A fixed point conjecture for
Borsuk continuous set-valued mappings

by

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Abstract. The main result of this paper is that for \( n = 3, 4, 5 \) and \( k = n - 2 \),
every Borsuk continuous set-valued map of the closed ball in the \( n \)-dimensional Euclidean
space with values which are one-point sets or sets homeomorphic to the \( k \)-sphere has a
fixed point. Our approach fails for \((k, n) = (1, 4)\). A relevant counterexample (for the
homological method, not for the fixed point conjecture) is indicated.

1. Introduction. The Lefschetz fixed point theorem holds for upper-
semicontinuous mappings with acyclic values and for their compositions [8],
[11]. On the other hand, even the Hausdorff continuity does not guaran-
tee an extension of the Brouwer theorem when the values are spheres [23].
Nevertheless, one can expect some fixed point results for mappings with
nonacyclic values provided a stronger kind of continuity is assumed. In 1954
Borsuk defined a distance \( g_c \) in the hyperspace \( K(M) \) of all nonempty comp-
act subsets of a metric space \((M, g)\) and called it the metric of continuity [1]. Let us recall that \( g_c(X, Y) = \max\{d_c(X, Y), d_c(Y, X)\} \), where
\( d_c(X, Y) = \inf\{\max\{g(x, g(x)) : x \in X\}\} \) and the infimum is taken over
all continuous functions \( g \) from \( X \) to \( Y \). We call a map into \( K(M) \) Borsuk
continuous if it is continuous with respect to \( g_c \). Let \( B^n \) denote the closed
unit ball in \( \mathbb{R}^n \). Górniewicz posed the following conjecture \(^{(1)}\):

\[(G.C.) \text{ Every Borsuk continuous map } f : B^n \to K(B^n) \text{ with connected values has a fixed point } x \in f(x).\]

\(^{(1)}\) Górniewicz published this conjecture in [13, 5.6] as an open problem, long after
communicating it to students and proving its two-dimensional case. It was called the
Górniewicz Conjecture in [21], in honour of Górniewicz’s 60th birthday. The proof for
\( n = 2 \) is based on [12] and can be found in [4].
In this paper we study a special case of G.C.:

**Conjecture 1.** Every Borsuk continuous map $f : B^n \to K(B^n)$ with values which are one-point sets or sets homeomorphic to the sphere $S^k$ ($k$ fixed) has a fixed point.

We will denote by $*$ the one-point space. The G.C. is confirmed for maps $f$ with the rational Čech cohomology group $\tilde{H}^*(f(x); \mathbb{Q})$ isomorphic to $\tilde{H}^*(S^{n-1}; \mathbb{Q})$ for $x \in B^n$ (see [13]). The latter case clearly implies Conjecture 1 for $k = n - 1$. The proof is based on the fact that the set $\tilde{f}(x) = f(x) \cup b(\mathbb{R}^n \setminus f(x))$ is acyclic, where $b(\mathbb{R}^n \setminus f(x))$ denotes the bounded component of $\mathbb{R}^n \setminus f(x)$. Since Čech homology spheres of codimension greater than 1 do not separate $\mathbb{R}^n$, it is clear that this approach cannot work for $1 \leq k \leq n - 2$ (2). Our purpose is to prove Conjecture 1 for $k = n - 2$ and $n = 3, 4, 5$. A different proof of the $n = 3$ case was given in [21]. In Preliminaries we give a brief exposition of some results from [20], which are basic for this paper.

**2. Preliminaries.** Let $f : B^n \to K(B^n)$ be an upper-semicontinuous map. From now on, $B = B^n$ and $S = \partial B$. For any $C \subset B$ we will denote by $\Gamma_C$ the set $\{(x, y) \in C \times B : y \in f(x)\}$, called the graph of $f|_C$. Let $F$ be a field.

**Definition 1 ([20]).** The map $f$ is called an $F$-Brouwer map if the homomorphism

$$\tilde{H}_n(\Gamma_B, \Gamma_S; F) \to \tilde{H}_n(B \times B, S \times B; F)$$

induced by inclusion is a nonzero homomorphism.

