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Recent developments in the theory of Borel reducibility

by

Greg Hjorth (Los Angeles, CA) and
Alexander S. Kechris (Pasadena, CA)

Abstract. Let E_0 be the Vitali equivalence relation and E_3 the product of countably many copies of E_0. Two new dichotomy theorems for Borel equivalence relations are proved. First, for any Borel equivalence relation E that is (Borel) reducible to E_3, either E is reducible to E_0 or else E_3 is reducible to E. Second, if E is a Borel equivalence relation induced by a Borel action of a closed subgroup of the infinite symmetric group that admits an invariant metric, then either E is reducible to a countable Borel equivalence relation or else E_3 is reducible to E.

We also survey a number of results and conjectures concerning the global structure of reducibility on Borel equivalence relations.

1. Introduction. In this paper we present the proofs of the results announced in [12] and survey the recent work bearing on the sweeping conjectures which were presented in that paper.

2. Definitions. We briefly recall the relevant definitions. This is only a skeleton of the introduction of [12], which also presents considerable motivation.

Definition 2.1. A topological space is said to be *Polish* if it is separable and the topology is generated by some complete metric. The *Borel* subsets of a Polish space are those contained in the σ-algebra generated by the open sets.

An equivalence relation $E \subseteq X \times X$ on a Polish space X is said to be *Borel* if it appears in the σ-algebra generated by the open sets in the product topology on $X \times X$. Here we have not really departed from our original use of the term "Borel", since $X \times X$ is a Polish space in this product topology.

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A function $f : X \to Y$ between Polish spaces is said to be *Borel* if the preimage of any open set is Borel. It follows from classical techniques (see [14], §18.C) that this is equivalent to requiring that the graph of f be Borel as a subset of $X \times Y$.

DEFINITION 2.2. For E and F Borel equivalence relations on Polish spaces X and Y, we say that E is *Borel reducible to* F, written

$$E \leq_{\mathrm{B}} F,$$

if there is a Borel function $f : X \to Y$ such that for all $x_1, x_2 \in X$,

$$x_1 E x_2 \ \Leftrightarrow\ f(x_1) F f(x_2).$$

This definition naturally gives rise to variations. We write $E \not\leq_{\mathrm{B}} F$ if it is not the case that $E \leq_{\mathrm{B}} F$. We write

$$E <_{\mathrm{B}} F$$

if we have both $E \leq_{\mathrm{B}} F$ *and* $F \not\leq_{\mathrm{B}} E$. We write

$$E \sim_{\mathrm{B}} F$$

if there is a reduction in both directions:

$$E \leq_{\mathrm{B}} F, \qquad F \leq_{\mathrm{B}} E.$$

We say that E and F are *Borel incomparable* if there is reduction in neither direction:

$$E \not\leq_{\mathrm{B}} F, \qquad F \not\leq_{\mathrm{B}} E.$$

For E an equivalence relation on a space X and $x \in X$, we let $[x]_E = \{y \in X : xEy\}$ denote the equivalence class of x. We can then let $X/E = \{[x]_E : x \in X\}$ indicate the collection of all equivalence classes.

The first comment that must be made about that partial order (Borel equivalence relations, $\leq_{\mathrm{B}}$) is that it is massively complicated and apparently resistant to any *global* structure theorems. For instance:

THEOREM 2.3 (Louveau–Veličković; see [18]). *There is an assignment $S \mapsto E_S$ of Borel equivalence relations to subsets of $\mathbb{N}$ such that for all $S, T \subseteq \mathbb{N}$, $E_S \leq E_T$ if and only if $S \setminus T$ is finite.*

DEFINITION 2.4. For X a Polish space, we let $\mathit{\Delta}(X)$ denote the equivalence relation of equality on X:

$$\mathit{\Delta}(X) = \{(x_1, x_2) \in X^2 : x_1 = x_2\}.$$

Since $X/\mathit{\Delta}(X)$ screams out to be identified with X, we will frequently slur over the distinction between X and $\mathit{\Delta}(X)$. In particular, we will use n to denote

$$\mathit{\Delta}(\{0, 1, \ldots, n-1\}),$$

the equality relation on the discrete space $\{0, 1, \ldots, n-1\}$ of size n, and $\mathbb{N}$ to denote equality on the countably infinite discrete space $\mathbb{N} = \{0, 1, \ldots\}$, and $\mathbb{R}$ to denote equality on the set of reals. Note that any countable discrete space is Polish, and thus

$$0, 1, \ldots, n, \ldots, \mathbb{N}, \mathbb{R},$$

can be thought of as the simplest examples of Borel equivalence relations on Polish spaces. It was shown in [19] that for every Borel equivalence relation E, either E is $\sim_{\mathrm{B}}$ to one of $1, 2, \ldots, n, \ldots, \mathbb{N}$ or else $\mathbb{R} \leq_{\mathrm{B}} E$.

Slightly more complicated is the equivalence relation of eventual agreement on infinite binary sequences. So for

$$x, y \in 2^{\mathbb{N}} := \{z \mid z : \mathbb{N} \to \{0, 1\}\},$$

we set

$$xE_0y \quad \text{iff} \quad \exists k\, \forall n > k\ (x(n) = y(n)).$$

We may view $2^{\mathbb{N}}$ as a Polish space by taking the discrete topology on $2 := \{0, 1\}$ and the resulting product topology on $2^{\mathbb{N}}$. It was shown in [8] that E_0 is the *next* Borel equivalence relation after $\mathbb{R}$.

After E_0 the ordering fans out. The first Borel equivalence relation here to be seriously studied was the equivalence relation of eventual agreement on sequences of *points* in the Cantor space $2^{\mathbb{N}}$. For

$$x, y \in (2^{\mathbb{N}})^{\mathbb{N}} := \{z \mid z : \mathbb{N} \to 2^{\mathbb{N}}\},$$

we set

$$xE_1y \quad \text{iff} \quad \exists k\, \forall n > k\ (x(n) = y(n)).$$

The paper [16] showed that $E_0 <_{\mathrm{B}} E_1$ and there is *no* Borel E with $E_0 <_{\mathrm{B}} E <_{\mathrm{B}} E_1$.

Also strictly above E_0 is the equivalence relation $(E_0)^{\mathbb{N}}$ obtained by taking its countable product. So for $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$ we set

$$xE_3y \quad \text{iff} \quad \forall n\ (x(n)E_0y(n)).$$

It is folklore that $E_0 <_{\mathrm{B}} E_3$ and it follows from [16] that E_1 and E_3 are Borel incomparable. We announced in [12] that there is no E with $E_0 <_{\mathrm{B}} E <_{\mathrm{B}} E_3$, and we will give the proof below.

In passing from E_1 to E_3 we skipped over E_2. In fact, its construction is less obvious than these other examples, and so we will postpone giving the usual definiton until we come to the subject of Polishable ideals. In the meantime it might be worth saying that, up to $\sim_{\mathrm{B}}$-equivalence, E_2 is given by the coset equivalence relation of

$$\ell^1 := \left\{ x \in \mathbb{R}^{\mathbb{N}} : \sum |x(i)| < \infty \right\}$$

thought of as a subgroup of $\mathbb{R}^{\mathbb{N}}$ in the usual way.

Definition 2.5. A Borel equivalence relation E on X is said to be *countable* if every equivalence class is countable. It is then said to be *treeable* if there is a symmetric Borel relation $R \subseteq X \times X$ which has no cycles and whose connected components form the equivalence classes of E; in other words, we may, in a Borel manner, place the structure of a tree on each $[x]_E$.

An equivalence relation E is said to be *smooth* or *concretely classifiable* if it is Borel reducible to $\mathbb{R}$.

It is a non-trivial fact that in the $\leq_{\mathrm{B}}$ ordering there is a maximal countable Borel equivalence relation, E_∞, and a maximal countable treeable Borel equivalence relation, $E_{\mathrm{T}\infty}$. For the sake of definiteness, we give each one an instantiation, but the reader should see [4] or [13] for a more detailed analysis.

Definition 2.6. Let $\mathbb{F}_2$ denote the free group on two generators and let $2^{\mathbb{F}_2}$ denote the space of all functions $f : \mathbb{F}_2 \to \{0,1\}$, equipped with the product topology (under which it is isomorphic to the Cantor space $2^{\mathbb{N}}$). We let $\mathbb{F}_2$ act on $2^{\mathbb{F}_2}$ by the shift action

$$(\sigma \cdot f)(\tau) = f(\sigma^{-1}\tau),$$

for $f \in 2^{\mathbb{F}_2}$ and $\sigma, \tau \in \mathbb{F}_2$. For any two $f_1, f_2 : \mathbb{F}_2 \to \{0,1\}$ we set

$$f_1 E_\infty f_2 \quad \text{iff} \quad \text{there is some } \sigma \in \mathbb{F}_2 \text{ with } \sigma \cdot f_1 = f_2.$$

We then obtain the universal treeable equivalence relation by restricting to the set on which the action is free. So first let $F(2^{\mathbb{F}_2})$ be the set of functions f for which, whenever $\sigma \in \mathbb{F}_2$ is not the identity,

$$\sigma \cdot f \neq f.$$

This set of points is a G_δ subset of $2^{\mathbb{N}}$ and hence a Polish space (see [14], 3.C). The relation $E_{\mathrm{T}\infty}$ is the restriction of E_∞ to the set $F(2^{\mathbb{F}_2})$.

The notation E_∞ is somewhat misleading, since it is $\geq_{\mathrm{B}} E_0$ but not $\geq_{\mathrm{B}} E_1$, E_2, or E_3.

Definition 2.7. We let $p(\mathbb{N})$ denote the collection of all subsets of the natural numbers. A collection $I \subseteq p(\mathbb{N})$ is said to be an *ideal* if it is closed under finite unions and the process of passing to a subset of a member of I. We can view $p(\mathbb{N})$ as a Polish space in the natural way, by identifying it with $2^{\mathbb{N}}$ via the association of the characteristic function to a subset of $\mathbb{N}$; in other words, we give it the topology generated by taking as basic open sets those of the form

$$\{A \subseteq \mathbb{N} : \forall i \in F_0\ (i \notin A),\ \forall i \in F_1\ (i \in A)\},$$

where F_0, F_1 are *finite* sets of natural numbers. It will be convenient to identify $p(\mathbb{N})$ with $2^{\mathbb{N}}$.

Any ideal can be viewed as an abelian group under the operation of symmetric difference. Thus for $A, B \subseteq \mathbb{N}$ we let

$$A + B = (A \setminus B) \cup (B \setminus A).$$

A Borel ideal I on $p(\mathbb{N})$ is said to be *Polishable* if there is a Polish topology τ on I such that

(i) $(I, \tau, +)$ is a Polish group, that is to say, the operation of symmetric difference is continuous with respect to τ;

(ii) τ gives rise to the original Borel structure on I, that is to say, a set $X \subseteq I$ appears in the σ-algebra generated by the τ-open sets if and only if it is Borel with respect to the above Polish topology on $2^{\mathbb{N}}$.

Sławomir Solecki in [20] has shown that all Polishable ideals are $F_{\sigma\delta}$, and that the F_σ Polishable ideals are those which may be represented as sets which are finite for some appropriately chosen "exhaustive" lower semicontinuous submeasure on $\mathbb{N}$.

If I is an ideal on $p(\mathbb{N})$, then we let E_I denote the corresponding coset equivalence relation on $2^{\mathbb{N}}$; thus

$$x E_I y \quad \text{iff} \quad \{n : x(n) \neq y(n)\} \in I.$$

Definition 2.8. We let $\mathcal{I}_{(1/n)}$ denote the *summable ideal*, where for $A \subseteq \mathbb{N}$ we have $A \in \mathcal{I}_{(1/n)}$ if

$$\sum_{n \in A} \frac{1}{n+1} < \infty.$$

With this in hand we can finally define E_2 to be equal to $\mathcal{I}_{(1/n)}$, the coset equivalence relation arising from $\mathcal{I}_{(1/n)}$ in $2^{\mathbb{N}}$.

Unlike E_1 and E_3, we are still only able to conjecture that there is no E strictly (in $<_{\mathrm{B}}$) between E_0 and E_2. However, [9] comes close to proving this.

Definition 2.9. A topological group is said to be *Polish* if the underlying topological space is Polish. If G is a Polish group equipped with a continuous (resp. Borel) action on a Polish space X, then we say X is a *Polish* (resp. *Borel*) *G-space*. We then denote the orbit equivalence relation by E_G^X, so that

$$x_1 E_G^X x_2 \quad \text{iff} \quad \text{there is some } g \in G \text{ with } g \cdot x_1 = x_2.$$

Many Borel equivalence relations arise in this form, or are at least Borel reducible to the orbit equivalence relation of some Polish group acting Borel on a Polish space. It was shown in [16] that whenever G is a Polish group and X is a Borel G-space, $E_1 \not\leq_{\mathrm{B}} E_G^X$. Consequently, whenever $E_1 \leq_{\mathrm{B}} E$ we deduce that E fails to be Borel reducible to any such E_G^X; it remains open whether this is the *only reason* a Borel equivalence relation may fail to be Borel reducible to a Borel Polish group action.

An important class of Polish group actions are those presented by S_∞, the group of all permutations of $\mathbb{N}$ equipped with the topology of pointwise convergence. Appropriately understood, the isomorphism relation on countable structures can be viewed as the orbit equivalence relation induced by an action of S_∞ (see for instance [10], §2.3). There is a fundamental kind of obstruction to reduction to the orbit equivalence relations of the form $E^X_{S_\infty}$.

Definition 2.10. A continuous action of a Polish group G on a Polish space X is said to be *turbulent* if:

(i) every orbit is dense;
(ii) every orbit is meager;
(iii) for all $x, y \in X$, $U \subseteq X$, $V \subseteq G$ open with $x \in U$, $1 \in V$, there exists $y_0 \in [y]_G := G \cdot y$ and $(g_i)_{i\in\mathbb{N}} \subseteq V$, $(x_i)_{i\in\mathbb{N}} \subseteq U$ with

$$x_0 = x, \qquad x_{i+1} = g_i \cdot x_i,$$

and for some subsequence $(x_{n(i)})_{i\in\mathbb{N}}$,

$$x_{n(i)} \to y_0.$$

In [10] it is shown that an orbit equivalence relation arising from a turbulent action of a Polish group is never reducible to the orbit equivalence relation $E^X_{S_\infty}$ arising from a Borel action of the infinite symmetric group on some Polish space X.

3. The countable equivalence relations. The paper [12] bemoaned the failure to find two $\leq_{\mathrm{B}}$-incomparable countable Borel equivalence relations. At the end of 1998 this was finally settled by Scott Adams and Alexander Kechris, who used the superrigidity theory of Zimmer [23], in the ergodic theory of higher-rank linear algebraic groups, to show that such examples exist and they exist in abundance. For instance, their methods were easily sufficient to obtain a Louveau–Veličković type result:

Theorem 3.1 (Adams–Kechris; see [1]). *There is an assignment $S \mapsto E_S$ of countable Borel equivalence relations to subsets of $\mathbb{N}$ such that for all $S, T \subseteq \mathbb{N}$ we have $E_S \leq E_T$ if and only if $S \setminus T$ is finite.*

The examples obtained by their methods are all non-treeable. Thus the original problem lives on in a weaker form:

Question 3.2. Do there exist $\leq_{\mathrm{B}}$-incomparable treeable countable Borel equivalence relations?

4. The summable ideal. Independently, Ilijas Farah and Boban Veličković refuted Conjecture 3 from [12] regarding the equivalence relation induced by the cosets of the summable ideal in $p(\mathbb{N})$:

THEOREM 4.1 (Farah, Veličković; see [5], [22]). *There is an F_σ Polishable ideal $\mathcal{I}_{\mathrm{Tsir}} \subseteq p(\mathbb{N})$ such that*

$$E_0 <_{\mathrm{B}} E_{\mathcal{I}_{\mathrm{Tsir}}} \quad \textit{but} \quad E_2 \not\leq_{\mathrm{B}} E_{\mathcal{I}_{\mathrm{Tsir}}}.$$

In both cases their proofs made striking use of ideas from Banach space theory by defining a kind of Polishable ideal analog of the *Tsirelson Banach space*. This idea has naturally become known as the *Tsirelson ideal*, and its variations have played an important role in further work by Farah on turbulence.

Conjecture 4 of [12] to the effect that there is no Borel equivalence relation E with $E_0 <_{\mathrm{B}} E <_{\mathrm{B}} E_2$ remains open and very likely true.

5. Dichotomies for turbulence. Knowing that the action of the Banach space c_0 by translation on $\mathbb{R}^{\mathbb{N}}$ is turbulent and gives rise to an equivalence relation $E_{c_0}^{\mathbb{R}^{\mathbb{N}}}$ which is $\leq_{\mathrm{B}}$-incomparable with E_2 (see Hjorth [9]), it was even speculated at Conjecture 7 of [12] that for any Polish group G and turbulent Polish G-space X, either

$$E_2 \leq E_G^X \quad \text{or} \quad E_{c_0}^{\mathbb{R}^{\mathbb{N}}} \leq_{\mathrm{B}} E_G^X.$$

This further gathers plausibility from an observation due to Kechris (for a proof see [9]) that $E_2 \sim_{\mathrm{B}} E_{\ell^1}^{\mathbb{R}^{\mathbb{N}}}$, and thus we might hope that $E_{\ell^1}^{\mathbb{R}^{\mathbb{N}}}$ and $E_{c_0}^{\mathbb{R}^{\mathbb{N}}}$ would stand like Adam and Eve at the very base of the turbulent equivalence relations.

