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A fixed point conjecture for Borsuk continuous set-valued mappings

by

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Abstract. The main result of this paper is that for $n = 3,4,5$ and $k = n-2$, every Borsuk continuous set-valued map of the closed ball in the n-dimensional Euclidean space with values which are one-point sets or sets homeomorphic to the k-sphere has a fixed point. Our approach fails for $(k,n) = (1,4)$. A relevant counterexample (for the homological method, not for the fixed point conjecture) is indicated.

1. Introduction. The Lefschetz fixed point theorem holds for upper-semicontinuous mappings with acyclic values and for their compositions [8], [11]. On the other hand, even the Hausdorff continuity does not guarantee an extension of the Brouwer theorem when the values are spheres [23]. Nevertheless, one can expect some fixed point results for mappings with nonacyclic values provided a stronger kind of continuity is assumed. In 1954 Borsuk defined a distance ϱ_{c} in the hyperspace $K(M)$ of all nonempty compact subsets of a metric space (M,ϱ) and called it the metric of continuity [1]. Let us recall that $\varrho_{\mathrm{c}}(X,Y) = \max\{d_{\mathrm{c}}(X,Y), d_{\mathrm{c}}(Y,X)\}$, where $d_{\mathrm{c}}(X,Y) = \inf\{\max\{\varrho(x,g(x)) : x \in X\}\}$ and the infimum is taken over all continuous functions g from X to Y. We call a map into $K(M)$ *Borsuk continuous* if it is continuous with respect to ϱ_{c}. Let B^n denote the closed unit ball in $\mathbb{R}^n$. Górniewicz posed the following conjecture ([1]):

(G.C.) Every Borsuk continuous map $f : B^n \to K(B^n)$ with connected values has a fixed point $x \in f(x)$.

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([1]) Górniewicz published this conjecture in [13, 5.6] as an open problem, long after communicating it to students and proving its two-dimensional case. It was called the Górniewicz Conjecture in [21], in honour of Górniewicz's 60th birthday. The proof for $n = 2$ is based on [12] and can be found in [4].

In this paper we study a special case of G.C.:

CONJECTURE 1. *Every Borsuk continuous map $f : B^n \to K(B^n)$ with values which are one-point sets or sets homeomorphic to the sphere S^k (k fixed) has a fixed point.*

We will denote by $*$ the one-point space. The G.C. is confirmed for maps f with the rational Čech cohomology group $\check{H}^\star(f(x);\mathbb{Q})$ isomorphic to $\check{H}^\star(*;\mathbb{Q})$ or to $\check{H}^\star(S^{n-1};\mathbb{Q})$ for $x \in B^n$ (see [13]). The latter case clearly implies Conjecture 1 for $k = n-1$. The proof is based on the fact that the set $\widetilde{f}(x) = f(x) \cup b(\mathbb{R}^n \setminus f(x))$ is acyclic, where $b(\mathbb{R}^n \setminus f(x))$ denotes the bounded component of $\mathbb{R}^n \setminus f(x)$. Since Čech homology spheres of codimension greater than 1 do not separate $\mathbb{R}^n$, it is clear that this approach cannot work for $1 \le k \le n-2$ (2). Our purpose is to prove Conjecture 1 for $k = n-2$ and $n = 3, 4, 5$. A different proof of the $n = 3$ case was given in [21]. In Preliminaries we give a brief exposition of some results from [20], which are basic for this paper.

2. Preliminaries. Let $f : B^n \to K(B^n)$ be an upper-semicontinuous map. From now on, $B = B^n$ and $S = \partial B$. For any $C \subset B$ we will denote by Γ_C the set $\{(x, y) \in C \times B : y \in f(x)\}$, called the graph of $f|_C$. Let F be a field.

DEFINITION 1 ([20]). The map f is called an *F-Brouwer map* if the homomorphism

$$\check{H}_n(\Gamma_B, \Gamma_S; F) \to \check{H}_n(B \times B, S \times B; F)$$

induced by inclusion is a nonzero homomorphism.

