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Categoricity of theories in $L_{\kappa^*,\omega}$, when κ^* is a measurable cardinal. Part 2

by

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Abstract. We continue the work of [2] and prove that for λ successor, a λ-categorical theory $\mathbf{T}$ in $L_{\kappa^*,\omega}$ is μ-categorical for every $\mu \le \lambda$ which is above the $(2^{\mathrm{LS}(\mathbf{T})})^+$-beth cardinal.

0. Introduction. We deal here with the categoricity spectrum of theories $\mathbf{T}$ in the logic $L_{\kappa^*,\omega}$ with κ^* measurable and more generally, continue the attempts to develop a classification theory of non-elementary classes, in particular non-forking. Makkai and Shelah [3] dealt with the case of κ^* a compact cardinal. So κ^* measurable is too high compared with the hope of dealing with $\mathbf{T} \subseteq L_{\omega_1,\omega}$ (or any $L_{\kappa,\omega}$) but seems quite small compared to the compact cardinal in [3]. Model-theoretically a compact cardinal ensures many cases of amalgamation, whereas a measurable cardinal ensures no maximal model. We continue [13], Makkai and Shelah [3], Kolman and Shelah [2]; try to imitate [3]; a parallel line of research is [16]. Earlier works are [8], [10], [11]; for later works on the upward Łoś conjecture, look at [5] and [4].

On the situation generally see more in [5].

This paper continues the tasks begun in Kolman and Shelah [2]. We use the results obtained therein to advance our knowledge of the categoricity spectrum of theories in $L_{\kappa^*,\omega}$ when κ^* is a measurable cardinal.

The main theorems are proved in Section 3; Section 1 treats types and Section 2 describes some constructions.

Note that we may expect to be able to develop a better, more informative classification theory, in particular stability theory, for $\mathbf{T} \subseteq L_{\kappa^*,\omega}$

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with κ^* measurable than without the measurablility assumption, and less informative than in the case of κ^* compact.

The notation follows [2], except in two important details: we reserve κ^* for a fixed measurable cardinal and $\mathbf{T}$ for a fixed λ-categorical theory in $L_{\kappa^*,\omega}$ in a given vocabulary L; κ is any infinite cardinal and T is usually some kind of tree. To recap briefly: $\mathbf{T}$ is a λ-categorical theory in $L_{\kappa^*,\omega}$, $\mathrm{LS}(\mathbf{T}) := \kappa^* + |\mathbf{T}|$, $\mathcal{K} = \langle K, \preceq_{\mathcal{F}}\rangle$ is the class of models of $\mathbf{T}$, where $\mathcal{F}$ is a fragment of $L_{\kappa^*,\omega}$ satisfying $\mathbf{T} \subseteq \mathcal{F}$ and $|\mathcal{F}| \leq \kappa^* + |\mathbf{T}|$, and for $M, N \in K$, $M \preceq_{\mathcal{F}} N$ means that M is an $\mathcal{F}$-elementary submodel of N. We take the minimal such $\mathcal{F}$ so $\mathbf{T}$ determines $\mathcal{F}$.

The principal relevant results from [2] are: $\mathcal{K}_{<\lambda}$ has the amalgamation property (5.5 there), and every member of $K_{<\lambda}$ is nice (5.4 there). But this assumption ($\mathbf{T}$ categorical in λ) or its consequences mentioned above will be mentioned in theorems when used.

Let $(M_1, M_0) \preceq_{\mathcal{F}} (M_3, M_2)$ mean $M_1 \preceq_{\mathcal{F}} M_3$ and $M_0 \preceq_{\mathcal{F}} M_2$.

(I_1, I_2) is a *Dedekind cut* of a linear order I if

$$I = I_1 \cup I_2, \qquad I_1 \cap I_2 = \emptyset, \qquad (\forall x \in I_1)(\forall y \in I_2)(x < y).$$

The *two-sided cofinality* of the Dedekind cut (I_1, I_2) of I, $\mathrm{cf}(I_1, I_2)$, is $(\mathrm{cf}(I_1), \mathrm{cf}(I_2^*))$, where I_2^* is the order I_2 inverted. The two-sided cofinality of I, $\mathrm{cf}(I, I) = \mathrm{dcf}(I)$, is $(\mathrm{cf}(I^*), \mathrm{cf}(I))$.

Writing proofs we also consider their possible role in the hopeful classification theory. But we have always been trying to be careful in stating the assumptions.