**Theorem 1** [20, Lemma 1]. Every $F$-Brouwer map has a fixed point.

Let $f$ satisfy the hypotheses of Conjecture 1 and define $U = \{x \in B : f(x) \cong S^k\}$. By the Chapman–Ferry–Jakobsche results on approximating homotopy equivalences by homeomorphisms [9, Theorem 3], [2, $\alpha$-approximation theorem], [16], [17] for $k \neq 4$ (3), $f|_U$ is continuous with respect to the distance $d_\infty$ which is defined similarly to $d_c$, but the infimum is taken over all homeomorphisms $g$ from $X$ onto $Y$. This continuity implies that the projection $p : \Gamma_U \to U$ is a completely regular mapping [6]. Consequently, $p$ is a locally trivial bundle with fibre $S^k$ (see [6]; cf. also [3, Theorem, p. 131] and [7, Corollary 1.1, p. 63]).

(2) Though $S^0$ is not connected, Conjecture 1 holds for $k = 0$. In this case $f$ may be identified with a continuous mapping into the second symmetric product of its domain. The fixed point theory of such mappings is developed in [25], [26], [22]. In spite of these results the G.C. cannot be extended to maps with disconnected values [18].

(3) The author does not know if the $\alpha$-approximation theorem holds in dimension 4.
**Definition 2.** We say that an \((n-1)\)-dimensional topological manifold \(N\) \(\varepsilon\)-approximates \(\partial U\) in \(U\) if there exists a compact \(n\)-dimensional topological manifold \(K\) with boundary such that \(K\) is a subcomplex of a simplicial decomposition of \(B\), \(\partial K = N\) and \(U \supset K \supset U \setminus O_\varepsilon(\partial U)\), where 
\[O_\varepsilon(\partial U) = \{ x \in U : \text{dist}(x, \partial U) < \varepsilon \}.\]

We begin with a triangulation of the interior of \(B\) with mesh \(\leq \varepsilon/2\). Then let \(K'\) be the union of all simplices intersecting \(U \setminus O_\varepsilon(\partial U)\) and let \(K\) be a small regular neighbourhood of \(K'\). It is clear that \(\partial K\) \(\varepsilon\)-approximates \(\partial U\) in \(U\). We can now rephrase [20, Theorem 1] as follows:

**Theorem 2.** Suppose that:

1. \(f : B \to K(B)\) is a Borsuk continuous map,
2. \(f\) is singlevalued on \(B \setminus U\) and takes values homeomorphic to \(S^k\) on \(U\), \(k \neq 4\),
3. \(U \subset \text{Int } B\),
4. For every \(\varepsilon > 0\) there exists a manifold \(N\) which \(\varepsilon\)-approximates \(\partial U\) in \(U\) and satisfies the inequality
\[\dim H_k(\Gamma_M; \mathbb{Z}_2) > \dim H_k(M; \mathbb{Z}_2)\]
for all components \(M\) of \(N\).

Then \(f\) is a \(\mathbb{Z}_2\)-Brouwer map.

This theorem follows directly from [20, proof of Theorem 1] and generalizations of [15, 3.4.3, 3.4.6] to \(\text{TOP}(S^k)\)-bundles, where \(\text{TOP}(S^k)\) denotes the group of all homeomorphisms \(S^k \to S^k\).

**3. A conjecture on homology of sphere bundles.** It is of interest to know when assumption 4 of Theorem 2 is satisfied. Since the projection from \(\Gamma_U\) onto \(U\) is a locally trivial bundle with fibre \(S^k\), so is \(p : \Gamma_M \to M\). In fact, \(p\) may be considered as a bundle with structural group \(\text{TOP}(S^k)\). Moreover, the diagram

\[
\begin{array}{ccc}
E & \longrightarrow & M \times \mathbb{R}^n \\
\downarrow p & & \downarrow \pi_1 \\
M & = & M \\
\end{array}
\]

