On approximation of homeomorphisms of a Cantor set

by

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Abstract. We continue the study of topological properties of the group $\mathrm{Homeo}(X)$ of all homeomorphisms of a Cantor set X with respect to the uniform topology τ, which was started by Bezuglyi, Dooley, Kwiatkowski and Medynets. We prove that the set of periodic homeomorphisms is τ-dense in $\mathrm{Homeo}(X)$ and deduce from this result that the topological group $(\mathrm{Homeo}(X), \tau)$ has the Rokhlin property, i.e., there exists a homeomorphism whose conjugacy class is τ-dense in $\mathrm{Homeo}(X)$. We also show that for any homeomorphism T the topological full group $[[T]]$ is τ-dense in the full group $[T]$.

1. Introduction. Many famous problems in ergodic theory involve the use of topologies on the group $\mathrm{Aut}(X, \mathcal{B}, \mu)$ of all measure-preserving transformations of a standard measure space. The first results on group topologies of $\mathrm{Aut}(X, \mathcal{B}, \mu)$ appeared in [Hal 1]. Halmos introduced two topologies d_{u} and d_{w}, which were called later the uniform and weak topologies, respectively. He defined the uniform topology d_{u} by saying that two automorphisms T and S are "close" to each other if the quantity $\mu(\{x \in X : Tx \neq Sx\})$ is small enough. The weak topology is generated by the sets of the form $N(T; E; \varepsilon) = \{S \in \mathrm{Aut}(X, \mathcal{B}, \mu) : \mu(SE \bigtriangleup TE) < \varepsilon\}$, where $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ and $E \in \mathcal{B}$. The use of these topologies turned out to be very fruitful and led to many outstanding results in ergodic theory (for references, see, for example, [B-K-M] and [C-F-S]). One of the most relevant results in the theory is the Rokhlin lemma [Ro] stating that the set of periodic automorphisms is d_{u}-dense in $\mathrm{Aut}(X, \mathcal{B}, \mu)$.

The idea of investigation of transformation groups by means of introducing various topologies into these groups was used in [B-D-K 1] and [B-D-K 2], where the authors considered the groups $\mathrm{Aut}(X, \mathcal{B})$ of all automorphisms of a standard Borel space and the group $\mathrm{Homeo}(X)$ of all

homeomorphisms of a Cantor set X with several topologies analogous to those in ergodic theory.

Following [B-D-K 2], we continue studying the group Homeo(X) of all homeomorphisms of a Cantor set X with the topology τ (cf. Definition 1.1), which is obviously a direct analog of the topology d_{u}. We show that the set of all periodic homeomorphisms is τ-dense in Homeo(X) (Corollary 2.2). This result can be treated as a topological version of the Rokhlin lemma. As a corollary, we prove that the set of topologically free homeomorphisms is τ-dense in Homeo(X) (Theorem 2.8). Recall that a homeomorphism is called *topologically free* if the set of aperiodic points is dense.

In [G-K], an interesting class of topological groups was defined: by definition, a topological group has the *Rokhlin property* if it has an element whose conjugacy class is dense. The authors raised the question which groups possess this property. At the moment, there is an extensive list of such groups, including the group Homeo(X) with the topology generated by the metric $D(T, S) = \sup_{x \in X} d(Tx, Sx)$, where d is a metric on X compatible with the topology [Gl-W], and the group Aut($X, \mathcal{B}, \mu$) with the weak topology [Hal 2]. See also the paper [Ke-Ros] for a general approach to the study of groups with dense conjugacy classes.

Motivated by this, we present a uniform approach allowing us to show that the topological groups (Aut($X, \mathcal{B}$), τ) and (Homeo(X), τ) have the Rokhlin property (Theorem 2.5).

Furthermore, inspired by the remarkable work [G-P-S], where full groups were indispensable in the study of orbit equivalence of Cantor minimal systems, we study the full groups $[T]$ with their dense subsets for arbitrary homeomorphisms $T \in$ Homeo(X).

In this context, we show that for any $T \in$ Homeo(X), the topological full group $[[T]]$ is τ-dense in the full group $[T]$ (Theorem 2.1).

In the last section, we give a description of homeomorphisms from the topological full group $[[T]]$ for T aperiodic (Theorem 3.3). We consider a subgroup Γ_Y of $[[T]]$, which is an increasing union of permutation groups, and find a criterion for Γ_Y to be τ-dense in $[[T]]$ (Theorem 3.4).

Background. Throughout the paper, X denotes a *Cantor set* and $\mathcal{B}$ stands for the σ-algebra of Borel subsets of X. A one-to-one Borel map T of X onto itself is called an *automorphism* of $(X, \mathcal{B})$. Denote by Aut($X, \mathcal{B}$) the group of all automorphisms of $(X, \mathcal{B})$ and by Homeo(X) the group of all homeomorphisms of X.

Following [B-K], recall the definition of the *uniform topology* on Aut($X, \mathcal{B}$). Let $\mathcal{M}_1(X)$ denote the set of all Borel probability measures on X. For $T, S \in$ Aut($X, \mathcal{B}$), define $E(T, S) = \{x \in X : Tx \neq Sx\}$.