THEOREM 1 [20, Lemma 1]. *Every F-Brouwer map has a fixed point.*

Let f satisfy the hypotheses of Conjecture 1 and define $U = \{x \in B : f(x) \cong S^k\}$. By the Chapman–Ferry–Jakobsche results on approximating homotopy equivalences by homeomorphisms [9, Theorem 3], [2, α-approximation theorem], [16], [17] for $k \ne 4$ (3), $f|_U$ is continuous with respect to the distance ϱ_{h} which is defined similarly to d_{c}, but the infimum is taken over all homeomorphisms g from X onto Y. This continuity implies that the projection $p : \Gamma_U \to U$ is a completely regular mapping [6]. Consequently, p is a locally trivial bundle with fibre S^k (see [6]; cf. also [3, Theorem, p. 131] and [7, Corollary 1.1, p. 63]).

(2) Though S^0 is not connected, Conjecture 1 holds for $k = 0$. In this case f may be identified with a continuous mapping into the second symmetric product of its domain. The fixed point theory of such mappings is developed in [25], [26], [22]. In spite of these results the G.C. cannot be extended to maps with disconnected values [18].

(3) The author does not know if the α-approximation theorem holds in dimension 4.

DEFINITION 2. We say that an $(n-1)$-dimensional topological manifold N *ε-approximates ∂U* in U if there exists a compact n-dimensional topological manifold K with boundary such that K is a subcomplex of a simplicial decomposition of B, $\partial K = N$ and $U \supset K \supset U \setminus O_\varepsilon(\partial U)$, where $O_\varepsilon(\partial U) = \{x \in U : \operatorname{dist}(x, \partial U) < \varepsilon\}$.

We begin with a triangulation of the interior of B with mesh $\le \varepsilon/2$. Then let K' be the union of all simplices intersecting $U \setminus O_\varepsilon(\partial U)$ and let K be a small regular neighbourhood of K'. It is clear that ∂K ε-approximates ∂U in U. We can now rephrase [20, Theorem 1] as follows:

THEOREM 2. *Suppose that*:

1. *$f : B \to K(B)$ is a Borsuk continuous map*,

2. *f is singlevalued on $B \setminus U$ and takes values homeomorphic to S^k on U*, $k \neq 4$,

3. $U \subset \operatorname{Int} B$ ([4]),

4. *For every $\varepsilon > 0$ there exists a manifold N which ε-approximates ∂U in U and satisfies the inequality*

$$\dim H_k(\Gamma_M; \mathbb{Z}_2) > \dim H_k(M; \mathbb{Z}_2)$$

for all components M of N.

Then f is a $\mathbb{Z}_2$-Brouwer map.

This theorem follows directly from [20, proof of Theorem 1] and generalizations of [15, 3.4.3, 3.4.6] to $\mathrm{TOP}(S^k)$-bundles, where $\mathrm{TOP}(S^k)$ denotes the group of all homeomorphisms $S^k \to S^k$.

3. A conjecture on homology of sphere bundles. It is of interest to know when assumption 4 of Theorem 2 is satisfied. Since the projection from Γ_U onto U is a locally trivial bundle with fibre S^k, so is $p : \Gamma_M \to M$. In fact, p may be considered as a bundle with structural group $\mathrm{TOP}(S^k)$. Moreover, the diagram

$$(\sharp) \qquad \begin{array}{ccc} E & \hookrightarrow & M \times \mathbb{R}^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ M & = & M \end{array}$$

commutes for $E = \Gamma_M$. In Section 6 we will prove the following

([4]) If f is not singlevalued on S, we extend it to $2B$ taking $\widetilde{f}(x) = (2 - \|x\|) f(\|x\|^{-1} x)$ for $1 \le \|x\| \le 2$. Of course, $\widetilde{f}$ is singlevalued on $2S$. Moreover, by [20, proof of Statement 6], if $\widetilde{f}$ is a $\mathbb{Z}_2$-Brouwer map, so is f. It would be nice to have Theorem 2 without the third assumption. Unfortunately, the author does not know if the fourth hypothesis for f implies the same for $\widetilde{f}$.