commutes for \(E = \Gamma_M\). In Section 6 we will prove the following

\((^4)\) If \(f\) is not singlevalued on \(S\), we extend it to \(2B\) taking \(\tilde{f}(x) = (2 - ||x||)f(||x||^{-1}x)\) for \(1 \leq ||x|| \leq 2\). Of course, \(\tilde{f}\) is singlevalued on \(2S\). Moreover, by [20, proof of Statement 6], if \(\tilde{f}\) is a \(\mathbb{Z}_2\)-Brouwer map, so is \(f\). It would be nice to have Theorem 2 without the third assumption. Unfortunately, the author does not know if the fourth hypothesis for \(f\) implies the same for \(\tilde{f}\).
**Theorem 3.** Let \( M \subset \mathbb{R}^n \) be a compact connected \((n - 1)\)-dimensional topological manifold without boundary, \( n \geq 3 \). Let \( p : E \to M \) be a bundle with fibre \( S^{n-2} \) and structural group \( O(n - 1) \) such that the diagram (\( \natural \)) commutes. Then

\[
\dim H_{n-2}(E; \mathbb{Z}_2) > \dim H_{n-2}(M; \mathbb{Z}_2).
\]

**Remark 1.** It suffices to assume in Theorem 3 that the structural group reduces to \( O(n - 1) \).

Recall that the structural group \( G \) reduces to the subgroup \( H \) if every bundle \( p : E \to M \) with structural group \( G \) is \( G \)-equivalent to a bundle \( \tilde{E} \to M \) with structural group \( H \) (in particular, there is a homeomorphism \( h : \tilde{E} \to E \) with \( p \circ h = \tilde{p} \)). Moreover, if the inclusion \( O(n - 1) \subset G \) is a homotopy equivalence, then \( G \) reduces to \( O(n - 1) \) [30, proof of 11.45].

**Conjecture 2.** Theorem 3 remains true with \( O(n - 1) \) replaced by \( \text{TOP}(S^{n-2}) \).

**Fact 1.** Conjecture 2 holds for \( n \in \{3, 4, 5\} \).

Fact 1 follows from the homotopy equivalences \( O(2) \simeq \text{TOP}(S^1) \) (see for instance [20, Fact 2]), \( O(3) \simeq \text{TOP}(S^2) \) ([19], see also [28]), \( O(4) \simeq \text{TOP}(S^3) \) [14, p. 606].

**4. A fixed point theorem.** This section contains the main result of this paper. According to Fact 1 and Theorem 1–3, we have the following

**Theorem 4.** Conjecture 1 is true for \((k, n) = (1, 3), (2, 4), (3, 5)\).

The case \((k, n) = (1, 3)\) (first proved in [21] using \(K\)-theory) has a nice geometric interpretation: the Brouwer fixed point theorem holds for Borsuk continuous maps whose values are points or knots. One thing that distinguishes the case \((k, n) = (1, 3)\) from other cases is the classification of all 1-sphere bundles over 2-manifolds up to weak bundle equivalence [27] (see also [24], [10]).

**5. Preparation for proving Theorem 3**

**Lemma 1.** Let \( M \subset \mathbb{R}^{n+1} \) be an \( n \)-dimensional compact connected topological manifold without boundary, \( n \geq 2 \). Then \( x^n = 0 \) for every \( x \in H^1(M; \mathbb{Z}_2) \). Here \( x^n \) denotes the cup product \( x^{n-1} \cup x \).

The situation described in the hypotheses of this lemma is very well known in the literature. Let us gather some facts before the proof. First, \( M \subset \mathbb{R}^{n+1} \cup \{\infty\} \cong S^{n+1} \) and \( S^{n+1} \setminus M = U \cup V \) (\( U, V \) connected). The closures \( A = \overline{U}, B = \overline{V} \) are ANR’s [5, VIII.4.8]. By the Alexander duality, \( H^n(A; \mathbb{Z}_2) = H^n(B; \mathbb{Z}_2) = 0 \). Let \( i : M \to A \) and \( j : M \to B \) be inclusions.
The Mayer–Vietoris exact sequence shows that $\varphi : H^s(A; \mathbb{Z}_2) \oplus H^s(B; \mathbb{Z}_2) \to H^s(M; \mathbb{Z}_2)$, $\varphi(\alpha, \beta) = i^*\alpha + j^*\beta$, is an isomorphism for $1 \leq s < n$. Moreover:

1. $\text{Sq}^{n-r} y = 0$ for every $y \in H^r(M; \mathbb{Z}_2)$, $1 \leq r < n$ [29, III.2.3];
2. $i^*\text{Sq}^1 a \cup j^* b = i^* a \cup j^* \text{Sq}^1 b$ for all $a \in H^*(A; \mathbb{Z}_2), b \in H^{n-1-r}(B; \mathbb{Z}_2)$ [29, III.2.4], see also [29, II.4, III.1.4];
3. $\text{Sq}^i u^k = \binom{k}{i} u^{k+i}$ if $\dim(u) = 1$ [29, I.2.4].