However:

THEOREM 5.1 (Oliver). *For any equivalence relation E_S obtained in the Louveau–Veličković construction from* 2.3 *we have*

$$E_S <_{\mathrm{B}} E_{c_0}^{\mathbb{R}^{\mathbb{N}}}.$$

Since these Louveau–Veličković equivalence relations are easily seen to arise from turbulent Polish group actions, we see in particular that there are many incomparable turbulent orbit equivalence relations $<_{\mathrm{B}}$-below $E_{c_0}^{\mathbb{R}^{\mathbb{N}}}$.

Not only was Conjecture 7 false as stated, the hope it expressed—that one would have a small basis of turbulent orbit equivalence relations, with at least one member of the basis reducing to any other example of an orbit equivalence relation arising from a turbulent Polish group action—was misguided. Since Mike Oliver's result, Farah has advanced steadily on the structure of the turbulent orbit equivalence relations, and used suitable refinements of the Tsirelson ideal and the Louveau–Veličković examples to refute every structural conjecture we might have ever entertained.

THEOREM 5.2 (Farah; see [6]). *There is no finite or even countably infinite sequence $E_{G_0}^{X_0}, E_{G_1}^{X_1}, \ldots, E_{G_n}^{X_n}, \ldots$ of orbit equivalence relations arising* *from turbulent actions of Polish groups such that for any other Polish group H and turbulent Polish H-space Y, there is some k with*

$$E_{G_k}^{X_k} \leq_{\mathrm{B}} E_H^Y.$$

THEOREM 5.3 (Farah; see [7]). *There is a Polish group G and a turbulent Polish G-space X which is above no* minimal *turbulent orbit equivalence relation*; *that is to say*, *for each Polish group H_0 and turbulent Polish H_0-space Y_0 with*

$$E_{H_0}^{Y_0} \leq_{\mathrm{B}} E_G^X,$$

we may find a Polish group H_1 and turbulent Polish H_1-space Y_1 with

$$E_{H_1}^{Y_1} <_{\mathrm{B}} E_{H_0}^{Y_0}.$$

6. What remains open? From the original sequence of conjectures the following remain open:

CONJECTURE 6.1 (Conjecture 1 of [12]). For E a Borel equivalence relation, either we have $E_1 \leq_{\mathrm{B}} E$ or there is some Polish group G and Polish G-space X with $E \sim_{\mathrm{B}} E_G^X$.

CONJECTURE 6.2 (Conjecture 4 of [12]). For E a Borel equivalence relation with $E \leq E_2$ we have either $E_2 \sim_{\mathrm{B}} E$ or $E \leq_{\mathrm{B}} E_0$.

CONJECTURE 6.3 (Conjecture 5 of [12]). For E a Borel equivalence relation of the form E_G^X for some Polish G-space X, where G is a closed subgroup of S_∞, we have either $E_3 \leq_{\mathrm{B}} E$ or $E \leq_{\mathrm{B}} E_\infty$.

Below we give the proof of the result announced in [12] that this conjecture holds when $E \leq_{\mathrm{B}} E_G^X$, for G a closed invariantly metrizable subgroup of S_∞ and X a Polish G-space.

Added in proof. Hjorth has now proved the full Conjecture 6.3. His proof appears in his preprint "A dichotomy theorem for isomorphism".

CONJECTURE 6.4 (Conjecture 6 of [12]). For G a Polish group and X a Polish G-space, either we have some turbulent Polish G-space Y with $E_G^Y \leq_{\mathrm{B}} E_G^X$ or there is some Polish S_∞-space Z with $E_G^X \sim_{\mathrm{B}} E_{S_\infty}^Z$.

In particular, these conjectures imply that for any Borel equivalence relation one of the following holds:

(i) $E_1 \leq_{\mathrm{B}} E$,
(ii) $E_G^X \leq_{\mathrm{B}} E$ for some turbulent G-space X,
(iii) $E_3 \leq_{\mathrm{B}} E$,
(iv) $E \leq_{\mathrm{B}} E_\infty$.

This consequence is also open at this time.

7. The Sixth Dichotomy Theorem. We present here the proof of the following result, labeled the *Sixth Dichotomy Theorem* in [12].

THEOREM 7.1. *Let E be a Borel equivalence relation. If $E \leq_{\mathrm{B}} E_3$, then*

$$E \leq_{\mathrm{B}} E_0 \quad \textit{or} \quad E \sim_{\mathrm{B}} E_3.$$

As discussed in §11 of [12] it is enough to prove the following two results.

THEOREM 7.2. *Let E be a Borel equivalence relation such that $E \leq_{\mathrm{B}} E_\infty^{\mathbb{N}}$. Then exactly one of the following holds*:

(i) $E \leq_{\mathrm{B}} E_\infty$,
(ii) $E_3 \sqsubseteq_{\mathrm{c}} E$,

where $\sqsubseteq_{\mathrm{c}}$ means that there is an injective continuous reduction.

THEOREM 7.3. *Let G_i, $i = 1, 2, \ldots$, be closed subgroups of S_∞, let $G = \prod_{i=1}^{\infty} G_i$ and let X be a Borel G-space. If $E \leq_{\mathrm{B}} E_G^X$ and $E \leq_{\mathrm{B}} E_\infty$, then for each n, there is a Borel G^n-space Z_n, where $G^n = \prod_{i\leq n} G_i$, such that*

$$E \leq_{\mathrm{B}} \bigoplus_n E_{G_n}^{Z_n},$$

where $\bigoplus$ denotes direct sum.

To see that 7.2 and 7.3 together imply 7.1, let $E \leq_{\mathrm{B}} E_3$. Then $E \leq_{\mathrm{B}} E_3 \leq_{\mathrm{B}} E_\infty^{\mathbb{N}}$, so by 7.1 either $E_3 \sqsubseteq_{\mathrm{c}} E$, so $E \sim_{\mathrm{B}} E_3$, or else $E \leq_{\mathrm{B}} E_\infty$. If the last alternative holds, we have $E \leq_{\mathrm{B}} E_3 = E_0^{\mathbb{N}}$ and $E \leq_{\mathrm{B}} E_\infty$.

As discussed in [4], $E_0 \sim_{\mathrm{B}} E_{\mathbb{Z}}^{2^{\mathbb{Z}}}$, where the action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ is the shift action. So $E \leq_{\mathrm{B}} E_0^{\mathbb{N}} \sim_{\mathrm{B}} (E_{\mathbb{N}}^{2^{\mathbb{Z}}})^{\mathbb{N}} \sim_{\mathrm{B}} E_{2^{\mathbb{N}}}^{X}$, where $X = (2^{\mathbb{Z}})^{\mathbb{N}}$ and $\mathbb{Z}^{\mathbb{N}}$ acts on $(2^{\mathbb{N}})^{\mathbb{N}}$ coordinatewise. Since $\mathbb{Z}$ is a closed subgroup of S_∞, 7.3 implies that E is Borel reduced to a direct sum of a sequence of equivalence relations of the form $E_{\mathbb{Z}^n}^{Y}$. As discussed in [13], it is a theorem of Weiss that any orbit equivalence relation associated with a Borel $\mathbb{Z}^n$-space is $\leq_{\mathrm{B}} E_0$. So $E \leq_{\mathrm{B}} E_0$.

We now present the proofs of 7.2 and 7.3.

Proof of 7.2. Since E_∞ can be realized in the form $E_G^{2^{\mathbb{N}}}$, where G is a countable group acting continuously on $2^{\mathbb{N}}$, it is enough to prove that (i) or (ii) hold for any $E \leq_{\mathrm{B}} (E_G^{2^{\mathbb{N}}})^{\mathbb{N}}$, where G is a countable group acting continuously on $2^{\mathbb{N}}$. We can clearly assume that E lives on $X = 2^{\mathbb{N}}$. We also claim that we can assume (by changing G to $\mathbb{Z}_2 \times G$ if necessary) that there is a Borel map $f : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ which is 1-1 and for which there is a continuous $f^* : (2^{\mathbb{N}})^{\mathbb{N}} \to X$ such that $f^* | f[X] = f^{-1}$, and $xEy \Leftrightarrow f(x)(E_G^{2^{\mathbb{N}}})^{\mathbb{N}} f(y)$. Indeed, if $g : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ is Borel such that $xEy \Leftrightarrow g(x)(E_G^{2^{\mathbb{N}}})^{\mathbb{N}} g(y)$, define $f : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ ($\equiv 2^{\mathbb{N}\times\mathbb{N}}$, via $x_i(j) = x(i,j)$) by

$$f(x)(i,0) = x(i), \qquad f(x)(i,j+1) = g(x)(i,j).$$

Then let $\mathbb{Z}_2 \times G$ act on $2^{\mathbb{N}}$ as follows:

$$(a,g) * x = (a + x(0))^\wedge g \cdot x', \qquad \text{where } x'(i) = x(i+1)$$

(here $g \cdot x$ is the given action that defines $E_G^{2^{\mathbb{N}}}$). Denoting by $E_{\mathbb{Z}_2 \times G}^{2^{\mathbb{N}}}$ the equivalence relation induced by this action, we clearly have

$$xEy \;\Leftrightarrow\; g(x)(E_G^{2^{\mathbb{N}}})^{\mathbb{N}} g(y) \;\Leftrightarrow\; f(x)(E_{\mathbb{Z}_2 \times G}^{2^{\mathbb{N}}})^{\mathbb{N}} f(y).$$

Finally, let f^* be defined by

$$f^*(z)(i) = z(i,0).$$

Then f is 1-1, Borel and reduces E to $(E_{\mathbb{Z}_2 \times G}^{2^{\mathbb{N}}})^{\mathbb{N}}$, and f^* is continuous and equal to f^{-1} on $f[X]$.

So fix G, X, E, f, f^* as above. By relativization, we can assume that $E, f \in \Delta_1^1$, and G and the action are recursive.

Notation. • For $x, y \in 2^{\mathbb{N} \times \mathbb{N}}$, let

$$x_{\leq n} = y_{\leq n} \;\Leftrightarrow\; \forall i \leq n \ (x_i = y_i).$$

Put

$$x \equiv_n y \;\Leftrightarrow\; x(E_G^{2^{\mathbb{N}}})^{\mathbb{N}} y \ \&\ x_{\leq n} = y_{\leq n}.$$

• For each n, let

$$V^{(n)} = \underbrace{\{1\} \times \ldots \times \{1\}}_{n+1} \times G^{\mathbb{N}}.$$

Then $V^{(n)}$ is an open subgroup of $G^{\mathbb{N}}$ and so has countable index in $G^{\mathbb{N}}$. Clearly, $(E_G^{2^{\mathbb{N}}})^{\mathbb{N}} = E_{G^{\mathbb{N}}}^{2^{\mathbb{N} \times \mathbb{N}}}$, where $G^{\mathbb{N}}$ acts by the product action on $(2^{\mathbb{N}})^{\mathbb{N}} \equiv 2^{\mathbb{N} \times \mathbb{N}}$. Moreover, $(\equiv_n) = E_{V^{(n)}}^{2^{\mathbb{N} \times \mathbb{N}}}$. It follows that every $E_{G^{\mathbb{N}}}^{2^{\mathbb{N} \times \mathbb{N}}}$-class contains only countably many $\equiv_n$-classes.

• Let $f[X] = X_0 \subseteq 2^{\mathbb{N} \times \mathbb{N}}$.

• Let $G = \{g_n\}_{n \in \mathbb{N}}$ (a recursive enumeration), with $g_0 = 1$.

For each $n < k$ and each p define

$$\mathcal{A}_{n,k,p} = \{A \in \Sigma_1^1, A \subseteq 2^{\mathbb{N} \times \mathbb{N}} : \\ \forall x, y \in X_0 \cap A \ (x \equiv_n y \Rightarrow \exists i \leq p \ (g_i \cdot x_k = y_k))\}.$$

Put

$$A_0 = \bigcup_n \bigcap_{k > n} \bigcup_p \{A : A \in \mathcal{A}_{n,k,p}\}.$$

Claim 1. $A \in \mathcal{A}_{n,k,p} \Rightarrow \exists B \supseteq A \ (B \in \Delta_1^1 \ \&\ B \in \mathcal{A}_{n,k,p})$.

Proof. The property "$A \in \mathcal{A}_{n,k,p}$" is Π_1^1 in the codes.

Claim 2. $A_0 \in \Pi_1^1$.

Proof. By Claim 1,

$$\begin{aligned} x \in A_0 &\Leftrightarrow \exists n\, \forall k > n\, \exists p\, \exists A \in \mathcal{A}_{n,k,p}\ (x \in A) \\ &\Leftrightarrow \exists n\, \forall k > n\, \exists p\, \exists A \in \Delta^1_1,\ A \in \mathcal{A}_{n,k,p}\ (x \in A). \end{aligned}$$

We now have 2 cases:

Case I: $X_0 \subseteq A_0$. We will then show that $E \leq_{\Delta^1_1} F$, where F is Δ^1_1 and countable. For that it is enough to find a Δ^1_1 function $g : X \to Y$, with Y a recursively presented Polish space, so that $g([x]_E)$ is countable and $\neg xEy \Rightarrow g([x]_E) \cap g([y]_E) = \emptyset$ (see, e.g., [10]).

Since $X_0 \subseteq A_0$, we have

$$\begin{aligned} &\forall x \in X_0\, \exists n\, \forall k > n\, \exists p\, \exists A \in \Delta^1_1 \\ &\qquad (x \in A\ \&\ \forall y, z \in X_0 \cap A\ (y \equiv_n z \Rightarrow \exists i \leq p\ (g_i \cdot y_k = z_k))). \end{aligned}$$

So there is a Δ^1_1 function $x \mapsto n(x)$ such that

$$\begin{aligned} &x \in X_0\ \Rightarrow\ \forall k > n(x)\, \exists p\, \exists A \in \Delta^1_1 \\ &\qquad (x \in A\ \&\ \forall y, z \in X_0 \cap A\ (y \equiv_n z\ \Rightarrow\ \exists i \leq p\ (g_i \cdot y_k = z_k))). \end{aligned}$$

Let $P_n = \{x : n(x) = x\}$ ($\in \Delta^1_1$) and

$$Q_n = P_n \cap X_0 \quad (\in \Delta^1_1).$$

Clearly, $X_0 = \bigcup_n Q_n$ and

$$\begin{aligned} &x \in Q_n\ \Rightarrow\ \forall k > n\, \exists p\, \exists A \in \Delta^1_1 \\ &\qquad (x \in A\ \&\ \forall y, z \in X_0 \cap A\ (y \equiv_n z\ \Rightarrow\ \exists i \leq p\ (g_i \cdot y_k = z_k))). \end{aligned}$$

We will find a Δ^1_1 function $f_n : 2^{\mathbb{N}\times\mathbb{N}} \to 2^{\mathbb{N}\times\mathbb{N}}$ such that

$$x, y \in Q_n\ \&\ x \equiv_n y\ \Rightarrow\ f_n(x) = f_n(y) \equiv_n x.$$

Then for $x \in X$, let

$$g(x) = \langle n, y \rangle\ \Leftrightarrow\ f(x) \in Q_n\ \&\ f_n(f(x)) = y.$$

This is Δ^1_1 and we claim that it works: To see that $g([x]_E)$ is countable, just use the fact that there are only countably many $\equiv_n$-classes in each $E^{2^{\mathbb{N}\times\mathbb{N}}}_{G^{\mathbb{N}}}$-class. Now assume $g(x) = g(y) = \langle n, z \rangle$. Then $f(x), f(y) \in Q_n$ and $f_n(f(x)) = f_n(f(y))$ $(= z)$, so $f(x) \equiv_n f(y)$, thus $f(x) E^{2^{\mathbb{N}\times\mathbb{N}}}_{G^{\mathbb{N}}} f(y)$, and so xEy.

We will now construct the f_n's:

Fix a $\equiv_n|Q_n$-class C. Note that for any $x \in C$ and any $k > n$, there are $p, A \in \Delta^1_1$ with $x \in A$ and

$$\forall y, z \in X_0 \cap A\ (y \equiv_n z \Rightarrow \exists i \leq p\ (g_i \cdot y_k = z_k)).$$

So for each $k > n$ let p_k, A_k be least such that $A_k \cap C \neq \emptyset$ and $\forall y, z \in X_0 \cap A_k$ $(y \equiv_n z \Rightarrow \exists i \leq p_k\ (g_i \cdot y_k = z_k))$. Then it is clear that $\{x_k : x \in C \cap A_k\}$

is finite, so let a_k^C be its least member (in the lexicographical ordering on $2^{\mathbb{N}\times\mathbb{N}} \equiv 2^{\mathbb{N}}$).