Theorem 3. *Let $M \subset \mathbb{R}^n$ be a compact connected $(n-1)$-dimensional topological manifold without boundary, $n \geq 3$. Let $p : E \to M$ be a bundle with fibre S^{n-2} and structural group $O(n-1)$ such that the diagram* ($\sharp$) *commutes. Then*

$$\dim H_{n-2}(E;\mathbb{Z}_2) > \dim H_{n-2}(M;\mathbb{Z}_2).$$

Remark 1. It suffices to assume in Theorem 3 that the structural group reduces to $O(n-1)$.

Recall that the structural group G *reduces to the subgroup H* if every bundle $p : E \to M$ with structural group G is G-equivalent to a bundle $\widetilde{p} : \widetilde{E} \to M$ with structural group H (in particular, there is a homeomorphism $h : \widetilde{E} \to E$ with $p \circ h = \widetilde{p}$). Moreover, if the inclusion $O(n-1) \subset G$ is a homotopy equivalence, then G reduces to $O(n-1)$ [30, proof of 11.45].

Conjecture 2. *Theorem* 3 *remains true with $O(n-1)$ replaced by* $\mathrm{TOP}(S^{n-2})$.

Fact 1. *Conjecture* 2 *holds for $n \in \{3,4,5\}$.*

Fact 1 follows from the homotopy equivalences $O(2) \simeq \mathrm{TOP}(S^1)$ (see for instance [20, Fact 2]), $O(3) \simeq \mathrm{TOP}(S^2)$ ([19], see also [28]), $O(4) \simeq \mathrm{TOP}(S^3)$ [14, p. 606].

4. A fixed point theorem. This section contains the main result of this paper. According to Fact 1 and Theorem 1–3, we have the following

Theorem 4. *Conjecture* 1 *is true for $(k,n) = (1,3), (2,4), (3,5)$.*

The case $(k,n) = (1,3)$ (first proved in [21] using K-theory) has a nice geometric interpretation: the Brouwer fixed point theorem holds for Borsuk continuous maps whose values are points or knots. One thing that distinguishes the case $(k,n) = (1,3)$ from other cases is the classification of all 1-sphere bundles over 2-manifolds up to weak bundle equivalence [27] (see also [24], [10]).

5. Preparation for proving Theorem 3

Lemma 1. *Let $M \subset \mathbb{R}^{n+1}$ be an n-dimensional compact connected topological manifold without boundary, $n \geq 2$. Then $x^n = 0$ for every $x \in H^1(M;\mathbb{Z}_2)$. Here x^n denotes the cup product $x^{n-1} \cup x$.*

The situation described in the hypotheses of this lemma is very well known in the literature. Let us gather some facts before the proof. First, $M \subset \mathbb{R}^{n+1} \cup \{\infty\} \cong S^{n+1}$ and $S^{n+1} \setminus M = U \cup V$ (U, V connected). The closures $A = \overline{U}$, $B = \overline{V}$ are ANR's [5, VIII.4.8]. By the Alexander duality, $H^n(A;\mathbb{Z}_2) = H^n(B;\mathbb{Z}_2) = 0$. Let $i : M \to A$ and $j : M \to B$ be inclusions.

The Mayer–Vietoris exact sequence shows that $\varphi : H^s(A;\mathbb{Z}_2)\oplus H^s(B;\mathbb{Z}_2) \to H^s(M;\mathbb{Z}_2)$, $\varphi(\alpha,\beta) = i^\star\alpha + j^\star\beta$, is an isomorphism for $1 \le s < n$. Moreover:

(1) $\mathrm{Sq}^{n-r}\, y = 0$ for every $y \in H^r(M;\mathbb{Z}_2)$, $1 \le r < n$ [29, III.2.3];

(2) $i^\star \mathrm{Sq}^1 a \cup j^\star b = i^\star a \cup j^\star \mathrm{Sq}^1 b$ for all $a \in H^r(A;\mathbb{Z}_2)$, $b \in H^{n-1-r}(B;\mathbb{Z}_2)$ ([29, III.2.4], see also [29, II.4, III.1.4]);

(3) $\mathrm{Sq}^i\, u^k = \binom{k}{i} u^{k+i}$ if $\dim(u) = 1$ [29, I.2.4].

