Finite-to-one continuous s-covering mappings

by

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Abstract. The following theorem is proved. Let $f : X \to Y$ be a finite-to-one map such that the restriction $f|f^{-1}(S)$ is an inductively perfect map for every countable compact set $S \subset Y$. Then $Y$ is a countable union of closed subsets $Y_i$ such that every restriction $f|f^{-1}(Y_i)$ is an inductively perfect map.

All spaces in this paper are supposed to be separable and metrizable and all the mappings $f : X \to Y$ to be continuous and “onto”.

We recall the following definitions:

$f$ is inductively perfect if there exists a closed subset $X' \subset X$ such that $f(X') = Y$ and the restriction $f|X'$ is perfect, i.e. $f|X'$ is a closed map with compact fibers $f^{-1}(y)$.

$f$ is s-covering if $f|f^{-1}(S)$ is inductively perfect for every countable and compact set $S \subset Y$ \(^{(1)}\).

The following main theorem is an obvious corollary of Theorem 6 below:

**Theorem 1.** If $f : X \to Y$ is a finite-to-one s-covering mapping, then $Y$ is a countable union of closed subsets $Z_i$ such that every restriction $f|f^{-1}(Z_i)$ is an inductively perfect mapping. If $Y \subset 2^\omega$, then the $Z_i$ are pairwise disjoint.

Under the assumption that for some integer $n$ all the fibers have at most $n$ points G. Debs and J. Saint Raymond proved that $f$ is inductively perfect, but the finiteness of the fibers does not suffice to ensure the same conclusion [1].

\(^{(1)}\) Since the inverse image of a compact set under a perfect mapping is always compact, a mapping $f : X \to Y$ is s-covering if and only if every countable compact subset $S \subset Y$ is the image of some compact $B \subset X$.

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[89]
1. Some properties of $s$-covering mappings. In this section, we use
the following property of $s$-covering mappings, which was proved by W. Just
and H. Wicke [2], as well as independently by the author [3].

**Proposition 2.** A mapping $f$ is $s$-covering if and only if in every fiber
$f^{-1}(y)$ there exists a nonempty family $\varepsilon_y$ of nonempty compact subsets such
that every open set containing $K \in \varepsilon_y$ also contains a set $K' \in \varepsilon_{y'}$ for any
point $y'$ from a neighborhood of $y$.

Throughout the paper, we keep the notation $\varepsilon_y$, $y \in Y$, for the above
families of compact subsets of the fibers $f^{-1}(y)$ of an $s$-covering map $f : X \to Y$.

**Lemma 3.** Let $f : X \to Y$ be an $s$-covering mapping. Set

$$M_y = \bigcap \{ K : K \in \varepsilon_y \}, \quad X_0 = \bigcup_{y \in Y} M_y, \quad Y_0 = f(X_0).$$

Then the restriction $f|X_0$ is a perfect mapping.

**Proof.** Let $y \in f(X_0)$ and $V \supseteq M_y$ be an open set. We will prove that
$f_0 = f|X_0$ is a closed mapping by applying the following characterization:
$f_0 : X_0 \to Y_0$ is closed if and only if for every $y \in Y_0$ and every open
$V \supseteq f_0^{-1}(y)$ there is an open $O \ni y$ such that $f_0^{-1}(y') \subset V$ for every $y' \in O$.

Since $M_y$ is compact, there are finitely many $K_i \in \varepsilon_y$ such that $\bigcap_i K_i \subset V$.
It follows from the normality of $X$ that there are open sets $V_i \supseteq K_i \setminus V$ such that
$\bigcap_i V_i = \emptyset$.

Since $V_i \cup V \supseteq K_i$ are open sets, for every $i$ there exists an open set
$O_i(y)$ such that for every $y' \in O_i(y)$ there is $B'_i \in \varepsilon_{y'}$ with $B'_i \subset V_i \cup V$.

Let $O(y) = \bigcap_i O_i(y)$. If $y' \in O(y) \cap f(X_0)$, then

$$M_{y'} = \bigcap \{ K : K \in \varepsilon_{y'} \} \subset \bigcap_i B'_i \subset \bigcap_i (V_i \cup V) = \left( \bigcap_i V_i \right) \cup V = V,$$

and hence $f|X_0$ is a closed mapping with compact fibers $M_y$.