Define α_C by

$$\begin{aligned}(\alpha_C)_i &= x_i && \text{for any (all) } x \in C, \text{ if } i \le n;\\ (\alpha_C)_k &= a_k^C && \text{for } k > n.\end{aligned}$$

Then $\alpha_C \equiv_n x$ for any $x \in C$.

For $x \in Q_n$, put

$$h_n(x) = \alpha_{[x]_{\equiv_n|Q_n}}.$$

Then h_n is C-measurable (where C is the smallest σ-algebra containing the open sets and closed under the Suslin operation $\mathcal{A}$) and

$$x, y \in Q_n \ \&\ x \equiv_n y \ \Rightarrow\ h_n(x) = h_n(y) \equiv_n x.$$

Let $\widetilde{Q}_n = [Q_n]_{\equiv_n}$ = the saturation of Q_n by $\equiv_n$ in $2^{\mathbb{N}\times\mathbb{N}}$. Clearly, $\widetilde{Q}_n \in \Sigma^1_1$ and there is $q_n : \widetilde{Q}_n \to Q_n$ which is C-measurable with $q_n(x) \equiv_n x$. If $r_n = h_n \circ q_n$, then $r_n : \widetilde{Q}_n \to 2^{\mathbb{N}\times\mathbb{N}}$ is C-measurable and

$$x, y \in \widetilde{Q}_n \ \&\ x \equiv_n y \ \Rightarrow\ r_n(x) = r_n(y) \equiv_n x.$$

Thus $\mu(\widetilde{Q}_n) = 0$ for every $\equiv_n$-ergodic, non-atomic probability Borel measure on $2^{\mathbb{N}\times\mathbb{N}}$. Since this is a Π^1_1 in the codes property of $\widetilde{Q}_n$, there is a Δ^1_1 set $R_n \supseteq \widetilde{Q}_n$, $R_n \in \Delta^1_1$, which still has this property. Let

$$\langle R_n\rangle = \{x : \forall y \equiv_n x\ (y \in R_n)\}.$$

Then $\langle R_n\rangle \in \Pi^1_1$, $\langle R_n\rangle$ is $\equiv_n$-invariant, and $\widetilde{Q}_n \subseteq \langle R_n\rangle \subseteq R_n$, so there is a $\Delta^1_1, \equiv_n$-invariant set S_n with $\widetilde{Q}_n \subseteq S_n \subseteq \langle R_n\rangle$. Then $\mu(S_n) = 0$ for any measure as above, so $\equiv_n|S_n$ is smooth. Since $\equiv_n|S_n$ is induced by a Δ^1_1 action of the Polish group $V^{(n)}$, it follows that $\equiv_n|S_n$ has a Δ^1_1-selector, i.e., there is a Δ^1_1 map $f_n : 2^{\mathbb{N}\times\mathbb{N}} \to 2^{\mathbb{N}\times\mathbb{N}}$ such that

$$x, y \in S_n \ \&\ x \equiv_n y \ \Rightarrow\ f_n(x) = f_n(y) \equiv_n x,$$

and since $Q_n \subseteq S_n$ we are done.

Case II: $X_0 \not\subseteq A_0$. We will then show that $E_0^{\mathbb{N}} \sqsubseteq_c E$. Notice that it is enough to show that $E_0^{\mathbb{N}} \sqsubseteq_c E_0^{\mathbb{N}}|X_0$, say by a continuous embedding e, because then $f^* \circ e$ is a continuous embedding of $E_0^{\mathbb{N}}$ into E.

Let $Y_0 = X_0 \setminus A_0$. Then $Y_0 \ne \emptyset$, $Y_0 \in \Sigma^1_1$. By definition, for any $x \in Y_0$,

$$\forall n\ \exists k > n\ \forall p\ \forall A \in \Sigma^1_1\ (x \in A \ \Rightarrow\ A \notin \mathcal{A}_{n,k,p}),$$

i.e., if $x \in Y_0$, then

$$\begin{aligned}&\forall n\ \exists k > n\ \forall p\ \forall A \in \Sigma^1_1\\ &\qquad (x \in A \Rightarrow \exists y, z \in X_0 \cap A\ (y \equiv_n z \ \&\ \forall i \le p\ (g_i \cdot y_k \ne z_k))).\end{aligned}$$

Let, for $n < k$,

$$\begin{aligned} Y_{n,k} &= \{x \in Y_0 : \forall p \ \forall A \in \Sigma^1_1 \ (x \in A \Rightarrow A \notin \mathcal{A}_{n,k,p})\} \\ &= \{x \in Y_0 : \forall p \ \forall A \in \Delta^1_1 \ (x \in A \Rightarrow A \notin \mathcal{A}_{n,k,p})\} \end{aligned}$$

(by Claim 1), so that $Y_{n,k} \in \Sigma^1_1$. Thus

$$\forall x \in Y_0 \ \forall n \ \exists k > n \ (x \in Y_{n,k}).$$

NOTATION. Below $\langle m, j\rangle$ denotes the usual Cantor bijection of $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$, given by

$$\langle m, j\rangle = (m + j)(m + j + 1)/2 + j.$$

Let $L(n) = \max\{k : \exists i \ (\langle k, i\rangle \le n)\}$. Note that $L(\langle m, 0\rangle) = m$ and $L(n) \le L(n + 1)$, while $L(n) = L(n - 1)$ if $n = \langle m, j\rangle$ with $j > 0$, and $L(n) = L(n - 1) + 1$ if $n = \langle m, 0\rangle > 0$.

We will define the following by induction on $n \ge 0$:

(i) Non-empty Σ^1_1 sets A_s, $s \in 2^{n+1}$. These will be chosen so that $A_\emptyset = Y_0$, $A_{s^\frown i} \subseteq A_s$, $\operatorname{diam}(A_s) \le 2^{-\mathrm{lh}(s)}$, and for each $x \in 2^{\mathbb{N}}$, $A_{x|i}$ "converges" in the Gandy–Harrington topology, so that $\bigcap_i A_{x|i} = \{\alpha_x\}$ and if $\alpha_i \in A_{x|i}$, then $\alpha_i \to \alpha_x$ in the usual topology.

(ii) $k_m \in \mathbb{N}$, $m \le L(n)$. These will be chosen so that $0 < k_0 < k_1 < \ldots$

(iii) We will also have

$$A_{0^{n+1}} \subseteq \bigcap_{r \le L(n)} Y_{r,k_r}.$$

(iv) $g_s \in G^{\mathbb{N}}$, $s \in 2^{n+1}$, such that $g_{0^{n+1}} = 1$, $(g_s)_i = 1$ if $i > k_{L(n)}$. (Thus, essentially, $g_s \in G^{k_{L(n)}+1}$.)

(v) *Links.* We will also have, for $s \in 2^{n+1}$,

$$\forall x \in A_{0^{n+1}} \exists y \in A_s \ (g_s \cdot x \equiv_{k_{L(n)}} y).$$

(vi) *Positive requirements.* For $s, t \in 2^{n+1}$, put $g_{s,t} = g_t \cdot g_s^{-1}$. If $\overline{n} < n, (\bar{s}, \bar{t}\,) \subseteq (s, t)$, $\bar{s}, \bar{t} \in 2^{\overline{n}+1}$, then for $l \le L(n)$ we must have

$$[\forall \bar{l} \ \le l \ \forall \langle \bar{l}, i\rangle \in (n + 1) \setminus (\overline{n} + 1) \ (s(\langle \bar{l}, i\rangle) = t(\langle \bar{l}, i\rangle))] \ \Rightarrow \ g_{s,t} \equiv_l g_{\bar{s},\bar{t}},$$

where for $g, h \in G^{\mathbb{N}}$, $l \in \mathbb{N}$ we let

$$g \equiv_l h \ \Leftrightarrow \ \forall i \le l \ (g_i = h_i).$$

(vii) *Negative requirements.* If $s, t \in 2^{n+1}, n = \langle m, j\rangle$, then we must have

$$s(n) \ne t(n) \ \Rightarrow \ \forall l \le n \ (x \in A_s \ \& \ y \in A_t \ \Rightarrow \ g_l \cdot x_{k_m} \ne y_{k_m}).$$

Assume all this can be done. Then we claim that

$$(*) \qquad x E_0^{\mathbb{N}} y \ \Leftrightarrow \ \alpha_x E_{G^{\mathbb{N}}}^{2^{\mathbb{N}\times\mathbb{N}}} \alpha_y,$$

which, since $\alpha_x \in Y_0$, proves what we want. (Notice that by (vii), $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$ (as $g_0 = 1$), so $x \mapsto \alpha_x$ is 1-1 and clearly continuous.)

Proof of $(*)$. $\Rightarrow$ Assume $xE_0^{\mathbb{N}}y$. Fix l. Choose $t_0, t_1, \ldots, t_l$ such that

$$x(\langle \bar{l}, i\rangle) = y(\langle \bar{l}, i\rangle) \quad \text{for } \bar{l} \le l \text{ and } i \ge t_{\bar{l}}.$$

Then there is $\overline{n}$ with $L(\overline{n}) \ge l$ such that for any $n > \overline{n}$ we have

$$\forall \bar{l} \le l\ \forall \langle \bar{l}, i\rangle \in (n+1) \setminus (\overline{n}+1)\ [x|(n+1)(\langle \bar{l}, i\rangle) = y|(n+1)(\langle \bar{l}, i\rangle)].$$

So, by (vi) for $n > \overline{n}$, we have

$$g_{x|(n+1),y|(n+1)} \equiv_l g_{x|(\overline{n}+1),y|(\overline{n}+1)}.$$

By (v), find $w_{x|(n+1)}, w_{y|(n+1)}$ in $A_{x|(n+1)}, A_{y|(n+1)}$, respectively, so that

$$g_{x|(n+1)} \cdot \alpha_{0^\infty} \equiv_{k_{L(n)}} w_{x|(n+1)}, \qquad g_{y|(n+1)} \cdot \alpha_{0^\infty} \equiv_{k_{L(n)}} w_{y|(n+1)}.$$

Thus

$$g_{y|(n+1)} g_{x|(n+1)}^{-1} \cdot w_{x|(n+1)} \equiv_{k_{L(n)}} w_{y|(n+1)},$$

and so, since $L(n) \ge L(\overline{n}) \ge l$ and therefore $k_{L(n)} \ge l$, we have

$$g_{x|(n+1),y|(n+1)} \cdot w_{x|(n+1)} \equiv_l w_{y|(n+1)}.$$

Hence

$$g_{x|(\overline{n}+1),y|(\overline{n}+1)} \cdot w_{x|(n+1)} \equiv_l w_{y|(n+1)}.$$

Taking the limit as $n \to \infty$, we get

$$(g_{x|(\overline{n}+1),y|(\overline{n}+1)} \cdot \alpha_x)_{\bar{l}} = (\alpha_y)_{\bar{l}}, \qquad \forall \bar{l} \le l,$$

so, in particular, there is $g_l \in G$ with $g_l \cdot (\alpha_x)_l = (\alpha_y)_l$. Since this is true for every l, we see that $\alpha_x E_{G^{\mathbb{N}}}^{2^{\mathbb{N}\times\mathbb{N}}} \alpha_y$.

$\Leftarrow$ Assume $\neg x E_0^{\mathbb{N}} y$. Fix m such that $x(\langle m, j\rangle) \ne y(\langle m, j\rangle)$ for infinitely many j. Assume, towards a contradiction, that $\alpha_x E_{G^{\mathbb{N}}}^{2^{\mathbb{N}\times\mathbb{N}}} \alpha_y$. Let l be such that $g_l \cdot (\alpha_x)_{k_m} = (\alpha_y)_{k_m}$. Fix $n = \langle m, j\rangle$ such that $x(n) \ne y(n)$ and $n \ge l$. Then $x|(n+1)(n) \ne y|(n+1)(n)$, so by (vi), as $\alpha_x \in A_{x|(n+1)}$, $y \in A_{y|(n+1)}$, we have

$$g_l \cdot (\alpha_x)_{k_m} \ne (\alpha_y)_{k_m},$$

a contradiction.

CONSTRUCTION

STEP 1. Let $y \in Y_0$. Then $\forall k\ \exists n > k\ (y \in Y_{n,k})$, so fix $k_0 > 0$ such that $y \in Y_{0,k_0}$, and so $\forall p\ \forall A \in \Sigma_1^1\ (y \in A \Rightarrow A \notin \mathcal{A}_{0,k_0,p})$. Recall that $g_0 = 1$. So for $p = 0$ and $A = Y_{0,k_0}$ we have $Y_{0,k_0} \notin \mathcal{A}_{0,k_0,0}$, so there are $x_{(0)}, x_{(1)} \in Y_{0,k_0}$ with $x_{(0)} \equiv_0 x_{(1)}$, but $g_0 \cdot (x_{(0)})_{k_0} = (x_{(0)})_{k_0} \ne (x_{(1)})_{k_0}$. Let

$g_{(0)} = 1$ and let $g'_{(1)} \in G^{\mathbb{N}}$ be such that $g'_{(1)} \cdot x_{(0)} = x_{(1)}$. Define $g_{(1)} \in G^{\mathbb{N}}$ by

$$(g_{(1)})_i = \begin{cases} (g'_{(1)})_i & \text{if } i \le k_0 = k_{L(0)}, \\ 1 & \text{if } i > k_0. \end{cases}$$

Then $g_{(1)} \cdot x_{(0)} \equiv_{k_0} x_{(1)}$. Let then $A_{(0)}, A_{(1)}$ be small enough Σ^1_1 subsets of Y_{0,k_0} ($\subseteq Y_0 = A_\emptyset$) so that the following are satisfied: (i), (ii) ($k_0 > 0$), (iii) ($A_{(0)} \subseteq Y_{0,k_0}$), (iv), (v), (vi) (vacuously), (vii) (if $(x_{(0)})_{k_0}(t) \neq (y_{(0)})_{k_0}(t)$, we just make sure that every $x \in A_{(0)}$ agrees with $x_{(1)}$ at (k_0, t)).

Step $n+1$ ($n > 0$). Assume the construction has been done up to level n, i.e., for $\bigcup_{k \le n} 2^n$, and k_m has been defined for $m \le L(n-1)$. We now consider 2^{n+1}. Let $n = \langle m, j \rangle$. We consider two cases, (A) and (B).

(A) $j > 0$. Then $L(n) = L(n-1)$, so k_m is already defined for all $m \le L(n)$.

First we shrink all $A_s,\ s \in 2^n$, to make sure that $\operatorname{diam}(A_s) < 2^{-\operatorname{lh}(s)-1}$, and "convergence" in the Gandy–Harrington topology is improved, and moreover, this is done so that (v) still remains valid (for $s \in 2^n$ of course). To avoid complicated notation we will still call these smaller sets A_s (we have not changed, by the way, the $g_s,\ s \in 2^n$). So all conditions (i)–(vii) are satisfied up to that point, and we took care of (i) at Step $n+1$. Also, as we pointed out, (ii) has been taken care of at Step $n+1$, and we have

$$A_{0^n} \subseteq \bigcap_{r \le L(n-1)} Y_{r,k_r} = \bigcap_{r \le L(n)} Y_{r,k_r}.$$

So choose $x_{0^{n+1}}, x_{0^n{}^\wedge 1} \in A_{0^n}$ so that $x_{0^{n+1}} \equiv_m x_{0^n{}^\wedge 1}$ (as $m \le L(n)$) and $g_{s_1}^{-1} g_l^{\pm 1} g_{t_1} \cdot (x_{0^{n+1}})_{k_m} \neq (x_{0^n{}^\wedge 1})_{k_m}$ for all $l \le n$ and all $s_1, t_1 \in 2^n$. This is possible since $A_{0^n} \subseteq Y_{m,k_m}$. Let $g_{0^{n+1}} = 1$ and $g'_{0^n{}^\wedge 1} \in G^{\mathbb{N}}$ be such that $g'_{0^n{}^\wedge 1} \cdot x_{0^{n+1}} = x_{0^n{}^\wedge 1}$. As $x_{0^{n+1}} \equiv_m x_{0^n{}^\wedge 1}$, we can assume that $(g'_{0^n{}^\wedge 1})_i = 1$ for $i \le m$.

Define $g_{0^n{}^\wedge 1}$ by

$$(g_{0^n{}^\wedge 1})_i = \begin{cases} (g'_{0^n{}^\wedge 1})_i & \text{for } i \le k_{L(n)}, \\ 1 & \text{for } i > k_{L(n)}. \end{cases}$$

Then for $s \in 2^{n+1},\ s = \bar{s}{}^\wedge i$ define g_s by

$$g_s = \begin{cases} g_{\bar{s}} & \text{if } i = 0, \\ g_{\bar{s}} g_{0^n{}^\wedge 1} & \text{if } i = 1. \end{cases}$$

(Notice that this is consistent with the previous definitions of $g_{0^{n+1}}, g_{0^n{}^\wedge 1}$.) Then (iv) is clearly satisfied, as $L(n) = L(n-1)$.

We next verify (vi).

Take $s, t \in 2^{n+1}$, say $s = s_1{}^\wedge i,\ t = t_1{}^\wedge j$. If $i = j$ then $g_{s,t} = g_{s_1,t_1}$. Moreover, as $g_{0^n{}^\wedge 1} \equiv_m 1$, we always clearly have $g_{s,t} \equiv_m g_{s_1,t_1}$.

