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On dimensionally restricted maps

by

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Abstract. Let $f : X \to Y$ be a closed n-dimensional surjective map of metrizable spaces. It is shown that if Y is a C-space, then: (1) the set of all maps $g : X \to \mathbb{I}^n$ with $\dim(f \triangle g) = 0$ is uniformly dense in $C(X, \mathbb{I}^n)$; (2) for every $0 \le k \le n-1$ there exists an F_σ-subset A_k of X such that $\dim A_k \le k$ and the restriction $f|(X \setminus A_k)$ is $(n-k-1)$-dimensional. These are extensions of theorems by Pasynkov and Toruńczyk, respectively, obtained for finite-dimensional spaces. A generalization of a result due to Dranishnikov and Uspenskij about extensional dimension is also established.

1. Introduction. All spaces are assumed to be completely regular and all maps continuous. This paper is concerned with the following two results. The first one was proved by Pasynkov [25] (see [24] for noncompact versions) and the second one by Toruńczyk [31]:

Theorem 1.1 (Pasynkov). *Let $f : X \to Y$ be an n-dimensional map with X and Y being finite-dimensional compact metric spaces. Then there exists $g : X \to \mathbb{I}^n$ such that $f \triangle g : X \to Y \times \mathbb{I}^n$ is 0-dimensional. Moreover, the set of all such g is dense and G_δ in $C(X, \mathbb{I}^n)$ with respect to the uniform convergence topology.*

Theorem 1.2 (Toruńczyk). *Let $f : X \to Y$ be a σ-closed map of separable metric spaces with $\dim f = n$ and $\dim Y < \infty$. Then for each $0 \le k \le n-1$ there exists an F_σ-subset A_k of X such that $\dim A_k \le k$ and the restriction $f|(X \setminus A_k)$ is $(n-k-1)$-dimensional.*

The above two theorems are equivalent in the realm of compact spaces (see [19] and [29]). However, the problem whether they hold without any dimensional restrictions on Y is still open. Sternfeld and Levin made a significant progress in solving this problem. In 1995, Sternfeld [29] proved that

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if $f : X \to Y$ is an n-dimensional map between compact metric spaces, then $\dim(f \triangle g) \le 1$ for almost all $g \in C(X, \mathbb{I}^n)$, where $f \triangle g$ denotes the diagonal product of the maps f and g; equivalently, there exists a σ-compact $(n-1)$-dimensional subset A of X such that $\dim(f|(X \setminus A)) \le 1$. Levin [19] improved Sternfeld's result showing that $\dim(f \triangle g) \le 0$ for almost all maps $g \in C(X, \mathbb{I}^{n+1})$, which is equivalent to the existence of a σ-compact n-dimensional set $A \subset X$ with $\dim(f|(X \setminus A)) \le 0$.

In the present paper we generalize Theorems 1.1 and 1.2 to arbitrary metrizable spaces by replacing the finite dimensionality of Y with the less restrictive condition of being a C-space. Recall that a space X is a *C-space* [1] if for any sequence $\{\omega_n : n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{\gamma_n : n \in \mathbb{N}\}$ of open disjoint families in X such that each γ_n refines ω_n and $\bigcup\{\gamma_n : n \in \mathbb{N}\}$ covers X. The C-space property was introduced by Haver [15] for compact metric spaces, and Addis and Gresham [1] extended Haver's definition to more general spaces. All countable-dimensional metrizable spaces (spaces which are countable unions of finite-dimensional subsets), in particular all finite-dimensional ones, form a proper subclass of the class of C-spaces because there exists a metric C-compactum which is not countable-dimensional [27].

Here is a generalized version of Theorem 1.1.

THEOREM 1.3. *Let $f : X \to Y$ be a closed map of metric spaces with $\dim f = n$ and Y a C-space. Then all maps $g : X \to \mathbb{I}^n$ such that $\dim(f \triangle g) = 0$ form a dense subset of $C(X, \mathbb{I}^n)$ with respect to the uniform convergence topology. Moreover, if f is σ-perfect, then this set is dense and G_δ in $C(X, \mathbb{I}^n)$ with respect to the source limitation topology.*

Theorem 1.3 answers affirmatively Pasynkov's question in [25] whether Theorem 1.1 is true for countable-dimensional spaces.

For any map $f : X \to Y$, $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$ is the *dimension* of f. We say that a surjective map $f : X \to Y$ is *σ-closed* (resp., *σ-perfect*) if X is the union of countably many closed sets X_i such that each restriction $f|X_i : X_i \to f(X_i)$ is a closed (resp., perfect) map and all $f(X_i)$ are closed in Y.

Using Theorem 1.3 we prove the following generalization of Theorem 1.2:

THEOREM 1.4. *Let $f : X \to Y$ be a σ-closed map of metric spaces with $\dim f = n$ and Y a C-space. Then for each $0 \le k \le n-1$ there exists an F_σ-subset A_k of X such that $\dim A_k \le k$ and $f|(X \setminus A_k)$ is $(n-k-1)$-dimensional.*

A few words about this note. In Section 2 we give a characterization of finite-dimensional proper maps (see Theorem 2.2), which is the main tool in the proof of Theorem 1.3. The proof of Theorem 2.2 is based on a selection theorem established by V. Gutev and the second author [14, Theorem 1.1]. Sections 3 and 4 are devoted to the proof of Theorems 1.3 and 1.4, respectively. In the last Section 5 we provide applications of the main results. One of them is a generalization of a result by Dranishnikov and Uspenskij [10] concerning maps which lower extensional dimension, another one is a parametric version of the Bogatyĭ representation theorem for n-dimensional metrizable spaces [2]. Some results in the spirit of Pasynkov's recent paper [24] are also obtained.

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2. Finite-dimensional maps. In this section we provide a characterization of n-dimensional perfect maps onto paracompact C-spaces (see Theorem 2.2 below).

For any spaces M and K, we denote by $C(K, M)$ the set of all continuous maps from K into M. If (M, d) is a metric space and K is any space, then the *source limitation topology* on $C(K, M)$ is defined in the following way: a subset $U \subset C(K, M)$ is open in $C(K, M)$ with respect to the source limitation topology provided for every $g \in U$ there exists a continuous function $\alpha : K \to (0, \infty)$ such that $\bar{B}(g, \alpha) \subset U$. Here, $\bar{B}(g, \alpha)$ denotes the set $\{h \in C(K, M) : d(g(x), h(x)) \le \alpha(x) \text{ for each } x \in K\}$.

The source limitation topology is also known as the *fine topology* and $C(K, M)$ with this topology has the Baire property provided (M, d) is a complete metric space [23]. We also need the following fact: if K is paracompact and $F \subset K$ closed, then the restriction map $p_F : C(K, M) \to C(F, M)$, $p_F(g) = g|F$, is continuous when both $C(K, M)$ and $C(F, M)$ are equipped with the source limitation topology; moreover p_F is open and surjective provided M is a closed convex subset of a Banach space and d is the metric on M generated by the norm. Finally, when K and M are metrizable, the source limitation topology on $C(K, M)$ does not depend on the metric on M.

Let ω be an open cover of the space M and $m \in \mathbb{N} \cup \{0\}$. A family γ of subsets of M is said to be (m, ω)-*discrete* in M if $\mathrm{ord}(\gamma) \le m+1$ (i.e., every point of M belongs to at most $m+1$ elements of γ) and γ refines ω; a subset of M which can be represented as the union of an open (m, ω)-discrete family in M is called (m, ω)-*discrete*; a map $g : M \to Z$ is (m, ω)-*discrete* if every $z \in g(M)$ has a neighborhood V_z in Z such that $g^{-1}(V_z)$ is (m, ω)-discrete in M.

We denote by $\mathrm{cov}(M)$ the family of all open covers of M. In case (M, d) is a metric space, $B_\varepsilon(x)$ (resp., $\overline{B}_\varepsilon(x)$) stands for the open (resp., closed) ball in (M, d) with center x and radius ε.

