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Finite-to-one continuous s-covering mappings

by

Alexey Ostrovsky (München)

Abstract. The following theorem is proved. Let $f : X \to Y$ be a finite-to-one map such that the restriction $f|f^{-1}(S)$ is an inductively perfect map for every countable compact set $S \subset Y$. Then Y is a countable union of closed subsets Y_i such that every restriction $f|f^{-1}(Y_i)$ is an inductively perfect map.

All spaces in this paper are supposed to be separable and metrizable and all the mappings $f : X \to Y$ to be continuous and "onto".

We recall the following definitions:

f is *inductively perfect* if there exists a closed subset $X' \subset X$ such that $f(X') = Y$ and the restriction $f|X'$ is perfect, i.e. $f|X'$ is a closed map with compact fibers $f^{-1}(y)$.

f is *s-covering* if $f|f^{-1}(S)$ is inductively perfect for every countable and compact set $S \subset Y$ ([1]).

The following main theorem is an obvious corollary of Theorem 6 below:

THEOREM 1. *If $f : X \to Y$ is a finite-to-one s-covering mapping, then Y is a countable union of closed subsets Z_i such that every restriction $f|f^{-1}(Z_i)$ is an inductively perfect mapping. If $Y \subset 2^\omega$, then the Z_i are pairwise disjoint.*

Under the assumption that for some integer n all the fibers have at most n points G. Debs and J. Saint Raymond proved that f is inductively perfect, but the finiteness of the fibers does not suffice to ensure the same conclusion [1].

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([1]) Since the inverse image of a compact set under a perfect mapping is always compact, a mapping $f : X \to Y$ is s-covering if and only if every countable compact subset $S \subset Y$ is the image of some compact $B \subset X$.

1. Some properties of s-covering mappings. In this section, we use the following property of s-covering mappings, which was proved by W. Just and H. Wicke [2], as well as independently by the author [3].

PROPOSITION 2. *A mapping f is s-covering if and only if in every fiber $f^{-1}(y)$ there exists a nonempty family ε_y of nonempty compact subsets such that every open set containing $K \in \varepsilon_y$ also contains a set $K' \in \varepsilon_{y'}$ for any point y' from a neighborhood of y* ([2]).

Throughout the paper, we keep the notation ε_y, $y \in Y$, for the above families of compact subsets of the fibers $f^{-1}(y)$ of an s-covering map $f : X \to Y$.

LEMMA 3. *Let $f : X \to Y$ be an s-covering mapping. Set*

$$M_y = \bigcap\{K : K \in \varepsilon_y\}, \qquad X_0 = \bigcup_{y \in Y} M_y, \qquad Y_0 = f(X_0).$$

Then the restriction $f|X_0$ is a perfect mapping.

Proof. Let $y \in f(X_0)$ and $V \supset M_y$ be an open set. We will prove that $f_0 = f|X_0$ is a closed mapping by applying the following characterization: $f_0 : X_0 \to Y_0$ is closed if and only if for every $y \in Y_0$ and every open $V \supset f_0^{-1}(y)$ there is an open $O \ni y$ such that $f_0^{-1}(y') \subset V$ for every $y' \in O$.

Since M_y is compact, there are finitely many $K_i \in \varepsilon_y$ such that $\bigcap_i K_i \subset V$. It follows from the normality of X that there are open sets $V_i \supset K_i \setminus V$ such that $\bigcap_i V_i = \emptyset$.

Since $V_i \cup V \supset K_i$ are open sets, for every i there exists an open set $O_i(y)$ such that for every $y' \in O_i(y)$ there is $B_i' \in \varepsilon_{y'}$ with $B_i' \subset V_i \cup V$.

Let $O(y) = \bigcap_i O_i(y)$. If $y' \in O(y) \cap f(X_0)$, then

$$M_{y'} = \bigcap\{K : K \in \varepsilon_{y'}\} \subset \bigcap_i B_i' \subset \bigcap_i (V_i \cup V) = \Big(\bigcap_i V_i\Big) \cup V = V,$$

and hence $f|X_0$ is a closed mapping with compact fibers M_y.