DEFINITION 1.1. The *uniform topology* τ on $\mathrm{Aut}(X, \mathcal{B})$ is defined by the base of neighborhoods $\mathcal{U} = \{U(T; \mu_1, \ldots, \mu_n; \varepsilon)\}$, where

$$U(T; \mu_1, \ldots, \mu_n; \varepsilon) = \{S \in \mathrm{Aut}(X, \mathcal{B}) : \mu_i(E(S, T)) < \varepsilon, \, i = 1, \ldots, n\}.$$

Here $T \in \mathrm{Aut}(X, \mathcal{B})$, $\mu_1, \ldots, \mu_n \in \mathcal{M}_1(X)$, and $\varepsilon > 0$.

As $\mathrm{Homeo}(X)$ is a subgroup of $\mathrm{Aut}(X, \mathcal{B})$, we also denote by τ the topology on $\mathrm{Homeo}(X)$ induced from $(\mathrm{Aut}(X, \mathcal{B}), \tau)$.

Observe that $\mathrm{Aut}(X, \mathcal{B})$ and $\mathrm{Homeo}(X)$ are Hausdorff topological groups with respect to the uniform topology τ. More results on topological properties of $\mathrm{Aut}(X, \mathcal{B})$ and $\mathrm{Homeo}(X)$ with respect to τ can be found in [B-D-K 1, B-D-K 2, B-D-M, B-K-M, B-M].

Let $T \in \mathrm{Aut}(X, \mathcal{B})$. A point $x \in X$ is called *periodic* of period $n > 0$ if $T^n x = x$ and $T^i x \neq x$ for $i = 1, \ldots, n - 1$. If $T^n x \neq x$ for $n \neq 0$, the point x is called *aperiodic*. We say that T is *aperiodic* if it has no periodic points. Note that for any $T \in \mathrm{Aut}(X, \mathcal{B})$ the set X can be decomposed into a disjoint union of Borel sets $X = X_\infty \cup \bigcup_{n \geq 1} X_n$, where X_n consists of all points of period n and X_∞ is formed by all aperiodic points. Notice that some X_n's may be empty. Moreover, for every X_n with $n < \infty$ there exists a Borel set $X_n^0 \subset X_n$ such that $X_n = \bigcup_{i=0}^{n-1} T^i X_n^0$ is a disjoint union. We call $\{X_\infty, X_1, X_2, \ldots\}$ the *canonical partition of X associated to T*.

Recall that a finite family of disjoint Borel sets $\xi = \{A, TA, \ldots, T^{n-1}A\}$ is called a *T-tower* with *base* $B(\xi) = A$ and *height* $h(\xi) = n$. A partition $\Xi = \{\xi_1, \xi_2, \ldots\}$ of X is called a *Kakutani–Rokhlin (K-R) partition* if every ξ_i is a T-tower. For a K-R partition Ξ, we denote $\bigcup_{n \geq 1} B(\xi_i)$ by $B(\Xi)$ and call it the *base* of the K-R partition. Notice that $T^{-1}B(\Xi) = \bigcup_{\xi \in \Xi} T^{h(\xi)-1}B(\xi)$.

For $T \in \mathrm{Aut}(X, \mathcal{B})$, let $\mathrm{Orb}_T(x) = \{T^n x : n \in \mathbb{Z}\}$ denote the T-orbit of x. To any $T \in \mathrm{Homeo}(X)$, we can assign two *full groups* $[T]_\mathrm{C}$ and $[T]_\mathrm{B}$, where

$$[T]_\mathrm{C} = \{S \in \mathrm{Homeo}(X) : \mathrm{Orb}_S(x) \subseteq \mathrm{Orb}_T(x), \, x \in X\}$$
$$[T]_\mathrm{B} = \{S \in \mathrm{Aut}(X, \mathcal{B}) : \mathrm{Orb}_S(x) \subseteq \mathrm{Orb}_T(x), \, x \in X\}.$$

Here the subscripts C and B indicate the Cantor and Borel dynamics, respectively. Clearly, $[T]_\mathrm{C}$ is a subgroup of $[T]_\mathrm{B}$. Observe that if $S \in [T]_\mathrm{B}$, then there is a Borel function $n_S : X \to \mathbb{Z}$ such that $Sx = T^{n_S(x)}x$ for all $x \in X$. The subgroup $[[T]] = \{S \in [T]_\mathrm{C} : n_S \text{ is continuous}\}$ is called the *topological full group* of T.

One of the main results in the approximation theory of Borel automorphisms is a Borel version of the Rokhlin lemma. The following τ-version of the Rokhlin lemma was proved in [B-D-K 1, Proposition 3.6]. We also refer the reader to [N, Section 7] and [W, Section 4] for measure-free versions of the result.

THEOREM 1.2. *Let T be an aperiodic automorphism of X. Then there exists a sequence of periodic automorphisms $P_n \in \text{Aut}(X, \mathcal{B})$ such that $P_n \xrightarrow{\tau} T$ as $n \to \infty$. Moreover, the P_n can be taken from $[T]_{\text{B}}$.*

Denote by Per_0 the set of all homeomorphisms P such that $P^n = \mathbb{I}$ for some $n \in \mathbb{N}$; and for $T \in \text{Homeo}(X)$, set $\text{Per}_0(T) = \text{Per}_0 \cap [[T]]$.