Now consider $\overline{n} < n, (\bar{s}, \bar{t}\,) \subseteq (s, t),\ \bar{s}, \bar{t} \in 2^{\overline{n}+1}$ and fix l such that

$$\forall \bar{l} \le l\ \forall \langle \bar{l}, i \rangle \in (n+1) \setminus (\overline{n}+1)\ (s(\langle \bar{l}, i \rangle) = t(\langle \bar{l}, i \rangle)).$$

CASE 1: $i = j$. If $\overline{n} = n-1$, then $\bar{s} = s_1, \bar{t} = t_1$ and $g_{s,t} = g_{s_1,t_1} = g_{\bar{s},\bar{t}}$, so we are done. If $\overline{n} < n-1$, then, by induction hypothesis, $g_{s_1,t_1} \equiv_l g_{\bar{s},\bar{t}}$, so $g_{s,t} (= g_{s_1,t_1}) \equiv_l g_{\bar{s},\bar{t}}$.

CASE 2: $i \ne j$. Then we must have $l < m$. Since $g_{s,t} \equiv_m g_{s_1,t_1}$, and thus $g_{s,t} \equiv_l g_{s_1,t_1}$, we are done as in Case 1.

Since $x_{0^{n+1}}, x_{0^n{}^\wedge 1} \in A_{0^n}$, by induction hypothesis ((v)—recall that $L(n) = L(n-1)$) we can find $x_{s_1{}^\wedge 0} \in A_{s_1}$ so that $g_{s_1} \cdot x_{0^{n+1}} \equiv_{k_{L(n)}} x_{s_1{}^\wedge 0}$ $(s_1 \in 2^n)$ and $x_{s_1{}^\wedge 1} \in A_{s_1}$ so that $g_{s_1} \cdot x_{0^n{}^\wedge 1} \equiv_{k_{L(n)}} x_{s_1{}^\wedge 1}$ $(s_1 \in 2^n)$. (Again $x_{0^{n+1}}, x_{0^n{}^\wedge 1}$ are consistently defined.) Then $g_{s_1{}^\wedge 0} \cdot x_{0^{n+1}} \equiv_{k_{L(n)}} x_{s_1{}^\wedge 0}$ and

$$g_{s_1{}^\wedge 1} \cdot x_{0^{n+1}} = g_{s_1} \cdot (g_{0^n{}^\wedge 1} \cdot x_{0^{n+1}}) = g_{s_1} \cdot x_{0^n{}^\wedge 1} \equiv_{k_{L(n)}} x_{s_1{}^\wedge 1}.$$

Next notice that, as $k_m \le k_{L(n)}$, we have $(g_{s_1{}^\wedge 0})_{k_m} \cdot (x_{0^{n+1}})_{k_m} = (x_{s_1{}^\wedge 0})_{k_m}$ and $(x_{t_1{}^\wedge 1})_{k_m} = (g_{t_1})_{k_m} \cdot x_{0^n{}^\wedge 1}$, so we cannot have $g_l \cdot (x_{s_1{}^\wedge 0})_{k_m} = (x_{t_1{}^\wedge 1})_{k_m}$, and similarly we cannot have $g_l \cdot (x_{s_1{}^\wedge 1})_{k_m} = (x_{t_1{}^\wedge 0})_{k_m}$, for any $l \le n$ and any $s_1, t_1 \in 2^n$. Clearly, this conclusion can be guaranteed by fixing only finitely many values of $(x_s)_{k_m}$, $s \in 2^{n+1}$.

Thus it is routine to define $A_s \in \Sigma^1_1$ with $x_s \in A_s,\ s \in 2^{n+1}$, so that all conditions (i)–(vii) are satisfied (for (vii) we just make sure that all $x \in A_s$ agree with x_s on enough, but finitely many, values).

(B) $j = 0$. Then $L(n) = L(n-1) + 1 = m$.

Again we may assume that we have shrunk the $A_s,\ s \in 2^n$, so that (i) will be satisfied at level $n+1$. Next fix $y_0 \in A_{0^n}$. Then $y_0 \in Y_0$, so $\forall n'\ \exists k' > n'\ (y_0 \in Y_{n',k'})$, and by taking $n' = k_{m-1} = k_{L(n-1)}$ we can find $k_m = k' > k_{m-1}$ with $y_0 \in Y_{k_{m-1},k_m}$, so, in particular, $y_0 \in Y_{m,k_m}$ as $k_{m-1} \ge m$. Hence, by shrinking again if necessary, we can assume that $A_{0^n} \subseteq \bigcap_{r \le L(n)} Y_{r,k_r} = \bigcap_{r \le L(n-1)} Y_{r,k_r} \cap Y_{m,k_m}$. Next choose $x_s \in A_s$, $s \in 2^n$, so that $g_s \cdot x_{0^n} \equiv_{k_{L(n-1)}} x_s$, and fix $\bar{g}_s \in G^{\mathbb{N}}$ so that $g_s \equiv_{k_{L(n-1)}} \bar{g}_s$, $(\bar{g}_s)_i = 0$ if $i > k_{L(n)}$ and $\bar{g}_s \cdot x_{0^n} \equiv_{k_{L(n)}} x_s$. Then, as $g_s \equiv_{k_{L(n-1)}} \bar{g}_s$, it is clear that (vi) is still satisfied if (for $s \in 2^n$) we replace g_s by $\bar{g}_s$. It follows that we can shrink A_s, $s \in 2^n$, to $\overline{A}_s$, $s \in 2^n$, so that all of (i)–(vii) are satisfied with $\overline{A}_s$, $\bar{g}_s$ replacing A_s, g_s, and moreover, $\overline{A}_s$ satisfies (i) for level $n+1$, $\overline{A}_{0^{n+1}} \subseteq \bigcap_{r \le L(n)} Y_{r,k_r}$, $(\bar{g}_s)_i = 0$ for all $i > k_{L(n)}$, and $\forall x \in \overline{A}_{0^n}\ \exists y \in \overline{A}_s$ $(\bar{g}_s \cdot x \equiv_{k_{L(n)}} y)$. So, to avoid complicated notation, we may as well assume that at step n we already have A_s, g_s $(s \in 2^n)$ satisfying all these conditions. But then we can repeat exactly the construction of the previous case (A). ■

Proof of 7.3. We will first deal with the case $E = E^X_G$, which is simpler. For each n fix a clopen basis $\{U^n_k\}_{k \in \mathbb{N}}$ for G^n closed under right multiplication. Identifying U_k^n with $U_k^n \times G_{n+1} \times G_{n+2} \times \ldots$, we derive that $\{U_k^n\}_{k\in\mathbb{N}}$ is closed under multiplication by elements of G and $\{U_k^n\}_{n,k\in\mathbb{N}}$ is a basis for G.

If E_∞ lives on Y, let h be a Borel function witnessing that $E_G^X \leq_{\mathrm{B}} E_\infty$. Fix $x \in X$, and let $[h(x)]_{E_\infty} = \{y_i\}_{i\in\mathbb{N}}$. Then $G = \bigcup_i \{g : h(g\cdot x) = y_i\}$, so for some $i, \{g : h(g\cdot x) = y_i\}$ is non-meager, thus comeager in some U_k^n. So

$$\forall x\ \exists n\ P(x,n) \quad \text{with} \quad P(x,n) \Leftrightarrow \exists k\ \exists y \in [h(x)]_{E_\infty} \forall^* g \in U_k^n\ (h(g\cdot x) = y).$$

Clearly, P is Borel. Moreover, it is E_G^X-invariant, i.e., if $P(x,n)$ & $x' E_G^X x$, then $P(x',n)$. To see this, let $x' = g_0^{-1}\cdot x$. Then

$$\exists k\ \exists y \in [h(x)]_{E_\infty}\ \forall^* g \in U_k^n\ (h(gg_0\cdot x') = y),$$

so

$$\exists k\ \exists y \in [h(x')]_{E_\infty}\ \forall^* g' \in U_k^n g_0\ (h(g'\cdot x') = y).$$

Since $U_k^n g_0 = U_{k'}^n$ for some k', we see that $P(x',n)$ holds. Thus, by the invariant uniformization theorem, there is an E_G^X-invariant Borel function $F : X \to \mathbb{N}$ with $P(x, F(x))$, $\forall x \in X$. Let $X_n = F^{-1}(\{n\})$. Then X_n is an E_G^X-invariant Borel set. We will find a Borel G^n-space Y_n so that $E_G^X|X_n \leq_{\mathrm{B}} E_{G^n}^{Y_n}$, which will complete the proof.

Fix $\infty \notin Y$ and let

$$Y_n = (Y \oplus \{\infty\})^{\{U_k^n\}_{k\in\mathbb{N}}}.$$

Then have G^n act on Y_n by right-shift, i.e.,

$$g\cdot H(U_k^n) = H(U_k^n g).$$

Clearly, this is a Borel action. Now define $Q_n : X_n \to Y_n$ as follows: $Q_n(x)(U_k^n) =$ the unique $y \in [h(x)]_{E_\infty}$ such that $\forall^* g \in U_k^n\ (h(g\cdot x) = y)$, if such exists; ∞ otherwise. Clearly, Q_n is Borel. We verify that for $x, x' \in X_n$,

$$x E_G^X x' \Leftrightarrow Q(x) E_{G_n}^{Y_n} Q(y).$$

$\Rightarrow$ Say $g_0\cdot x = x'$. Let $g_0 = (g_1, g_2, \ldots)$ and $g^n = (g_1, \ldots, g_n)$. We will check that $g^n \cdot Q_n(x) = Q_n(x')$, i.e., $Q_n(x)(U_k^n g^n) = Q(x')(U_k^n)$. Indeed, for $y \in [h(x)]_{E_\infty} = [h(x')]_{E_\infty}$, we have $Q(x)(U_k^n g^n) = y$ iff $\forall^* g' \in U_k^n$ $(h(g'\cdot x') = y)$ iff $Q(x')(U_k^n) = y$.

$\Leftarrow$ Now assume that $Q(x) E_{G^n}^{Y_n} Q(x')$. Since $x \in X_n$, fix k so that $\forall^* g \in U_k^n\ (h(g\cdot x) = y)$, for some $y \in [h(x)]_{E_\infty}$. Thus $Q(x)(U_k^n) = y$. Let $g^n \in G^n$ be such that $g^n \cdot Q(x) = Q(x')$. Then $Q(x')(U_l^n) = Q(x)(U_l^n g^n)$ for any l, so if l is so chosen that $U_k^n = U_l^n g^n$, we have $Q(x')(U_l^n) = Q(x)(U_k^n) = y$, thus $y \in [h(x')]_{E_\infty}$ and $h(x) E_\infty h(x')$, and $x E_G^X x'$.

Now consider the general case. Assume that E lives on Z and that the functions f, h witnessing, resp., that $E \leq_{\mathrm{B}} E_G^X$ and $E \leq_{\mathrm{B}} E_\infty$ are continuous. Fix a basis $\{W_m\}$ for Z and a basis $\{N_p\}$ for Y. Put

$$P(x,g,z,y) \Leftrightarrow f(z)=g\cdot x \,\&\, h(z)=y.$$

This is closed in $X\times G\times Z\times Y$. For each $x\in[f(z)]_{E_G^X}$, $\mathrm{proj}_Y(P(x))$ is $\mathbf{\Sigma}_1^1$, non-empty and countable, so there are n, k, m, p such that $\mathrm{proj}_Y(P(x)\cap(U_k^n\times W_m\times N_p))$ is a singleton, say $y\in[h(x)]_{E_\infty}$. Then for $N_q\subseteq N_p$,

$$y\in N_q \Leftrightarrow \mathrm{proj}_Y(P(x)\cap(U_k^n\times W_m\times N_q))\neq\emptyset.$$

Put

$$R(x,n,k,m,p) \Leftrightarrow \mathrm{proj}_Y(P(x)\cap(U_k^n\times W_m\times N_p))=\emptyset.$$

This is $\mathbf{\Pi}_1^1$ and invariant under the $\mathbf{\Sigma}_1^1$ equivalence relation

$$\begin{aligned}&(x,n,k,m,p)\sim(x',n',k',m',p')\\&\qquad\Leftrightarrow\ n=n',\ m=m',\ p=p' \,\&\, \exists g_0\ (g_0^{-1}\cdot x=x' \,\&\, U_k^n g_0=U_{k'}^n).\end{aligned}$$

So, by Solovay's Theorem (see [14], 34.6(ii)), there is an $\sim$-invariant $\mathbf{\Pi}_1^1$-rank $\varphi:R\to\omega_1$ on R. As usual, let $\varphi=\omega_1$ off R. We have

$$\begin{aligned}&\forall z\ \exists y\in[h(z)]_{E_\infty}\ \exists n,k,m,p\ [y\in N_p \,\&\, \forall q\ (N_q\subseteq N_p\ \Rightarrow\\&\qquad\qquad\bullet\ y\in N_q\ \Rightarrow\ \varphi(f(z),n,k,m,q)=\omega_1,\\&\qquad\qquad\bullet\ y\notin N_q\ \Rightarrow\ \varphi(f(z),n,k,m,q)<\alpha<\omega_1^{z,a})],\end{aligned}$$

where $a\in\mathbb{N}^{\mathbb{N}}$ is an appropriate fixed parameter, independent of z. So

$$\begin{aligned}&\forall z\ \exists\alpha<\omega_1^{z,a}\ \exists y\in[h(z)]_{E_\infty}\exists n,k,m,p\\&\qquad[y\in N_p \,\&\, \forall q\ (N_q\subseteq N_p\Rightarrow(y\in N_q\Leftrightarrow\varphi(f(z),n,k,m,q)\geq\alpha))].\end{aligned}$$

By boundedness there is some fixed $\alpha_0<\omega_1$ so that

$$\begin{aligned}(*)\quad&\forall z\ \exists n\ \exists y\in[h(z)]_{E_\infty}\ \exists\alpha<\alpha_0\ \exists k,m,p\\&\quad[y\in N_p \,\&\, \forall q\ (N_q\subseteq N_p\Rightarrow(y\in N_q\Leftrightarrow\varphi(f(z),n,k,m,q)\geq\alpha))].\end{aligned}$$

Let, for $\alpha<\alpha_0,n,m,p\in\mathbb{N}$,

$$Z_{n,\alpha,m,p}=\{z:\exists y\in[h(z)]_{E_\infty}\ \exists k\ [y\in N_p \,\&\, \forall q\ (N_q\subseteq N_p\Rightarrow(*))]\}.$$

Then $Z_{n,\alpha,m,p}$ is Borel and E-invariant, and $Z=\bigcup_{n,\alpha,m,p}Z_{n,\alpha,m,p}$, so, in the notation of the special case, it is enough to show that $E|Z_{n,\alpha,m,p}\leq_{\mathrm{B}} E_{G^n}^{Y_n}$. For that define $Q_{n,\alpha,m,p}:Z_{n,\alpha,m,p}\to Y_n$ by $Q(z)(U_k^n)=$ the unique $y\in[h(z)]_{E_\infty}$ such that $y\in N_p \,\&\, \forall q\ (N_q\subseteq N_p\Rightarrow(*))$, if such exists; ∞ otherwise. This works as in the special case. ■

8. The Seventh Dichotomy Theorem. Finally, we prove the following result, labeled the *Seventh Dichotomy Theorem* in [12].

THEOREM 8.1. *Let $G\subseteq S_\infty$ be a closed subgroup of S_∞, admitting an invariant metric. If X is a Borel G-space and E_G^X is Borel*, *then for any*

$E \leq_{\mathrm{B}} E_G^X$,

$$E \leq_{\mathrm{B}} E_\infty \quad or \quad E_3 \leq_{\mathrm{B}} E.$$

Proof. We start with the following:

LEMMA 1. *If $G \subseteq S_\infty$ is a closed subgroup of S_∞ admitting an invariant metric, there is a sequence G_n of countable (discrete) groups so that G is isomorphic to a closed subgroup of $\prod_n G_n$.*

Proof. Fix a conjugation invariant nbhd basis $\{V_n\}$ at the identity. Let U_n be the subgroup generated by V_n. Since G has a nbhd basis at the identity consisting of open subgroups, it follows that $\{U_n\}$ is a nbhd basis consisting of normal open subgroups. Fix an invariant metric d for G and assume without loss of generality that $d(U_n) < 2^{-n}$.