LEMMA 4. *Let $f : X \to Y$ be an s-covering mapping, let X_0, Y_0 be as in Lemma 3, and define inductively*

$$Y_i = \Big\{y \in Y \setminus \bigcup_{k=0}^{i-1} Y_k : \exists K_y^1, \ldots, K_y^{i+1} \in \varepsilon_y \text{ such that } K_y^1 \cap \cdots \cap K_y^{i+1} = \emptyset\Big\} \quad \text{for } i \geq 1.$$

Then $Y = \bigcup_{i=0}^{\infty} Y_i$ and Y_i are pairwise disjoint F_σ-sets.

([2]) It is easy to see that $|\varepsilon_y| = 1$ for all $y \in Y$ if and only if f is inductively perfect.

Proof. Note that if $y \in \bigcup_{i=1}^{n} Y_i$, then there exist $i \in \{1,\ldots,n\}$ and $K_y^1,\ldots,K_y^{i+1} \in \varepsilon_y$ such that $K_y^1 \cap \cdots \cap K_y^{i+1} = \emptyset$. By the normality of X, there are open sets $O_j \supset K_y^j$ $(j = 1,\ldots,i+1)$ such that $\bigcap_j O_j = \emptyset$, and by the definition of ε_y (Proposition 2) there is an open set $O \ni y$ such that for every $y' \in O$ one has $K_{y'}^j \subset O_j$ for some $K_{y'}^j \in \varepsilon_{y'}$. Since $\bigcap_j O_j = \emptyset$ we obtain $\bigcap_j K_{y'}^j = \emptyset$ and hence $y' \in \bigcup_{i=1}^{n} Y_i$ for $y' \in O$. This implies that $\bigcup_{i=1}^{n} Y_i$ is open in Y and $Y_n = \bigcup_{i=0}^{n} Y_i \setminus \bigcup_{i=0}^{n-1} Y_i$ is of type F_σ, for each $n > 0$.

Suppose $y \in Y \setminus Y_0$. Then $\bigcap\{K : K \in \varepsilon_y\} = \emptyset$. Since the sets K are compact, there are finitely many $K^j \in \varepsilon_y$ such that $\bigcap_j K^j = \emptyset$. Hence, y belongs to some Y_i and $Y = \bigcup_{i=0}^{\infty} Y_i$.

2. s-covering mappings with finite families ε_y

LEMMA 5. *Let $f : X \to Y$ be an s-covering mapping with finite families ε_y, and let Y_i be as in Lemma 4. Then for every $y \in Y_i$ $(i = 1, 2, \ldots)$ there is an open subset $O(y)$ of Y such that the restriction of f to $f^{-1}(O(y) \cap Y_i)$ is an s-covering map onto $O(y) \cap Y_i$ with a family $\varepsilon_y^1 \subsetneq \varepsilon_y$, hence*, $\mathrm{card}(\varepsilon_y^1) \le \mathrm{card}(\varepsilon_y) - 1$.

Proof. As in the proof of Lemma 4 there are open sets $O_j \supset K_y^j \in \varepsilon_y$ such that $\bigcap_{j=1}^{i+1} O_j = \emptyset$ and, hence,

$$O_1 \cap \bigcap_{j=2}^{i+1} O_j = \emptyset. \tag{1}$$

Let $O(y)$ be an open set such that for every $y' \in O(y) \cap Y_i$ and every O_j there is $K_{y'}^j \subset O_j$ for which $K_{y'}^j \in \varepsilon_{y'}$ $(j = 1,\ldots,i+1)$. Since $y' \in Y_i$, and hence $y' \notin Y_{i-1}$, we have

$$\bigcap_{j=2}^{i+1} K_{y'}^j \ne \emptyset. \tag{2}$$

CLAIM. *There is $j > 1$ such that $K_{y'}^j \not\subset O_1$.*

Suppose not; then $K_{y'}^j \subset O_1$ for all $j = 2,\ldots,i+1$, and hence

$$\bigcap_{j=2}^{i+1} K_{y'}^j \subset O_1. \tag{3}$$

Since $K_{y'}^j \subset O_j$, it follows that

$$\bigcap_{j=2}^{i+1} K_{y'}^j \subset \bigcap_{j=2}^{i+1} O_j. \tag{4}$$

The conditions (2), (3), (4) contradict (1).