**Proof of Lemma 1. Case 1.** Let $n \neq 2^m - 1$ for every natural $m$. Since $0 = \text{Sq}^{n-r} x^r = \binom{r}{n-r} x^n$ by (1), (3), it suffices to find $r$ such that $\binom{r}{n-r}$ is odd and $1 \leq n - r < n$. If $n = 2t$ then $r = t$ satisfies the above conditions. If $n = 2t - 1$ then $t \neq 2^m - 1$ for every $m$. Thus $t = 2^{i-1} + j$ for some $i \geq 2$ and $1 \leq j \leq 2^{i-1} - 1$. It is easy to check that $\binom{2^i-1}{s}$ is odd (5) for every $k = 0, 1, \ldots, 2^i - 1$ and $r = 2^i - 1$ satisfies $1 < n - r < n$.

**Case 2.** Let $n = 2^m - 1$. Then

$$x^n = (i^*\alpha + j^*\beta)^n = \sum_{k=0}^n \binom{n}{k} i^*\alpha^k \cup j^*\beta^{n-k} = \sum_{k=1}^{n-1} i^*\alpha^k \cup j^*\beta^{n-k}$$

$$= \sum_{p=1}^{(n-1)/2} \left( i^*\alpha^{2p-1} \cup j^*\beta^{n-2p+1} + i^*\alpha^{2p} \cup j^*\beta^{n-2p} \right)$$

$$= \sum_{p=1}^{(n-1)/2} \left( i^*\alpha^{2p-1} \cup j^* \left( \binom{n-2p}{1} \beta^{n-2p+1} + \binom{2p-1}{1} \alpha^{2p} \cup j^*\beta^{n-2p} \right) \right)$$

$$= \sum_{p=1}^{(n-1)/2} \left( i^*\alpha^{2p-1} \cup j^* \text{Sq}^1\beta^{n-2p} + i^* \text{Sq}^1\alpha^{2p-1} \cup j^*\beta^{n-2p} \right) = 0,$$

by (2), which proves the lemma.

We now recall some properties of Stiefel–Whitney classes. The first fact generalizes the well known Borsuk–Ulam theorem:

**Fact 2.** Let $E$ be a compact space, $T : E \to E$ a fixed point free involution (or equivalently, the generator of a free $\mathbb{Z}_2$-action on $E$), $c \in H^1(E/T; \mathbb{Z}_2)$ the first Stiefel–Whitney class of the 0-sphere bundle $\pi : E \to E/T$ and $g : E \to \mathbb{R}^n$ a continuous function. Suppose that $c^n \neq 0$. Then there is $x \in E$ such that $g(x) = g(Tx)$.

We give no reference here, because Fact 2 is an immediate consequence of the naturality of Stiefel–Whitney classes. Now, let $p : E \to M$ be a $k$-sphere bundle with structural group $O(k + 1)$. The antipodal map of $S^k$ induces a

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(5) By induction on $i$, $(x + y)^q \mod 2 = x^q + y^q$ for $q = 2^i$, so $\binom{2^i}{k}$ is even for $k = 1, \ldots, 2^i - 1$. 

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fibre preserving fixed point free involution $T : E \to E$, $p \circ T = p$. We have a projection $q : E/T \to M$ with $q \circ \pi = p$ ($\pi, c$ are defined in Fact 2).

