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A Ramsey-style extension of a theorem of Erdős and Hajnal

by

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Abstract. If n, t are natural numbers, μ is an infinite cardinal, G is an n-chromatic graph of cardinality at most μ, then there is a graph X with $X \to (G)^1_\mu$, $|X| = \mu^+$, such that every subgraph of X of cardinality $< t$ is n-colorable.

The Erdős-Hajnal result we would like to generalize states that there are arbitrarily large chromatic graphs omitting short odd circuits ([1]). We reformulate this as follows. If μ is a cardinal, t is a natural number, then there exists a graph X with $X \to (K_2)^1_\mu$ such that every subgraph of X on at most t points is bipartite. Here K_2 is the one-edge graph and $X \to (Y)^1_\mu$ denotes that every vertex coloring of the graph X with μ colors admits a monocolored (but not necessarily induced) copy of Y. If we take into consideration another result of Erdős and Hajnal (also in [1]), namely, that if $X \to (K_2)^1_\omega$ then X contains every finite bipartite graph, we get the somewhat stronger statement that if $K_{t,t}$ is the complete bipartite graph with bipartition classes of size t, then there exists, for every cardinal μ and every finite t, a graph X with $X \to (K_{t,t})^1_\mu$ such that all subgraphs of X on less than t vertices are bipartite. This is the result we want to generalize; for any natural number n, we take an arbitrary n-chromatic graph in place of $K_{t,t}$ and we require that all subgraphs on less than t vertices should be n-chromatic. For the proof, we somewhat modify the original Erdős–Hajnal construction (no surprise), we use another Specker-style graph.

Notation. We use the standard set theory notation. If S is a set and μ a cardinal, then $[S]^\mu = \{X \subseteq S : |X| = \mu\}$. We write $\mathbf{x} = \{x_1, \ldots, x_n\}_<$ to denote that we enumerate the elements of $\mathbf{x}$ in increasing order.

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Theorem. *If n, t are natural numbers, μ is an infinite cardinal, and G is an n-chromatic graph of cardinality at most μ, then there is a graph X with $X \to (G)^1_\mu$, $|X| = \mu^+$, such that every subgraph of X of cardinality $< t$ is n-colorable.*

Proof. Set $a = 2t$, $b = 2nt + 1$. The ground set of the graph X will be $[\mu^+]^{ab}$, the set of ab-element subsets of μ^+. We join two elements $\mathbf{x} = \{x_1, \dots, x_{ab}\}_<$ and $\mathbf{y} = \{y_1, \dots, y_{ab}\}_<$ if there is some $1 \le k < n$ such that

$$x_{ka} < x_{ka+1} < y_1 < x_{ka+2} < y_2 < \ldots < x_{ab} < y_{(b-k)a}$$

or the other way round.

Claim 1. *If $\mathbf{x} = \{x_1, \dots, x_{ab}\}_<$ and $\mathbf{y} = \{y_1, \dots, y_{ab}\}_<$ are joined and the interval $[x_\alpha, x_{\alpha+1}]$ of $\mathbf{x}$ intersects the interval $[y_\beta, y_{\beta+1}]$ of $\mathbf{y}$ then either $\alpha = \beta + ka$, or $\alpha = \beta + ka + 1$, or the other way round.*

Proof. Straightforward. ∎

Claim 2. *Every subgraph of X with less than t vertices can be n-colored.*

Proof. Let H be such a graph. We can assume that H is connected (as otherwise we can split H into connected components). Then the vertices of H can be enumerated as $\{\mathbf{v}_0, \dots, \mathbf{v}_s\}$ (for some s) so that every $\mathbf{v}_i$ with $i > 0$ is joined to some $\mathbf{v}_j$ with $0 \le j < i$. Set $\mathbf{v}_i = \{v^i_1, \dots, v^i_{ab}\}_<$ and define $\xi = v^0_{2nt^2+t}$. For every $0 \le i \le s$ let $c(i)$ be the largest index for which $v^i_{c(i)} \le \xi$. Clearly, $c(0) = 2nt^2 + t$. We claim that there exist numbers $1 \le d_i \le 2nt - 1$ (for $0 \le i \le s$) such that

$$ad_i + t - i \le c(i) \le ad_i + t + i.$$

In fact, we show by induction on i that there exists such a d_i with $|d_i - nt| \le in$. For $i = 0$, $d_0 = nt$ does the job. By induction, if we have the statement up to and including i, then using Claim 1, we find that for some $0 \le j \le i$, $c(i+1)$ assumes the value $c(j) + ka$, or $c(j) + ka + 1$, or $c(j) - ka$, or $c(j) - ka - 1$ for some $1 \le k < n$. Then we can choose $d_{i+1} = d_j \pm k$.

We now show that $f(\mathbf{v}_i) = d_i \pmod n$ is a good coloring of H. Assume that $f(\mathbf{v}_i) = f(\mathbf{v}_j)$ and they are joined. Again, by Claim 1, for some $1 \le k < n$, $c(i) = c(j) + ka$, or $c(i) = c(j) + ka + 1$ (or vice versa). From this, we get $|a(d_i - d_j - k)| \le 2t - 1$, so $d_i = d_j + k$, and so they do not get the same color. ∎

Claim 3. $X \to (G)^1_\mu$.