Proof of Lemma 1. Case 1. Let $n \ne 2^m - 1$ for every natural m. Since $0 = \mathrm{Sq}^{n-r}\, x^r = \binom{r}{n-r} x^n$ by (1), (3), it suffices to find r such that $\binom{r}{n-r}$ is odd and $1 \le n - r \le r < n$. If $n = 2t$ then $r = t$ satisfies the above conditions. If $n = 2t-1$ then $t \ne 2^{m-1}$ for every m. Thus $t = 2^{i-1} + j$ for some $i \ge 2$ and $1 \le j \le 2^{i-1} - 1$. It is easy to check that $\binom{2^i-1}{k}$ is odd (5) for every $k = 0, 1, \ldots, 2^i - 1$ and $r = 2^i - 1$ satisfies $1 < n - r < r < n$.

Case 2. Let $n = 2^m - 1$. Then

$$\begin{aligned} x^n &= (i^\star\alpha + j^\star\beta)^n = \sum_{k=0}^{n} \binom{n}{k} i^\star\alpha^k \cup j^\star\beta^{n-k} = \sum_{k=1}^{n-1} i^\star\alpha^k \cup j^\star\beta^{n-k} \\ &= \sum_{p=1}^{(n-1)/2} (i^\star\alpha^{2p-1} \cup j^\star\beta^{n-2p+1} + i^\star\alpha^{2p} \cup j^\star\beta^{n-2p}) \\ &= \sum_{p=1}^{(n-1)/2} \left(i^\star\alpha^{2p-1} \cup j^\star \binom{n-2p}{1} \beta^{n-2p+1} + i^\star \binom{2p-1}{1} \alpha^{2p} \cup j^\star\beta^{n-2p} \right) \\ &= \sum_{p=1}^{(n-1)/2} (i^\star\alpha^{2p-1} \cup j^\star \mathrm{Sq}^1 \beta^{n-2p} + i^\star \mathrm{Sq}^1 \alpha^{2p-1} \cup j^\star\beta^{n-2p}) = 0, \end{aligned}$$

by (2), which proves the lemma.

We now recall some properties of Stiefel–Whitney classes. The first fact generalizes the well known Borsuk–Ulam theorem:

Fact 2. *Let E be a compact space, $T : E \to E$ a fixed point free involution (or equivalently, the generator of a free $\mathbb{Z}_2$-action on E), $c \in H^1(E/T;\mathbb{Z}_2)$ the first Stiefel–Whitney class of the 0-sphere bundle $\pi : E \to E/T$ and $g : E \to \mathbb{R}^n$ a continuous function. Suppose that $c^n \ne 0$. Then there is $x \in E$ such that $g(x) = g(Tx)$.*

We give no reference here, because Fact 2 is an immediate consequence of the naturality of Stiefel–Whitney classes. Now, let $p : E \to M$ be a k-sphere bundle with structural group $O(k+1)$. The antipodal map of S^k induces a

(5) By induction on i, $(x+y)^q \bmod 2 = x^q + y^q$ for $q = 2^i$, so $\binom{2^i}{k}$ is even for $k = 1, \ldots, 2^i - 1$.

fibre preserving fixed point free involution $T : E \to E$, $p \circ T = p$. We have a projection $q : E/T \to M$ with $q \circ \pi = p$ (π, c are defined in Fact 2).

FACT 3 [15, 16.2.5]. *The group* $H^\star(E/T;\mathbb{Z}_2)$ *is an* $H^\star(M;\mathbb{Z}_2)$*-module freely generated by* $\{1, c, \ldots, c^k\}$ *with the multiplication*

$$H^\star(M;\mathbb{Z}_2) \times H^\star(E/T;\mathbb{Z}_2) \ni (\alpha,\beta) \mapsto \alpha\beta = q^\star(\alpha) \cup \beta.$$

Moreover, $c^{k+1} = \sum_{j=1}^{k+1} w_j c^{k+1-j}$, *where* w_j *are the Stiefel–Whitney classes of the bundle* p.