Put $\Omega_n = G/U_n$. Clearly, Ω_n is countable. G acts on Ω_n by $g \cdot hU_n = ghU_n$. It is easily seen that $g \cdot U_n = g' \cdot U_n$ implies $g \cdot hU_n = ghU_n = gU_nh = g'U_nh = g'hU_n = g' \cdot hU_n$, so $\pi_g^n(hU_n) = g \cdot hU_n$ is completely determined by $\pi_g^n(U_n)$ and so $\pi_n(g) = \pi_g^n$ is a homomorphism of G onto a countable subgroup G_n of the symmetric group of Ω_n. If we define $\pi : G \to \prod_n G_n$ by $\pi(g)_n = \pi_n(g)$, clearly π is a continuous injection of G into $\prod_n G_n$. It remains to show that it has continuous inverse on $\pi(G)$. So assume $\pi(g_n) \to \pi(g)$ in the product topology of $\prod_n G_n$. Then for any fixed k, for all large enough n, we have $\pi(g_n)_k = \pi(g)_k$, i.e., $\pi_k(g_n) = \pi_k(g)$, so $g_nU_k = gU_k$ and then $g^{-1}g_n \in U_k$. So $d(g^{-1}g_n, 1) = d(g_n, g) < 2^{-k}$, thus $g_n \to g$. ∎

By [3, 2.3.5], if G is a closed subgroup of H, then for any Borel G-space X there is a Borel H-space Y with $E_G^X \sim_{\mathrm{B}} E_H^Y$. So it is enough to prove the theorem for G a countable product of countable (discrete) groups and since every such group is a homomorphic image of $\mathbb{F}_{\aleph_0}$ (the free group on $\aleph_0$ generators) which in turn is a homomorphic image of $H = \bigoplus_n \mathbb{F}_{\aleph_0}$ = the direct sum of countably many copies of $\mathbb{F}_{\aleph_0}$, it is enough to prove it for $G = H^{\mathbb{N}}$. Note that $H^{n+1} \cong H$ for each $n \in \mathbb{N}$.

We will next describe a countable structure with automorphism group G. Let

$$\mathcal{A}_0 = \left\langle \bigcup_{n=0}^{\infty} H^{n+1}, \{Q_n^{\mathcal{A}_0}\}_{n\in\mathbb{N}}, \{F_h^{\mathcal{A}_0}\}_{h\in H}, \{p_{ij}^{\mathcal{A}_0}\}_{0\leq i<j} \right\rangle$$

be defined as follows: Putting $A_0 = \bigcup_{n=0}^{\infty} H^{n+1}$, Q_n are unary relations such that

$$Q_n^{\mathcal{A}_0}(a) \;\Leftrightarrow\; a \in H^{n+1}.$$

Next, using $H^{n+1} \cong H$ for each n, we fix an isomorphism $\varrho_n : H \to H^{n+1}$, say $\varrho_n(h) = (h_0^n, \ldots, h_n^n)$. Then each $F_h^{\mathcal{A}_0}$ is the unary function such that

$$F_h^{\mathcal{A}_0}((g_0, \ldots, g_n)) = (g_0(h_0^n)^{-1}, \ldots, g_n(h_n^n)^{-1}).$$

Thus
$$F_{h_1}^{\mathcal{A}_0} \circ F_{h_2}^{\mathcal{A}_0} = F_{h_1 h_2}^{\mathcal{A}_0}$$
and $F_h^{\mathcal{A}_0}$ is a permutation of each $(Q_n)^{\mathcal{A}_0}$, and in fact $h \cdot (g_0, \ldots, g_n) = F_h^{\mathcal{A}_0}((g_0, \ldots, g_n))$ is a free transitive action of H on $(Q_n)^{\mathcal{A}_0}$.

Finally, $p_{ij}^{\mathcal{A}_0}$ is the unary function defined by
$$p_{ij}^{\mathcal{A}_0}((g_0, \ldots, g_n)) = \begin{cases} (g_0, \ldots, g_n) & \text{if } j \neq n, \\ (g_0, \ldots, g_i) & \text{if } j = n. \end{cases}$$

We deduce, using $F_h^{\mathcal{A}_0}, p_{ij}^{\mathcal{A}_0}$, that every $a \in (Q_n)^{\mathcal{A}_0}$ is definable (by a term) from any $b \in (Q_k)^{\mathcal{A}_0}$, if $k \geq n$.

It is clear that every $g = (h_0, h_1, \ldots) \in H^{\mathbb{N}}$ gives rise to an automorphism ϱ_g of $\mathcal{A}_0$ by
$$\varrho_g(g_0, \ldots, g_n) = (h_0 g_0, \ldots, h_n g_n)$$
and it is easy to check that every automorphism π of $\mathcal{A}_0$ is of the form ϱ_g. Thus $g \mapsto \varrho_g$ is an isomorphism of G with $\mathrm{Aut}(\mathcal{A}_0)$.

By a simple coding we can assume that the universe of $\mathcal{A}_0$ is $A_0 = \mathbb{N}$. By identifying g with ϱ_g we identify $G = H^{\mathbb{N}}$ with $\mathrm{Aut}(\mathcal{A}_0)$. Denote by L_0 the language of $\mathcal{A}_0$.

Suppose now X is a Borel G-space with E_G^X Borel and $E \leq_{\mathrm{B}} E_G^X$. As in the proof of Theorem 7.2 we can assume that the Borel reduction is actually 1-1. By [3, pp. 31–32], the G-space X is Borel embeddable in the relativized logic action $J_{L_0 \cup L}^{\mathcal{A}_0}$ (where L is a countable relational language disjoint from L_0) of $\mathrm{Aut}(\mathcal{A}_0) = G$ on $Y_{L_0 \cup L}^{\mathcal{A}_0} = \{\mathcal{M} \in X_{L_0 \cup L} : \mathcal{M}|L_0 = \mathcal{A}_0\}$, with $X_{L_0 \cup L}$ denoting the Polish space of all $L_0 \cup L$-structures with universe $\mathbb{N}$. The equivalence relation associated with $J_{L_0 \cup L}^{\mathcal{A}_0}$ is $\cong|Y_{L_0 \cup L}^{\mathcal{A}_0}$. Let $Y \subseteq Y_{L_0 \cup L}^{\mathcal{A}_0}$ be the range of this G-embedding, so Y is an $\cong|Y_{L_0 \cup L}^{\mathcal{A}_0}$-invariant subset of $Y_{L_0 \cup L}^{\mathcal{A}_0}$. Clearly, E_G^X is Borel isomorphic to $\cong|Y$. Put
$$Z = \{\mathcal{M} \in X_{L_0 \cup L} : \exists \mathcal{B} \in Y \ (\mathcal{B} \cong \mathcal{M})\}.$$
We claim that Z is Borel. This is because
$$\mathcal{M} \in Z \ \Leftrightarrow\ \mathcal{M}|L_0 \cong \mathcal{A}_0 \ \& \ \forall g \in S_\infty \ (g \cdot \mathcal{M}|L_0 = \mathcal{A}_0 \Rightarrow g \cdot \mathcal{M} \in Y).$$

We now claim that $\cong|Z$ is also Borel. This is because for $\mathcal{M}, \mathcal{N} \in Z$,
$$\mathcal{M} \cong \mathcal{N}$$
$$\Leftrightarrow\ \forall g \in S_\infty \ \forall h \in S_\infty \ (g \cdot \mathcal{M}|L_0 = \mathcal{A}_0 \ \& \ g \cdot \mathcal{N}|L_0 = \mathcal{A}_0 \ \Rightarrow\ g \cdot \mathcal{M} \cong g \cdot \mathcal{N}).$$
Of course, there is a sentence $\sigma \in (L_0 \cup L)_{\omega_1}$ such that $Z = \mathrm{Mod}(\sigma)$.

In summary: We have the countable structure $\mathcal{A}_0$ with universe $\mathbb{N}$ in the language L_0 such that:

(i) $L_0 = \{Q_n : n \in \mathbb{N}\} \cup \{F_g : g \in H\} \cup \{p_{ij} : 0 \leq i < j\}$.

(ii) $\mathbb{N}$ is the disjoint union of $(Q_n)^{\mathcal{A}_0}$.

(iii) $F_g^{\mathcal{A}_0} \circ F_h^{\mathcal{A}_0} = F_{gh}^{\mathcal{A}_0}$; F_g maps each $(Q_n)^{\mathcal{A}_0}$ 1-1 and onto itself; $F_g(a)=a \Leftrightarrow g=1$, for any $a \in \mathbb{N}$, $g \in H$.

(iv) If $a \in (Q_n)^{\mathcal{A}_0}$, then $p_{i,n}^{\mathcal{A}_0}(a) \in (Q_i)^{\mathcal{A}_0}$.

Moreover, we have a sentence $\sigma \in (L_0 \cup L)_{\omega_1}$ such that $\cong|\mathrm{Mod}(\sigma)$ is Borel, and every $\mathcal{M} \in \mathrm{Mod}(\sigma)$ is isomorphic to an expansion of $\mathcal{A}_0$. Finally, there is a Borel injection $f : W \to \mathrm{Mod}(\sigma)$, where E lives on W, such that

$$xEy \Leftrightarrow f(x) \cong f(y) \text{ and } f[W] = X_0 \subseteq \{\mathcal{A} \in \mathrm{Mod}(\sigma) : \mathcal{A}|L_0 = \mathcal{A}_0\}.$$

By relativization, we can assume that $L_0 \cup L$ is recursive, $\sigma \in L_{\omega_1^{ck}}$, $\cong|\mathrm{Mod}(\sigma)$ is Δ_1^1, also $f, X_0 \in \Delta_1^1$, and $\mathrm{Aut}(\mathcal{A}_0)$ has a dense subgroup consisting of recursive elements. (Notice that $\mathcal{A}_0, L_0$ are also recursive.)

Fix now for each n an element $p_n \in (Q_n)^{\mathcal{A}_0}$. Using this we can define an action of H on $(Q_n)^{\mathcal{A}_0}$ by

$$h \cdot F_{h_1}(p_n) = F_{h_1 h^{-1}}(p_n).$$

(Notice that every $a \in (Q_n)^{\mathcal{A}_0}$ is of the form $F_{h_1}(p_n)$ for a unique $h_1 \in H$.) This defines a homomorphism $\pi : G(= H^{\mathbb{N}}) \to S_\infty$ by letting $\pi((h_0, h_1, \ldots))$ act on $(Q_n)^{\mathcal{A}_0}$ via h_n. It is now easy to check that $\pi(G) \supseteq \mathrm{Aut}(\mathcal{A}_0)$. We put $\pi_g = \pi(g)$. By relativization, we can also assume that the closed set $\{g \in G : \pi_g \in \mathrm{Aut}(\mathcal{A}_0)\}$ admits a countable dense set consisting of recursive elements.

Below, *fragment* of a language $\mathcal{L}_{\omega_1\omega}$ means *countable fragment*. (The definitions concerning the theory of $\mathcal{L}_{\omega_1\omega}$ are as in [2].) For $\mathcal{M} = \langle M, - \rangle$ an $\mathcal{L}$-structure, F a fragment and $\overline{a} \in M^{<\mathbb{N}}$, let $\mathrm{Th}_F(M, \overline{a}) = \{\varphi \in F : \mathcal{M} \models \varphi(\overline{a})\}$.

If $A \subseteq \mathrm{Mod}(\sigma)$, $\mathcal{M} \in \mathrm{Mod}(\sigma)$, $\overline{a} \in M^{<\mathbb{N}}$ then put

$$(\mathcal{M}, \overline{a}) \models A \Leftrightarrow \exists g \in S_\infty\ (g(a_i) = a_i \text{ and } g \cdot \mathcal{M} \in A).$$

Definition. Let $F \subseteq (L_0 \cup L)_{\omega_1\omega}$ be a fragment, $A \subseteq \mathrm{Mod}(\sigma)$, $\mathcal{M} \in \mathrm{Mod}(\sigma)$ and $a \in (Q_n)^{\mathcal{M}}$, $b^0 \in (Q_{k_0})^{\mathcal{M}}, \ldots, b^i \in (Q_{k_i})^{\mathcal{M}}$ with $n < k_0 < \ldots < k_i$. We say that A *isolates* $\mathrm{Th}_F(\mathcal{M}, a, \overline{b})$ *over* $(\mathcal{M}, a)$ if

(i) $(\mathcal{M}, a, \overline{b}) \models A$,

(ii) for all $\mathcal{M}_0 \in \mathrm{Mod}(\sigma)$ and all $g \in S_\infty$ with $g(a) = a$, we have

$$(\mathcal{M}_0, a, \overline{b}) \models A \ \&\ (g \cdot \mathcal{M}_0, a, \overline{b}) \models A \Rightarrow \mathrm{Th}_F(\mathcal{M}_0, a, \overline{b}) = \mathrm{Th}_F(g \cdot \mathcal{M}_0, a, \overline{b}).$$

If $\psi \in (L_0 \cup L)_{\omega_1\omega}$, we say that ψ *isolates* $\mathrm{Th}_F(\mathcal{M}, a, \overline{b})$ *over* $(\mathcal{M}, a)$ if

(i) $(\mathcal{M}, a, \overline{b}) \models \psi$ (i.e., $\mathcal{M} \models \psi(a, \overline{b})$),

(ii) for all $\mathcal{M}_0 \in \mathrm{Mod}(\sigma)$, all $a_0 \in (Q_n)^{\mathcal{M}_0}$ and all $b_0^i, (b_0^i)' \in (Q_{k_i})^{\mathcal{M}_0}$, we have

$$(\mathcal{M}_0, a_0, \overline{b}_0) \models \psi \ \&\ (\mathcal{M}_0, a_0, \overline{b}_0') \models \psi \Rightarrow \mathrm{Th}_F(\mathcal{M}_0, a_0, \overline{b}_0) = \mathrm{Th}_F(\mathcal{M}_0, a_0, b_0').$$

Let us note that if $A \subseteq \mathrm{Mod}(\sigma)$ is $\mathbf{\Delta}^1_1$ and invariant under all $g \in S_\infty$ with $g(a) = a,\ g(b^i) = b^i$, then by [21], there is a formula $\psi \in (L_0 \cup L)_{\omega_1\omega}$ with

$$(\mathcal{N}, a, \bar{b}) \models \psi \ \Leftrightarrow\ (\mathcal{N}, a, \bar{b}) \models A \ \Leftrightarrow\ \mathcal{N} \in A,$$

for all $\mathcal{N} \in \mathrm{Mod}(\sigma)$. We check then that if A isolates $\mathrm{Th}_F(\mathcal{M}, a, \bar{b})$ over $(\mathcal{M}, a)$, so does ψ.

Indeed, assume $\mathcal{M}_0, a_0, \bar{b}_0, \bar{b}'_0$ are as in (ii) of the definition of isolation for ψ and assume $(\mathcal{M}_0, a_0, \bar{b}_0) \models \psi, (\mathcal{M}_0, a_0, \bar{b}'_0) \models \psi$. Choosing $\widetilde{g} \in S_\infty$ with $\widetilde{g}(a_0) = a$, $\widetilde{g}(b^i_0) = b^i$, we see that by replacing $\mathcal{M}_0$ by $\widetilde{g} \cdot \mathcal{M}_0$, a_0 by $\widetilde{g}(a_0) = a$, b^i_0 by $\widetilde{g}(b^i_0) = b^i$, it is enough to assume that $a_0 = a$, $\bar{b}_0 = \bar{b}$. Let then $g \in S_\infty$ be such that $g(a) = a$, $g((b^i_0)') = b^i$. Then we have $(\mathcal{M}_0, a, \bar{b}) \models \psi$, so $(\mathcal{M}_0, a, \bar{b}) \models A$ and $(g \cdot \mathcal{M}_0, a, \bar{b}) \models \psi$, so $(g \cdot \mathcal{M}_0, a, \bar{b}) \models A$, thus, since A isolates $\mathrm{Th}_F(\mathcal{M}, a, \bar{b})$ over $(\mathcal{M}, a)$ we get $\mathrm{Th}_F(\mathcal{M}_0, a_0, \bar{b}_0) = \mathrm{Th}_F(g \cdot \mathcal{M}_0, a, \bar{b}) = \mathrm{Th}_F(\mathcal{M}_0, a, \bar{b}'_0)$, so we are done.