**Lemma 4.** Let $f : X \to Y$ be an $s$-covering mapping, let $X_0, Y_0$ be as in
Lemma 3, and define inductively

$$Y_i = \left\{ y \in Y \setminus \bigcup_{k=0}^{i-1} Y_k : \exists K^1_y, \ldots, K^{i+1}_y \in \varepsilon_y \right.$$ 

such that $K^1_y \cap \ldots \cap K^{i+1}_y = \emptyset \} \text{ for } i \geq 1.$

Then $Y = \bigcup_{i=0}^{\infty} Y_i$ and $Y_i$ are pairwise disjoint $F_\sigma$-sets.

(2) It is easy to see that $|\varepsilon_y| = 1$ for all $y \in Y$ if and only if $f$ is inductively perfect.
Proof. Note that if \( y \in \bigcap_{i=1}^n Y_i \), then there exist \( i \in \{1, \ldots, n\} \) and \( K_y^1, \ldots, K_y^{i+1} \in \mathcal{E}_y \) such that \( K_y^1 \cap \cdots \cap K_y^{i+1} = \emptyset \). By the normality of \( X \), there are open sets \( O_j \supset K_y^j \) (\( j = 1, \ldots, i + 1 \)) such that \( \bigcap_j O_j = \emptyset \), and by the definition of \( \mathcal{E}_y \) (Proposition 2) there is an open set \( O \ni y \) such that for every \( y' \in O \) one has \( K_{y'}^j \subset O_j \) for some \( K_{y'}^j \in \mathcal{E}_{y'} \). Since \( \bigcap_j O_j = \emptyset \), we obtain \( \bigcap_j K_{y'}^j = \emptyset \) and hence \( y' \in \bigcup_{i=1}^n Y_i \) for \( y' \in O \). This implies that \( \bigcup_{i=1}^n Y_i \) is open in \( Y \) and \( Y_n = \bigcup_{i=0}^n Y_i \setminus \bigcup_{i=0}^{n-1} Y_i \) is of type \( F_\sigma \), for each \( n > 0 \).

Suppose \( y \in Y \setminus Y_0 \). Then \( \bigcap \{ K : K \in \mathcal{E}_y \} = \emptyset \). Since the sets \( K \) are compact, there are finitely many \( K^j \in \mathcal{E}_y \) such that \( \bigcap_j K^j = \emptyset \). Hence, \( y \) belongs to some \( Y_i \) and \( Y = \bigcup_{i=0}^\infty Y_i \).

2. \( s \)-covering mappings with finite families \( \mathcal{E}_y \)

**Lemma 5.** Let \( f : X \to Y \) be an \( s \)-covering mapping with finite families \( \mathcal{E}_y \), and let \( Y_i \) be as in Lemma 4. Then for every \( y \in Y_i \) (\( i = 1, 2, \ldots \)) there is an open subset \( O(y) \) of \( Y \) such that the restriction of \( f \) to \( f^{-1}(O(y) \cap Y_i) \) is an \( s \)-covering map onto \( O(y) \cap Y_i \) with a family \( \mathcal{E}^1_y \subset \mathcal{E}_y \), hence, \( \text{card}(\mathcal{E}^1_y) \leq \text{card}(\mathcal{E}_y) - 1 \).

**Proof.** As in the proof of Lemma 4 there are open sets \( O_j \supset K_{y}^j \in \mathcal{E}_y \) such that \( \bigcap_{j=1}^{i+1} O_j = \emptyset \) and, hence,

\[
(1) \quad O_1 \cap \bigcap_{j=2}^{i+1} O_j = \emptyset.
\]

Let \( O(y) \) be an open set such that for every \( y' \in O(y) \cap Y_i \) and every \( O_j \) there is \( K_{y'}^j \subset O_j \) for which \( K_{y'}^j \in \mathcal{E}_{y'} \) (\( j = 1, \ldots, i + 1 \)). Since \( y' \in Y_i \), and hence \( y' \not\in Y_{i-1} \), we have

\[
(2) \quad \bigcap_{j=2}^{i+1} K_{y'}^j \neq \emptyset.
\]

**Claim.** There is \( j > 1 \) such that \( K_{y'}^j \not\subset O_1 \).

Suppose not; then \( K_{y'}^j \subset O_1 \) for all \( j = 2, \ldots, i + 1 \), and hence

\[
(3) \quad \bigcap_{j=2}^{i+1} K_{y'}^j \subset O_1.
\]