Note that [2] improves some of the results of [3]; but they do not fully recapture the results on the compact case to the measurable case. E.g. there categoricity in successor λ implies that categoricity starts at the relevant Hanf number of omitting types so in general we deduce categoricity in larger cardinals. For a good understanding of this work, the reader is expected to know well [2]. Now it will be helpful for the reader to beware of some "black boxes" [6] (or [13] for less good source) and to have some knowledge of [5] or [3] but usually proofs are repeated.

We thank Oren Kolman for writing and ordering notes from lectures on the subject from Spring 1990 on which the paper is based (you can see his style in the parts with good language) and Andres Villaveces for corrections.

1. Knowing the right types. The classical notion of type relates to the satisfaction of sets of formulas in a model. We shall define a post-classical type (following [13], [7] which was followed by Makkai and Shelah [3], or see [5, §0], but here niceness is involved) and use this to define notions of freeness and non-forking appropriate in the context of a λ-categorical theory in $L_{\kappa^*,\omega}$.

The definitions try to locate a notion which under the circumstances behaves as in [14] and, if you accept some inevitable limitations, succeed.

CONTEXT 1.1. $\mathbf{T} \subseteq L_{\kappa^*,\omega}$ in the vocabulary L, $K = \{M : M$ a model of $\mathbf{T}\}$, $\preceq_{\mathcal{F}}$ as in the introduction.

$K_\mu = \{M \in K : \|M\| = \mu\}$, $K_{<\kappa} = \bigcup_{\mu<\kappa} K_\mu$, $\mathcal{K} = (K, \preceq_{\mathcal{F}})$, and we stipulate $K_{<\kappa^*} = \emptyset$, hence, e.g., $K_{<\kappa} = \bigcup\{K_\mu : \mu < \kappa$ but $\mu \geq \kappa^*\}$. (Why? Models of cardinality $< \kappa^*$ are the parallel of finite ones for first order logic: such models may have no $\prec_{L_{\kappa^*,\omega}}$ proper extensions, and using our main tool, ultrapowers, we can say little on them. So instead of excluding them many times, we ignore them always.) We let $\mathrm{LS}(\mathcal{K}) = |\mathcal{F}| + \kappa^*$.

We assume that if $A \subseteq N \in K$, $\|N\| \geq \lambda$, and $\mu = |A| \in [\kappa^* + |\mathbf{T}|, \lambda)$, then for some nice $N \in K_\mu$, $A \subseteq M \preceq_{\mathcal{F}} N$. This is reasonable as by [2, 5.4, p. 238] every $M \in K_{<\lambda}$ is nice. The reader may simplify assuming every $M \in K_{<\lambda}$ is nice.

Remember "$M \in K$ is nice" is defined in [2], Definitions 3.2, 1.8; nice implies being an amalgamation base in $K_{<\lambda}$ (see [2, 3.5]). Here for simplicity we mean "amalgamation" to include the JEP (the joint embedding property).

DEFINITION 1.2. Suppose that $M \in K_{<\lambda}$ is a nice model of $\mathbf{T}$. Define a binary relation, $E_M = E_M^{<\lambda}$, as follows: $(\overline{a}_1, N_1)E_M(\overline{a}_2, N_2)$ if and only if for $l = 1, 2$, $N_l \in K_{<\lambda}$ is nice and $M \preceq_{\mathcal{F}} N_l$, $\overline{a}_l \in N_l$ (i.e., $\overline{a}_l$ a finite sequence of members of N_l), and there exist a model N and embeddings h_l such that

$$M \preceq_{\mathcal{F}} N, \quad h_l : N_l \underset{\mathcal{F}}{\rightarrow} N, \quad \mathrm{id}_M = h_1 \restriction M = h_2 \restriction M,$$

and $h_1(\overline{a}_1) = h_2(\overline{a}_2)$.

REMARK. This definition, in fact a generalization for amalgamation bases and more general, are important in [13], [5], [4], but here we restrict ourselves to nice models.

FACT 1.3. (1) *E_M is an equivalence relation.*

(2) *Let $M \in K_{<\lambda}$ be nice, $M \preceq_{\mathcal{F}} N$, $\overline{a} \in N$, and for $l = 1, 2$, $M \cup \overline{a} \subseteq N_l \preceq_{\mathcal{F}} N$, N_l nice and $\|N_l\| < \lambda$. Then $(\overline{a}, N_1)E_M(\overline{a}, N_2)$.*