Proof. Assume that we are given a coloring $F : [\mu^+]^{ab} \to \mu$ of the ground set. We are going to show that for some color $\tau < \mu$ the complete n-partite graph $K_{\mu,\dots,\mu}$, with color classes of cardinality μ, is embeddable into the subgraph of those vertices which get color τ.

We define the formulas φ_0^τ, $\varphi_1^\tau(x_1)$, $\ldots$, $\varphi_i^\tau(x_1,\ldots,x_i)$ for all $i \le ab$ as follows. $\varphi_{ab}^\tau(x_1,\ldots,x_{ab})$ denotes that $x_1 < \ldots < x_{ab}$ and $F(x_1,\ldots,x_{ab}) = \tau$. If φ_{i+1}^τ is defined, we let $\varphi_i^\tau(x_1,\ldots,x_i)$ stand for

$$\exists^* y\ \varphi_{i+1}^\tau(x_1,\ldots,x_i,y)$$

where the quantifier $\exists^* y$ reads "there exist μ^+ elements y such that …".

SUBCLAIM. *There is a $\tau < \mu$ such that φ_0^τ holds.*

Proof. Assume, for a contradiction, that φ_0^τ is false for every $\tau < \mu$. Spelled out, this means that, for every $\tau < \mu$ there is a bound α_τ such that $\varphi_1^\tau(x_1)$ is false for $x > \alpha_\tau$. There is a common bound for all these bounds, so there is an x_1 for which every $\varphi_1^\tau(x_1)$ is false. Fix such an x_1 and proceed. We can find an $x_2 > x_1$ such that $\varphi_2^\tau(x_1,x_2)$ is false for every $\tau < \mu$. Fix such an x_2, and continue. Eventually we get a sequence $\{x_1,\ldots,x_{ab}\}_<$ such that $F(x_1,\ldots,x_{ab}) = \tau$ is false for every $\tau < \mu$, which is clearly impossible. ∎

For the rest of the proof fix $\tau < \mu$ as in the Subclaim. Using induction we can find an increasing sequence of ordinals $0 = \delta_0 < \delta_1 < \ldots < \delta_r$ $(r < \omega)$ such that if $i < ab$, $r < \omega$, $x_1 < \ldots < x_i < \delta_r$, and $\varphi_i^\tau(x_1,\ldots,x_i)$ holds then there are μ ordinals $\delta_r < y < \delta_{r+1}$ such that $\varphi_{i+1}^\tau(x_1,\ldots,x_i,y)$ is true. That is, there are μ elements $x_1 < \delta_1$ for which $\varphi_1^\tau(x_1)$ holds, for each such x_1 there are μ elements $\delta_1 < x_2 < \delta_1$ for which $\varphi_2^\tau(x_1,x_2)$ holds, and so forth. (In fact, we could easily continue for μ^+ steps, but we will not need it.)

For $1 \le i \le n$, $1 \le j \le b$, $1 \le m \le a$ we let

$$A_{jm}^i = [\delta_r, \delta_{r+1})$$

where

$$r = (i+j)(n+1)(a+1) + m(n+1) + i,$$

i.e., the pairwise disjoint sets A_{jm}^i are ordered by $(i+j)$ first, then according to the value of m, and finally by i. In this order, we select from each of these sets μ elements: $x_\alpha^{ijm} \in A_{jm}^i$ $(\alpha < \mu)$ in such a way that

$$\varphi_{(j-1)a+m}^\tau(x_\alpha^{i11}, x_\alpha^{i12}, \ldots, x_\alpha^{i1a}, x_\alpha^{i21}, \ldots, x_\alpha^{i2a}, \ldots, x_\alpha^{ij1} \ldots, x_\alpha^{ijm})$$

holds. This can be done by the selection of the δ's and the way the A_{jk}^i's were ordered. Finally, put $\mathbf{x}_\alpha^i = \{x_\alpha^{ijm} : 1 \le j \le b,\ 1 \le m \le a\}$.

Now consider $1 \le i_0 < i_1 \le n$, say $i_1 = i_0 + k$. The relative order of the sets in the families $\{A_{jm}^{i_0} : 1 \le j \le b,\ 1 \le m \le a\}$ and $\{A_{jm}^{i_1} : 1 \le j \le b,\ 1 \le m \le a\}$ will be the following (we replace superscript i_0 by a prime and superscript i_1 by two primes):

$$A'_{11} < A'_{12} < \ldots < A'_{1a} < A'_{21} < \ldots < A'_{k+1,1} < A''_{11} < A'_{k+1,2} < A''_{12} < \ldots,$$

i.e., they are in one of the interlacing positions we defined connectivity in X. From this we conclude that if $1 \le i_0 < i_1 \le n$, α, $\beta < \mu$, then $\mathbf{x}_\alpha^{i_0}$ and $\mathbf{x}_\beta^{i_1}$ are joined, so the vertices $\{\mathbf{x}_\alpha^i : 1 \le i \le n,\ \alpha < \mu\}$ give a $K_{\mu,\ldots,\mu}$ in color τ. ■

References

[1] P. Erdős and A. Hajnal, *On chromatic number of graphs and set systems*, Acta Math. Acad. Sci. Hungar. 17 (1966), 61–99.

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