Lemma 2. *If F, F' are fragments, $a \in (Q_n)^{\mathcal{M}}$ and for all $k > n$ and all $b \in (Q_k)^{\mathcal{M}}$ there is $\psi \in F'$ isolating* $\mathrm{Th}_F(\mathcal{M}, a, b)$ *over $(\mathcal{M}, a)$, then for any $\bar{b}$ as in the preceding definition there is $\psi \in F'$ isolating* $\mathrm{Th}_F(\mathcal{M}, a, \bar{b})$ *over $(\mathcal{M}, a)$.*

Proof. Fix $\bar{b} = (b^0, \ldots, b^i)$ and put $b = b^i$. Fix terms $t_0, \ldots, t_{i-1}$ with $t^{\mathcal{M}}_j(b) = b^j$. Let $\psi \in F'$ isolate $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$. Put

$$\psi'(x, y^0, \ldots, y^i) \ \Leftrightarrow\ \psi(x, y^i) \ \&\ \bigwedge_{j<i} (t_j(y^i) = y^j).$$

Then $(\mathcal{M}, a, b) \models \psi'$. We claim that ψ' isolates $\mathrm{Th}_F(\mathcal{M}, a, \bar{b})$ over $(\mathcal{M}, a)$. To check this fix $\mathcal{M}_0, a_0, \bar{b}_0, \bar{b}'_0$ so that

$$(\mathcal{M}_0, a_0, \bar{b}_0) \models \psi', \qquad (\mathcal{M}_0, a_0, \bar{b}'_0) \models \psi'.$$

Thus $b^j_0 = t^{\mathcal{M}_0}_j(b^i_0)$, $(b'_0)^j = t^{\mathcal{M}_0}_j((b'_0)^i)$ and $(\mathcal{M}_0, a_0, b^i_0) \models \psi$, $(\mathcal{M}_0, a_0, (b'_0)^i) \models \psi$, so $\mathrm{Th}_F(\mathcal{M}_0, a_0, b^i_0) = \mathrm{Th}_F(\mathcal{M}_0, a_0, (b'_0)^i)$. If now $\theta \in \mathrm{Th}_F(\mathcal{M}_0, a_0, \bar{b}_0)$, then $\theta(x, t_0(y), \ldots, t_{i-1}(y), y) \in \mathrm{Th}_F(\mathcal{M}_0, a_0, b^i_0)$, so

$$\theta(x, t_0(y), \ldots, t_{i-1}(y), y) \in \mathrm{Th}_F(\mathcal{M}_0, a_0, (b'_0)^i),$$

thus $\theta \in \mathrm{Th}_F(\mathcal{M}_0, a_0, \bar{b}'_0)$ and so $\mathrm{Th}_F(\mathcal{M}_0, a_0, \bar{b}_0) = \mathrm{Th}_F(\mathcal{M}_0, a_0, \bar{b}'_0)$. ■

Lemma 3. *If F, F' are fragments $a \in (Q_n)^{\mathcal{M}}$ and for all $k > n$ and all $b \in (Q_k)^{\mathcal{M}}$ there is $\psi \in F'$ isolating* $\mathrm{Th}_F(\mathcal{M}, a, b)$ *over $(\mathcal{M}, a)$, then for any $n' > n$, $a' \in (Q_{n'})^{\mathcal{M}}$ and every $k' > n'$, $b \in (Q_{k'})^{\mathcal{M}}$ there is $\psi \in F'$ isolating* $\mathrm{Th}_F(\mathcal{M}, a', b)$ *over $(\mathcal{M}, a')$.*

Proof. Fix such n', a', b. By the preceding lemma there is a formula $\psi \in F'$ isolating $\mathrm{Th}_F(\mathcal{M}, a, a', b)$ over $(\mathcal{M}, a)$. Let t be a term with $t^{\mathcal{M}}(a') = a$. Put

$$\psi'(x, y) \ \Leftrightarrow\ \psi(t(x), x, y).$$

Then $(\mathcal{M}, a', b) \models \psi$ and we claim that ψ' isolates $(\mathcal{M}, a', b)$ over $(\mathcal{M}, a')$. Indeed, let $(\mathcal{M}_0, a'_0, b_0) \models \psi'$, $(\mathcal{M}_0, a'_0, b'_0) \models \psi'$. If $a_0 = t^{\mathcal{M}_0}(a'_0)$, then $(\mathcal{M}_0, a_0, a'_0, b_0) \models \psi$ and $(\mathcal{M}_0, a_0, a'_0, b'_0) \models \psi$, so $\mathrm{Th}_F(\mathcal{M}_0, a_0, a'_0, b_0) = \mathrm{Th}_F(\mathcal{M}_0, a_0, a'_0, b_0)$, thus $\mathrm{Th}_F(\mathcal{M}_0, a'_0, b_0) = \mathrm{Th}_F(\mathcal{M}_0, a'_0, b'_0)$. ∎

We now consider 2 cases:

Case I: $\forall$ fragments $F \subseteq (L_0 \cup L)_{\omega_1\omega}$, $F \in L_{\omega_1^{ck}}$ $\forall \mathcal{M} \in X_0$ $\exists n$ $\exists a \in (Q_n)^{\mathcal{M}}$ $\forall k > n$ $\forall b \in (Q_k)^{\mathcal{M}}$ $\exists A \in \Sigma^1_1$ (A isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$). We will then show that $E \leq_{\mathrm{B}} E_\infty$.

Lemma 4. *Assume $F \in L_{\omega_1^{ck}}$ and $A \in \Sigma^1_1$ isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$. Then there is $\psi \in L_{\omega_1^{ck}}$ which isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$.*

Proof. By reflection, as property (ii) in the definition of isolation is Π^1_1 in the codes for A, there is $\mathrm{Mod}(\sigma) \supseteq A^*_0 \supseteq A, A^*_0 \in \Delta^1_1$ satisfying (ii), and so having the same isolation property. Let $A_0 = \{\mathcal{M} : \exists g \in S_\infty$ $(g(a) = a \;\&\; g(b) = b \;\&\; g \cdot \mathcal{M} \in A^*_0)\}$. Then $A^*_0 \subseteq A_0$ and A^*_0 still satisfies this property (ii). Clearly, $A^*_0 \in \Sigma^1_1$. We can repeat effectively this process to get $A \subseteq A^*_0 \subseteq A_0 \subseteq A^*_1 \subseteq A_1 \subseteq \ldots$ with $A_n \in \Delta^1_1$ uniformly in n, A_n isolating $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$, and A_n closed under all $g \in S_\infty$ fixing a, b. Let $A^* = \bigcup_n A_n$. Then $A^* \in \Delta^1_1$ is invariant under all such g and isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$. If ψ is the corresponding formula, then $\psi \in (L_0 \cup L)_{\omega_1\omega}$, $\psi \in L_{\omega_1^{ck}}$, and ψ isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$. ∎

Thus $\forall F \in L_{\omega_1^{ck}}$ $\forall \mathcal{M} \in X_0$ $\exists n$ $\exists a \in (Q_n)^{\mathcal{M}}$ $\forall k > n$ $\forall b \in (Q_k)^{\mathcal{M}}$ $\exists \psi \in L_{\omega_1^{ck}}$ (ψ isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$). By reflection we can then find, for each fragment $F \in L_{\omega_1^{ck}}$, a fragment $F^+ \in L_{\omega_1^{ck}}$ so that $F \subseteq F^+$ and $F \mapsto F^+$ is $\Delta_1(L_{\omega_1^{ck}})$ such that $\forall \mathcal{M} \in X_0$ $\exists n$ $\exists a \in (Q_n)^{\mathcal{M}}$ $\forall k > n$ $\forall b \in (Q_k)^{\mathcal{M}}$ $\exists \psi \in F^+$ (ψ isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$).

Define then recursively $(F_\alpha)_{\alpha < \omega_1^{ck}}$ by

$$\begin{aligned} F_0 &= \text{the fragment generated by } \sigma, \\ F_{<\alpha} &= \Big(\bigcup_{\beta<\alpha} F_\beta\Big), \\ F_\alpha &= (F_{<\alpha})^+, \end{aligned}$$

so that each $F_\alpha \in L_{\omega_1^{ck}}$.

For convenience, given $\mathcal{M} \in \mathrm{Mod}(\sigma)$, $a \in (Q_n)^{\mathcal{M}}$, $\gamma < \omega_1^{ck}$, we say that $(\mathcal{M}, a)$ is γ-*good* if for all $k > n$ and all $b \in (Q_k)^{\mathcal{M}}$ there is $\psi \in F_\gamma$ isolating $\mathrm{Th}_{F_{<\gamma}}(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$. Thus for any $\gamma < \omega_1^{ck}, \mathcal{M} \in X_0$, there are n and $a \in (Q_n)^{\mathcal{M}}$ such that $(\mathcal{M}, a)$ is γ-good, so that by Lemma 3, this is also true for any $n' > n$.

Below, *rank* means *quantifier rank,* as in [2].

LEMMA 5. *Fix* $\gamma < \omega_1^{ck}$. *If* $\mathcal{M}, \mathcal{M}_0 \in X_0$, $a \in (Q_n)^{\mathcal{M}}$, $a_0 \in (Q_n)^{\mathcal{M}_0}$, $(\mathcal{M}, a)$, $(\mathcal{M}_0, a_0)$ *are* γ*-good and* $\mathrm{Th}_{F_\gamma}(\mathcal{M}, a) = \mathrm{Th}_{F_\gamma}(\mathcal{M}_0, a_0)$, *then* $(\mathcal{M}, a)$, $(\mathcal{M}_0, a_0)$ *satisfy the same formulas of rank* γ.

Proof. By induction on γ. For $\gamma = 0$ this is obvious as $F_0 \supseteq$ quantifier-free formulas. Let now γ be limit and φ be of rank γ with $(\mathcal{M}, a) \models \varphi$. Without loss of generality we can assume that $\varphi = \bigvee_i \varphi_i$ with each φ_i of rank $\gamma_i < \gamma$. Thus $(\mathcal{M}, a) \models \varphi_i$ for some i. Fix $n' > n$ and $b \in (Q_{n'})^{\mathcal{M}}$ so that $(\mathcal{M}, b)$ is γ_i-good. As $(\mathcal{M}, a)$ is γ-good there is $\psi \in F_\gamma$ isolating $\mathrm{Th}_{F_{\gamma_i}}(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$. Since $\mathrm{Th}_{F_\gamma}(\mathcal{M}, a) = \mathrm{Th}_{F_\gamma}(\mathcal{M}_0, a_0)$, there is $b_0 \in (Q_{n'})^{\mathcal{M}}$ with $(\mathcal{M}_0, a_0, b_0) \models \psi$. We claim that $(\mathcal{M}_0, b_0)$ is γ_i-good and that $\mathrm{Th}_{F_{\gamma_i}}(\mathcal{M}, a, b) = \mathrm{Th}_{F_{\gamma_i}}(\mathcal{M}_0, a_0, b_0)$ so that, in particular, $\mathrm{Th}_{F_{\gamma_i}}(\mathcal{M}, b) = \mathrm{Th}_{F_{\gamma_i}}(\mathcal{M}_0, b_0)$. Then by induction hypothesis $(\mathcal{M}, b)$ and $(\mathcal{M}_0, b_0)$ satisfy the same formulas of rank γ_i. But there is a term t such that $t^{\mathcal{M}}(b) = a$ and $t^{\mathcal{M}_0}(b_0) = a_0$, so as $(\mathcal{M}, a) \models \varphi_i$, $(\mathcal{M}, b) \models \varphi_i(t)$ and so we have $(\mathcal{M}_0, b_0) \models \varphi_i(t)$, i.e., $(\mathcal{M}_0, a_0) \models \varphi_i$, and we are done.

First we check that $\mathrm{Th}_{F_{\gamma_i}}(\mathcal{M}, a, b) = \mathrm{Th}_{F_{\gamma_i}}(\mathcal{M}_0, a_0, b_0)$. Suppose $(\mathcal{M}, a, b) \models \theta$, $\theta \in F_{\gamma_i}$. Then there is $b_0' \in (Q_{n'})^{\mathcal{M}_0}$ with $(\mathcal{M}_0, a_0, b_0') \models \theta$ and $(\mathcal{M}_0, a_0, b_0') \models \psi$, so, as then $\mathrm{Th}_{F_{\gamma_i}}(\mathcal{M}_0, a_0, b_0) = \mathrm{Th}_{F_{\gamma_i}}(\mathcal{M}_0, a_0, b_0')$, we have $(\mathcal{M}_0, a_0, b_0) \models \theta$.

Next we check that $(\mathcal{M}_0, b_0)$ is γ_i-good. Fix $k > n'$, $d_0 \in (Q_k)^{\mathcal{M}_0}$. Take any $d \in (Q_k)^{\mathcal{M}}$ and, since $(\mathcal{M}, b)$ is γ_i-good, let $\varrho' \in F_{\gamma_i}$ isolate $\mathrm{Th}_{F_{<\gamma_i}}(\mathcal{M}, b, d)$ over $(\mathcal{M}, b)$. Then there is $d_0' \in (Q_k)^{\mathcal{M}_0}$ with $(\mathcal{M}_0, b_0, d_0') \models \varrho'$. So ϱ' isolates $\mathrm{Th}_{F_{<\gamma_i}}(\mathcal{M}_0, b_0, d_0')$ over $(\mathcal{M}_0, b_0)$. If $\varrho' = \varrho'(x, y)$ and we fix $h \in H$ with $F_h(d_0) = d_0'$, we see that $\varrho = \varrho'(x, F_h(y))$ isolates $\mathrm{Th}_{F_{<\gamma_i}}(\mathcal{M}_0, b_0, d_0)$ over $(\mathcal{M}_0, b_0)$ and $\varrho \in F_{\gamma_i}$, so we are done.

Finally, consider the successor case $\gamma = \delta + 1$. Without loss of generality, let $\varphi = \exists x\ \psi(x)$ be of rank $\delta + 1$ and assume that $(\mathcal{M}, a) \models \varphi$. Then fix $n' > n$ and $b \in (Q_{n'})^{\mathcal{M}}$ so that $(\mathcal{M}, a) \models \psi(t^{\mathcal{M}}(b))$ for some term t, and $(\mathcal{M}, b)$ is δ-good. As before, we can find $b_0 \in (Q_{n'})^{\mathcal{M}_0}$ which is δ-good and $\mathrm{Th}_{F_\gamma}(\mathcal{M}, a, b) = \mathrm{Th}_{F_\gamma}(\mathcal{M}_0, a_0, b_0)$. Then by induction hypothesis $(\mathcal{M}, b), (\mathcal{M}_0, b_0)$ satisfy the same formulas of rank δ, so (as also $a = t_1^{\mathcal{M}}(b_0)$, $a_0 = t_1^{\mathcal{M}_0}(b_0)$ for some term t_1), we have $(\mathcal{M}_0, a_0) \models \psi(t^{\mathcal{M}_0}(b_0))$, thus $(\mathcal{M}, a_0) \models \varphi$ and we are done. ■

Since $\cong|\mathrm{Mod}(\sigma)$ is Δ_1^1, it follows from Vaught's work (see, e.g., [14, 16.9]) that there is $\gamma_0 < \omega_1^{ck}$ such that if $\mathcal{M}, \mathcal{M}_0$ are in $\mathrm{Mod}(\sigma)$ and satisfy the same formulas of rank γ_0, then $\mathcal{M} \cong \mathcal{M}_0$.

By Kreisel Selection, there is a Δ_1^1 function which assigns to each $\mathcal{M} \in X_0$ some $n_{\mathcal{M}} \in \mathbb{N}$ and $a_{\mathcal{M}} \in (Q_{n_{\mathcal{M}}})^{\mathcal{M}}$ so that $(\mathcal{M}, a_{\mathcal{M}})$ is γ_0-good. Put, for $\mathcal{M} \in X_0$,

$$U(\mathcal{M}) = \mathrm{Th}_{F_{\gamma_0}}(\mathcal{M}, a_{\mathcal{M}}).$$

Then U is a Δ^1_1 function (from X_0 into $2^{F_{\gamma_0}}$) and, by the preceding lemma, if $\mathcal{M}, \mathcal{M}_0 \in X_0$ and $U(\mathcal{M}) = U(\mathcal{M}_0)$, then $\mathcal{M} \cong \mathcal{M}_0$. Moreover, for each $\mathcal{M} \in X_0, \{U(\mathcal{M}_0) : \mathcal{M}_0 \in X_0 \ \&\ \mathcal{M} \cong \mathcal{M}_0\}$ is countable, since it is contained in $\{\mathrm{Th}_{F_{\gamma_0}}(\mathcal{M}, a) : a \in \mathcal{M}\}$. It follows that U maps each $\cong|X_0$-class into a countable set and distinct isomorphic classes are mapped to disjoint sets, so $\cong|X_0 \le_{\mathrm{B}} E_\infty$. As E^X_G is Borel isomorphic to $\cong|X_0$, this shows that $E \le_{\mathrm{B}} E^X_G \le_{\mathrm{B}} E_\infty$, so $E \le_{\mathrm{B}} E_\infty$.

Case II: There are $F \in L_{\omega_1^{ck}}$ and $\mathcal{M} \in X_0$ so that for all $n, a \in (Q_n)^{\mathcal{M}}$ there are $k > n$ and $b \in (Q_k)^{\mathcal{M}}$ such that no $A \in \Sigma^1_1$ isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$. Fix such an F from now.