2. Rokhlin lemma. In this section, we prove a topological version of the Rokhlin lemma, namely, we show that the set of periodic homeomorphisms is τ-dense in $\text{Homeo}(X)$. Then we deduce several corollaries of this result. In particular, we prove that the topological group $(\text{Homeo}(X), \tau)$ has the Rokhlin property and the topological full group $[[T]]$ is τ-dense in $[T]_{\text{B}}$ for any $T \in \text{Homeo}(X)$.

THEOREM 2.1.

(1) *The set $\mathcal{P}er_0$ is τ-dense in $\text{Homeo}(X)$.*
(2) *Let $T \in \text{Homeo}(X)$. Then for any automorphism $S \in [T]_{\text{B}}$ and any τ-neighborhood $U = U(S; \mu_1, \dots, \mu_p; \varepsilon)$ of S there exists a periodic homeomorphism $P \in [[T]]$ such that $P \in U$.*

Proof. Notice that statement (1) is an immediate corollary of (2). By Theorem 1.2, it is enough to prove (2) for S periodic.

Let us sketch the main stages of the proof:

(i) We find a finite number of disjoint S-towers consisting of closed sets and "almost" covering the entire space X with respect to the measures μ_i such that on each level of these S-towers the automorphism S coincides with a power of T.
(ii) We extend the S-towers found in (i) to clopen ones constructed by means of powers of T.
(iii) Using the clopen towers, we define a periodic homeomorphism P which belongs to $U(S; \mu_1, \dots, \mu_p; \varepsilon)$.

(i) Let $\Xi = \{X_1, X_2, \dots\}$ be the canonical Borel partition of X associated to S. Without loss of generality, we will assume that the sets X_i are nonempty for all $i \in \mathbb{N}$.

We first find $N \in \mathbb{N}$ such that

$$(2.1) \qquad \mu_j(X_1 \cup \dots \cup X_N) > 1 - \varepsilon/3 \quad \text{for all } j = 1, \dots, p.$$

For $n \geq 1$, set $Z_n = \{x \in X : Sx = T^i x \text{ for some } -n \leq i \leq n\}$. For $i \geq 1$ define

$$X_i^0(n) = \bigcap_{j=0}^{i-1} S^{-j}(S^j X_i^0 \cap Z_n),$$

where $X_i = X_i^0 \cup S X_i^0 \cup \cdots \cup S^{i-1} X_i^0$ is a disjoint union. Since $S \in [T]_{\mathrm{B}}$, we have $X_i = \bigcup_{n \geq 1} X_i(n)$, where $X_i(n) = \bigcup_{j=0}^{i-1} S^j X_i^0(n)$. Then find $K \in \mathbb{N}$ such that

$$(2.2) \qquad \mu_j \Big(\bigcup_{i=1}^{N} (X_i \setminus X_i(K)) \Big) < \varepsilon/3 \quad \text{for all } j = 1, \ldots, p.$$

Denote by $\mathcal{S}_i$ the set of all maps from $\{0, \ldots, i-1\}$ to $\{-K, \ldots, K\}$. For $\sigma \in \mathcal{S}_i$, set

$$X_i^0(K, \sigma) = \bigcap_{j=0}^{i-1} S^{-j} (\{x \in S^j X_i^0(K) : Sx = T^{\sigma(j)} x\}).$$

Thus, we get a finite cover $X_i^0(K) = \bigcup_{\sigma \in \mathcal{S}_i} X_i^0(K, \sigma)$. Applying the standard argument, make the $X_i^0(K, \sigma)$'s disjoint and denote the resulting sets by $X_i^0(K, \sigma)$ again. Some of the $X_i^0(K, \sigma)$'s may be empty, but, without loss of generality, we will assume they are not. Observe that S restricted to $S^j X_i^0(K, \sigma)$ is equal to $T^{\sigma(j)}$ for $i \geq 1$, $j = 0, \ldots, i-1$, and $\sigma \in \mathcal{S}_i$. This means that S is a homeomorphism on $S^j X_i^0(K, \sigma)$.

For every $X_i(K, \sigma) = \bigcup_{j=0}^{i-1} S^j X_i^0(K, \sigma)$, find a closed set $A_i^0(\sigma) \subset X_i^0(K, \sigma)$ such that

$$(2.3) \qquad \mu_j \Big(\bigcup_{i=1}^{N} \bigcup_{\sigma \in \mathcal{S}_i} (X_i(K, \sigma) \setminus A_i(\sigma)) \Big) < \varepsilon/3 \quad \text{for } j = 1, \ldots, p,$$

where $A_i(\sigma) = \bigcup_{j=0}^{i-1} S^j A_i^0(\sigma)$.