FACT 4 [21, Lemma 1]. *Let* M *be a compact ANR. Then* $\dim H_k(E;\mathbb{Z}_2) > \dim H_k(M;\mathbb{Z}_2)$ *if and only if* $w_{k+1} = 0$.

6. Proof of Theorem 3. We begin by extending the diagram (♯):

$$\begin{array}{ccccc} E & \xrightarrow{\ i\ } & M \times \mathbb{R}^n & \xrightarrow{\ \pi_2\ } & \mathbb{R}^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_1} & & \\ M & = & M & & \end{array}$$

Suppose that $c^n \neq 0$. From Fact 2 with $g = \pi_2 \circ i$, we obtain points x and $y = T(x)$ such that $\pi_2 \circ i(x) = \pi_2 \circ i(y)$. Since $\pi_1 \circ i(y) = p(y) = p(x) = \pi_1 \circ i(x)$, it follows that $i(x) = i(y)$ and $x = y$, which contradicts the fact that T is fixed point free. Thus $c^n = 0$. Fact 3 for $k = n-2$ leads to $c^{n-1} = \sum_{j=1}^{n-1} w_j c^{n-1-j}$. Hence

$$\begin{aligned} 0 = c^{n-1} \cup c &= \sum_{j=1}^{n-1} w_j c^{n-j} = w_1 c^{n-1} + \sum_{j=2}^{n-1} w_j c^{n-j} \\ &= w_1 \sum_{j=1}^{n-1} w_j c^{n-1-j} + \sum_{j=1}^{n-2} w_{j+1} c^{n-j-1} \\ &= (w_1 \cup w_{n-1})1 + \sum_{j=1}^{n-2} (w_1 \cup w_j + w_{j+1}) c^{n-j-1}. \end{aligned}$$

By Fact 3, $w_1 \cup w_j + w_{j+1} = 0$ for $j = 1, \ldots, n-2$. This gives $w_{n-1} = [w_1]^{n-1} = 0$, by Lemma 1. Fact 4 completes the proof.

The same proof with Fact 2 applied to $g = \pi_2 \circ i \circ h$ ($h : \widetilde{E} \to E$ a bundle equivalence) yields Remark 1.

5. A counterexample. In this section it is shown that the notion of F-Brouwer mapping is not suitable for proving Conjecture 1 for $(k, n) = (1, 4)$. It is worth pointing out that our example does have an obvious fixed point.

THEOREM 5. *There is a Borsuk continuous mapping $f : B^4 \to K(B^4)$ with values homeomorphic to $*$ or S^1, which is an F-Brouwer map for no field F.*

Proof. Part 1. Write $x = \sum_{i=1}^4 x_i e_i \in \mathbb{R}^4$, $(e_i)_{i=1}^4$ the standard basis in $\mathbb{R}^4$, $\mathbb{R}^i = \operatorname{span}\{e_1, \dots, e_i\}$ for $i \le 4$, S^3 the unit sphere in $\mathbb{R}^4$, $S^{i-1} = S^3 \cap \mathbb{R}^i$, $E_x = \operatorname{span}\{x, e_3, e_4\}$ for $x \in S^1$, $S_x = S^3 \cap E_x$. Define $\varphi_x : S^2 \to S_x$ by $\varphi_x(y) = y_1 x + y_2 e_3 + y_3 e_4$ for $x \in S^1$ and $\varphi : S^1 \times S^2 \to \mathbb{R}^4$ by $\varphi(x, y) = \frac{1}{2}x + \frac{1}{4}\varphi_x(y)$. The map φ is an embedding of $S^1 \times S^2$ in $\mathbb{R}^4$. Set

$$K = \bigl\{\tfrac{1}{2}x + r \cdot \tfrac{1}{4}\varphi_x(y) : (x, y) \in S^1 \times S^2,\ 0 \le r \le 1\bigr\}.$$