Let

$$\begin{aligned} X_0' &= \{\mathcal{M} \in \mathrm{Mod}(\sigma) : \mathcal{M}|L_0 = \mathcal{A}_0 \text{ and } \exists \mathcal{N} \in X_0\ (\mathcal{M} \cong \mathcal{N})\} \\ Y_0' &= \{\mathcal{M} \in \mathrm{Mod}(\sigma) : \mathcal{M}|L_0 = \mathcal{A}_0 \text{ and } \forall n\ \forall a \in (Q_n)^{\mathcal{M}}\ \exists k > n\ \exists b \in (Q_k)^{\mathcal{M}} \\ &\qquad (\text{no } A \in \Sigma^1_1 \text{ isolates } \mathrm{Th}_F(\mathcal{M}, a, b) \text{ over } (\mathcal{M}, a))\} \\ &= \{\mathcal{M} \in \mathrm{Mod}(\sigma) : \mathcal{M}|L_0 = \mathcal{A}_0 \text{ and } \forall n\ \forall a \in (Q_n)^{\mathcal{M}}\ \exists k > n\ \exists b \in (Q_k)^{\mathcal{M}} \\ &\qquad (\text{no } \psi \in L_{\omega_1^{ck}} \text{ isolates } \mathrm{Th}_F(\mathcal{M}, a, b) \text{ over } (\mathcal{M}, a))\}. \end{aligned}$$

Then X_0', Y_0' are invariant under the action of $\mathrm{Aut}(\mathcal{A}_0)$ (i.e., under $\cong|Y^{\mathcal{A}_0}_{L_0 \cup L}$) and by the Case II assumption

$$Y_0 = X_0' \cap Y_0' \neq \emptyset$$

and clearly $Y_0 \in \Sigma^1_1$. We will show that

$$(*) \qquad E_0^{\mathbb{N}} \le_{\mathrm{c}} (\cong|Y_0).$$

Then $E_0^{\mathbb{N}} \le_{\mathrm{c}} (\cong|X_0')$. Since clearly $(\cong|X_0') \le_{C\text{-meas}} (\cong|X_0) \sqsubseteq_{\mathrm{B}} E$, it follows that $E_0^{\mathbb{N}} \le_{C\text{-meas}} E$. So there is a comeager set D with $E_0^{\mathbb{N}}|D \le_{\mathrm{c}} E$. We then claim that $E_0^{\mathbb{N}} \sqsubseteq_{\mathrm{c}} E_0^{\mathbb{N}}|D$, which completes the proof. To see this notice that by the Sixth Dichotomy Theorem it is enough to show that $E_0^{\mathbb{N}}|D \not\le_{\mathrm{B}} E_\infty$. Identifying, as usual, $(2^{\mathbb{N}})^{\mathbb{N}}$ with $2^{\mathbb{N}\times\mathbb{N}}$ we see that $E_0^{\mathbb{N}} = E_{I_3}$. If $E_0^{\mathbb{N}}|D = E_{I_3}|D \le_{\mathrm{B}} E_\infty$, then, by [17], $I_3 \in \mathbf{\Sigma}^0_2$, which is a contradiction, as I_3 is complete $\mathbf{\Pi}^0_3$.

So it remains to prove $(*)$. We keep the notation $\langle m, j\rangle$, $L(n)$ from the proof of Theorem 7.2. We assume without loss of generality that $\{0, \ldots, n-1\} \subseteq \bigcup_{i<n} (Q_i)^{\mathcal{A}_0}$, and for any $\mathcal{M} \in \mathrm{Mod}(\sigma)$ we let $\mathcal{M}|n$ be the restriction of $\mathcal{M}$ to n (for this we view $\mathcal{M}$ as relational by replacing functions by their graphs). We also fix a recursive free $T_\emptyset$ such that

$$\mathcal{M} \in Y_0 \ \Leftrightarrow\ \exists y\ \forall n\ (\mathcal{M}|n, y|n) \in T_\emptyset.$$

Let also, for each n,

$$V_n = \{g = (h_0, h_1, \ldots) \in H^{\mathbb{N}} : \pi_g \in \mathrm{Aut}(\mathcal{A}_0)\ \&\ h_0 = \ldots = h_n = 1\}.$$

We will define the following, by induction on $n \geq 0$:

(i) Non-empty Σ^1_1 sets $A_s, s \in 2^{n+1}$, with $A_\emptyset = Y_0, A_{s^\frown i} \subseteq A_s$.

(ii) $k_m \in \mathbb{N},\ m \leq L(n)$, chosen so that $0 < k_0 < k_1 < \ldots$

(iii) We will also have

$$\mathcal{M} \in A_{0^{n+1}} \ \Rightarrow\ \forall r \leq L(n)\ \forall a \in (Q_r)^{\mathcal{M}}\ \exists b \in (Q_{k_r})^{\mathcal{M}}$$
$$(\text{no } A \in \Sigma^1_1 \text{ isolates } \mathrm{Th}_F(\mathcal{M}, a, b) \text{ over } (\mathcal{M}, a)).$$

(iv) Each $A_s,\ s \in 2^{n+1}$, will be invariant under $\pi(V_{k_{L(n)}})$.

(v) For each $s \in 2^{n+1}$ we will also have μ_s, y_s of length $k_{L(n)} + 1$ and $p_s \in H^{k_{L(n)}+1}$ such that $(\mathcal{M}|(k_{L(n)}+1) = \mu_s)$ & $\forall \mathcal{M} \in A_s\ \exists g \in H^{\mathbb{N}}$, $g \supseteq p_s$ $\exists y \supseteq y_s$ $[(\pi_g \cdot \mathcal{M}, y) \in [T_\emptyset]]$. Moreover, $s \subseteq t \ \Rightarrow\ \mu_s \subseteq \mu_t$, $y_s \subseteq y_t$, $p_s \subseteq p_t$.

(v)$'$ We can view (v) as a requirement concerning A_s relative to $A_\emptyset = Y_0$. We will also impose a similar requirement relative to each A_{s_0}, with $s_0 \in 2^{n_0+1}$, $n_0 = \langle m_0, 0\rangle$, for all A_s, $s \in 2^{n+1}$, $n > n_0$, $s \supseteq s_0$, i.e., we will fix a free T_{s_0} for A_{s_0} and define $\mu^{s_0}_s$, $y^{s_0}_s$, $p^{s_0}_s$ with similar properties, with the stipulation that $\mu^{s_0}_s = \mu_s$ and $p^{s_0}_s(i) = 1$ if $i \leq k_{L(n_0)}$.

(vi) $g_s \in H^{\mathbb{N}}$, for each $s \in 2^{n+1}$, with $g_{0^{n+1}} = 1$, g_s recursive, and $\pi_{g_s} \in \mathrm{Aut}(\mathcal{A}_0)$.

(vii) *Links.* We will also have, for $s \in 2^{n+1}$,

$$\pi_{g_s} \cdot A_{0^{n+1}} = A_s.$$

(viii) *Positive requirements.* For $s, t \in 2^{n+1}$, put $g_{s,t} = g_t g_s^{-1}$. If $\overline{n} < n$, $(\bar{s}, \bar{t}) \subseteq (s, t)$, $\bar{s}$, $\bar{t} \in 2^{\overline{n}+1}$, then we must have, for $l \leq L(n)$,

$$[\forall \bar{l} \leq l\ \forall \langle \bar{l}, i\rangle \in (n+1) \setminus (\overline{n}+1)\ (s(\langle \bar{l}, i\rangle) = t(\langle \bar{l}, i\rangle))] \Rightarrow g_{s,t} \equiv_l g_{\bar{s},\bar{t}},$$

where for $g, h \in H^{\mathbb{N}}$, $l \in \mathbb{N}$ we let

$$g \equiv_l h \ \Leftrightarrow\ \forall i \leq l\ (g_i = h_i).$$

(ix) *Negative requirements.* If $s, t \in 2^{n+1}$, $n = \langle m, j\rangle$, and $s(n) \neq t(n)$, then

$$\mathcal{M} \in A_s \ \&\ g \in H^{\mathbb{N}} \ \&\ \pi_g \in \mathrm{Aut}(\mathcal{A}_0) \ \&\ g(m), g(k_m) \in \{\widetilde{h}_0, \ldots, \widetilde{h}_n\}$$
$$\Rightarrow\ \pi_g \cdot \mathcal{M} \notin A_t,$$

where $\{\widetilde{h}_0, \widetilde{h}_1, \ldots\}$ is a recursive enumeration of H with $\widetilde{h}_0 = 1$.

Assume all this can be done. For each $x \in 2^{\mathbb{N}}$, let

$$\mathcal{M}_x = \bigcup_n \mu_{x|n+1}.$$

We first claim that $\mathcal{M}_x \in Y_0$. To see this let $y_x = \bigcup_n y_{x|n+1}$ and $g_x = \bigcup_n p_{x|n+1} \in H^{\mathbb{N}}$. For each n, fix $\mathcal{M}^n \in A_{x|n+1}$. Then $\mathcal{M}^n|(k_{L(n)} + 1) = \mathcal{M}_x|(k_{L(n)} + 1)$, so $\mathcal{M}^n \to \mathcal{M}_x$ (in the usual topology of $X_{L_0 \cup L}$).

Also, fix $g^n \in H^{\mathbb{N}}, g^n \supseteq p_{x|n+1}$ and $y^n \supseteq y_{x|n+1}$ with $(\pi_{g^n} \cdot \mathcal{M}^n, y^n) \in [T_\emptyset]$. Notice that $\pi_{g^n} \in \mathrm{Aut}(\mathcal{A}_0)$, as $\mathcal{M}^n, \pi_{g^n} \cdot \mathcal{M}_n \in Y_0$, so they are both expansions of $\mathcal{A}_0$. As $\pi_{g^n} \to \pi_{g_x}$, $\mathcal{M}^n \to \mathcal{M}_x$, $y^n \to y$ we have $(\pi_{g_x} \cdot \mathcal{M}_x, y) \in [T_\emptyset]$, so $\pi_{g_x} \cdot \mathcal{M}_x \in Y_0$, and $\pi_{g_x} \in \mathrm{Aut}(\mathcal{A}_0)$, so as Y_0 is invariant under $\mathrm{Aut}(\mathcal{A}_0)$, we have $\mathcal{M}_x \in Y_0$.

By (v), $x \mapsto \mathcal{M}_x$ is clearly continuous. Finally, we check that

$$(**) \qquad xE_0^{\mathbb{N}}y \ \Leftrightarrow\ \mathcal{M}_x \cong \mathcal{M}_y.$$

Indeed, assume $xE_0^{\mathbb{N}}y$. By (viii), there is an element $(h_0, h_1, \ldots) \in H^{\mathbb{N}}$ such that $g_{x|n+1,y|n+1} \to (h_0, h_1, \ldots)$. Clearly,

$$\pi_{g_{x|n+1,y|n+1}} \cdot A_{x|n+1} = A_{y|n+1},$$

so if $\mathcal{M}^n \in A_{x|n+1}$, we have

$$\pi_{g_{x|n+1,y|n+1}} \cdot \mathcal{M}^n = \mathcal{N}^n \in A_{y|n+1}.$$

Since $\mathcal{M}^n \to \mathcal{M}_x$, $\mathcal{N}^n \to \mathcal{M}_y$ it follows that $\pi_{(h_0,h_1,\ldots)} \cdot \mathcal{M}_x = \mathcal{M}_y$, so $\mathcal{M}_x \cong \mathcal{M}_y$.

Conversely, assume $\neg xE_0^{\mathbb{N}}y$. Fix m such that $x(\langle m,j\rangle) \neq y(\langle m,j\rangle)$ for infinitely many j. Assume, towards a contradiction, that $\mathcal{M}_x \cong \mathcal{M}_y$, and let $g = (h_0, h_1, \ldots) \in H^{\mathbb{N}}$ be such that $\pi_g \cdot \mathcal{M}_x = \mathcal{M}_y$. Let $n = \langle m,j\rangle$ be large enough so that $h_m, h_{k_m} \in \{\widetilde{h}_0, \ldots, \widetilde{h}_n\}$. Then by (ix), $\pi_g \cdot \mathcal{M}_{x|n+1} \cap \mathcal{M}_{y|n+1} = \emptyset$. Now by (v)$'$, exactly as in the argument that $\mathcal{M}_x \in Y_0$ (and using the fact that each A_s, $s \in 2^{n+1}$, is $\pi(V_{k_{L(n)}})$-invariant), we conclude that $\mathcal{M}_x \in \bigcap_n A_{x|n+1}$, $\mathcal{M}_y \in \bigcap_n A_{y|n+1}$, thus $\pi_g \cdot \mathcal{M}_x \neq \mathcal{M}_y$, a contradiction.

CONSTRUCTION

We start with $A_\emptyset = Y_0$. Let also $k_{-1} = 0, L(-1) = -1$. Assume now the construction of A_s has been done up to level n, i.e., for $s \in \bigcup_{k\le n} 2^n$ $(n \ge 0)$, k_i has been defined for $i \le L(n-1)$ and consider 2^{n+1}; put $n = \langle m,j\rangle$. We consider cases as $j = 0$ or $j > 0$.

(A) $j = 0$, i.e., $n = \langle m, 0\rangle$. Thus $L(n) = L(n-1)+1 = m$. We first choose $k_m = k_{L(n)} > k_{m-1}$ so that there is $\mathcal{M} \in A_{0^n}$ such that for $a \in (Q_{k_{m-1}})^{\mathcal{M}}$, $b \in (Q_{k_m})^{\mathcal{M}}$, no Σ_1^1 set isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$. (This can be done as $A_{0^n} \subseteq Y_0$. Note that if this is true for *some* $a \in (Q_{k_{m-1}})^{\mathcal{M}}$, $b \in (Q_{k_m})^{\mathcal{M}}$ then it is true for *all* such a, b.)

Note also that for such an $\mathcal{M}$ and any $a \in (Q_m)^{\mathcal{M}}$, there is $b \in (Q_{k_m})^{\mathcal{M}}$ with no Σ_1^1 set isolating $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$. This is because $m \le k_{m-1}$. Indeed, fixing such an $a \in (Q_m)^{\mathcal{M}}$, let $a' \in (Q_{k_{m-1}})^{\mathcal{M}}$ and a term t be such that $t^{\mathcal{M}}(a') = a$. Then let $b \in (Q_{k_m})^{\mathcal{M}}$ be such that no Σ_1^1 set isolates $\mathrm{Th}_F(\mathcal{M}, a', b)$ over $(\mathcal{M}, a')$ and let s be a term such that $s(b) = a'$. If, towards a contradiction, $A \in \Sigma_1^1$ isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$, then there is a formula $\psi \in L_{\omega_1^{ck}}$ isolating $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$ (by

Lemma 4). Then the formula $\psi(t(x), y) \wedge s(y) = x$ isolates $\mathrm{Th}_F(\mathcal{M}, a', b)$ over $(\mathcal{M}, a')$, a contradiction.

So

$$C = \{\mathcal{M} \in A_{0^n} : \text{for } a \in (Q_m)^{\mathcal{M}},\ b \in (Q_{k_m})^{\mathcal{M}} \\ \text{no } A \in \Sigma^1_1 \text{ isolates } \mathrm{Th}_F(\mathcal{M}, a, b) \text{ over } (\mathcal{M}, a)\}$$

is Σ^1_1 and non-empty. Since $C \subseteq A_{0^n}$ it follows that any $\mathcal{M} \in C$ satisfies condition (iii) as well.

We will next find a recursive $h \in H^{\mathbb{N}}$ with $h \in V_{k_{L(n)}}$, $\pi_h \in \mathrm{Aut}(\mathcal{A}_0)$ and a set $\widehat{A}_{0^{n+1}} \subseteq C$, $\widehat{A}_{0^{n+1}} \neq \emptyset$, $\widehat{A}_{0^{n+1}} \in \Sigma^1_1$, $\widehat{A}_{0^{n+1}}$ invariant under $\pi(V_{k_{L(n)}})$, such that $\pi_h \cdot \widehat{A}_{0^{n+1}} \subseteq A_{0^n}$, and such that if we let $g_{t^\wedge 0} = g_t,\ g_{t^\wedge 1} = g_t h$ for $t \in 2^n$ and $\widehat{A}_{t^\wedge 0} = \pi_{g_t} \cdot \widehat{A}_{0^{n+1}} = \pi_{g_{t^\wedge 0}} \cdot \widehat{A}_{0^{n+1}}$ and $\widehat{A}_{t^\wedge 1} = \pi_{g_t} \cdot (\pi_h \cdot \widehat{A}_{0^{n+1}}) = \pi_{g_{t^\wedge 1}} \cdot \widehat{A}_{0^{n+1}}$, then $\widehat{A}_s,\ s \in 2^{n+1}$, satisfy the negative requirements (ix).

Then it is clear that (vi) is satisfied and (ix) will be satisfied even if we shrink each $\widehat{A}_s$. It is also clear, as in the proof of Theorem 7.2, that since $h \in V_{k_{m-1}} \subseteq V_m = V_{k_{L(n)}}$ (as $k_{m-1} \geq m$), the positive requirements (viii) are satisfied. Notice that we also have (vii) for the $\widehat{A}_s$, and (iv) for $\widehat{A}_s$ (as $\pi(V_{k_{L(n)}}) \cdot \widehat{A}_s = \pi(V_{k_{L(n)}} g_s) \cdot \widehat{A}_{0^{n+1}} = \pi(g_s V_{k_{L(n)}}) \cdot \widehat{A}_{0^{n+1}} = g_s \cdot (\pi(V_{k_{L(n)}}) \cdot \widehat{A}_{0^{n+1}}) = g_s \cdot \widehat{A}_{0^{n+1}} = \widehat{A}_s$), and also (iii) for $\widehat{A}_{0^{n+1}}$ (and thus any subset of it), since $\widehat{A}_{0^{n+1}} \subseteq C$.

It remains to shrink $A_s,\ s \in 2^{n+1}$, to $\widehat{A}_s$ to achieve also (v), (v)$'$ and make sure that (iv), (vii) are preserved.