**Fact 3** [15, 16.2.5]. The group $H^*(E/T; \mathbb{Z}_2)$ is an $H^*(M; \mathbb{Z}_2)$-module freely generated by $\{1, c, \ldots, c_k\}$ with the multiplication

$$H^*(M; \mathbb{Z}_2) \times H^*(E/T; \mathbb{Z}_2) \ni (\alpha, \beta) \mapsto \alpha \beta = q^*(\alpha) \cup \beta.$$ 

Moreover, $c^{k+1} = \sum_{j=1}^{k+1} w_j c^{k+1-j}$, where $w_j$ are the Stiefel–Whitney classes of the bundle $p$.

**Fact 4** [21, Lemma 1]. Let $M$ be a compact ANR. Then $\dim H_k(E; \mathbb{Z}_2) > \dim H_k(M; \mathbb{Z}_2)$ if and only if $w_{k+1} = 0$.

6. **Proof of Theorem 3.** We begin by extending the diagram ($\sharp$):

$$
\begin{array}{ccc}
E & \xrightarrow{i} & M \times \mathbb{R}^n \\
\downarrow p & & \downarrow \pi_1 \\
M & \xrightarrow{\pi_1} & M
\end{array}
$$

Suppose that $c^n \neq 0$. From Fact 2 with $g = \pi_2 \circ i$, we obtain points $x$ and $y = T(x)$ such that $\pi_2 \circ i(x) = \pi_2 \circ i(y)$. Since $\pi_1 \circ i(y) = p(y) = p(x) = \pi_1 \circ i(x)$, it follows that $i(x) = i(y)$ and $x = y$, which contradicts the fact that $T$ is fixed point free. Thus $c^n = 0$. Fact 3 for $k = n - 2$ leads to $c^{n-1} = \sum_{j=1}^{n-1} w_j c^{n-1-j}$. Hence

$$0 = c^{n-1} \cup c = \sum_{j=1}^{n-1} w_j c^{n-j} = w_1 c^{n-1} + \sum_{j=2}^{n-1} w_j c^{n-j}$$

$$= w_1 \sum_{j=1}^{n-1} w_j c^{n-1-j} + \sum_{j=1}^{n-2} w_{j+1} c^{n-j-1}$$

$$= (w_1 \cup w_{n-1}) 1 + \sum_{j=1}^{n-2} (w_1 \cup w_j + w_{j+1}) c^{n-j-1}.$$ 

By Fact 3, $w_1 \cup w_j + w_{j+1} = 0$ for $j = 1, \ldots, n - 2$. This gives $w_{n-1} = [w_1]^{n-1} = 0$, by Lemma 1. Fact 4 completes the proof.

The same proof with Fact 2 applied to $g = \pi_2 \circ i \circ h$ ($h : \tilde{E} \to E$ a bundle equivalence) yields Remark 1.

5. **A counterexample.** In this section it is shown that the notion of $F$-Brouwer mapping is not suitable for proving Conjecture 1 for $(k, n) = (1, 4)$. It is worth pointing out that our example does have an obvious fixed point.
Theorem 5. There is a Borsuk continuous mapping \( f : B^4 \to K(B^4) \) with values homeomorphic to \(*\) or \( S^1 \), which is an F-Brouwer map for no field \( F \).

Proof. Part 1. Write \( x = \sum_{i=1}^4 x_i e_i \in \mathbb{R}^4 \), \((e_i)_{i=1}^4\) the standard basis in \( \mathbb{R}^4 \), \( \mathbb{R}^i = \text{span}\{e_1, \ldots, e_i\} \) for \( i \leq 4 \), \( S^3 \) the unit sphere in \( \mathbb{R}^4 \), \( S^{i-1} = S^3 \cap \mathbb{R}^i \), \( E_x = \text{span}\{x, e_3, e_4\} \) for \( x \in S^1 \), \( S_x = S^3 \cap E_x \). Define \( \varphi_x : S^2 \to S_x \) by \( \varphi_x(y) = y_1 x + y_2 e_3 + y_3 e_4 \) for \( x \in S^1 \) and \( \varphi : S^1 \times S^2 \to \mathbb{R}^4 \) by \( \varphi(x, y) = \frac{1}{2} x + \frac{1}{4} \varphi_x(y) \). The map \( \varphi \) is an embedding of \( S^1 \times S^2 \) in \( \mathbb{R}^4 \). Set

\[ K = \left\{ \frac{1}{2} x + r \cdot \frac{1}{4} \varphi_x(y) : (x, y) \in S^1 \times S^2, 0 \leq r \leq 1 \right\}. \]