(3) *E_M is preserved by isomorphism.*

Proof. (1) Let us look at transitivity. Suppose $(\overline{a}_l, N_l)E_M(\overline{a}_{l+1}, N_{l+1})$, $l = 1, 2$. Now M, being nice, is an amalgamation base in $K_{<\lambda}$, thus there are models N^l and embeddings h_0^l, h_1^l of N_l, N_{l+1} over M into N^l, with $h_0^l(\overline{a}_l) = h_1^l(\overline{a}_{l+1})$, $l = 1, 2$. Without loss of generality, $N^l \in K_{<\lambda}$ (by the Downward Loewenheim–Skolem Theorem). By assumption N_2 is nice, hence by [2, 3.5] it is an amalgamation base for $\mathcal{K}_{<\lambda}$, i.e., there is an amalgam $N^* \in K_{<\lambda}$ and embeddings $g_l : N^l \xrightarrow{\mathcal{F}} N^*$, amalgamating N^1, N^2 over N^2 with respect to h_1^1, h_0^2. In other words, the following diagram commutes:

$$\begin{tikzcd}
 & & N^* & & \\
 & N^1 \arrow[ur, "g_1"] & & N^2 \arrow[ul, "g_2"'] & \\
\overline{a}_1 \in N_1 \arrow[ur, "h^1_0"] & & \overline{a}_2 \in N_2 \arrow[ul, "h^1_1"'] \arrow[ur, "h^2_0"] & & \overline{a}_3 \in N_3 \arrow[ul, "h^2_1"'] \\
 & & M \arrow[ull, "\mathrm{id}"] \arrow[u, "\mathrm{id}"] \arrow[urr, "\mathrm{id}"'] & &
\end{tikzcd}$$

Now just notice that N^*, $g_1 h^1_0$, $g_2 h^2_1$ witness that $(\overline{a}_1, N_1) E_M (\overline{a}_3, N_3)$, since

$$g_1 h^1_0(\overline{a}_1) = g_1(h^1_1(\overline{a}_2)) = g_2 h^2_0(\overline{a}_2) = g_2 h^2_1(\overline{a}_3).$$

(2), (3) Left to the reader. ∎

Definition 1.4. Suppose that $M, N \in K_{<\lambda}$ are nice, $a \in N$ and $M \preceq_{\mathcal{F}} N$.

(1) $p := \mathrm{tp}(a, M, N)$, the *type* of a over M in N, is the E_M-equivalence class of (a, N),

$$(a, N)/E_M = \{(b, N^1) : (a, N) E_M (b, N^1)\}.$$

We also say "$a \in N$ realizes p". If $\|N\| \geq \lambda$ define $\mathrm{tp}(\overline{a}, M, N)$ by 1.3(2) (using the hypothesis).

(2) If $M' \preceq_{\mathcal{F}} M \in K_{<\lambda}$ and $p \in S(M)$ (see below) is $(a^-, N)/E_M$, then $p \restriction M' = (a^-, N)/E_{M'}$; clearly the representation (a, N) does not matter.

(3) If $\mathrm{LS}(\mathbf{T}) < \kappa \leq \mu \leq \lambda$, we call $M \in K_\mu$ *κ-saturated* if for every nice $N \preceq_{\mathcal{F}} M$ with $\|N\| < \kappa$ and $p \in S(N)$, some $\overline{a} \in M$ realizes p (in M so necessarily M is nice) or at least for some nice N' with $N \preceq_{\mathcal{F}} N' \preceq_{\mathcal{F}} M$, some $a' \in N'$ realizes p in N'.

(4) $S^m(N) = \{p : p = \mathrm{tp}(\overline{a}, N, N_1)$ for any nice N_1 and $\overline{a}$ satisfying $N \preceq_{\mathcal{F}} N_1$ and $\|N_1\| \leq \|N\| + \mathrm{LS}(\mathcal{K})$ and $\overline{a} \in {}^m(N_1)\}$ and

$$S(N) = S^{<\omega}(N) = \bigcup_{m<\omega} S^m(N).$$

(5) $\mathbf{T}$ is *μ-stable* if $N \in K_{\leq\mu} \Rightarrow |S(N)| \leq \mu$.

(6) We say N is *μ-universal* over M when $M \preceq_{\mathcal{F}} N$, $N \in K_\mu$ and if $M \preceq_{\mathcal{F}} N' \in K_{\leq\mu}$ then there is a $\preceq_{\mathcal{F}}$-embedding of N' into N over M.

(7) We say N is (μ,κ)-*saturated* over M if there is a $\preceq_{\mathcal{F}}$-increasing continuous sequence $\langle M_i : i < \kappa\rangle$ such that $M_0 = M$, $N = \bigcup_{i<\kappa} M_i$, $M_i \in K_\mu$ and M_{i+1} is μ-universal over M_i. We say N is *saturated* over M if for some $\mu \in [\mathrm{LS}(\mathbf{T}), \lambda)$ and some $\kappa \leq \mu$, N is (μ,κ)-saturated over M. So (μ,κ)-saturated over M implies universal over M.