To do this, we fix $\mathcal{M}_s \in \widehat{A}_s$ so that $\pi_{g_s} \cdot \mathcal{M}_{0^{n+1}} = \mathcal{M}_s$. By applying (v) to $n-1$, for each $s \in 2^{n+1}$ we have $\mu_{s|n} \subseteq \mathcal{M}_s$, $y_{s|n}, p_{s|n} \in H^{k_{L(n-1)}+1}$, so that for some $g \in H^{\mathbb{N}}$, $g \supseteq p_{s|n}$, $y \supseteq y_s$, we get $(\pi_g \cdot \mathcal{M}_s, y) \in [T_\emptyset]$. Let then $\mu_s = \mathcal{M}_s|(k_{L(n)} + 1) \supseteq \mu_{s|n}$, $p_s = g|(k_{L(n)} + 1) \supseteq p_{s|n}$, $y_s = y|(k_{L(n)} + 1) \supseteq y_{s|n}$. Then let $\widehat{\widehat{A}}_s = \{x \in \widehat{A}_s : \mathcal{M}|(k_{L(n)} + 1) = \mu_s \ \&\ \exists g \in H^{\mathbb{N}},\ g \supseteq p_s\ \exists y \supseteq y_s\ ((\pi_g \cdot \mathcal{M}, y) \in [T_\emptyset])\}$. So $\widehat{\widehat{A}}_s \in \Sigma^1_1$ and $\mathcal{M}_s \in \widehat{\widehat{A}}_s \subseteq \widehat{A}_s$.

Put $A'_{0^{n+1}} = \bigcap_{s \in 2^{n+1}} \pi^{-1}_{g_s} \cdot \widehat{\widehat{A}}_s$ and $A'_s = \pi_{g_s} \cdot A'_{0^{n+1}}$. Then (v), (vii) are satisfied for A'_s.

We have dealt only with (v) for notational simplicity, but it is clear that fixing witnesses for each $\mathcal{M}_s$ with respect to all relevant A_{s_0}, T_{s_0} we can make sure that actually both (v), (v)$'$ are satisfied by $\widehat{\widehat{A}}_s$, and hence A'_s. So it only remains to modify A'_s to $A_s \subseteq \widehat{A}_s$ to satisfy (iv) without affecting (v), (v)$'$, (vii). But this is clear if we just take $A_s = \pi(V_{k_{L(n)}}) \cdot A'_s$ and notice that by arranging that (v), (v)$'$ remain unaffected, since $g \in V_{k_{L(n)}} \Rightarrow (\pi_g \cdot \mathcal{M})(|k_{L(n)} + 1) = \mathcal{M}|(k_{L(n)} + 1)$ and $hg|(k_{L(n)} + 1) = h|(k_{L(n)} + 1)$ for any $h \in H^{\mathbb{N}}$. But also

$$\begin{aligned}\pi_{g_s} \cdot A_{0^{n+1}} &= \pi_{g_s} \cdot (\pi(V_{k_{L(n)}}) \cdot A'_{0^{n+1}}) \\ &= \pi(g_s V_{k_{L(n)}}) \cdot A'_{0^{n+1}} = \pi(V_{k_{L(n)}} g_s) \cdot A'_{0^{n+1}} \\ &= \pi(V_{k_{L(n)}}) \cdot (\pi_{g_s} \cdot A'_{0^{n+1}}) = \pi(V_{k_{L(n)}}) \cdot A'_s = A_s,\end{aligned}$$

so (vii) remains true as well.

So it only remains to find h, $\widehat{A}_{0^{n+1}}$ satisfying the earlier specifications. The key claim is the following:

LEMMA 6. *Fix a finite set $S \subseteq H$. There are $\mathcal{M} \in C$, $b \in (Q_{k_m})^{\mathcal{M}}$, $h \in V_m$ such that for all $g_1, g_2 \in S$,*

$$\mathrm{Th}_F(\mathcal{M}, F_{g_1}(p_m), F_{g_2}(b)) \neq \mathrm{Th}_F(\mathcal{M}, p_m, p_{k_m}),$$

$\pi_h \cdot \mathcal{M} \in C$, and $\pi_h(b) = p_{k_m}$. (Recall that we have previously fixed $p_n \in (Q_n)^{\mathcal{A}_0}$.)

Let us assume this and proceed to complete the construction. Let

$$S_n = \{g_{t_1}(m)\widetilde{h}_i^{\pm 1} g_{s_1}(m)^{-1}, g_{t_1}(k_m)\widetilde{h}_i^{\pm 1} g_{s_1}(k_m)^{-1} : t_1, s_1 \in 2^n,\ i \leq n\}.$$

Let $\mathcal{M}$, b, h come from Lemma 6 for this S_n. For $g_1, g_2 \in S_n$ fix a formula $\psi_{g_1,g_2}(x,y) \in F$ with $\mathcal{M} \models \psi_{g_1,g_2}(p_m, p_{k_m})$ but $\mathcal{M} \models \neg\psi_{g_1,g_2}(F_{g_1}(p_m), F_{g_2}(b))$. Let

$$\psi(x,y) = \Big(\bigwedge_{g_1,g_2 \in S_n} \psi_{g_1,g_2}(x,y) \Big)$$

and $\theta(x,y) = \bigwedge_{g_1,g_2 \in S_n} \neg\psi(F_{g_1}(x), F_{g_2}(y))$. Then we have $\theta(x,y) \models \neg\psi(F_{g_1}(x), F_{g_2}(y))$ for all $g_1, g_2 \in S_n$ and $\mathcal{M} \models \psi(p_m, p_{k_m})$, $\mathcal{M} \models \theta(p_m, b)$. Now notice that if $\mathcal{M}$, b, h satisfy the lemma, so do $\mathcal{M}$, b, h' for any $h' \in hV_{k_m}$. Since hV_{k_m} is an open set in $\{g \in H^{\mathbb{N}} : \pi_g \in \mathrm{Aut}(\mathcal{A}_0)\}$, there is a recursive $h' \in hV_{k_m}$, so we can assume without loss of generality that h itself is recursive.

Now let

$$\widehat{A}_{0^{n+1}} = \{\mathcal{M} \in C : \pi_h \cdot \mathcal{M} \in C \ \&\ \mathcal{M} \models \psi(p_m, p_{k_m}) \wedge \theta(p_m, b)\},$$

and define the corresponding $\widehat{A}_s, g_s$ for $s \in 2^{n+1}$, as described earlier. All the other required properties are true, so it is enough to verify that they satisfy the negative requirements (ix).

Assume not, towards a contradiction, and fix $s_1, t_1 \in 2^n$, $\mathcal{M}_0 \in \widehat{A}_{s_1{}^\frown 0}$, $\pi_g \in \mathrm{Aut}(\mathcal{A}_0)$, with $g(m), g(k_m) \in \{\widetilde{h}_0^{\pm 1}, \dots, \widetilde{h}_n^{\pm 1}\}$, and $\pi_g \cdot \mathcal{M}_0 \in \widehat{A}_{t_1{}^\frown 1}$. Then there is $\mathcal{M} \in \widehat{A}_{0^{n+1}}$ such that $\pi_{g_{t_1}^{-1} g g_{s_1}} \cdot \mathcal{M} \in \pi_h \cdot \widehat{A}_0$, so there is $\mathcal{M}' \in \widehat{A}_{0^{n+1}}$ such that $\mathcal{N} = \pi_{g_{t_1}^{-1} g g_{s_1}} \cdot \mathcal{M} = \pi_h \cdot \mathcal{M}'$. Let $g_{t_1}^{-1} g g_{s_1}(m) = g_1^{-1}$, $g_{t_1}^{-1} g g_{s_1}(k_m) = g_2^{-1}$, so that $g_1, g_2 \in S_n$, $\pi_{g_{t_1}^{-1} g g_{s_1}}(p_m) = F_{g_1}(p_m)$,

$\pi_{g_{t_1}^{-1} g g_{s_1}}(p_{k_m}) = F_{g_2}(p_{k_m})$. Then, as $\mathcal{M} \models \psi(p_m, p_{k_m})$, we have $\mathcal{N} \models \psi(F_{g_1}(p_m), F_{g_2}(p_{k_m}))$. But also $\mathcal{M}' \models \theta(p_m, b)$ and $\pi_h(p_m) = p_m$, $\pi_h(b) = p_{k_m}$, so $\mathcal{N} \models \theta(p_m, p_{k_m})$, contradicting $\theta(x,y) \models \neg\psi(F_{g_1}(x), F_{g_2}(y))$.

So it remains to give the

Proof of Lemma 6. Assume it fails, towards a contradiction. Then for any given $\mathcal{M} \in C$, $b \in (Q_{k_m})^{\mathcal{M}}$, $h \in V_m$ with $\pi_h \cdot \mathcal{M} \in C$, $\pi_h(b) = p_{k_m}$, we have

$$\mathrm{Th}_F(\mathcal{M}, F_{g_1}(p_m), F_{g_2}(b)) = \mathrm{Th}_F(\mathcal{M}, p_m, p_{k_m})$$

for some $g_1, g_2 \in S$, and thus

$$\mathrm{Th}_F(\mathcal{M}, p_m, b) = \mathrm{Th}_F(\mathcal{M}, F_{g_1^{-1}}(p_m), F_{g_2^{-1}}(p_{k_m})),$$

so for any fixed $\mathcal{M} \in C$, if $B_{\mathcal{M}} = \{b \in (Q_{k_m})^{\mathcal{M}} : \exists h \in V_m\ (\pi_h \cdot \mathcal{M} \in C,\ \pi_h(b) = p_{k_m})\}$ then $\{\mathrm{Th}_F(\mathcal{M}, p_m, b) : b \in B_{\mathcal{M}}\}$ is finite. Enumerating $F = \{\varphi_0, \varphi_1, \ldots\}$, we see then that for every $\mathcal{M} \in C$, there must be some $N \in \mathbb{N}$ such that for any $b \in B_{\mathcal{M}}$,

$$\forall i \le N\ (\varphi_i \in \mathrm{Th}_F(\mathcal{M}, p_m, b) \Leftrightarrow \varphi_i \in \mathrm{Th}_F(\mathcal{M}, p_m, p_{k_m}))$$
$$\Rightarrow \mathrm{Th}_F(\mathcal{M}, p_m, b) = \mathrm{Th}_F(\mathcal{M}, p_m, p_{k_m}).$$

For each $b \in B_{\mathcal{M}}$, we let

$$\psi_b = \bigwedge_{i \le N} \{\varphi_i : \mathcal{M} \models \varphi_i(p_m, b)\} \wedge \bigwedge_{i \le N} \{\neg\varphi_i : \mathcal{M} \models \neg\varphi_i(p_m, b)\},$$

so that $\mathcal{M} \models \psi_b(p_m, b)$. If then $\{\psi_0, \ldots, \psi_k\} = \{\psi_b : b \in B_{\mathcal{M}}\}$, we see that if $b \in B_{\mathcal{M}}$ and $h \in V_m$ with $\pi_h \cdot \mathcal{M} \in C$, $\pi_h(b) = p_{k_m}$, then

$$\pi_h \cdot \mathcal{M} \models \psi_0(p_m, p_{k_m}) \vee \ldots \vee \psi_k(p_m, p_{k_m}),$$

together with

$$\pi_h \cdot \mathcal{M} \models \psi_i(p_m, p_{k_m}), \qquad \mathcal{M} \models \psi_i(p_m, p_{k_m})$$

for some $i \le k$, implies that

$$\mathrm{Th}_F(\mathcal{M}, p_m, b) = \mathrm{Th}_F(\mathcal{M}, p_m, p_{k_m}).$$

Thus we see that

$$\begin{aligned}
&\forall \mathcal{M} \in C\ \exists \{\psi_0, \ldots, \psi_k\} \subseteq F\ \Big[\forall b \in B_{\mathcal{M}}\ \forall h \in V_m \\
&\qquad \Big(\pi_h \cdot \mathcal{M} \in C\ \&\ \pi_h(b) = p_{k_m} \Rightarrow \pi_h \cdot \mathcal{M} \models \bigvee_{i \le k} \psi_i(p_m, p_{k_m})\Big) \\
&\qquad \wedge\ \forall i \le k\ \forall b \in B_{\mathcal{M}}\ \forall h \in V_m\ [(\pi_h \cdot \mathcal{M} \in C\ \&\ \pi_h(b) = p_{k_m} \\
&\qquad \&\ \pi_h \cdot \mathcal{M} \models \psi_i(p_m, p_{k_m})\ \&\ \mathcal{M} \models \psi_i(p_m, p_{k_m})) \\
&\qquad \Rightarrow \mathrm{Th}_F(\mathcal{M}, p_m, b) = \mathrm{Th}_F(\mathcal{M}, p_m, p_{k_m}))]\Big],
\end{aligned}$$

so by Π^1_1-uniformization on $\mathbb{N}$ we can find $C_0 \subseteq C$, C_0 non-empty Σ^1_1 on which these $\{\psi_0, \ldots, \psi_k\}$ are constant, say $\{\overline{\psi}_0, \ldots, \overline{\psi}_{k_0}\}$. Taking $b = p_{k_m}$, $h = 1$, we see that for any $\mathcal{M} \in C_0$ there is some $i \leq k_0$ with $\mathcal{M} \models \overline{\psi}_i(p_m, p_{k_m})$, so fix $i_0 \leq k_0$ such that

$$C_1 = \{\mathcal{M} \in C_0 : \mathcal{M} \models \overline{\psi}_{i_0}(p_m, p_{k_m})\}$$

is non-empty and clearly Σ^1_1. If $\mathcal{M} \in C_1$ then we claim that C_1 isolates $\mathrm{Th}_F(\mathcal{M}, p_m, p_{k_m})$ over $(\mathcal{M}, p_m)$, violating the fact that $\mathcal{M} \in C$. Indeed, first $(\mathcal{M}, p_m, p_{k_m}) \models C_1$ as $\mathcal{M} \in C_1$. Now fix $\mathcal{M}_0 \in \mathrm{Mod}(\sigma)$, $g \in S_\infty$ with $g(p_m) = p_m$, and $(\mathcal{M}_0, p_m, p_{k_m}) \models C_1$, $(g \cdot \mathcal{M}_0, p_m, p_{k_m}) \models C_1$, in order to show that $\mathrm{Th}_F(\mathcal{M}_0, p_m, p_{k_m}) = \mathrm{Th}_F(g \cdot \mathcal{M}_0, p_m, p_{k_m})$. Then $h_1 \cdot \mathcal{M}_0 \in C_1$, $h_2 g \cdot \mathcal{M}_0 = h_2 g h_1^{-1} \cdot (h_1 \cdot \mathcal{M}_0) \in C_1$ for some $h_1, h_2 \in S_\infty$ that fix p_m, p_{k_m}, so it is enough to show that if $\mathcal{M} \in C_1$, $h' \cdot \mathcal{M} \in C_1$, h' fixes p_m then $\mathrm{Th}_F(\mathcal{M}, p_m, p_{k_m}) = \mathrm{Th}_F(h' \cdot \mathcal{M}, p_m, p_{k_m})$. As $\mathcal{M}, h' \cdot \mathcal{M}$ are expansions of $\mathcal{A}_0$, we see that $h' \in \mathrm{Aut}(\mathcal{A}_0)$, so $h' = \pi_h$ for some $h \in H^{\mathbb{N}}$. As h' fixes p_m and every $a \in (Q_i)^{\mathcal{A}_0}$ for $i \leq m$ is definable from p_m, h' fixes all such $(Q_i)^{\mathcal{A}_0}$, so $h \in V_m$. Let $b = (\pi_h)^{-1}(p_{k_m})$. Then we have $\pi_h \cdot \mathcal{M} \models \overline{\psi}_{i_0}(p_m, p_{k_m})$ as $\pi_h \cdot \mathcal{M} \in C_1$, and $\mathcal{M} \models \overline{\psi}_{i_0}(p_m, p_{k_m})$ as $\mathcal{M} \in C_1$. So $\mathrm{Th}_F(\pi_h \cdot \mathcal{M}, p_m, p_{k_m}) = \mathrm{Th}_F(\mathcal{M}, p_m, b) = \mathrm{Th}_F(\mathcal{M}, p_m, p_{k_m})$. ■

(B) $n = \langle m, j\rangle$ with $j > 0$. Then $L(n) = L(n-1)$. Thus k_r has already been defined for all $r \leq L(n)$ and by (iii) for A_{0^n} we have

$$\mathcal{M} \in A_{0^n} \;\Rightarrow\; \forall r \leq L(n)\ \forall a \in (Q_r)^{\mathcal{M}}\ \exists b \in (Q_{k_r})^{\mathcal{M}}$$
$$(\text{no } A \in \Sigma^1_1 \text{ isolates } \mathrm{Th}_F(\mathcal{M}, a, b) \text{ over } (\mathcal{M}, a)).$$

So the proof in this case proceeds by defining C exactly as before, i.e.,

$$C = \{\mathcal{M} \in A_{0^n} : \text{for } a \in (Q_m)^{\mathcal{M}},\ b \in (Q_{k_m})^{\mathcal{M}}$$
$$\text{no } A \in \Sigma^1_1 \text{ isolates } \mathrm{Th}_F(\mathcal{M}, a, b) \text{ over } (\mathcal{M}, a)\}$$

and then literally repeating the rest of the proof of Case (A). ■

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Department of Mathematics
UCLA
Los Angeles, CA 90095-1555, U.S.A.
E-mail: greg@math.ucla.edu

Department of Mathematics
Caltech
Pasadena, CA 91125, U.S.A.
E-mail: kechris@caltech.edu

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