Since \( K_{y'}^j \subset O_j \), it follows that

\[
(4) \quad \bigcap_{j=2}^{i+1} K_{y'}^j \subset \bigcap_{j=2}^{i+1} O_j.
\]

The conditions (2), (3), (4) contradict (1).
For every \( y' \in O(y) \cap Y_i \), there is \( j \) such that \( K^j_{y'} \not\subset O_1 \). It follows that the restriction of \( f \) to \( O_1 \cap f^{-1}(O(y) \cap Y_i) \) is an \( s \)-covering map onto \( O(y) \cap Y_i \) with a family \( \varepsilon^1_y \) such that \( \text{card}(\varepsilon^1_y) \leq \text{card}(\varepsilon_y) - 1 \).

**Theorem 6.** Let \( f : X \to Y \) be an \( s \)-covering mapping with finite families \( \varepsilon_y \). Then \( Y \) is a countable union of closed subsets \( Z_i \) such that every restriction \( f|f^{-1}(Z_i) \) is an inductively perfect mapping.

Indeed, it follows from Lemma 4 that every set \( O(y) \cap Y_i \) is \( F_\sigma \) in \( Y_i \). If \( Y \subset 2^\omega \), it is well known that the open cover \( \{O(y)\}_{y \in Y_i} \) of the zero-dimensional space \( Y_i \) has a refinement consisting of clopen (in \( Y_i \)) pairwise disjoint sets \( F_{i_r} \). Hence, \( Y_i = \bigcup_{i_r} F_{i_r} \) is a countable union of pairwise disjoint subsets closed in \( Y_i \). In the general case (\( Y \not\subset 2^\omega \)), the open cover \( \{O(y)\}_{y \in Y_i} \) of \( Y_i \) has a locally finite open refinement and the sets \( F_{i_r} \) are only closed in \( Y \) and not pairwise disjoint.

Now Theorem 6 results from step-by-step application of Lemma 5 to the sets \( F_{i_r} \), etc.

### 3. Application to Borel sets

**Theorem 7.** If \( f : X \to Y \) is an \( s \)-covering finite-to-one mapping of a Borel set \( X \subset 2^\omega \) of additive or multiplicative class \( \alpha \geq 1 \) onto \( Y \subset 2^\omega \), then \( Y \) is a Borel set of the same class.

**Proof.** If \( X \) is of additive class \( \alpha \), then, by Theorem 1 and by the theorem on preservation of the Borel class under perfect mappings \((3)\), every \( Z_i \) is of additive class \( \alpha \). It is obvious that \( Y \) is of additive class \( \alpha \) because it is the countable union of the \( Z_i \).

Let \( X \) be of multiplicative class \( \alpha \). If \( \alpha = 1 \), then by \([4, \text{Main result}]\), \( Y \) is of multiplicative class 1.

For \( \alpha > 1 \) we consider in \( C = 2^\omega \) according to Theorem 1 the sets \( L_i = [Z_i]_C \setminus Z_i \) of additive class \( \alpha \). Obviously,

\[
Y = \bigcup_i Z_i = \bigcup_i ([Z_i]_C \setminus L_i) = \bigcup_i [Z_i]_C \setminus \bigcup_i L_i,
\]

where \( \bigcup_i [Z_i]_C \) is of multiplicative class 2 and \( \bigcup_i L_i \) is of additive class \( \alpha \). This implies that \( Y \) is of multiplicative class \( \alpha \).

**Question.** I do not know whether the conclusion of Theorem 6 is still true if the condition that \( f \) is an \( s \)-covering mapping with finite families \( \varepsilon_y \) is replaced by the condition that \( f \) is an \( s \)-covering mapping with compact fibers and each set in any family \( \varepsilon_y \) is finite.

\[(3)\] A. D. Taimanov proved \([6, \text{Theorem 6}]\) that the image of a Borel set of class \( \xi \) under a perfect mappings is of the same class if \( \xi \geq \omega_0 \), and of class \( \xi + 1 \) if \( 1 < \xi < \omega_0 \). J. Saint Raymond proved the preservation in the case \( 1 < \xi < \omega_0 \) \([5]\).
References


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