(ii) Summing up the above, we see that $\{A_i(\sigma) : 1 \leq i \leq N, \ \sigma \in \mathcal{S}_i\}$ is a family of disjoint closed S-towers such that S restricted to $S^j A_i^0(\sigma)$ is equal to $T^{\sigma(j)}$. Furthermore, it follows from (2.1)–(2.3) that

$$(2.4) \qquad \mu_j \Big(\bigcup_{i=1}^{N} \bigcup_{\sigma \in \mathcal{S}_i} A_i(\sigma) \Big) > 1 - \varepsilon.$$

As the closed S-towers $A_i(\sigma)$ are disjoint, we can find clopen sets $\overline{A}_i^0(\sigma) \supset A_i^0(\sigma)$ so that all the sets $\overline{A}_i^0(\sigma)$ and $T^{\sigma(0)+\cdots+\sigma(j)} \overline{A}_i^0(\sigma)$ are mutually disjoint for $i = 1, \ldots, N$, $j = 0, \ldots, i-2$, and $\sigma \in \mathcal{S}_i$.

(iii) Define the periodic homeomorphism P as follows:

$$Px = \begin{cases} T^{\sigma(0)} x & \text{if } x \in \overline{A}_i^0, \\ T^{\sigma(j+1)} x & \text{if } x \in T^{\sigma(0)+\cdots+\sigma(j)} \overline{A}_i^0(\sigma), & \sigma \in \mathcal{S}_i, \\ T^{-\sigma(0)-\cdots-\sigma(i-2)} x & \text{if } x \in T^{\sigma(0)+\cdots+\sigma(i-2)} \overline{A}_i^0(\sigma), & 0 \leq j \leq i-3, \\ x & \text{otherwise.} \end{cases}$$

Clearly, P is well-defined and belongs to $[[T]]$. By the definition of P,

$$\{x \in X : Px = Sx\} \supset \bigcup_{i=1}^{N} \bigcup_{\sigma \in \mathcal{S}_i} A_i(\sigma).$$

Hence, (2.4) shows that $P \in U(S; \mu_1, \ldots, \mu_p; \varepsilon)$, completing the proof. ∎

REMARK. After this work was submitted, B. Miller showed how using ideas of the proof above one can generalize Theorem 2.1(2) to any countable group acting by homeomorphisms on a zero-dimensional Polish space [Mil].

Rokhlin property. We give several immediate corollaries of Theorem 2.1, which have the well-known analogs in ergodic theory.

COROLLARY 2.2. *Let* $T \in \mathrm{Homeo}(X)$. *Then, for every* τ-*neighborhood* U *of* T, *there exists a homeomorphism* $P \in \mathrm{Per}_0(T) \cap U$ *whose associated canonical partition is clopen.*

The next statement generalizes Theorem 4.5 of [B-K] proved originally for minimal homeomorphisms.

COROLLARY 2.3. *Let* $T \in \mathrm{Homeo}(X)$. *The topological full group* $[[T]]$ *of* T *is* τ-*dense in* $[T]_{\mathrm{C}}$.

As $\mathrm{Homeo}(X)$ is not τ-closed in $\mathrm{Aut}(X, \mathcal{B})$, in [B-D-K 2] the authors brought up the question of describing the closure of $[[T]]$ in $(\mathrm{Aut}(X, \mathcal{B}), \tau)$. They answered it for minimal homeomorphisms (see Theorem 2.8 of [B-D-K 2]) and we generalize it to an arbitrary homeomorphism.

COROLLARY 2.4. *Let* $T \in \mathrm{Homeo}(X)$. *Then* $\overline{[[T]]}^{\tau} = \overline{[T]}_{\mathrm{C}}^{\tau} = [T]_{\mathrm{B}}$.

DEFINITION. A topological group G has the *Rokhlin property* if the action of G on itself by conjugation is topologically transitive, i.e. there is an element of G whose conjugacy class is dense.

The following proposition extends the list of topological groups that have the Rokhlin property. See also [Gl-W] and [Ke-Ros] for other examples.

THEOREM 2.5. *The topological groups* $(\mathrm{Aut}(X, \mathcal{B}), \tau)$ *and* $(\mathrm{Homeo}(X), \tau)$ *have the Rokhlin property.*

Proof. We prove the theorem for $(\mathrm{Homeo}(X), \tau)$ only, the other case being similar.

Take a decomposition $X = \{x_0\} \cup \bigcup_{i \geq 1} X_i$ such that the X_i's are non-empty clopen sets with $\mathrm{diam}(X_i \cup \{x_0\}) \to 0$ as $i \to \infty$. Let S be a homeomorphism such that $Sx_0 = x_0$ and $S^i x = x$, $S^j x \neq x$ for any $x \in X_i$, $j = 1, \ldots, i-1$. Our goal is to show that we can approximate any $T \in \mathrm{Homeo}(X)$ by elements from the conjugacy class of S. By Corollary 2.2, it suffices to approximate periodic homeomorphisms whose canonical partitions are clopen. So, suppose T has a clopen partition $X = \bigcup_{i=1}^{k} Y_i$, where

Y_i is the set of all points having T-period n_i for some $n_i \geq 0$. Observe that there exists a clopen set Y_i^0 such that $Y_i = \bigcup_{j=0}^{n_i-1} T^j Y_i^0$ is a disjoint union (see Lemma 3.2 of [B-D-K 2]). Analogously, there exists a clopen set $X_{n_i}^0$ with $X_{n_i} = \bigcup_{j=0}^{n_i-1} S^j X_{n_i}^0$ being a disjoint union.