Lemma 2.1. *If $\omega \in \operatorname{cov}(M)$ and $K \subset M$ is compact, then every functionally open (m,ω)-discrete subset of K can be extended to an (m,ω)-discrete subset of M.*

Proof. Let $U \subset K$ be functionally open and (m,ω)-discrete in K and $\gamma = \{U_s : s \in A\}$ an open (m,ω)-discrete family in K whose union is U. Since U is paracompact (being functionally open in K), we can suppose that γ is locally finite and there exists a partition of unity $\{f_s : s \in A\}$ in U such that $U_s = f_s^{-1}((0,1])$ for each $s \in A$. Denote by $\mathcal{N}$ the nerve of γ with the Whitehead topology and define $f : U \to \mathcal{N}$ by $f(x) = \sum_{s\in A} f_s(x)s$. Observe that $\mathcal{N}$ is at most m-dimensional because $\operatorname{ord}(\gamma) \le m+1$. Let W be a functionally open subset of βM with $W \cap K = U$. Then, by [11], there exists an open set $V \subset W$ containing U and an extension $g : V \to \mathcal{N}$ of f. The map g generates maps $g_s : V \to [0,1]$ such that each g_s extends f_s. We finally choose $G_s \in \omega$ with $U_s \subset G_s$, $s \in A$, and define $V_s = G_s \cap g_s^{-1}((0,1])$. Then the family $\{V_s : s \in A\}$ is (m,ω)-discrete in M and $\bigcup_{s\in A} V_s$ is the required (m,ω)-discrete extension of U. ∎

Throughout the paper $\mathbb{I}^k$ denotes the k-dimensional cube equipped with the Euclidean metric d_k, and D_k denotes the uniform convergence metric on $C(X,\mathbb{I}^k)$ generated by d_k. If $f : X \to Y$, we denote by $C(X, Y \times \mathbb{I}^k, f)$ the set of all maps $h : X \to Y \times \mathbb{I}^k$ such that $\pi_Y \circ h = f$, where $\pi_Y : Y \times \mathbb{I}^k \to Y$ is the projection. For any $\omega \in \operatorname{cov}(X)$ and closed $K \subset X$, $C_{(m,\omega)}(X|K, Y\times\mathbb{I}^k, f)$ stands for the set of all $h \in C(X, Y\times\mathbb{I}^k, f)$ with $h|K$ being (m,ω)-discrete (as a map from K into $Y \times \mathbb{I}^k$) and $C_{(m,\omega)}(X|K,\mathbb{I}^k)$ consists of all $g \in C(X,\mathbb{I}^k)$ such that $f \triangle g \in C_{(m,\omega)}(X|K, Y\times\mathbb{I}^k, f)$. In case $K = X$ we write simply $C_{(m,\omega)}(X, Y\times\mathbb{I}^k, f)$ (resp., $C_{(m,\omega)}(X,\mathbb{I}^k)$) instead of $C_{(m,\omega)}(X|X, Y\times\mathbb{I}^k, f)$ (resp., $C_{(m,\omega)}(X|X,\mathbb{I}^k)$).

Now we can establish the following characterization of n-dimensional perfect maps:

Theorem 2.2. *Let $f : X \to Y$ be a perfect surjection between paracompact spaces with Y being a C-space. Then $\dim f \le n$ if and only if for any $\omega \in \operatorname{cov}(X)$ and $0 \le k \le n$ the set $C_{(n-k,\omega)}(X,\mathbb{I}^k)$ is open and dense in $C(X,\mathbb{I}^k)$ with respect to the source limitation topology.*

Sufficiency is a consequence of the following observation: if $C_{(0,\omega)}(X,\mathbb{I}^n)$ is not empty for all $\omega \in \operatorname{cov}(X)$, then every open cover of $f^{-1}(y)$, $y \in Y$, admits an open refinement of order $\le n+1$, i.e. $\dim f^{-1}(y) \le n$. Indeed, let γ be a family of open subsets of X covering $f^{-1}(y)$. Then $\omega = \gamma \cup \{X \setminus f^{-1}(y)\} \in \operatorname{cov}(X)$, so there exists $g \in C_{(0,\omega)}(X,\mathbb{I}^n)$. Obviously, $g|f^{-1}(y)$ is $(0,\omega)$-discrete. Hence, every $z \in H = g(f^{-1}(y))$ has a neighborhood G_z in $\mathbb{I}^n$ with $g^{-1}(G_z) \cap f^{-1}(y)$ being the union of a disjoint family μ_z of sets open in $f^{-1}(y)$ which refines ω. Take finitely many $z(i) \in H$,

$i = 1, \ldots, p$, such that $\lambda = \{G_{z(i)} : i = 1, \ldots, p\}$ covers H. Since $\dim H \le n$, we can suppose that $\mathrm{ord}(\lambda) \le n + 1$. Then $\mu = \bigcup\{\mu_{z(i)} : i = 1, \ldots, p\}$ is an open cover of $f^{-1}(y)$ refining γ and $\mathrm{ord}(\mu) \le n + 1$.

To prove necessity we need a few lemmas; the proof will be completed by Lemma 2.9. In all these lemmas we suppose that X and Y are given paracompact spaces and $f : X \to Y$ a perfect surjective map with $\dim f \le n$, where $n \in \mathbb{N}$. We also fix $\omega \in \mathrm{cov}(X)$, an integer k such that $0 \le k \le n$, and an arbitrary $m \in \mathbb{N} \cup \{0\}$.

Lemma 2.3. *Let $g \in C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$ for some $y \in Y$. Then there exists a neighborhood U of y in Y such that $g|f^{-1}(U)$ is (m, ω)-discrete.*

Proof. Obviously, $g \in C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$ implies that $g|f^{-1}(y)$ is an (m, ω)-discrete map. Hence, for every $x \in f^{-1}(y)$ there exists an open neighborhood $V_{g(x)}$ of $g(x)$ in $\mathbb{I}^k$ such that $g^{-1}(V_{g(x)}) \cap f^{-1}(y)$ is an (m, ω)-discrete set in $f^{-1}(y)$. Since $V_{g(x)}$ is functionally open in $\mathbb{I}^k$, so is $g^{-1}(V_{g(x)}) \cap f^{-1}(y)$ in $f^{-1}(y)$. Then, by Lemma 2.1, there is an (m, ω)-discrete subset W_x in X extending $g^{-1}(V_{g(x)}) \cap f^{-1}(y)$. Therefore, for every $x \in f^{-1}(y)$ we have $(f \triangle g)^{-1}(f(x), g(x)) = f^{-1}(y) \cap g^{-1}(g(x)) \subset W_x$ and, since $f \triangle g$ is a closed map, there exists an open neighborhood $H_x = U_y^x \times G_x$ of $(y, g(x))$ in $Y \times \mathbb{I}^k$ with $S_x = (f \triangle g)^{-1}(H_x) \subset W_x$. Next, choose finitely many points $x(i) \in f^{-1}(y)$, $i = 1, \ldots, p$, such that $f^{-1}(y) \subset \bigcup_{i=1}^p S_{x(i)}$. As f is a closed map we can find a neighborhood U_y of y in Y such that $U_y \subset \bigcap_{i=1}^p U_y^{x(i)}$ and $f^{-1}(U_y) \subset \bigcup_{i=1}^p S_{x(i)}$.