Clearly, $K \cong S^1 \times B^3$. Let $q : S^3 \to S^2$ be the Hopf fibration. We define $f : B^4 \to K(B^4)$ by the formula

$$\begin{cases} f\bigl(\tfrac{1}{2}x + r \cdot \tfrac{1}{4}\varphi_x(y)\bigr) = r(1-r) \cdot q^{-1}(y) & \text{on } K, \\ f(z) = 0 & \text{on } B^4 \setminus K. \end{cases}$$

Part 2. Suppose, contrary to our claim, that there is a field F making f an F-Brouwer map. Set $B = B^4$, $S = \partial B$ and $H_\star(\cdot) = H_\star(\cdot; F)$. Note that $f|_S = 0$. The commutative diagram

$$\begin{array}{ccc} H_4(\Gamma_B, \Gamma_S) & \xrightarrow{\ne 0} & H_4(B \times B, S \times B) \\ {\scriptstyle p_\star}\downarrow & & \uparrow{\scriptstyle \cong\, j_\star} \\ H_4(B, S) & = & H_4(B, S) \end{array}$$

with $j(x) = (x, 0)$ yields $p_\star \ne 0$. The diagram

$$\begin{array}{ccccc} H_4(\Gamma_B, \Gamma_S) & \longrightarrow & H_3\Gamma_S & \xrightarrow{i_\star} & H_3\Gamma_B \\ {\scriptstyle p_\star}\downarrow & & {\scriptstyle p_\star}\downarrow{\scriptstyle \cong} & & \\ H_4(B, S) & \xrightarrow{\cong} & H_3 S & = & F \end{array}$$

with the first row exact shows that $i_\star = 0$. Let $C = B \setminus \operatorname{Int}(K)$. Consider the segment of the Mayer–Vietoris exact sequence:

$$H_4 B \to H_3(\partial K) \to H_3 C \oplus H_3 K \to H_3 B.$$

Since $H_3 K = H_3(S^1 \times B^3) = 0$ and $H_3(\partial K) = H_3(S^1 \times S^2) = F$, we have $H_3 C = F$. Take $v \in \operatorname{Int}(K)$. Since S is a strong deformation retract of $B \setminus \{v\}$, the composition

$$F = H_3 S \xrightarrow{\eta} H_3 C \to H_3(B \setminus \{v\})$$

of homomorphisms induced by inclusions is an isomorphism. Therefore η is a monomorphism. Now, the equality $H_3 C = F$ shows that η is an isomorphism. Since $\Gamma_S = S \times 0$ and $\Gamma_C = C \times 0$, also $\overline{\eta} : H_3\Gamma_S \to H_3\Gamma_C$ is an isomorphism. It follows that $j_1 : H_3\Gamma_C \to H_3\Gamma_B$ is zero, because $0 = i_\star = j_1 \circ \overline{\eta}$. Summarizing, we have: $j_1 = 0$, $H_3\Gamma_C = H_3C = F$, $H_3\Gamma_{\partial K} = H_3(\partial K) = F$.

Part 3. Our next goal is to determine the group $H_3\Gamma_K$. Note that $K = L \cup N$, where

$$L = \left\{\tfrac{1}{2}x + r \cdot \tfrac{1}{4}\varphi_x(y) : (x,y) \in S^1 \times S^2,\ 0 \le r \le 1/2\right\},$$
$$N = \left\{\tfrac{1}{2}x + r \cdot \tfrac{1}{4}\varphi_x(y) : (x,y) \in S^1 \times S^2,\ 1/2 \le r \le 1\right\}$$

and $L \cap N = \partial(L)$. For abbreviation, we let

$$\Omega = \left(\tfrac{1}{2}x + r \cdot \tfrac{1}{4}\varphi_x(y), r(1-r)z\right)$$

for $z \in q^{-1}(y)$. The homotopy

$$G_t(\Omega) = \left(\tfrac{1}{2}x + [t + (1-t)r] \cdot \tfrac{1}{4}\varphi_x(y), [t + (1-t)r][1 - t - (1-t)r]z\right)$$