(8) We say $\mathcal{K}$ (or $\mathbf{T}$) is *stable* in μ if for every $M \in K_\mu$, M is nice and $|S(M)| \leq \mu$.

DEFINITION 1.5. We shall write

$$M_1 \mathop{\biguplus}_{M_0}^{M_3} M_2$$

to mean

$$M_0 \preceq_{\mathcal{F}} M_1 \preceq_{\mathcal{F}} M_3, \quad M_0 \preceq_{\mathcal{F}} M_2 \preceq_{\mathcal{F}} M_3$$

and there exist a suitable operation (I, D, G) and an embedding

$$h : M_3 \xrightarrow{\mathcal{F}} \mathrm{Op}(M_1, I, D, G)$$

such that $h{\restriction}M_1 = \mathrm{id}_{M_1}$ and $\mathrm{Rang}(h{\restriction}M_2) \subseteq \mathrm{Op}(M_0, I, D, G)$ (remember that $\mathrm{Op}(M, I, D, G)$ is the limit ultrapower of M with respect to (I, D, G); see [2, 1.7.4]). We say that M_1, M_2 *do not fork* in M_3 over M_0 if

$$M_1 \mathop{\biguplus}_{M_0}^{M_3} M_2.$$

If

$$M_1 \mathop{\biguplus}_{M_0}^{M_3} M_2$$

does not hold, we write

$$M_1 \mathop{\not\biguplus}_{M_0}^{M_3} M_2$$

and say that M_1, M_2 *forks* in M_3 over M_0.

THEOREM 1.6. (i) *Suppose that*

$$M_1 \mathop{\biguplus}_{M_0}^{M_3} M_2 \quad \textit{and} \quad M_2 \mathop{\not\biguplus}_{M_0}^{M_3} M_1$$

(*failure of* $\biguplus$-*symmetry*) *and* $M_0 \preceq_{\mathrm{nice}} M_3$. *Let* $\mu = \kappa^* + |\mathbf{T}| + \|M_2\| + \|M_1\|$. *Then for every linear order* $(I, <)$ *there exists an Ehrenfeucht–Mostowski model* $N = \mathrm{EM}(I, \Phi)$ *with* μ (*individual*) *constants* $\{\tau_i^0 : i < \mu\}$ *and unary function symbols* $\{\tau_i^1(x_i) : i < \mu\}$, $\{\tau_i^2(x_i) : i < \mu\}$ *such that*, *for* $M = (N{\restriction}L){\restriction}\{\tau_i^0 : i < \mu\}$ (*i.e.*, M *is a submodel of* N *with the same vocabulary as* $\mathbf{T}$ *and universe* $\{\tau_i^0 : i < \mu\}$, *i.e.*, *the set of interpretations of these individual constants*) *and for every* $t \in I$, $l = 1, 2$,

$$M_t^l = (N{\restriction}L){\restriction}\{\tau_i^l(x_t) : i < \mu\},$$

one has $M \preceq_{\mathcal{F}} N$, $M_t^l \preceq_{\mathcal{F}} N$ *and for* $s \neq t \in I$, $t < s$ *iff* $M_t^1 \mathop{\biguplus}_{M}^{N} M_s^2$.

(2) *Assume*:

(a) $\mu \geq \mathrm{LS}(\mathbf{T})$ *and* $M \in K_\mu$ *is nice*,
(b) *for* $l = 1, 2$, Op_l *is defined by* (I_l, D_l, G_l), $f_{l,\alpha} \in {}^I M$ *for* $\alpha < \alpha_l$ *with* $\mathrm{eq}(f_{l,\alpha}) \in G_l$, *i.e., such that* $\mathrm{eq}(f_{l,\alpha)}/D \in M^I_D|G$,
(a) *for* $l = 1, 2$ *we have* $M^l_0 = M$, $M^l_1 = \mathrm{Op}_l(M^l_0)$, $M^l_2 = \mathrm{Op}_{3-l}(M^l_1)$, $a^{l,1}_\alpha = f_{l,\alpha}/D_1 \in (M^l_0)^{I_l}_{D_l}|G_l = M^l_1$ *and* $a^{l,2}_\beta = f_{3-l,\alpha}/D_2 \in (M^l_2)^{I_{3-l}}_{D_{3-l}}|G_{3-l} = M^l_2$.