Let us show that $g|f^{-1}(U_y)$ is (m, ω)-discrete. Indeed, if $z \in f^{-1}(U_y)$, then $z \in S_{x(j)}$ for some j and $g(z) \in G_{x(j)}$ because $S_{x(j)} = f^{-1}(U_y^{x(j)}) \cap g^{-1}(G_{x(j)})$. Consequently, $f^{-1}(U_y) \cap g^{-1}(G_{x(j)}) \subset S_{x(j)} \subset W_{x(j)}$. Therefore, $G_{x(j)}$ is a neighborhood of $g(z)$ such that $f^{-1}(U_y) \cap g^{-1}(G_{x(j)})$ is (m, ω)-discrete in $f^{-1}(U_y)$ as a subset of the (m, ω)-discrete set $W_{x(j)}$ in X. ∎

Corollary 2.4. *If $g \in C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$ for every $y \in Y$, then $g \in C_{(m,\omega)}(X, \mathbb{I}^k)$.*

Proof. We need to show that $f \triangle g$ is (m, ω)-discrete, i.e. for any $x \in X$ there exist neighborhoods U_y of $y = f(x)$ in Y and G_x of $g(x)$ in $\mathbb{I}^k$ such that $f^{-1}(U_y) \cap g^{-1}(G_x)$ is (m, ω)-discrete in X. Indeed, by Lemma 2.3, there exists a neighborhood U_y of y in Y such that $g|f^{-1}(U_y)$ is (m, ω)-discrete. Therefore, we can find a neighborhood G_x of $g(x)$ in $\mathbb{I}^k$ with $f^{-1}(U_y) \cap g^{-1}(G_x)$ being (m, ω)-discrete in $f^{-1}(U_y)$. Consequently, $f^{-1}(U_y) \cap g^{-1}(G_x)$ is (m, ω)-discrete in X. ∎

Lemma 2.5. *The set $C_{(m,\omega)}(X|K, \mathbb{I}^k)$ is open in $C(X, \mathbb{I}^k)$ with respect to the source limitation topology for any closed $K \subset X$.*

Proof. Let $g_0 \in C_{(m,\omega)}(X|K, \mathbb{I}^k)$. We are going to find $\alpha \in C(X, (0, \infty))$ with $\bar{B}(g_0, \alpha) \subset C_{(m,\omega)}(X|K, \mathbb{I}^k)$. Since each restriction $g_0|(f^{-1}(y) \cap K)$, $y \in H = f(K)$, is (m, ω)-discrete, by Lemma 2.3, for every $y \in H$ there exists a neighborhood U_y of y in Y such that $g_0|(f^{-1}(U_y) \cap K)$ is (m, ω)-discrete. Then $\omega_1 = \{U_y : y \in H\} \cup \{Y \setminus H\}$ is an open cover of Y. As Y is paracompact, we can find a metric space (M, d), a surjection $p : Y \to M$ and $\mu \in \mathrm{cov}(M)$ such that $p^{-1}(\mu)$ refines ω_1. Hence, every $z \in p(H)$ has a neighborhood W_z in M such that $g_0|(p\circ f)^{-1}(W_z)\cap K$ is (m, ω)-discrete. The last condition implies that $h_0|K$ is (m, ω)-discrete, where $h_0 = (p \circ f) \triangle g_0$. Now we need the following:

Claim. *There exists an open family γ in $M \times \mathbb{I}^k$ covering $h_0(K)$ such that every $g \in C(X, \mathbb{I}^k)$ belongs to $C_{(m,\omega)}(X|K, \mathbb{I}^k)$ provided $h|K$ is γ-close to $h_0|K$, where $h = (p \circ f) \triangle g$.*

Proof of the claim. Since $h_0|K$ is (m, ω)-discrete, every $t \in h_0(K)$ has an open neighborhood V_t in $M \times \mathbb{I}^k$ such that $h_0^{-1}(V_t) \cap K$ is (m, ω)-discrete in K. Then $\nu = \{V_t : t \in h_0(K)\}$ forms an open cover of $h_0(K)$. Take γ to be a locally finite open cover of $V = \bigcup \nu$ such that $\{\mathrm{St}(W, \gamma) : W \in \gamma\}$ refines ν. Let $h|K$ be a map γ-close to $h_0|K$, where $h = (p\circ f)\triangle g$ with $g \in C(X, \mathbb{I}^k)$. If $W \in \gamma$, then $h_0(h^{-1}(W)\cap K) \subset \mathrm{St}(W, \gamma)$. But $\mathrm{St}(W, \gamma)$ is contained in V_t for some $t \in h_0(K)$. Consequently, $h^{-1}(W)\cap K \subset h_0^{-1}(V_t)\cap K$. The last inclusion implies that $h^{-1}(W) \cap K$ is (m, ω)-discrete in K because $h_0^{-1}(V_t) \cap K$ is. Therefore, $h|K$ is (m, ω)-discrete. To finish the proof of the claim observe that $h|K$ being (m, ω)-discrete implies $(f \triangle g)|K$ is also (m, ω)-discrete, i.e. $g \in C_{(m,\omega)}(X|K, \mathbb{I}^k)$.

We continue with the proof of Lemma 2.5. Let ϱ be the metric on $M \times \mathbb{I}^k$ defined by $\varrho(t_1, t_2) = d(z_1, z_2) + d_k(w_1, w_2)$, where $t_i = (z_i, w_i)$, $i = 1, 2$. Let $\alpha_1 : K \to (0, \infty)$ be the function $\alpha_1(x) = 2^{-1} \sup\{\varrho(h_0(x), V \setminus W) : W \in \gamma\}$. Since $h_0(K) \subset V$ and γ is a locally finite open cover of V, α_1 is continuous. Moreover, if $h = (p\circ f)\triangle g$ with $g \in C(X, \mathbb{I}^k)$ and $\varrho(h_0(x), h(x)) \le \alpha_1(x)$ for every $x \in K$, then $h|K$ is γ-close to $h_0|K$. According to the claim, the last relation yields $g \in C_{(m,\omega)}(X|K, \mathbb{I}^k)$. We finally take a continuous extension $\alpha : X \to (0, \infty)$ of α_1. Observe that $d_k(g_0(x), g(x)) = \varrho(h_0(x), h(x))$ for every $x \in X$. Therefore, $\bar{B}(g_0, \alpha) \subset C_{(m,\omega)}(X|K, \mathbb{I}^k)$. ■

Lemma 2.6. *If $C(X, \mathbb{I}^k)$ is equipped with the uniform convergence topology, then the set-valued map $\psi_{(m,\omega)} : Y \to 2^{C(X,\mathbb{I}^k)}$, defined by the formula $\psi_{(m,\omega)}(y) = C(X, \mathbb{I}^k) \setminus C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$, has a closed graph.*

Proof. Let $G = \bigcup\{y \times \psi_{(m,\omega)}(y) : y \in Y\} \subset Y \times C(X, \mathbb{I}^k)$ be the graph of $\psi_{(m,\omega)}$ and $(y_0, g_0) \in (Y \times C(X, \mathbb{I}^k)) \setminus G$. We are going to show that (y_0, g_0) has a neighborhood in $Y \times C(X, \mathbb{I}^k)$ which does not meet G.

Since $(y_0, g_0) \notin G$, $g_0 \notin \psi_{(m,\omega)}(y_0)$. Hence, $g_0 \in C_{(m,\omega)}(X|f^{-1}(y_0), \mathbb{I}^k)$ and, by Lemma 2.3, there exists a neighborhood U of y_0 in Y with $g_0|f^{-1}(U)$ being (m,ω)-discrete, in particular, $g_0 \in C_{(m,\omega)}(X|f^{-1}(U), \mathbb{I}^k)$. We can assume that $U \subset Y$ is closed, hence so is $f^{-1}(U)$ in X. Then, according to Lemma 2.5, $C_{(m,\omega)}(X|f^{-1}(U), \mathbb{I}^k)$ is open in $C(X, \mathbb{I}^k)$ with respect to the source limitation topology. Consequently, there exists a continuous positive function α on X such that $\overline{B}(g_0, \alpha)$ is contained in $C_{(m,\omega)}(X|f^{-1}(U), \mathbb{I}^k)$. Since $f^{-1}(y_0)$ is compact, $2\delta = \min\{\alpha(x) : x \in f^{-1}(y_0)\} > 0$ and $H = \{x \in f^{-1}(U) : \alpha(x) > \delta\}$ is a neighborhood of $f^{-1}(y_0)$. Therefore, there exists a closed neighborhood V of y_0 in Y with $f^{-1}(V) \subset H$ (we use again the fact that f is a closed map). Let $B_\delta(g_0)$ be the open ball in $C(X, \mathbb{I}^k)$ (with respect to the uniform metric D_k) with center g_0 and radius δ. Since $W = V \times B_\delta(g_0)$ is a neighborhood of (y_0, g_0) in $Y \times C(X, \mathbb{I}^k)$, the following claim completes the proof.

CLAIM. $W \cap G = \emptyset$.

