde{f}_n(x) = \min_{1 \le k \le p} \{f_n(y_i^n) + d(x, y_k^n)\}.$$

Then $\widetilde{f}_n \in E(X, p)$, and one checks easily, since $y_k^n \to y_k$ for all $k \le p$, that $(\widetilde{f}_n)$ converges uniformly to $\widetilde{f}$, where

$$\widetilde{f}(x) = \min_{1 \le k \le p} \{f(y_k) + d(x, y_k)\}.$$

If $p = i$ then we are finished; otherwise, notice that, using again the induction hypothesis, we may assume that there is $\eta > 0$ such that

$$(***) \qquad \forall n \ \forall j > p \quad f_n(y_j^n) < \widetilde{f}_n(y_j^n) - \eta.$$

Now define

$$\widetilde{g}_n(x) = \min_{j>p}\{f_n(y_j^n) + d(x, y_j^n)\}.$$

Choose M such that $n, m \geq M \Rightarrow d(f_n, f_m) < \eta/4$ and $d(\widetilde{f}_n, \widetilde{f}_m) < \eta/4$. Then $(***)$ shows that for, all $n, m \geq M$ and all $j > p$,

$$f_m(y_j^n) \leq f_n(y_j^n) + \eta/4 \leq \widetilde{f}_n(y_j^n) - 3\eta/4 \leq \widetilde{f}_m(y_j^n) - \eta/2,$$

so that $f_m(y_j^n) = f_m(y_k^m) + d(y_j^n, y_k^m)$ for some $k > p$. Consequently, for $m, n \geq M$ and $j > p$, $f_m(y_j^n) = \widetilde{g}_m(y_j^n)$; by definition, $f_m(y_j^m) = \widetilde{g}_m(y_j^m)$.

This proves that for all $n, m \geq M$ one has $d(\widetilde{g}_n, \widetilde{g}_m) \leq d(f_n, f_m)$, so that $(\widetilde{g}_n)$ is Cauchy in $E(X, i-p)$, hence has a limit $\widetilde{g} \in E(X, i-p)$ by the induction hypothesis. But then $(**)$ shows that $f(x) = \min(\widetilde{f}(x), \widetilde{g}(x))$ for all $x \in X$, and this concludes the proof. ∎

If Y is a nonempty, closed and bounded subset of a metric space X, define

$$E(X, Y) = \{f \in E(X) \colon \exists d \in \mathbb{R}^+ \ \forall x \in X \ \ f(x) = d + d(x, Y)\}.$$

Then $E(X, Y)$ is closed in $E(X)$, and is isometric to $\mathbb{R}^+$.

Proof of Theorem 1.1. Essential to our proof is the fact that for every Polish group G there exists a Polish space (X, d) such that G is isomorphic to the group of isometries of X (this result was proved by Gao and Kechris, see [2]).

So, let G be a Polish group, and X be a metric space such that G is isomorphic to $\mathrm{Iso}(X)$. One can assume that X contains more than two points, and (X, d) is bounded, of diameter $d_0 \leq 1$. (If not, define $d'(x, y) = d(x, y)/(1 + d(x, y))$. Then (X, d') is a bounded Polish metric space with the same topology as X, and the isometries of (X, d') are exactly those of (X, d).)

Let $X_0 = X$, and define inductively bounded Polish metric spaces X_i, of diameter d_i, by

$$X_{i+1} = \Big\{f \in E(X_i, i) \cup \bigcup_{j<i} E(X_i, X_j) \colon \forall x \in X_i \ \ f(x) \leq 2d_i\Big\}$$

(we endow X_{i+1} with the sup-metric; since X_i canonically embeds isometrically in X_{i+1} via the Kuratowski map, we assume that $X_i \subseteq X_{i+1}$).

Note that $d_i \to \infty$ as $i \to \infty$, and that each X_i is a Polish metric space. Let Y be the completion of $\bigcup_{i \geq 0} X_i$. The definition of $\bigcup X_i$ makes it easy to see that it is finitely injective, so that Y is isometric to $\mathbb{U}$.

Also, any isometry $g \in G$ extends to an isometry of X_i, and for any i and $g \in G$ there is a unique isometry g^i of X_i such that $g^i(X_j) = X_j$ for all $j \leq i$ and $g^i|_{X_0} = g$ (same proof as in [4]).

Observe also that the mappings $g \mapsto g^i$, from G to $\mathrm{Iso}(X_i)$, are continuous (see [9]).

All this enables us to assign to each g an isometry g^* of Y, given by $g^*|_{X_i} = g^i$, and this defines a continuous embedding of G into $\mathrm{Iso}(Y)$ (see again [9] for details).

It is important to remark here that, if $f \in X_{i+1}$ is defined by $f(x) = d + d(x, X_j)$ for some $d \geq 0$ and some $j < i$, then $g^*(f) = f$ for all $g \in G$ (this was the aim of the definition of X_i: adding "many" points that are fixed by the action of G).

Notice that an isometry φ of Y is equal to g^* for some $g \in G$ if, and only if, $\varphi(X_n) = X_n$ for all n. The idea of the construction is then simply to construct a closed set F such that $\varphi(F) = F$ if, and only if, $\varphi(X_n) = X_n$ for all n. To achieve this, we will build F as a set of carefully chosen "witnesses".