Let $U = U(T; \mu_1, \ldots, \mu_p; \varepsilon)$ be a τ-neighborhood of T. Take a non-empty clopen T-invariant set Z with $\mu_i(Z) < \varepsilon$ for $i = 1, \ldots, p$. Without loss of generality, we may assume that $Y_i^0 \setminus Z$ is not empty for $i = 1, \ldots, k$. Let R_i be any homeomorphism from $X_{n_i}^0$ onto $Y_i^0 \setminus Z$. Define a homeomorphism R as follows: let $R(x)$ be equal to $T^j R_i S^{-j}(x)$ whenever $x \in S^j X_{n_i}^0$ for $i = 1, \ldots, k$, $j = 0, \ldots, n_i - 1$ and let R map the rest of X onto Z. It is not hard to check that $RSR^{-1} \in U$.

In the setting of Borel dynamics, we need to produce a periodic transformation that has uncountably many orbits of any finite length. Then, the application of the Rokhlin lemma shows that its conjugacy class is dense. ∎

REMARK. Let p be the topology on $\mathrm{Homeo}(X)$ generated by the metric $D(T, S) = \sup_{x \in X} d(Tx, Sx)$, where d is a metric on X compatible with the topology. In [Gl-W], it is shown that $(\mathrm{Homeo}(X), p)$ has the Rokhlin property. Moreover, the elements whose conjugacy classes are dense form a residual set with respect to p.

Topologically free homeomorphisms. It is interesting to compare the topological properties of the set Ap of all aperiodic homeomorphisms with respect to both topologies τ and p. The following statement is proved in [B-D-K 2, Theorem 2.1].

THEOREM 2.6. *The set* Ap *is dense in* $(\mathrm{Homeo}(X), p)$.

However, the situation in $(\mathrm{Homeo}(X), \tau)$ is completely different. The set Ap is nowhere dense with respect to τ. To see this, one can check that Ap is τ-closed in $\mathrm{Homeo}(X)$. Then an application of Theorem 2.1 implies the result.

The question we investigate in this section is: "How can we extend the class of aperiodic homeomorphisms to produce a τ-dense class?". Apparently, the most natural extension of aperiodic homeomorphisms is the class of topologically free homeomorphisms.

DEFINITION. A homeomorphism is *topologically free* if the set of all aperiodic points is dense.

In Theorem 2.8, we prove that the set of topologically free homeomorphisms is τ-dense. To begin with, we need the following lemma on homeomorphism extensions proved in [Kn-R]. We will need the arguments used in its proof. Thus, we give a sketch of the proof, but without going into the details.

LEMMA 2.7. *Let A and B be closed nowhere dense subsets of Cantor sets X and Y, respectively. Suppose there is a homeomorphism $h : A \to B$. Then h can be extended to a homeomorphism $h^* : X \to Y$ such that $h^*|_A = h$.*

Sketch of proof. Find clopen sets U_i and V_j such that $X \setminus A = \bigsqcup_{i \geq 1} U_i$, $Y \setminus B = \bigsqcup_{j \geq 1} V_j$, and their diameters tend to zero. Find points $a_i \in A$ such that $\mathrm{dist}(U_i, A) = \mathrm{dist}(U_i, a_i)$ and $b_j \in B$ with $\mathrm{dist}(V_j, B) = \mathrm{dist}(V_j, b_j)$.

Set $I = J = \mathbb{N}$. There exist injective functions $f : I \to J$ and $g : J \to I$ such that

$$\mathrm{dist}(U_i, a_i) > \mathrm{dist}(V_{f(i)}, h(a_i)) \qquad \text{for } i \in I,$$
$$\mathrm{dist}(V_j, b_j) > \mathrm{dist}(U_{g(j)}, h^{-1}(b_j)) \quad \text{for } j \in J.$$

Applying the usual Schröder–Bernstein argument to f and g, find disjoint partitions $I = I' \sqcup I''$ and $J = J' \sqcup J''$ such that $f(I') = J'$ and $g(J'') = I''$.

Let ϕ be an arbitrary homeomorphism of $U' = \bigcup_{i \in I'} U_i$ onto $V' = \bigcup_{j \in J'} V_j$ such that $\phi(U_i) = V_{f(i)}$. Analogously, let ψ be a homeomorphism of $V'' = \bigcup_{j \in J''} V_j$ onto $U'' = \bigcup_{i \in I''} U_i$ such that $\psi(V_j) = U_{g(j)}$.

Define

$$h^*(x) = \begin{cases} \phi(x), & x \in U', \\ \psi^{-1}(x), & x \in U'', \\ h(x), & x \in A. \end{cases}$$

For the verification of continuity of h^*, we refer the reader to [Kn-R]. ∎

THEOREM 2.8. *The set of topologically free homeomorphisms is τ-dense in $\mathrm{Homeo}(X)$.*

Proof. By Corollary 2.2, it suffices to approximate the homeomorphisms from Per_0 that have clopen canonical partitions. Assume that R belongs to Per_0 and its canonical partition $X = X_{n_1} \cup \cdots \cup X_{n_m}$ is clopen. Recall that X_{n_i} consists of all points with period n_i. Consider a τ-neighborhood $U = U(R; \mu_1, \ldots, \mu_k; \varepsilon)$. Since the X_{n_i}'s are R-invariant and clopen, we will prove the theorem under the assumption that $X = X_{n_i}$ for some i and leave the generalization to the reader.