Suppose $(y, g) \in W \cap G$ for some $(y, g) \in Y \times C(X, \mathbb{I}^k)$. Then $y \in V$ and

$$d_k(g(x), g_0(x)) \le \delta < \alpha(x) \quad \text{for every } x \in f^{-1}(V). \tag{1}$$

Let us show that the existence of a function $g_1 \in C(X, \mathbb{I}^k)$ such that

$$g_1 \in \overline{B}(g_0, \alpha) \quad \text{and} \quad g_1|f^{-1}(V) = g|f^{-1}(V) \tag{2}$$

contradicts the assumption $(y, g) \in W \cap G$. Indeed, $g_1 \in \overline{B}(g_0, \alpha)$ yields $g_1 \in C_{(m,\omega)}(X|f^{-1}(U), \mathbb{I}^k)$ and, since $f^{-1}(y) \subset f^{-1}(U)$, we have $g_1 \in C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$. So, $g \in C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$ because $g_1|f^{-1}(y) = g|f^{-1}(y)$ (recall that $f^{-1}(y) \subset f^{-1}(V)$). On the other hand, $(y, g) \in G$ implies $g \in \psi_{(m,\omega)}(y)$, i.e. $g \notin C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$.

Therefore, the proof is reduced to finding g_1 satisfying (2). And this can be done by using the convex-valued selection theorem of Michael [21]. Define the set-valued map $\Phi : X \to \mathcal{F}_c(\mathbb{I}^k)$ by $\Phi(x) = g(x)$ if $x \in f^{-1}(V)$ and $\Phi(x) = \overline{B}_{\alpha(x)}(g_0(x))$ otherwise. Here, $\mathcal{F}_c(\mathbb{I}^k)$ denotes the convex closed subsets of $\mathbb{I}^k$ and $\overline{B}_{\alpha(x)}(g_0(x))$ is the closed ball in $\mathbb{I}^k$ with center $g_0(x)$ and radius $\alpha(x)$. By (1), $g(x) \in \overline{B}_{\alpha(x)}(g_0(x))$ for all $x \in f^{-1}(V)$. The last condition, together with the definition of Φ outside $f^{-1}(V)$, implies that Φ is lower semicontinuous (i.e., $\{x \in X : \Phi(x) \cap O \neq \emptyset\}$ is open in X for any open set $O \subset \mathbb{I}^k$). Then, by the above mentioned Michael theorem, Φ admits a continuous selection g_1. Since $g_1(x) \in \Phi(x)$ for any $x \in X$, we have $g_1|f^{-1}(V) = g|f^{-1}(V)$ and $g_1 \in \overline{B}(g_0, \alpha)$. ∎

LEMMA 2.7. *Let K and M be compact spaces such that* $\dim K \le n$ *and M is metrizable. Then for every $\gamma \in$* $\mathrm{cov}(K)$ *and* $0 \le k \le n$ *the set of all maps $h \in C(M \times K, \mathbb{I}^k)$ with each $h|(\{z\} \times K)$, $z \in M$, being $(n-k, \gamma)$-*

discrete (*as a map from* K *into* $\mathbb{I}^k$) *is dense in* $C(M \times K, \mathbb{I}^k)$ *with respect to the uniform convergence topology.*

Proof. Suppose first that K is metrizable and let $p_M : M \times K \to M$ and $p_K : M \times K \to K$ be the projections. Then, by Hurewicz's theorem [18], there exists a 0-dimensional map $h^* : K \to \mathbb{I}^n$. Consequently, $g^* = h^* \circ p_K$ is a map from $M \times K$ into $\mathbb{I}^n$ such that $p_M \triangle g^* : M \times K \to M \times \mathbb{I}^n$ is also 0-dimensional. According to Levin's [19] and Sternfeld's [29] results, the existence of such a map g^* implies that the set $\mathcal{M}_n$ of all maps $g \in C(M \times K, \mathbb{I}^n)$ with $\dim(p_M \triangle g) \le 0$ is dense in $C(M \times K, \mathbb{I}^n)$ with respect to the uniform convergence topology. If $q : \mathbb{I}^n \to \mathbb{I}^k$ is the projection generated by the first k coordinates, then the map $g \mapsto q \circ g$ is a continuous surjection from $C(M \times K, \mathbb{I}^n)$ onto $C(M \times K, \mathbb{I}^k)$ (both equipped with the uniform convergence topology), so $\mathcal{M}_k = \{q \circ g : g \in \mathcal{M}_n\}$ is dense in $C(M \times K, \mathbb{I}^k)$. Moreover, since $\dim q = n - k$ and each $p_M \triangle g$, $g \in \mathcal{M}_n$, is 0-dimensional, $\dim(p_M \triangle h) \le n - k$ for any $h \in \mathcal{M}_k$ (the last conclusion follows from the Hurewicz theorem on closed maps which lower dimension [16]). Therefore, $h_z = h|(\{z\} \times K)$ is an $(n-k)$-dimensional map for every $z \in M$ and $h \in \mathcal{M}_k$.

Let us show that any such h_z is $(n-k, \gamma)$-discrete. Indeed, for fixed $y \in h_z(K)$ we have $\dim h_z^{-1}(y) \le n - k$. So, there exists $\nu \in \operatorname{cov}(h_z^{-1}(y))$ refining γ such that $\operatorname{ord}(\nu) \le n - k + 1$. Applying Lemma 2.1, we obtain an $(n-k, \gamma)$-discrete set W_y in K which contains $h_z^{-1}(y)$. Finally, choose a neighborhood V_y of y in $\mathbb{I}^k$ such that $h_z^{-1}(V_y) \subset W_y$ and observe that $h_z^{-1}(V_y)$ is $(n-k, \gamma)$-discrete.

Suppose now K is not metrizable and fix $\delta > 0$ and $h_0 \in C(M \times K, \mathbb{I}^k)$. We are going to find $h \in C(M \times K, \mathbb{I}^k)$ satisfying the requirement of the lemma and such that h is δ-close to h_0. To this end, represent K as the limit space of a σ-complete inverse system $\mathcal{S} = \{K_\beta, \pi_\beta^{\beta+1} : \beta \in B\}$ such that each K_β is a metrizable compactum with $\dim K_\beta \le n$. Applying standard inverse spectra arguments (see [4]), we can find $\theta \in B$, $\gamma_1 \in \operatorname{cov}(K_\theta)$ and $h_\theta \in C(M \times K_\theta, \mathbb{I}^k)$ such that $h_\theta \circ (\mathrm{id}_M \times \pi_\theta) = h_0$ and $\pi_\theta^{-1}(\gamma_1)$ refines γ, where $\pi_\theta : K \to K_\theta$ denotes the θth limit projection. Then, by the previous case, there exists a map $h_1 \in C(M \times K_\theta, \mathbb{I}^k)$ which is δ-close to h_θ and $h_1|(\{z\} \times K_\theta)$ is $(n-k, \gamma_1)$-discrete. It follows from our construction that $h = h_1 \circ (\mathrm{id}_M \times \pi_\theta)$ is δ-close to h_0 and $h|(\{z\} \times K)$ is $(n-k, \gamma)$-discrete. ■

Recall that a closed subset F of the metrizable space M is said to be a *Z-set* in M if the set $C(Q, M \setminus F)$ is dense in $C(Q, M)$ with respect to the uniform convergence topology, where Q denotes the Hilbert cube. If, in the above definition, Q is replaced by $\mathbb{I}^m$, $m \in \mathbb{N} \cup \{0\}$, we say that F is a Z_m-*set* in M.

LEMMA 2.8. *Let $\alpha : X \to (0,\infty)$ be a positive continuous function and $g_0 \in C(X,\mathbb{I}^k)$. Then $\psi_{(n-k,\omega)}(y) \cap \overline{B}(g_0,\alpha)$ is a Z-set in $\overline{B}(g_0,\alpha)$ for every $y \in Y$, where $\overline{B}(g_0,\alpha)$ is considered as a subspace of $C(X,\mathbb{I}^k)$ with the uniform convergence topology.*

Proof. In this proof all function spaces are equipped with the uniform convergence topology. Since, by Lemma 2.6, $\psi_{(n-k,\omega)}$ has a closed graph, each $\psi_{(n-k,\omega)}(y)$ is closed in $C(X,\mathbb{I}^k)$. Hence, $\psi_{(n-k,\omega)}(y) \cap \overline{B}(g_0,\alpha)$ is closed in $\overline{B}(g_0,\alpha)$. We need to show that, for fixed $y \in Y$, $\delta > 0$ and a map $u : Q \to \overline{B}(g_0,\alpha)$ there exists a map $v : Q \to \overline{B}(g_0,\alpha) \setminus \psi_{(n-k,\omega)}(y)$ which is δ-close to u with respect to the uniform metric D_k.