Then there are Φ, τ^l_i $(l = 0,\ i < \mu$ *or* $l \in \{1, 2\},\ i < \alpha_l)$ *such that*:

(α) Φ *is a blueprint for E.M. models*, $|L_\Phi| \leq \mu$ *where* L_Φ *is the vocabulary of* Φ *so* $L \subseteq L_\Phi$,
(β) *for any linear order* I, $\mathrm{EM}(I, \Phi) = \mathrm{EM}_L(I, \Phi)$ *is the* L*-reduct of* $\mathrm{EM}_{L_\Phi}(I, \Phi)$ (*an* L_Φ*-model*) *which is a model of* $\mathbf{T}$ *of cardinality* $\mu + |I|$ *and*

$$I \subseteq J \ \Rightarrow\ \mathrm{EM}(I, \Phi) \preceq_{\mathcal{F}} \mathrm{EM}(J, \Phi),$$

(γ) τ^l_i *are unary function symbols in* L_Φ,
(δ) $\mathrm{EM}(\emptyset, \Phi)$ *is* M,
(ε) *for any linear order* I *and* $s < t$ *in* I *we have: the type which*

(i) $\langle \tau^1_\alpha(x_s) : \alpha < \alpha_1\rangle^\wedge\langle \tau^2_\beta(x_t) : \beta < \alpha_2\rangle$ *realizes over* M *in* $\mathrm{EM}(I, \Phi)$ *is the same as the type that* $\langle a^{1,1}_\alpha : \alpha < \alpha_1\rangle^\wedge\langle a^{1,2}_\alpha : \alpha < \alpha_2\rangle$ *realizes over* M *in* M^1_2,
(ii) $\langle \tau^1_\alpha(x_t) : \alpha < \alpha_1\rangle^\wedge\langle \tau^2_\beta(x_s) : \beta < \alpha_2\rangle$ *realizes over* M *in* $\mathrm{EM}(I, \Phi)$ *is the same as the type that* $\langle a^{2,2}_\alpha : \alpha < \alpha_1\rangle^\wedge\langle a^{2,1}_\beta : \beta < \alpha_2\rangle$ *realizes over* M *in* M^2_2.

Remark. Note $M_0 \preceq_{\mathrm{nice}} M_3$ is automatic in the interesting case since $M_0 \in K_{<\lambda}$ and every element of $K_{<\lambda}$ is nice by [2, 5.4]. On the operations see [2].

Proof of Theorem 1.6. (1) Without loss of generality $\|M_3\| = \mu$. Let M^+_0 be an expansion of M_0 by $\leq \mathrm{LS}(\mathbf{T})$ functions such that M^*_0 has Skolem functions for the formulas in $\mathcal{F}$. We know that $M_0 \preceq_{\mathrm{nice}} M_3$. So there is Op^1 such that $M_0 \preceq_{\mathcal{F}} M_1 \preceq_{\mathcal{F}} \mathrm{Op}^1(M_0)$ and as $M_1 \bigcup\limits_{M_0}^{M_3} M_2$ there is Op^2 such that $M_1 \preceq_{\mathcal{F}} M_3 \preceq_{\mathcal{F}} \mathrm{Op}^2(M_1)$ and $M_2 \preceq_{\mathcal{F}} \mathrm{Op}^2(M_0)$. Let $\mathrm{Op} = \mathrm{Op}^2 \circ \mathrm{Op}^1$. For each $t \in I$, let $\mathrm{Op}_t = \mathrm{Op}$. Let N be the iterated ultrapower of M_0 with respect to $\langle \mathrm{Op}_t : t \in I\rangle$. For each $t \in I$, there is a canonical $\mathcal{F}$-elementary embedding $F_t : \mathrm{Op}_t(M_0) \xrightarrow{\mathcal{F}} N$. Let $M = M_0$ and $M^l_t = F_t(M_l)$ for $l = 1, 2$, $t \in I$.

For each $t < s$, we can let $M_s^+ = \langle \mathrm{Op}_v : v < s\rangle(M_0)$, so $M_0 \preceq_{\mathcal{F}} M_t^+ \preceq_{\mathcal{F}} M_s^+ \preceq_{\mathcal{F}} \mathrm{Op}^1(M_s^+)$ and we can extend $F_t{\restriction}M_1$ to an embedding of $\mathrm{Op}^2(M_1)$ into $\mathrm{Op}_s^2(\mathrm{Op}_s^1(M_s^+))$, so $(F_t{\restriction}M_1) \cup (F_s{\restriction}M_2)$ can be extended to a $\preceq_{\mathcal{F}}$-embedding of M_3 into N. From the definition of the iterated ultrapower and non-forking it follows that for $s \neq t \in I$, $t < s$ implies $M_t^1 \underset{M_0}{\overset{N}{\bigcup\!\!\!\!\bigcup}} M_s^2$. On the other hand, similarly, if $s, t \in I$ and $s < t$ then $(F_s{\restriction}M_1) \cup (F_t{\restriction}M_2)$ can be extended to a $\preceq_{\mathcal{F}}$-embedding of M_3 into N, and hence by assumption $M_t^1 \underset{M_0}{\overset{N}{\biguplus}} M_s^2$.