The construction proceeds as follows. First, let $(k_i)_{i\geq 1}$ be an enumeration of the nonnegative integers where every number appears infinitely many times. Using the definition of the sets X_i, we choose recursively for all $i \geq 1$ points $a_i \in \bigcup_{n\geq 1} X_n$ (the witnesses), nonnegative reals e_i, and a nondecreasing sequence (j_i) of integers such that:

- $e_1 \geq 4$ and $\forall i \geq 1\ e_{i+1} > 4e_i$.
- $\forall i \geq 1\ \ j_i \geq k_i$, $a_i \in X_{j_i+1}$ and $\forall x \in X_{j_i}\ \ d(a_i, x) = e_i + d(x, X_{k_i-1})$.
- $\forall i \geq 1\ \forall g \in G\ g^*(a_i) = a_i$.

(This is possible, since at step i it is enough to fix $e_i > \max(4e_{i-1}, \mathrm{diam}(X_{k_i}))$, then find $j_i \geq \max(1 + j_{i-1}, k_i)$ such that $\mathrm{diam}(X_{j_i}) \geq e_i$, and define $a_i \in X_{j_i+1}$ by the equation above; then, by definition of g^* and of a_i, one has $g^*(a_i) = a_i$ for all $g \in G$.)

Let now $F = X_0 \cup \{a_i\}_{i\geq 1}$; since X_0 is complete, and $d(a_i, X_0) = e_i \to \infty$, F is closed. We claim that for all $\varphi \in \mathrm{Iso}(Y)$, one has

$$(\varphi(F) = F) \ \Leftrightarrow\ (\varphi \in G^*).$$

The definition of F makes one implication obvious.

To prove the converse, we need a lemma:

Lemma 3.2. *If $\varphi \in \mathrm{Iso}(F)$, then $\varphi(X_0) = X_0$, so that $\varphi(a_i) = a_i$ for all i. Moreover, there exists $g \in G$ such that $\varphi = g^*|_F$.*

Admitting this lemma for a moment, it is now easy to conclude the proof. Notice that Lemma 3.2 implies that G is isomorphic to the isometry group of F, and that any isometry of F extends to Y. Thus, to finish the proof of Theorem 1.1, we only need to show that the extension of a given isometry of F to Y is unique. As explained before, it is enough to show that, if $\varphi \in \mathrm{Iso}(Y)$ is such that $\varphi(F) = F$, then $\varphi(X_n) = X_n$ for all $n \geq 0$.

So, let $\varphi \in \mathrm{Iso}(Y)$ be such that $\varphi(F) = F$. It is enough to prove that $\varphi(X_n) \supseteq X_n$ for all $n \in \mathbb{N}$ (since this will also be true for φ^{-1}), so assume that this is not true, i.e. there is some $n \in \mathbb{N}$ and $x \notin X_n$ such that $\varphi(x) \in X_n$. Let $\delta = d(x, X_n) > 0$ (since X_n is complete), and pick $y \in \bigcup X_m$ such that $d(x, y) \leq \delta/4$. Then $y \in X_m \setminus X_n$ for some $m > n$; now choose i such that $k_i = n + 1$ and $j_i \geq m$. Then we know that

$$d(\varphi(y), \varphi(a_i)) = d(y, a_i) = e_i + d(y, X_n) \geq e_i + 3\delta/4,$$
$$d(a_i, \varphi(y)) \leq d(a_i, \varphi(x)) + d(x, y) \leq e_i + \delta/4,$$

so that $d(\varphi(a_i), a_i) \geq \delta/2$, and this contradicts Lemma 3.2. ∎

It only remains to give

Proof of Lemma 3.2. Since we assumed that X_0 has more than two points and $\operatorname{diam}(X_0) \leq 1$, the definition of F makes it clear that

$$\forall x \in F \quad (x \in X_0) \Leftrightarrow (\exists y \in F\colon\ 0 < d(x, y) \leq 1).$$

The right part of the equivalence is invariant under isometries of F, so this proves that $\varphi(X_0) = X_0$ for any $\varphi \in \operatorname{Iso}(F)$. In turn, this easily implies that $\varphi(a_i) = a_i$ for all $i \geq 1$.

Thus, if one lets $g \in G$ be such that $g|_{X_0} = \varphi|_{X_0}$, we have shown that $\varphi = g^*|_F$. ∎

References

[1] H. Becker and A. S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Math. Soc. Lecture Note Ser. 232, Cambridge Univ. Press, 1996.

[2] S. Gao and A. S. Kechris, *On the classification of Polish metric spaces up to isometry*, Mem. Amer. Math. Soc. 766 (2003).

[3] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, 1998.

[4] M. Katětov, *On universal metric spaces*, in: General Topology and its Relations to Modern Analysis and Algebra, VI (Prague, 1986), Z. Frolík (ed). Helderman, Berlin, 1988, 323–330.

[5] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.

[6] —, *Actions of Polish groups and classification problems*, in: Analysis and Logic (Mons, 1997), London Math. Soc. Lecture Note Ser. 262, Cambridge Univ. Press, 2000, 115–187.

[7] P. S. Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math. 51 (1927), 43–64 and 74–96.

[8] V. V. Uspenskij, *On the group of isometries of the Urysohn universal metric space*, Comment. Math. Univ. Carolin. 31 (1990), 181–182.

[9] —, *Compactifications of topological groups*, in: Proc. 9th Prague Topological Symposium (2001), Topology Atlas, Toronto, 2002, 331–346.

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