Suppose $X = \bigcup_{i=0}^{p-1} R^i F$ is a clopen partition and $R^p x = x$ for all $x \in X$. Using the standard Cantor argument, find a closed nowhere dense set $P \subset R^{p-1} F$ such that $\mu_i(P) > 1 - \varepsilon$ for $i = 1, \ldots, k$. Repeating the proof of Lemma 2.7, we extend the homeomorphism $R : P \to RP$ to a homeomorphism $T : R^{p-1}F \to F$ so that the homeomorphism $T^* \in \mathrm{Homeo}(X)$ defined as $T^*|_{R^{p-1}F} = T|_{R^{p-1}F}$ and $T^* = P$ elsewhere is topologically free. To do this, it suffices to choose the functions ψ and ϕ so that $\phi(x) \neq Rx$ and $\psi^{-1}(x) \neq Rx$ for $x \in R^{p-1}F \setminus P$. Since $E(T^*, R) = R^{p-1}F \setminus P$, we conclude that $T^* \in U$. ∎

3. Structure of homeomorphisms from a topological full group.
In this section, we discuss the structure of homeomorphisms from the topological full group $[[T]]$ for any aperiodic $T \in \mathrm{Homeo}(X)$.

Consider a Cantor aperiodic system (X, T). A Borel set $Y \subset X$ is called *wandering* if $T^n Y \cap Y = \emptyset$ for all $n \geq 1$.

DEFINITION. We say that a closed wandering set Y is *basic* if every clopen neighborhood of Y meets every T-orbit.

THEOREM 3.1. *Every Cantor aperiodic system has a basic set.*

Sketch of proof. Applying the argument developed in [B-D-M, Theorem 2], we can find a decreasing sequence $\{U_n\}$ of clopen sets such that: $U_{n+1} \subset U_n$; $T^i U_n \cap U_n = \emptyset$ for $i = 1, \ldots, n-1$; and U_n meets every T-orbit. Then $Y = \bigcap_n U_n$ is a basic set. ∎

REMARK. For more results on basic sets and their interaction with Bratteli diagrams, see [M].

Fix a triple (X, T, Y), where (X, T) is a Cantor aperiodic system and Y is a basic set. Consider a clopen neighborhood U of Y. It is not hard to check that for every $x \in U$, there is $n = n(x) > 0$ such that $T^n x \in U$. Therefore, it follows from the definition of a basic set that, by applying the first return function, we can construct a clopen K-R partition $\varXi$ of X with base $B(\varXi) = U$.

Take a decreasing sequence $\{U_n\}$ of clopen sets such that $Y = \bigcap_n U_n$. Constructing clopen K-R partitions for the U_n's and refining them, we prove the following:

THEOREM 3.2. *Let (X, T, Y) be a Cantor aperiodic system with a basic set Y. There exists a sequence $\{\mathcal{P}_n\}$ of clopen K-R partitions of X such that for all $n \geq 1$:*

   (i) $\mathcal{P}_{n+1}$ *refines* $\mathcal{P}_n$;
   (ii) $h_{n+1} > h_n$, *where* h_n *is the height of the lowest T-tower in* $\mathcal{P}_n$;
   (iii) $B(\mathcal{P}_n) \supset B(\mathcal{P}_{n+1})$;
   (iv) *the sequence* $\{\mathcal{P}_n\}$ *generates the clopen topology of* X;
   (v) $\bigcap_n B(\mathcal{P}_n) = Y$.

We will follow here the method developed in [B-K] for minimal homeomorphisms (see also [K-W]). Let $\mathcal{P}$ be a clopen K-R partition with towers $\mathcal{P}(i)$, $i = 1, \ldots, k$. We will define two partitions $\alpha = \alpha(\mathcal{P})$ and $\alpha' = \alpha'(\mathcal{P})$ of $\{1, \ldots, k\}$. We say that J is an atom of α if there exists a subset J' (then said to belong to α') such that

$$(3.1) \qquad T\Big(\bigcup_{i \in J} T^{h(i)-1} D_i \Big) = \bigcup_{i' \in J'} D_{i'}$$

and for every proper subset J_0 of J, the T-image of $\bigcup_{i \in J_0} T^{h(i)-1} D_i$ is not a union of atoms from $\mathcal{P}$. It follows from (3.1) that J' is uniquely defined by J and T.