(2) A similar proof. ∎

COROLLARY 1.7. *Assume* $\mathbf{T}$ *is categorical in* λ *or just* $I(\lambda, \mathbf{T}) < 2^\lambda$. *Then* $\bigcup_{\mu^+<\lambda} K_\mu$ *obeys* $\bigcup\!\!\!\!\bigcup$*-symmetry, i.e., for* $M_0, M_1, M_2, M_3 \in \bigcup_{\mu^+<\lambda} K_\mu$,

$$\textit{if}\quad M_1 \underset{M_0}{\overset{M_3}{\bigcup\!\!\!\!\bigcup}} M_2 \quad \textit{then}\quad M_2 \underset{M_0}{\overset{M_3}{\bigcup\!\!\!\!\bigcup}} M_1.$$

Proof. If $\mu^+ < \lambda$, $M_1 \underset{M_0}{\overset{M_3}{\bigcup\!\!\!\!\bigcup}} M_2$ and $M_2 \underset{M_0}{\overset{M_3}{\biguplus}} M_2$, then Theorem 1.6 gives the assumptions of the results at the end of Section 3 of [13, III] (or better [6, III, §3]). These yield a contradiction to the λ-categoricity of $\mathbf{T}$ and even 2^λ pairwise non-isomorphic models.

But we give a self-contained proof of the version of 1.7 needed here, i.e. for $\mathbf{T}$ categorical in λ, allowing ourselves to use the rest of this section (which does not rely on 1.7 except 1.24), really use just 1.16, 1.18, 1.20 here. Let Φ be as in 1.6(2); we can assume it is as used in 1.18, 1.19. Choose an increasing continuous sequence $\langle I_\alpha : \alpha \le \mu^+ + 1\rangle$ of linear orders each of cardinality μ^+, $|I_{\alpha+1} \setminus I_\alpha| = \mu^+$, $t^* \in I_{\mu^++1} \setminus I_{\mu^+}$, s_α^+, $s_\alpha^- \in I_{\alpha+1} \setminus I_\alpha$ for $\alpha < \mu$ such that

$$\alpha < \beta \;\Rightarrow\; s_\alpha^- < s_\beta^- < t^* < s_\beta^+ < s_\alpha^+,$$

and s_α^+, s_α^- realize the same Dedekind cut of I_α. Let $M_\alpha = \mathrm{EM}(I_\alpha, \Phi)$ for $\alpha \le \mu^+$, so $\langle M_\alpha : \alpha \le \mu^+ + 1\rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous, $M_\alpha \in K_{\mu^+}$, $M_{\alpha+1}$ is (μ, μ)-saturated over M_α, $\overline{a}_t = \langle \tau_i^1(x_t) : i\rangle$, $\overline{b}_t = \langle \tau_i^2(x_t) : i\rangle$ for $t \in I_{\mu^++1}$. Clearly $\mathrm{tp}(\overline{a}_{s_\alpha^-}, M_\alpha, M_{\mu^++1}) = \mathrm{tp}(\overline{a}_{s_\alpha^+}, M_\alpha, M_{\mu^++1})$ for $\alpha < \mu$ but

$$(*)\qquad \mathrm{tp}(\overline{b}_t^{*\wedge}\overline{a}_{s_\alpha^-}, M_\alpha, M_{\mu^++1}) \neq \mathrm{tp}(\overline{b}_t^{*\wedge}\overline{a}_{s_\alpha^+}, M_\alpha, M_{\mu^++1}).$$

We now choose enough sequences of models; first we define a linear order J with set of elements

$$\{t_i : i < \kappa^*\} \cup \{s_\gamma : \gamma < \mu^+ \times (\mu^+ + 1)\}$$

such that

$$i < j < \kappa^* \ \& \ \beta < \gamma < \mu^+ \times (\mu^+ + 1) \ \Rightarrow \ t_i < t_j < s_\beta < s_\gamma.$$