To this end, observe first that u generates $h \in C(Q \times X, \mathbb{I}^k)$, $h(z,x) = u(z)(x)$, such that $d_k(h(z,x), g_0(x)) \le \alpha(x)$ for any $(z,x) \in Q \times X$. Since $f^{-1}(y)$ is compact, we can find $\lambda \in (0,1)$ such that $\lambda \sup\{\alpha(x) : x \in f^{-1}(y)\} < \delta/2$. Now, define $h_1 \in C(Q \times f^{-1}(y), \mathbb{I}^k)$ by $h_1(z,x) = (1-\lambda)h(z,x) + \lambda g_0(x)$. Then, for every $(z,x) \in Q \times f^{-1}(y)$, we have

$$d_k(h_1(z,x), g_0(x)) \le (1-\lambda)\alpha(x) < \alpha(x) \tag{3}$$

and

$$d_k(h_1(z,x), h(z,x)) \le \lambda\alpha(x) < \delta/2. \tag{4}$$

Let $q < \min\{r, \delta/2\}$, where r is the positive number $\inf\{\alpha(x) - d_k(h_1(z,x), g_0(x)) : (z,x) \in Q \times f^{-1}(y)\}$. Since $\dim f^{-1}(y) \le n$, by Lemma 2.7 (applied to the product $Q \times f^{-1}(y)$), there is a map $h_2 \in C(Q \times f^{-1}(y), \mathbb{I}^k)$ such that $d_k(h_2(z,x), h_1(z,x)) < q$ and $h_2|(\{z\} \times f^{-1}(y))$ is an $(n-k,\omega)$-discrete map for each $(z,x) \in Q \times f^{-1}(y)$. Then, by (3) and (4), for all $(z,x) \in Q \times f^{-1}(y)$ we have

$$d_k(h_2(z,x), h(z,x)) < \delta \quad \text{and} \quad d_k(h_2(z,x), g_0(x)) < \alpha(x). \tag{5}$$

Because both Q and $f^{-1}(y)$ are compact, $u_2(z)(x) = h_2(z,x)$ defines the map $u_2 : Q \to C(f^{-1}(y), \mathbb{I}^k)$. The required map v will be obtained as a lifting of u_2. The restriction map $\pi : \overline{B}(g_0,\alpha) \to C(f^{-1}(y),\mathbb{I}^k)$, $\pi(g) = g|f^{-1}(y)$, is obviously continuous (with respect to the uniform convergence topology).

CLAIM. *$\pi : \overline{B}(g_0,\alpha) \to \pi(\overline{B}(g_0,\alpha))$ is an open map.*

It is enough to show that

$$\pi(\overline{B}(g_0,\alpha) \cap B_\varepsilon(g)) = \pi(\overline{B}(g_0,\alpha)) \cap B_\varepsilon(\pi(g)) \tag{6}$$

for every $g \in \overline{B}(g_0,\alpha)$ and $\varepsilon > 0$, where $B_\varepsilon(g)$ and $B_\varepsilon(\pi(g))$ are open balls, respectively, in $C(X,\mathbb{I}^k)$ and $C(f^{-1}(y),\mathbb{I}^k)$, both with the uniform metric generated by d_k. Let $p \in \pi(\overline{B}(g_0,\alpha)) \cap B_\varepsilon(\pi(g))$. Then $d_k(p(x), g_0(x)) \le \alpha(x)$ and $d_k(p(x), g(x)) < \eta < \varepsilon$ for every $x \in f^{-1}(y)$ and some positive number η. Define the closed-and-convex-valued map $\Phi : X \to \mathcal{F}_{\mathrm{c}}(\mathbb{I}^k)$ by $\Phi(x) = p(x)$ if $x \in f^{-1}(y)$ and $\Phi(x) = \overline{B_{\alpha(x)}(g_0(x)) \cap B_\eta(g(x))}$ if $x \notin f^{-1}(y)$ (recall that $B_{\alpha(x)}(g_0(x))$ and $B_{\eta(x)}(g(x))$ are open balls in $\mathbb{I}^k$). Since $g \in \overline{B}(g_0, \alpha)$, $\Phi(x) \neq \emptyset$ for every $x \in X$. Moreover, since α, g and g_0 are continuous, Φ is lower semicontinuous. Therefore, by Michael's convex-valued selection theorem, Φ admits a selection $g_1 \in C(X, \mathbb{I}^k)$. Then $\pi(g_1) = p$ and $g_1 \in \overline{B}(g_0, \alpha) \cap B_\varepsilon(g)$. So, $\pi(\overline{B}(g_0, \alpha)) \cap B_\varepsilon(\pi(g)) \subset \pi(\overline{B}(g_0, \alpha) \cap B_\varepsilon(g))$ and, because the converse inclusion is trivial, we are done.

Before completing the final step of our proof, note that $u_2(z) \in \pi(\overline{B}(g_0, \alpha))$ for every $z \in Q$ (indeed, consider the lower semicontinuous set-valued map $\phi : X \to \mathcal{F}_c(\mathbb{I}^k)$, $\phi(x) = u_2(z)(x)$ for $x \in f^{-1}(y)$ and $\phi(x) = \overline{B_{\alpha(x)}(g_0(x))}$ for $x \notin f^{-1}(y)$, and take any continuous selection g of ϕ). Now, we are going to lift the map u_2 to a map $v : Q \to \overline{B}(g_0, \alpha)$ such that v is δ-close to u.

To this end, define $\theta : Q \to \mathcal{F}_c(C(X, \mathbb{I}^k))$ by $\theta(z) = \overline{\pi^{-1}(u_2(z)) \cap B_\delta(u(z))}$. The first inequality in (5) implies that $u_2(z) \in B_\delta(\pi(u(z)))$ for every $z \in Q$. Since each $u_2(z)$ belongs to $\pi(\overline{B}(g_0, \alpha))$, by (6), $\theta(z) \neq \emptyset$, $z \in Q$. On the other hand, since π is open, by [21, Example 1.1* and Proposition 2.5], θ is lower semicontinuous. Obviously, every image $\theta(z)$ is convex and closed in $C(X, \mathbb{I}^k)$, which is, in its turn, closed and convex in the Banach space of all bounded continuous functions from X into $\mathbb{R}^k$. Therefore, using again the Michael selection theorem [21, Theorem 3.2″], we can find a continuous selection $v : Q \to C(X, \mathbb{I}^k)$ for θ. Then v maps Q into $\overline{B}(g_0, \alpha)$ and v is δ-close to u. Moreover, for any $z \in Q$ we have $\pi(v(z)) = u_2(z)$, and $u_2(z)$, being the restriction $h_2|(\{z\} \times f^{-1}(y))$, is $(n-k, \omega)$-discrete. Hence, $v(z) \in C_{(n-k,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$, $z \in Q$, i.e. $v(Q) \subset \overline{B}(g_0, \alpha) \setminus \psi_{(n-k,\omega)}(y)$. ■

LEMMA 2.9. *If Y is a C-space, then $C_{(n-k,\omega)}(X, \mathbb{I}^k)$ is dense in $C(X, \mathbb{I}^k)$ with respect to the source limitation topology.*

Proof. It suffices to show that, for fixed $g_0 \in C(X, \mathbb{I}^k)$ and a positive continuous function $\alpha : X \to (0, \infty)$, there exists $g \in \overline{B}(g_0, \alpha) \cap C_{(n-k,\omega)}(X, \mathbb{I}^k)$. We equip $C(X, \mathbb{I}^k)$ with the uniform convergence topology and consider the constant (and hence, lower semicontinuous) map $\phi : Y \to \mathcal{F}_c(C(X, \mathbb{I}^k))$, $\phi(y) = \overline{B}(g_0, \alpha)$. According to Lemma 2.8, $\overline{B}(g_0, \alpha) \cap \psi_{(n-k,\omega)}(y)$ is a Z-set in $\overline{B}(g_0, \alpha)$ for every $y \in Y$. So, we have a lower semicontinuous map $\phi : Y \to \mathcal{F}_c(E)$ and a map $\psi_{(n-k,\omega)} : Y \to 2^E$ such that $\psi_{(n-k,\omega)}$ has a closed graph (see Lemma 2.6) and $\phi(y) \cap \psi_{(n-k,\omega)}(y)$ is a Z-set in $\phi(y)$ for each $y \in Y$, where E is the Banach space of all bounded continuous maps from Y into $\mathbb{R}^k$. Therefore, we can apply [14, Theorem 1.1] to obtain a continuous map $h : Y \to E$ with $h(y) \in \phi(y) \setminus \psi_{(n-k,\omega)}(y)$ for every $y \in Y$ (Theorem 1.1 of [14] was proved under the assumption that $\psi_{(n-k,\omega)}$ has nonempty values, but the proof works without this restriction). Observe that h is a map from Y into $\overline{B}(g_0, \alpha)$ such that $h(y) \notin \psi_{(n-k,\omega)}(y)$ for every $y \in Y$, i.e.