Let $S \in [[T]]$. Then there are a finite set $K \subset \mathbb{Z}$ and a clopen partition $\mathcal{E} = \{E_k : k \in K\}$ of X such that $Sx = T^k x$ for $x \in E_k$ and $k \in K$. Denote by $\mathcal{E}(K)$ the clopen partition $\{S^k E_k : k \in K\}$. By Theorem 3.2, find a K-R partition $\mathcal{P} = \{\mathcal{P}(i) : i = 1, \ldots, k\}$ with $\mathcal{P}(i) = \{D_{0,i}, \ldots, D_{h(i)-1,i}\}$ and $D_{j+1,i} = TD_{j,i}$ that refines $\mathcal{E}$ and $\mathcal{E}(K)$ and so that $K \subset (-h, h)$, where h is the height of the lowest T-tower in $\mathcal{P}$.

Let $\mathcal{F} = \{(j,i) : i = 1, \ldots, k, \ j = 0, \ldots, h(i)-1\}$. Observe that for every pair $(j,i) \in \mathcal{F}$ there is a unique $l = l(j,i) \in K$ such that

$$(3.2) \qquad\qquad S(D_{j,i}) = T^l D_{j,i}.$$

Divide $\mathcal{F} = \mathcal{F}(\mathcal{P})$ into three disjoint sets $\mathcal{F}_{\mathrm{in}}, \mathcal{F}_{\mathrm{top}}$ and $\mathcal{F}_{\mathrm{bot}}$ as follows:

(a) $(j,i) \in \mathcal{F}_{\mathrm{in}}$ if $S(D_{j,i}) \subset \mathcal{P}(i)$, i.e. $0 \le l + j \le h(i)-1$;
(b) $(j,i) \in \mathcal{F}_{\mathrm{top}}$ if $S(D_{j,i})$ goes through the top of $\mathcal{P}(i)$, i.e. $l + j \ge h(i)$;
(c) $(j,i) \in \mathcal{F}_{\mathrm{bot}}$ if $S(D_{j,i})$ goes through the bottom of $\mathcal{P}(i)$, i.e. $l+j < 0$;

here l is taken from (3.2).

Let α and α' be the partitions of $\{1, \ldots, k\}$ defined by T and $\mathcal{P}$. For $J \subset \{1, \ldots, k\}$, set $h_J = \min\{h(i) : i \in J\}$. For $J \in \alpha$ and $J' \in \alpha'$, let

$$F_1(r, J) = \bigcup_{i \in J} D_{h(i)-h_J+r,i}, \qquad F_2(r', J') = \bigcup_{i' \in J'} D_{r,i'},$$

where $r = 0, \ldots, h_J - 1$ and $r' = 0, \ldots, h_{J'} - 1$.

DEFINITION. We say that $S \in [[T]]$ belongs to $\Gamma(\mathcal{P})$ if for each pair $(j,i) \in \mathcal{F}$ the following conditions hold:

(a) if $(j,i) \in \mathcal{F}_{\mathrm{top}}$ and $D_{j,i} \subset E_l$, then $F_1(h_J - h(i) + j, J) \subset E_l$, where J is the atom of α containing i;
(b) if $(j,i) \in \mathcal{F}_{\mathrm{bot}}$ and $D_{j,i} \subset E_l$, then $F_2(j, J') \subset E_l$, where J' is the atom of α' containing i.

Condition (a) means that whenever the set $D_{j,i}$ goes through the top of $\mathcal{P}(i)$ under the action of S, then the entire level $F_1(r, J)$ containing $D_{j,i}$ also goes through the top of $\mathcal{P}$. Similarly, one can explain condition (b) by taking the $D_{j,i}$'s and levels $F_2(j, J')$ containing them that go through the bottom of $\mathcal{P}$. Observe that if $(j,i) \in \mathcal{F}_{\mathrm{in}}$, then the entire levels $F_1(r, J)$ and $F_2(j, J')$ containing $D_{j,i}$ remain "within" $\mathcal{P}$.

Clearly, $\Gamma(\mathcal{P})$ is a finite set. The following theorem reveals the structure of homeomorphisms from $[[T]]$ for an arbitrary aperiodic homeomorphism T. Notice that this structure was found earlier for minimal homeomorphisms (see Theorem 2.2 in [B-K]). Since our proof is similar to that in [B-K, Theorem 2.2], we omit it.

THEOREM 3.3. *Let (X, T, Y) be a Cantor aperiodic system with a basic set Y and a sequence $\{\mathcal{P}_n\}$ of K-R partitions satisfying the conditions of Theorem 3.2. Then $[[T]] = \bigcup_n \Gamma(\mathcal{P}_n)$ with $\Gamma(\mathcal{P}_n) \subset \Gamma(\mathcal{P}_{n+1})$.*

The subgroup Γ_Y. Let (X, T) be a Cantor aperiodic system with a basic set Y. Define the subgroup Γ_Y of $[[T]]$ as follows: $S \in \Gamma_Y$ if $S \in \Gamma(\mathcal{P}_n)$ (and hence $S \in \Gamma(\mathcal{P}_m)$ for $m > n$) implies that $\mathcal{F}(\mathcal{P}_n) = \mathcal{F}_{\text{in}}$. In other words, $S \in \Gamma_Y$ if no level from $\mathcal{P}_n$ goes over the top or through the bottom under the action of S. This means that S acts as a permutation on each T-tower from $\mathcal{P}_n$. Therefore, the group Γ_Y is an increasing union of permutation groups.

Our objective is to find a criterion for Γ_Y to be dense in $[T]$.