For $\alpha \le \mu^+ + 1$ let $J_\alpha = \{t_i : i < \kappa^*\} \cup \{t_\gamma : \gamma < \mu^+ \times (1 + \alpha)\}$ and $J^* = J_{\mu^+ + 1} \setminus J_\mu$. Let $N_\alpha = \mathrm{EM}(J_\alpha, \Phi)$. Again $\langle N_\alpha : \alpha \le \mu^+ + 1\rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous in K_{μ^+} and $N_{\alpha+1}$ is (μ^+, μ^+)-saturated over N_α. Hence there is an isomorphism f^* from $M_{\mu^+ + 1}$ onto $N_{\mu^+ + 1}$ mapping each M_α onto N_α. Now, $\overline{b}^* = f(\overline{b}_{t^*}) = \langle f(\tau_i^2(x_t)) : i\rangle$ is a sequence of $\le \mu$ members of $\mathrm{EM}(J_{\mu^+ + 1}, \Phi)$, hence for some $\alpha < \mu^+$ we have $\overline{b}^* \subseteq \mathrm{EM}(J', \Phi)$ where $J' = \{t_i : i < \kappa^*\} \cup J^* \cup J_\alpha$. However by [2, 2.6] we have $J_{\mu^+} \underset{J_\alpha}{\overset{J}{\biguplus}} J'$. Hence [2, 2.5]

$$(*) \qquad \mathrm{EM}(J_{\mu^+}, \Phi) \underset{\mathrm{EM}(J_\alpha, \Phi)}{\overset{\mathrm{EM}(J, \Phi)}{\biguplus}} \mathrm{EM}(J', \Phi).$$

Now clearly there is an automorphism f of $\mathrm{EM}(J_{\mu^+}, \Phi)$ over $\mathrm{EM}(J_\alpha, \Phi)$ which maps $\overline{a}_{s_\alpha^-}$ to $\overline{a}_{s_\alpha^+}$. The Op which witnesses $(*)$ extends f to an automorphism of $\mathrm{Op}(\mathrm{EM}(J_{\mu^+}, \Phi))$ which is the identity over $\mathrm{EM}(J', \Phi)$, contradicting $(*)$, so we are done. ∎

It may be helpful, though somewhat vague, to add the remark that $\biguplus$-asymmetry enables one to define order and to build many complicated models; so 1.7 removes a potential obstacle to a categoricity theorem. Note that we could have put 3.11(2) here.

Definition 1.8. Let A be a set. We write

$$M_1 \underset{M_0}{\overset{M_3}{\biguplus}} A$$

(where $A \subseteq M_3$, $M_0 \preceq_{\mathcal{F}} M_1 \preceq_{\mathcal{F}} M_3$) to mean that there exist M_2, M_3' such that $A \subseteq |M_2|$, $M_3 \preceq_{\mathcal{F}} M_3'$ and $M_1 \underset{M_0}{\overset{M_3'}{\biguplus}} M_2$. In this situation we say that $A/M_1 = \mathrm{tp}(A, M_1, M_3)$ *does not fork* over M_0 in M_3.

We write $M_1 \underset{M_0}{\overset{M_3}{\biguplus}} a$ to mean $M_1 \underset{M_0}{\overset{M_3}{\biguplus}} \{a\}$; we then say $\mathrm{tp}(a, M_1, M_3)$ does not fork over M_0.

We write $A_1 \underset{M_0}{\overset{M_3}{\biguplus}} A_2$ if for some M_3, $M_3 \preceq_{\mathcal{F}} M_3' \in K_{<\lambda}$, and for some M_1', $A_2 \subseteq M_1' \preceq_{\mathcal{F}} M_3'$, and $M_1' \underset{M_0}{\overset{M_3'}{\biguplus}} A_2$.

REMARK 1.9. (1) Of particular importance is the case where A is finite. Let us explain the reason. We wish to prove a result of the form:

(∗) if $\langle M_i : i \leq \delta + 1\rangle$ is a continuous $\prec_{\mathcal{F}}$-chain and $a \in M_{\delta+1}$, then there is $i < \delta$ such that $M_\delta \underset{M_i}{\overset{M_{\delta+1}}{\bigcup}} a$.

This says roughly that the type $\mathrm{tp}(a, M_\delta, M_{\delta+1})$ is definable over a finite set (or at least in some sense has finite character). In general the former relation is not obtained. However its properties are correct. Hence it will be possible to define the rank of a over M_0, $\mathrm{rk}(a, M_0)$, as an ordinal, so that for large enough M_3, if $M_1 \underset{M_0}{\overset{M_3}{\biguplus}} a$, then $\mathrm{rk}(a, M_1) < \mathrm{rk}(a, M_0)$.

(2) If A is an infinite set, then we cannot prove (∗) in general. For example, suppose that $\langle M_i : i \leq \omega\rangle$ is (strictly) increasing continuous, $a_i \in M_{i+1} \setminus M_i$ and $A = \{a_i : i < \omega\}$. Then for every $i < \omega$, $(\bigcup_{j<\omega} M_j) \underset{M_i}{\overset{M_\omega}{\biguplus}} A$ as the operation Op we use in the definition increases M_i and increases $\bigcup_{j<\omega} M_j$, but $\mathrm{Op}(M_i) \cap \bigcup_{j<\omega} M_j = M_i$. Still we can restrict ourselves to δ of cofinality $> |A|$.