Clearly, \( K \cong S^1 \times B^3 \). Let \( q : S^3 \to S^2 \) be the Hopf fibration. We define \( f : B^4 \to K(B^4) \) by the formula

\[
\begin{align*}
 f\left( \frac{1}{2} x + r \cdot \frac{1}{4} \varphi_x(y) \right) &= r(1 - r) \cdot q^{-1}(y) \quad \text{on } K, \\
 f(z) &= 0 \quad \text{on } B^4 \setminus K.
\end{align*}
\]

Part 2. Suppose, contrary to our claim, that there is a field \( F \) making \( f \) an F-Brouwer map. Set \( B = B^4 \), \( S = \partial B \) and \( H_\star(\cdot) = H_\star(\cdot ; F) \). Note that \( f|_S = 0 \). The commutative diagram

\[
\begin{array}{ccc}
H_4(\Gamma_B, \Gamma_S) & \xrightarrow{\neq 0} & H_4(B \times B, S \times B) \\
p_* \downarrow & & \uparrow \cong j_* \\
H_4(B, S) & \xrightarrow{\cong} & H_4(B, S)
\end{array}
\]

with \( j(x) = (x, 0) \) yields \( p_* \neq 0 \). The diagram

\[
\begin{array}{ccc}
H_4(\Gamma_B, \Gamma_S) & \xrightarrow{i_*} & H_3 \Gamma_B \\
p_* \downarrow & & \uparrow \cong p_* \\
H_4(B, S) & \xrightarrow{\cong} & H_3 S \xrightarrow{\cong} F
\end{array}
\]

with the first row exact shows that \( i_* = 0 \). Let \( C = B \setminus \text{Int}(K) \). Consider the segment of the Mayer–Vietoris exact sequence:

\[ H_4 B \to H_3(\partial K) \to H_3 C \oplus H_3 K \to H_3 B. \]

Since \( H_3 K = H_3(S^1 \times B^3) = 0 \) and \( H_3(\partial K) = H_3(S^1 \times S^2) = F \), we have \( H_3 C = F \). Take \( v \in \text{Int}(K) \). Since \( S \) is a strong deformation retract of \( B \setminus \{v\} \), the composition

\[ F = H_3 S \xrightarrow{\eta} H_3 C \to H_3(B \setminus \{v\}) \]

of homomorphisms induced by inclusions is an isomorphism. Therefore \( \eta \) is a monomorphism. Now, the equality \( H_3 C = F \) shows that \( \eta \) is an isomorphism. Since \( \Gamma_S = S \times 0 \) and \( \Gamma_C = C \times 0 \), also \( \overline{\eta} : H_3 \Gamma_S \to H_3 \Gamma_C \) is an iso-
morphism. It follows that \( j_1 : H_3 \Gamma_C \to H_3 \Gamma_B \) is zero, because \( 0 = i_* = j_1 \circ \eta \). Summarizing, we have: \( j_1 = 0, H_3 \Gamma_C = H_3 C = F, H_3 \Gamma_{\partial K} = H_3 (\partial K) = F \).

Part 3. Our next goal is to determine the group \( H_3 \Gamma_K \). Note that \( K = L \cup N \), where

\[
\begin{align*}
L &= \left\{ \frac{1}{2} x + r \cdot \frac{1}{4} \varphi_x(y) : (x, y) \in S^1 \times S^2, 0 \leq r \leq 1/2 \right\}, \\
N &= \left\{ \frac{1}{2} x + r \cdot \frac{1}{4} \varphi_x(y) : (x, y) \in S^1 \times S^2, 1/2 \leq r \leq 1 \right\}
\end{align*}
\]

and \( L \cap N = \partial (L) \). For abbreviation, we let

\[
\Omega = \left( \frac{1}{2} x + r \cdot \frac{1}{4} \varphi_x(y), r(1-r)z \right)
\]