REMARK. (1) Denote by $[[T]]_Y$ the subgroup of $[[T]]$ consisting of the homeomorphisms that preserve the forward T-orbit of every $y \in Y$, i.e., $S \in [[T]]_Y$ if $S(\{T^n y : n \geq 0\}) = \{T^n y : n \geq 0\}$. Observe that $\Gamma_Y \subset [[T]]_Y$.

(2) The subgroup $[[T]]_Y$ is not τ-dense in $[T]$. Therefore, so is Γ_Y. Indeed, take any $z \in T^{-1}Y$ and the Dirac measure δ_z. Consider $S \in U := U(T; \delta_z; 1/2)$. As $z \notin Y$ and $Sz = Tz \in Y$, S does not preserve the forward T-orbit of Tz. Therefore, U contains no elements from $[[T]]_Y$.

The fact that Γ_Y is not τ-dense in $[[T]]$ is mainly caused by the presence of discrete measures. We can partly overcome this obstacle by considering only continuous measures in the definition of the topology τ. Denote by τ_0 the topology defined by continuous measures as in Definition 1.1. One can check that τ_0 is a Hausdorff group topology on $\text{Homeo}(X)$. The next theorem answers the question when Γ_Y is τ_0-dense in $[T]$.

THEOREM 3.4. *Suppose we have a Cantor aperiodic system (X, T) with a basic set Y. Then the subgroup Γ_Y is τ_0-dense in $[T]$ if and only if the basic set Y is at most countable.*

Proof. (1) Assume that Y is uncountable. Take any continuous measure μ supported by $T^{-1}Y$. Then for every $S \in U := U(T; \mu; 1/2)$ there is at least one $z \in T^{-1}Y$ such that $Sz = Tz$. This implies that $\{T^n(Tz) : n \geq 0\}$ is not S-invariant. Therefore, by (1) of the remark above we conclude that $\Gamma_Y \cap U = \emptyset$.

(2) Now, assume that Y is countable. Observe that by Corollary 2.3, it is enough to approximate homeomorphisms from $[[T]]$ with elements of Γ_Y. Consider $R \in [[T]]$ and a τ_0-neighborhood $U = U(R; \mu_1, \ldots, \mu_p; \varepsilon)$ of R. By definition of R, the sets $E_k = \{x \in X : Rx = T^k x\}$, $k \in K$, $|K| < \infty$, form a clopen partition of X. Let $k_0 = \sup\{|k| : k \in K\}$. As Y is countable, $\mu(T^n Y) = 0$ for any continuous measure μ and integer n. Therefore, by Theorem 3.2 we can find a K-R partition $\mathcal{P}_n$ refining $\{E_k\}_{k \in K}$ such that

$2k_0 < h_n$, where h_n is the height of the lowest T-tower in $\mathcal{P}_n$, and

$$(3.3) \qquad \mu_j\Big(\bigcup_{i=-k_0}^{k_0} T^i B(\mathcal{P}_n) \Big) < \varepsilon \quad \text{for } j = 1, \ldots, p.$$

Define a homeomorphism $S \in \Gamma_Y \cap U$ as follows. Take a T-tower (say $\lambda = \{D, \ldots, T^{h(\lambda)-1}D\}$) from $\mathcal{P}_n$. Consider an atom $T^l D$ of λ. We have two possibilities:

  (i) The R-orbit of the set $T^l D$ does not leave the T-tower λ. In this case, we define S to be equal to R on the R-orbit of $T^l D$.
  (ii) The set $T^l D$ leaves λ under the action of R. Then there exist integers $q < 0 < d$ such that $R^{d+1}T^l D$ and $R^{q-1}T^l D$ do not lie in λ entirely, whereas the sets $R^j T^l D$, $j = q, \ldots, d$, are contained in λ. In this case, we set $S = R$ on $R^i T^l D$, $i = q, \ldots, d-1$, and $S = R^{-d+q}$ on $R^d T^l D$.

Observe that the choice of $\mathcal{P}_n$ guarantees that

$$R^d T^l D \subset \bigcup_{i=0}^{k_0} T^i D \cup \bigcup_{i=h(\lambda)-1-k_0}^{h(\lambda)-1} T^i D.$$

Clearly, the homeomorphism S constructed in (i) and (ii) is periodic, but it is not yet defined on the entire space. To extend its domain, we consider an atom $T^w D$ of λ on which S is not defined yet, and repeat (i) and (ii) with $T^w D$.

Repeating this procedure with every atom of λ, we define S on λ. Moreover, λ is S-invariant. By construction, S coincides with P everywhere, except maybe on the set $\bigcup_{i=0}^{k_0} T^i D \cup \bigcup_{i=h(\lambda)-1-k_0}^{h(\lambda)-1} T^i D$.

To finish constructing S, we need to repeat the argument for every T-tower of $\mathcal{P}_n$.

The definition of S implies that $S \in \Gamma_Y$ and

$$\{x \in X : Sx \neq Rx\} \subset \bigcup_{i=-k_0}^{k_0} T^i B(\mathcal{P}_n).$$

Therefore, by (3.3) we conclude that $S \in U$. ∎

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