$h(y) \in \overline{B}(g_0, \alpha) \cap C_{(n-k,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$, $y \in Y$. Then $g(x) = h(f(x))(x)$, $x \in X$, defines a map $g \in \overline{B}(g_0, \alpha)$ such that $g \in C_{(n-k,\omega)}(X|f^{-1}(y), \mathbb{I}^k)$ for every $y \in Y$. Hence, by Corollary 2.4, $g \in C_{(n-k,\omega)}(X, \mathbb{I}^k)$. ∎

3. Proof of Theorem 1.3. The following proposition proves Theorem 1.3 in the special case when f is σ-perfect.

PROPOSITION 3.1. *Let $f : X \to Y$ be a σ-perfect map of metrizable spaces with* $\dim f \leq n$ *and Y being a C-space. Then the set of all maps $g : X \to \mathbb{I}^n$ such that* $\dim(f \triangle g) = 0$ *is dense and G_δ in $C(X, \mathbb{I}^n)$ with respect to the source limitation topology.*

Proof. All function spaces in this proof are considered with the source limitation topology. Let X be the union of closed sets X_i, $i = 1, 2, \ldots$, such that each restriction $f_i = f|X_i$ is perfect and $Y_i = f(X_i)$ is closed in Y. Fix a sequence $\{\omega_q\}$ of open covers of X with $\mathrm{mesh}(\omega_q) < q^{-1}$. Every Y_i is a C-space (being a closed set in Y), so we can apply Lemma 2.9 to the maps $f_i : X_i \to Y_i$ and conclude that $H_i = \bigcap_{q=1}^{\infty} C_{(0,\omega_q)}(X_i, \mathbb{I}^n)$ is dense and G_δ in $C(X_i, \mathbb{I}^n)$, $i \in \mathbb{N}$. Here, $C_{(0,\omega_q)}(X_i, \mathbb{I}^n)$ consists of all $h \in C(X_i, \mathbb{I}^n)$ such that $f_i \triangle h$ is $(0, \omega_q)$-discrete. Since all restriction maps $p_i : C(X, \mathbb{I}^n) \to C(X_i, \mathbb{I}^n)$, $p_i(g) = g|X_i$, are continuous, open and surjective, the sets $C_i = p_i^{-1}(H_i)$ are dense and G_δ in $C(X, \mathbb{I}^n)$, and hence so is the intersection $\bigcap_{i=1}^{\infty} C_i$. It remains to observe that $g \in \bigcap_{i=1}^{\infty} C_i$ if and only if $\dim(f_i \triangle g_i) = 0$ for every i, where $g_i = g|X_i$. Hence, by the countable sum theorem, $g \in \bigcap_{i=1}^{\infty} C_i$ if and only if $\dim(f \triangle g) = 0$. ∎

We now continue with the proof of the first part of Theorem 1.3. Suppose $f : X \to Y$ is a closed n-dimensional surjection with both X and Y metrizable and Y a C-space. By the Vainstein lemma [12], the boundary $\mathrm{Fr}\, f^{-1}(y)$ of every $f^{-1}(y)$ is compact. Defining $F(y)$ to be $\mathrm{Fr}\, f^{-1}(y)$ if $\mathrm{Fr}\, f^{-1}(y) \neq \emptyset$, and an arbitrary point from $f^{-1}(y)$ otherwise, we obtain the set $X_0 = \bigcup\{F(y) : y \in Y\}$ such that $X_0 \subset X$ is closed and the restriction $f|X_0 : X_0 \to Y$ is a perfect surjection. Moreover, each $f^{-1}(y) \setminus X_0$ is open in X, so $\dim(X \setminus X_0) \leq n$.

Represent $X \setminus X_0$ as the union of countably many closed sets $X_i \subset X$ and for each $i = 0, 1, 2, \ldots$ let $p_i : C(X, \mathbb{I}^n) \to C(X_i, \mathbb{I}^n)$ be the restriction map. By Proposition 3.1, the set C_0 consisting of all $g \in C(X, \mathbb{I}^n)$ with $(f \triangle g)|X_0$ being 0-dimensional is dense and G_δ in $C(X, \mathbb{I}^n)$ with respect to the source limitation topology. Consequently, C_0 is uniformly dense in $C(X, \mathbb{I}^n)$. On the other hand, since $\dim X_i \leq n$ for every $i = 1, 2, \ldots$, the set $H_i \subset C(X_i, \mathbb{I}^n)$ of all uniformly 0-dimensional maps is dense and G_δ in $C(X_i, \mathbb{I}^n)$ with respect to the uniform convergence topology [17] (recall that a map $h : X_i \to \mathbb{I}^n$ is *uniformly* 0-*dimensional* if for every $\varepsilon > 0$ there exists

$\eta > 0$ such that if $T \subset \mathbb{I}^n$ and $\operatorname{diam}(T) \leq \eta$, then $h^{-1}(T)$ is covered by a disjoint open family in X_i consisting of sets with diameter $\leq \varepsilon$). Because p_i are open and continuous surjections when $C(X, \mathbb{I}^n)$ and $C(X_i, \mathbb{I}^n)$ carry the uniform convergence topology, all $C_i = p_i^{-1}(H_i)$, $i = 1, 2, \ldots$, are uniformly dense and G_δ in $C(X, \mathbb{I}^n)$. Therefore, $C_\infty = \bigcap_{i=0}^{\infty} C_i$ is G_δ in $C(X, \mathbb{I}^n)$ with respect to the source limitation topology. Moreover, $f \triangle g$ is 0-dimensional for every $g \in C_\infty$.

It remains to show that C_∞ is uniformly dense in $C(X, \mathbb{I}^n)$. For every $g \in C_0$ let $H(g) = \{h \in C(X, \mathbb{I}^n) : h|X_0 = g|X_0\}$. Obviously, $C_0 = \bigcup\{H(g) : g \in C_0\}$ and each $H(g)$ is uniformly closed in $C(X, \mathbb{I}^n)$. So, C_∞ is the union of the sets $A(g) = \bigcap_{i=1}^{\infty} C_i \cap H(g)$, $g \in C_0$. For fixed $g \in C_0$ and $i = 1, 2, \ldots$, let $p_i(g) = p_i|H(g)$. Using the fact that X_0 and X_i are closed disjoint subsets of X, one can show that every $p_i(g) : H(g) \to C(X_i, \mathbb{I}^n)$ is a uniformly continuous open surjection. Hence, $H(g) \cap C_i$ is dense and G_δ in $H(g)$ with respect to the uniform convergence topology, being the preimage of H_i under $p_i(g)$. Therefore, $A(g)$ is uniformly dense in $H(g)$ (recall that $H(g)$ is uniformly closed in $C(X, \mathbb{I}^n)$, so it has the Baire property). We finally observe that the uniform density of C_0 in $C(X, \mathbb{I}^n)$ and the uniform density of $A(g)$ in $H(g)$ for each $g \in C_0$ yield the uniform density of C_∞ in $C(X, \mathbb{I}^n)$.