for \( z \in q^{-1}(y) \). The homotopy

\[
G_t(\Omega) = \left( \frac{1}{2} x + [t + (1-t)r] \cdot \frac{1}{4} \varphi_x(y), [t + (1-t)r][1-t-(1-t)r]z \right)
\]

shows that \( \Gamma_{\partial K} \) is a strong deformation retract of \( \Gamma_N \). Another homotopy

\[
H_t(\Omega) = \left( \frac{1}{2} x + tr \cdot \frac{1}{4} \varphi_x(y), tr(1-tr)z \right)
\]

with \( H_0(\Omega) = (\frac{1}{2} x, 0) \) gives \( \Gamma_L \simeq S^1 \). We also have a homeomorphism \( h : \Gamma_{\partial L} \to S^1 \times S^3 \) which sends \( \left( \frac{1}{2} x + \frac{1}{8} \varphi_x(y), \frac{1}{4} z \right) \) to \((x, z)\) for \( z \in q^{-1}(y) \).

Consider the segment of the Mayer–Vietoris exact sequence:

\[
H_3 \Gamma_{\partial L} \xrightarrow{\lambda} H_3 \Gamma_L \oplus H_3 \Gamma_N \xrightarrow{\psi} H_3 \Gamma_K \to H_2 \Gamma_{\partial L}.
\]

Since \( H_2 \Gamma_{\partial L} = H_2 (S^1 \times S^3) = 0 \), \( \psi \) is an epimorphism. Clearly, \( H_3 \Gamma_L = H_3 S^1 = 0 \). If \( \lambda = 0 \) then \( \psi \) is an isomorphism and \( H_3 \Gamma_K \cong H_3 \Gamma_N \cong H_3 \Gamma_{\partial K} \cong H_3 (\partial K) = F \). What is left is to show that \( \lambda = 0 \) or equivalently, that the inclusion \( \omega : \Gamma_{\partial L} \to \Gamma_N \) induces the zero homomorphism on \( H_3 \)-groups. This is equivalent to \( 0 = \xi_* : H_3 (S^1 \times S^3) \to H_3 (S^1 \times S^2) \) for \( \xi = \varphi^{-1} \circ G_1 \circ \omega \circ h^{-1} \) where \( G_1 (\Gamma_N) = \Gamma_{\partial K} = \partial (K) \times 0 \) is identified with \( \partial (K) = \varphi (S^1 \times S^2) \). It is easy to check that \( \xi (x, z) = (x, q(z)) \). Thus \( \xi = \text{id} \times q \). By the Künneth theorem, the diagram

\[
\begin{array}{ccc}
H_3 (S^1 \times S^3) & \xleftarrow{\cong} & \bigoplus_{i=0}^3 H_i S^1 \otimes H_{3-i} S^3 \\
\downarrow & & \downarrow \\
H_3 (S^1 \times S^2) & \xleftarrow{\cong} & \bigoplus_{i=0}^3 H_i S^1 \otimes H_{3-i} S^2
\end{array}
\]

commutes. The \( i \)th component of the direct sum is nonzero only for \( i = 0 \) in the first row and only for \( i = 1 \) in the second row of the above diagram. Hence \( \xi_* = 0 \).
Part 4. Consider the segment of the Mayer–Vietoris exact sequence:

\[
H_3 \Gamma \partial K \xrightarrow{\alpha} H_3 \Gamma_C \oplus H_3 \Gamma_K \xrightarrow{\beta} H_3 \Gamma_B
\]

where \( \alpha(x) = (i_1x, i_2x) \) and \( \beta(x, y) = j_2y - j_1x = j_2y \) (see Part 2). Since \( i_2 \) is a composition

\[
H_3 \Gamma \partial K \xrightarrow{\cong} H_3 \Gamma_N \xrightarrow{\psi} H_3 \Gamma_K,
\]

\( i_2 \) is an isomorphism (see Part 3). Now, \( \dim \ker \alpha = 1 = \dim \ker \beta \). Thus \( \dim \ker j_2 = \dim \ker \beta = 2 - \dim \ker \beta = 1 \). But \( 0 = \beta \circ \alpha = j_2 \circ i_2 \). Therefore \( j_2 = 0 \), a contradiction.

References


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