For every $y' \in O(y) \cap Y_i$, there is j such that $K^j_{y'} \not\subset O_1$. It follows that the restriction of f to $O_1 \cap f^{-1}(O(y) \cap Y_i)$ is an s-covering map onto $O(y) \cap Y_i$ with a family ε^1_y such that $\mathrm{card}(\varepsilon^1_y) \le \mathrm{card}(\varepsilon_y) - 1$.

THEOREM 6. *Let $f : X \to Y$ be an s-covering mapping with finite families ε_y. Then Y is a countable union of closed subsets Z_i such that every restriction $f|f^{-1}(Z_i)$ is an inductively perfect mapping.*

Indeed, it follows from Lemma 4 that every set $O(y) \cap Y_i$ is F_σ in Y. If $Y \subset 2^\omega$, it is well known that the open cover $\{O(y)\}_{y \in Y_i}$ of the zero-dimensional space Y_i has a refinement consisting of clopen (in Y_i) pairwise disjoint sets F_{i_r}. Hence, $Y_i = \bigcup_{i_r} F_{i_r}$ is a countable union of pairwise disjoint subsets closed in Y. In the general case ($Y \not\subset 2^\omega$), the open cover $\{O(y)\}_{y \in Y_i}$ has a locally finite open refinement and the sets F_{i_r} are only closed in Y and not pairwise disjoint.

Now Theorem 6 results from step-by-step application of Lemma 5 to the sets F_{i_r}, etc.

3. Application to Borel sets

THEOREM 7. *If $f : X \to Y$ is an s-covering finite-to-one mapping of a Borel set $X \subset 2^\omega$ of additive or multiplicative class $\alpha \ge 1$ onto $Y \subset 2^\omega$, then Y is a Borel set of the same class.*

Proof. If X is of additive class α, then, by Theorem 1 and by the theorem on preservation of the Borel class under perfect mappings (3), every Z_i is of additive class α. It is obvious that Y is of additive class α because it is the countable union of the Z_i.

Let X be of multiplicative class α. If $\alpha = 1$, then by [4, Main result], Y is of multiplicative class 1.

For $\alpha > 1$ we consider in $\mathbf{C} = 2^\omega$ according to Theorem 1 the sets $L_i = [Z_i]_{\mathbf{C}} \setminus Z_i$ of additive class α. Obviously,

$$Y = \bigcup_i Z_i = \bigcup_i ([Z_i]_{\mathbf{C}} \setminus L_i) = \bigcup_i [Z_i]_{\mathbf{C}} \setminus \bigcup_i L_i,$$

where $\bigcup_i [Z_i]_{\mathbf{C}}$ is of multiplicative class 2 and $\bigcup_i L_i$ is of additive class α. This implies that Y is of multiplicative class α.

QUESTION. I do not know whether the conclusion of Theorem 6 is still true if the condition that f is an s-covering mapping with finite families ε_y is replaced by the condition that f is an s-covering mapping with compact fibers and each set in any family ε_y is finite.

(3) A. D. Taimanov proved [6, Theorem 6] that the image of a Borel set of class ξ under a perfect mappings is of the same class if $\xi \ge \omega_0$, and of class $\xi + 1$ if $1 < \xi < \omega_0$. J. Saint Raymond proved the preservation in the case $1 < \xi < \omega_0$ [5].

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Bundeswehr University Munich
D-85577 Neubiberg, Germany
E-mail: Alexey.Ostrovskiy@UniBw.de

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