4. Proof of Theorem 1.4. It suffices to prove this theorem for closed maps, so we suppose that $f : X \to Y$ is a closed surjection. If A_{n-1} is constructed, then for $k < n - 1$, we can find an F_σ-subset $A_k \subset A_{n-1}$ with $\dim A_k \leq k$ and $\dim(A_{n-1} \setminus A_k) \leq n - k - 2$ (this can be accomplished by induction, the first step is to represent A_{n-1} as the union of 0-dimensional G_δ-subsets B_j, $j = 1, \ldots, n$ and to set $A_{n-2} = \bigcup_{j=1}^{n-1} B_j$). Therefore, we only need to construct A_{n-1}. To this end, we first establish the following analogue of Sternfeld's [29, Lemma 1] which was proved for compact metrizable spaces.

LEMMA 4.1. *Let M be metrizable and K a compact metric space with $\dim K \leq n$. Then there exists a F_σ-subset $B \subset M \times K$ such that $\dim B \leq n - 1$ and $\pi_M|(M \times K) \setminus B$ is 0-dimensional, where $\pi_M : M \times K \to M$ is the projection.*

Proof. As in [29], the proof can be reduced to the case $n = 1$ and $K = [0, 1]$. So, we are going to show the existence of a 0-dimensional F_σ-subset B of $M \times \mathbb{I}$ such that each set $(\{y\} \times \mathbb{I}) \setminus B$, $y \in M$, is 0-dimensional, and that will complete the proof.

Let $h : Z \to M$ be a perfect surjection with Z being a 0-dimensional metrizable space. Then, by [24, Proposition 9.1], there exists a map $g : Z \to Q$ such that $h \triangle g : Z \to M \times Q$ is a closed embedding. Next, let D be the Cantor set and take a surjection $p : D \to Q$ admitting an averaging operator between the function spaces $C(D)$ and $C(Q)$ [26] (such maps are called *Milyutin maps*). According to [6], there exists a lower semicontinuous compact-valued map $\phi : Q \to 2^D$ with $\phi(y) \subset p^{-1}(y)$ for every $y \in Q$. We can apply Michael's 0-dimensional selection theorem [22] to obtain a continuous selection q for the map $\phi \circ g$. Obviously $h \triangle q : Z \to M \times D$ is a closed embedding, so $Z_0 = (h \triangle q)(Z)$ is a 0-dimensional closed subset of $M \times D$. Finally, considering D as a subset of $\mathbb{I}$, let $Z_r = \{(h(z), q(z) + r) : z \in Z\} \subset M \times \mathbb{I}$ for every rational $r \in \mathbb{I}$, where addition $q(z) + r$ is taken in $\mathbb{R}$ mod 1. Then each Z_r is a closed subset of $M \times \mathbb{I}$ homeomorphic to Z, so $B = \bigcup\{Z_r : r \text{ is rational}\}$ is 0-dimensional and F_σ in $M \times \mathbb{I}$. Moreover, $(\{y\} \times \mathbb{I}) \setminus B$ is also 0-dimensional for every $y \in M$. ∎

Let us continue the proof of Theorem 1.4. As in the proof of Theorem 1.3, there are closed subsets $X_i \subset X$, $i = 0, 1, 2, \ldots$, such that $f|X_0$ is a perfect map onto Y, each X_i, $i \geq 1$, is at most n-dimensional and $X \setminus X_0 = \bigcup_{i=1}^{\infty} X_i$. For every $i \geq 1$ we choose an $(n-1)$-dimensional F_σ-set $H_i \subset X_i$ with $\dim(X_i \setminus H_i) \leq 0$.

A similar subset of X_0 can also be found. Indeed, let $f_0 = f|X_0$ and take $g : X_0 \to \mathbb{I}^n$ such that $f_0 \triangle g : X_0 \to Y \times \mathbb{I}^n$ is 0-dimensional (see Theorem 1.3). By Lemma 4.1, there exists an F_σ-set $B \subset Y \times \mathbb{I}^n$ with $\dim B \leq n-1$ and each $(\{y\}\times\mathbb{I}^n)\setminus B$, $y \in Y$, being 0-dimensional. Then $H_0 = (f_0 \triangle g)^{-1}(B)$ is F_σ in X_0. Since $f_0 \triangle g$ is perfect, by the generalized Hurewicz theorem on closed maps lowering dimension [28], we have $\dim H_0 \leq n-1$ and $\dim(f_0^{-1}(y) \setminus H_0) \leq 0$ for every $y \in Y$.

Finally, set $A_{n-1} = \bigcup_{i=0}^{\infty} H_i$. Obviously, $\dim A_{n-1} \leq n-1$. On the other hand, each $f^{-1}(y) \setminus A_{n-1}$, $y \in Y$, is the union of its closed sets $F_i(y) = f^{-1}(y) \cap X_i \setminus A_{n-1}$, $i \geq 0$. But $F_0(y) = f_0^{-1}(y) \setminus H_0$ and $F_i(y) \subset f^{-1}(y) \cap (X_i \setminus H_i)$ for $i \geq 1$, so all $F_i(y)$ are 0-dimensional. Consequently, $\dim(f^{-1}(y) \setminus A_{n-1}) \leq 0$ for every $y \in Y$, i.e. $f|(X \setminus A_{n-1})$ is 0-dimensional.

5. Some applications. Our first application deals with extensional dimension introduced by Dranishnikov [7] (see also [3] and [8]). Let K be a CW-complex and X a normal space. We say that the extensional dimension of X does not exceed K, written $\text{e-dim}\, X \leq K$, if every map $h : A \to K$, where $A \subset X$ is closed, can be extended to a map from X into K provided there exist a neighborhood U of A in X and a map $g : U \to K$ extending h. Obviously, if K is an absolute neighborhood extensor for X, then $\text{e-dim}\, X \leq K$ iff K is an absolute extensor for X. In this notation, $\dim X \leq n$ is equivalent to $\text{e-dim}\, X \leq \mathbb{S}^n$. We also write $\text{e-dim}\, X \leq \text{e-dim}\, Y$ if $\text{e-dim}\, Y \leq K$ implies $\text{e-dim}\, X \leq K$ for any CW-complex K.

Dranishnikov and Uspenskij [10] provided a generalization of the Hurewicz theorem on dimension lowering maps: if $f : X \to Y$ is an n-dimensional surjection between compact finite-dimensional spaces, then $\text{e-dim}\, X \leq \text{e-dim}(Y \times \mathbb{I}^n)$; moreover, this statement holds for any compact spaces (not necessarily finite-dimensional) when $n = 0$. We can improve this result as follows (see also [5] and [9] for other extension-dimensional variants of Hurewicz's theorem):

Theorem 5.1. *If $f : X \to Y$ is a perfect n-dimensional surjection such that Y is a paracompact C-space, then* $\text{e-dim}\, X \leq \text{e-dim}(Y \times \mathbb{I}^n)$.

Theorem 5.1 follows from Theorem 2.2 and the next proposition which can be extracted from the Dranishnikov and Uspenskij proof of their [10, Lemma 2.1 and Theorem 1.4].

Proposition 5.2. *Let K be a CW-complex and X paracompact. If for any $\omega \in \text{cov}(X)$ there exist a paracompact space Z_ω with* $\text{e-dim}\, Z_\omega \leq K$ *and a perfect $(0,\omega)$-discrete map $g : X \to Z_\omega$, then* $\text{e-dim}\, X \leq K$.

Corollary 5.3. *Let $f : X \to Y$ be a σ-closed n-dimensional surjection between metrizable spaces with Y being a C-space. Then* $\text{e-dim}\, X \leq \text{e-dim}(Y \times \mathbb{I}^n)$.