shows that $\Gamma_{\partial K}$ is a strong deformation retract of Γ_N. Another homotopy

$$H_t(\Omega) = \left(\tfrac{1}{2}x + tr \cdot \tfrac{1}{4}\varphi_x(y), \mathrm{tr}(1 - tr)z\right)$$

with $H_0(\Omega) = (\frac{1}{2}x, 0)$ gives $\Gamma_L \simeq S^1$. We also have a homeomorphism $h : \Gamma_{\partial L} \to S^1 \times S^3$ which sends $\left(\frac{1}{2}x + \frac{1}{8}\varphi_x(y), \frac{1}{4}z\right)$ to (x, z) for $z \in q^{-1}(y)$. Consider the segment of the Mayer–Vietoris exact sequence:

$$H_3\Gamma_{\partial L} \xrightarrow{\lambda} H_3\Gamma_L \oplus H_3\Gamma_N \xrightarrow{\psi} H_3\Gamma_K \to H_2\Gamma_{\partial L}.$$

Since $H_2\Gamma_{\partial L} = H_2(S^1 \times S^3) = 0$, ψ is an epimorphism. Clearly, $H_3\Gamma_L = H_3S^1 = 0$. If $\lambda = 0$ then ψ is an isomorphism and $H_3\Gamma_K \cong H_3\Gamma_N \cong H_3\Gamma_{\partial K} \cong H_3(\partial K) = F$. What is left is to show that $\lambda = 0$ or equivalently, that the inclusion $\omega : \Gamma_{\partial L} \to \Gamma_N$ induces the zero homomorphism on H_3-groups. This is equivalent to $0 = \xi_\star : H_3(S^1 \times S^3) \to H_3(S^1 \times S^2)$ for $\xi = \varphi^{-1} \circ G_1 \circ \omega \circ h^{-1}$ where $G_1(\Gamma_N) = \Gamma_{\partial K} = \partial(K) \times 0$ is identified with $\partial(K) = \varphi(S^1 \times S^2)$. It is easy to check that $\xi(x, z) = (x, q(z))$. Thus $\xi = \mathrm{id} \times q$. By the Künneth theorem, the diagram

$$\begin{array}{ccc}
H_3(S^1 \times S^3) & \xleftarrow{\cong} & \bigoplus_{i=0}^{3} H_iS^1 \otimes H_{3-i}S^3 \\
\downarrow{\scriptstyle \xi_\star} & & \downarrow \\
H_3(S^1 \times S^2) & \xleftarrow{\cong} & \bigoplus_{i=0}^{3} H_iS^1 \otimes H_{3-i}S^2
\end{array}$$

commutes. The ith component of the direct sum is nonzero only for $i = 0$ in the first row and only for $i = 1$ in the second row of the above diagram. Hence $\xi_\star = 0$.

Part 4. Consider the segment of the Mayer–Vietoris exact sequence:

$$\begin{array}{ccccc} H_3\mathit{\Gamma}_{\partial K} & \xrightarrow{\alpha} & H_3\mathit{\Gamma}_C \oplus H_3\mathit{\Gamma}_K & \xrightarrow{\beta} & H_3\mathit{\Gamma}_B \\ \| & & \| & & \\ F & & F^2 & & \end{array}$$

where $\alpha(x) = (i_1 x, i_2 x)$ and $\beta(x, y) = j_2 y - j_1 x = j_2 y$ (see Part 2). Since i_2 is a composition

$$H_3\mathit{\Gamma}_{\partial K} \stackrel{\cong}{\to} H_3\mathit{\Gamma}_N \stackrel{\psi}{\to} H_3\mathit{\Gamma}_K,$$

i_2 is an isomorphism (see Part 3). Now, $\dim \operatorname{im} \alpha = 1 = \dim \ker \beta$. Thus $\dim \operatorname{im} j_2 = \dim \operatorname{im} \beta = 2 - \dim \ker \beta = 1$. But $0 = \beta \circ \alpha = j_2 \circ i_2$. Therefore $j_2 = 0$, a contradiction.

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