Proof. Let K be a CW-complex with $\text{e-dim}(Y \times \mathbb{I}^n) \leq K$. It suffices to show that $\text{e-dim}\, X \leq K$. Since extension dimension satisfies the countable sum theorem, the proof reduces to the case of f closed. We can also assume that K is an open subset of a normed space because every CW-complex is homotopy equivalent to such a set. Represent X as the union of closed sets $X_i \subset X$, $i \geq 0$, such that $f|X_0$ is a perfect map onto Y and $\dim X_i \leq n$ for each $i \geq 1$ (see the proof of Theorem 1.3). Then, by Theorem 5.1, $\text{e-dim}\, X_0 \leq K$. On the other hand, $\text{e-dim}(Y \times \mathbb{I}^n) \leq K$ implies that $\text{e-dim}\, \mathbb{I}^n \leq K$, in particular, every map from $\mathbb{S}^{n-1}$ into K is extendable to a map from $\mathbb{I}^n$ into K. In other words, K is C^{n-1} and, as an open subset of a normed space, K is also LC^{n-1}. It is well known that LC^{n-1} and C^{n-1} metrizable spaces are precisely the absolute extensors for n-dimensional metrizable spaces. Hence, $\text{e-dim}\, X_i \leq K$ for every $i \geq 1$. Finally, by the countable sum theorem for extensional dimension, we have $\text{e-dim}\, X \leq K$. ∎

Another application is a parametric version of the Bogatyĭ decomposition theorem for n-dimensional metrizable spaces [2]: For every metrizable n-dimensional space M there exist countably many 0-dimensional G_δ-subsets $M_k \subset M$ such that $M = \bigcup_{i=1}^{n+1} M_{k(i)}$ for all pairwise distinct $k(1), \ldots, k(n+1)$ in $\mathbb{N}$.

Proposition 5.4. *Let $f : X \to Y$ be a closed n-dimensional surjection between metrizable spaces with Y a C-space. Then there exists a sequence $\{A_k\}$ of G_δ-subsets of X such that every restriction $f|A_k$ is 0-dimensional and for any $P \subset \mathbb{N}$ of cardinality $n+1$ we have $X = \bigcup_{k \in P} A_k$.*

Proof. Take closed sets $X_i \subset X$, $i \geq 0$, and a map $g : X \to \mathbb{I}^n$ such that $f|X_0$ is perfect, $X \setminus X_0 = \bigcup_{i\geq 1} X_i$, $\dim(f \triangle g) = 0$ and each $g|X_i$, $i \geq 1$, is uniformly 0-dimensional (see the proof of Theorem 1.3). According to the Bogatyĭ theorem, there exists a sequence of 0-dimensional G_δ-subsets $B_k \subset \mathbb{I}^n$ such that $\mathbb{I}^n$ is the union of any $n+1$ elements of this sequence. Let $A_k = (f \triangle g)^{-1}(Y \times B_k)$, $k \in \mathbb{N}$. The only nontrivial condition we need to check is that each restriction $f|A_k$ is 0-dimensional, i.e. $\dim f^{-1}(y) \cap A_k \leq 0$ for all $y \in Y$ and $k \geq 1$. For fixed y and k we have $f^{-1}(y) \cap A_k = \bigcup_{i\geq 0} g_i^{-1}(B_k)$, where g_i denotes the restriction $g|(f^{-1}(y) \cap X_i)$. Since every $g_i^{-1}(B_k)$ is closed in $f^{-1}(y) \cap A_k$, it suffices to show that the sets $g_i^{-1}(B_k)$, $i \geq 0$, are 0-dimensional. For $i = 0$ this follows from the Hurewicz lowering dimension theorem [16] because g_0 is a perfect 0-dimensional map. For $i \geq 1$ we use the fact that $g|X_i$ is uniformly 0-dimensional and the preimage of any 0-dimensional set under a uniformly 0-dimensional map is again 0-dimensional. ■

A map $f : X \to Y$ is said to be of *countable functional weight* [24] (notation $W(f) \leq \aleph_0$) if there exists a map $h : X \to Q$, where Q is the Hilbert cube, such that $f \triangle h : X \to Y \times Q$ is an embedding. In [24] Pasynkov has shown that his results from [25] remain valid for maps $f : X \to Y$ between finite-dimensional completely regular spaces X and Y such that $W(f) \leq \aleph_0$ and both f and its Čech–Stone extension have the same dimension (the last condition holds, for example, if X is normal, Y paracompact and f closed). We are going to show that Theorem 2.2 implies a similar result with Y being a C-space.

THEOREM 5.5. *Let $f : X \to Y$ be a σ-closed n-dimensional surjection of countable functional weight such that X is normal and Y a paracompact C-space. Then the set $\mathcal{G}$ of all maps $g \in C(X, \mathbb{I}^n)$ with* $\dim(f \triangle g) = 0$ *is uniformly dense in $C(X, \mathbb{I}^n)$. If, in addition, X is paracompact and f is σ-perfect, then $\mathcal{G}$ is dense and G_δ in $C(X, \mathbb{I}^n)$ with respect to the source limitation topology.*

Proof. Since $W(f) \leq \aleph_0$, there exists a map $h : X \to Q$ such that $f \triangle h$ is an embedding. For every $k \in \mathbb{N}$ let γ_k be an open cover of Q of mesh $\leq k^{-1}$. Suppose f is σ-closed and represent X and Y as the union of closed sets X_i and Y_i, respectively, such that each $f_i = f|X_i$ is a closed map onto Y_i. Let $Z_i = (\beta f_i)^{-1}(Y_i)$ and $\overline{f}_i = (\beta f_i)|Z_i$, $i \in \mathbb{N}$, where βf_i denotes the Čech–Stone extension of f_i. Because X is normal, each Z_i is a closed subset of $Z = (\beta f)^{-1}(Y)$ and $\overline{f}_i : Z_i \to Y_i$ is a perfect n-dimensional map. Moreover, Z is paracompact as the preimage of Y. We consider the extension $\overline{h} : Z \to Q$ of h and the covers $\omega_k = \overline{h}^{-1}(\gamma_k) \in \mathrm{cov}(Z)$. By Theorem 2.2 (applied to the maps $\overline{f}_i$), the sets $\mathcal{H}_{i,k}$ consisting of all $g \in C(Z, \mathbb{I}^n)$ such that $f_i \triangle g$

is $(0,\omega_k)$-discrete, $i,k\in\mathbb{N}$, are open and dense in $C(Z,\mathbb{I}^n)$ with respect to the source limitation topology, and hence so is $\mathcal{H}=\bigcap_{i,k=1}^{\infty}\mathcal{H}_{i,k}$. Moreover, $\mathcal{H}$ is uniformly dense in $C(Z,\mathbb{I}^n)$. Therefore, the set $\mathcal{G}_0=\{g|X : g\in\mathcal{H}\}$ is uniformly dense in $C(X,\mathbb{I}^n)$. Since h is a homeomorphism on every fiber of f, $\mathcal{G}_0\subset\mathcal{G}$. Hence, $\mathcal{G}$ is also uniformly dense in $C(X,\mathbb{I}^n)$.

Let f be σ-perfect and X paracompact. Then substituting $\bar{f}_i=f_i$, $Z_i=X_i$ and $Z=X$ in the previous proof, we deduce that $\mathcal{G}$ coincides with $\mathcal{H}$. ∎

COROLLARY 5.6. *Let $f : X\to Y$ be a σ-closed n-dimensional surjection having second countable fibers. If X is metrizable and Y a paracompact C-space, then the set of all $g\in C(X,\mathbb{I}^n)$ with* $\dim(f\triangle g)=0$ *is uniformly dense in $C(X,\mathbb{I}^n)$.*

Proof. Since f is of countable functional weight (see [24, Proposition 9.1]), this corollary follows from Theorem 5.5. ∎

We finally formulate the following result; its proof is similar to that of Theorem 2.2.

THEOREM 5.7. *Let $f : X\to Y$ be a perfect surjection of countable functional weight with Y a paracompact C-space. Then all maps $g\in C(X,Q)$ such that $f\triangle g$ is an embedding form a dense G_δ subset of $C(X,Q)$ with respect to the source limitation topology.*

COROLLARY 5.8. *Let $f : X\to Y$ be a perfect surjection between metrizable spaces. If Y is a C-space, then the set of all $g\in C(X,Q)$ with $f\triangle g$ being an embedding is dense and G_δ in $C(X,Q)$ with respect to the source limitation topology.*

Addendum. In the special case when both X and Y are compact metric spaces, Theorem 1.3 (the existence of a map $g : X\to\mathbb{I}^n$ such that $\dim(f\triangle g)=0$), Theorem 1.4 and Corollary 5.3 have recently been obtained by Y. Turygin in his preprint *Approximation of k-dimensional maps*. He also proved that if $f : X\to Y$ is a k-dimensional map between metric compacta, then the following two conditions are equivalent: (i) there exists a map $g : X\to\mathbb{I}^k$ with $\dim(f\triangle g)=0$; (ii) f admits an approximation by k-dimensional simplicial maps (see [32] for this notion).

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