(3) Notice that quite generally, $N_1 \underset{N_0}{\overset{N_3}{\bigcup}} N_2$ implies that $N_1 \cap N_2 = N_0$ (see above).

DEFINITION 1.10. We define

$$\boldsymbol{\kappa}_\mu(\mathbf{T}) = \boldsymbol{\kappa}_\mu(\mathcal{K}) = \{\kappa : \mathrm{cf}(\kappa) = \kappa \leq \mu \text{ and there exist a continuous } \prec_{\mathcal{F}}\text{-chain } \langle M_i : i \leq \kappa + 1\rangle \subseteq K_{\leq\mu} \text{ and } a \in M_{\kappa+1} \text{ such that for all } i < \kappa,\ a/M_\kappa \text{ forks over } M_i \text{ in } M_{\kappa+1}\}.$$

That is, for $\kappa \in \boldsymbol{\kappa}_\mu(\mathbf{T})$ there are $\langle M_i \in K_{\leq\mu} : i \leq \kappa + 1\rangle$ and $a \in M_{\kappa+1}$ such that $i < \kappa \Rightarrow M_\kappa \underset{M_i}{\overset{M_{\kappa+1}}{\biguplus}} a$.

EXAMPLE 1.11. Fix μ and $\alpha \leq \mu$. Let $({}^\mu\omega, E_\beta)_{\beta<\alpha}$ be the structure with universe

$${}^\mu\omega = \{\eta : \eta \text{ is a function from } \mu \text{ to } \omega\},$$

and $\eta E_\beta \nu$ iff $\eta{\restriction}\beta = \nu{\restriction}\beta$. Let $\mathbf{T} = \mathrm{Th}({}^\mu\omega, E_\beta)_{\beta<\alpha}$. Then

$$\boldsymbol{\kappa}_\mu(\mathbf{T}) = \{\kappa : \mathrm{cf}(\kappa) = \kappa \leq \alpha\}.$$

Why? If $\mathrm{cf}(\kappa) = \kappa \leq \alpha$, then there are M_i $(i \leq \kappa + 1)$, $a \in M_{\kappa+1}$ and $a_i \in M_{i+1} \setminus M_i$ for $i < \kappa$ such that $a_i/E_{i+1} \notin M_i$ (that is to say, no element of M_i is E_{i+1}-equivalent to a_i) and $aE_{i+1}a_i$.

Definition 1.12. The class $\mathcal{K} = \langle K, \preceq_{\mathcal{F}} \rangle$ is χ-*based* if for every pair of continuous $\prec_{\mathcal{F}}$-chains $\langle N_i \in K_{\leq \chi} : i < \chi^+ \rangle$, $\langle M_i \in K_{\leq \chi} : i < \chi^+ \rangle$ with $M_i \preceq_{\mathcal{F}} N_i$, there is a club C of χ^+ such that

$$(\forall i \in C)\left(M_{i+1} \underset{M_i}{\overset{N_{i+1}}{\bigcup}} N_i \right).$$

Replacing χ^+ by regular χ we write $(<\chi)$-based. We say synonymously that $\mathbf{T}$ is χ-based.

Definition 1.13. The class $\mathcal{K} = \langle K, \preceq_{\mathcal{F}} \rangle$ *has continuous non-forking* in (μ, κ) if

(α) whenever $\langle M_i \in K_{\leq \mu} : i \leq \delta \rangle$ is a continuous $\prec_{\mathcal{F}}$-chain, $|\delta| \leq \mu$, $\mathrm{cf}(\delta) = \kappa$,

$$M_0 \preceq_{\mathcal{F}} N_0 \preceq_{\mathcal{F}} N^*, \quad M_\delta \preceq_{\mathcal{F}} N^* \quad \text{and} \quad (\forall i < \delta)\left(M_i \underset{M_0}{\overset{N^*}{\bigcup}} N_0 \right),$$

then $M_\delta \underset{M_0}{\overset{N^*}{\bigcup}} N_0$;

(β) whenever $\langle M_i \in K_{\leq \mu} : i \leq \delta + 1 \rangle$, $\langle N_i \in K_{\leq \mu} : i \leq \delta + 1 \rangle$ are continuous $\prec_{\mathcal{F}}$-chains, $M_i \preceq_{\mathcal{F}} N_i$, $|\delta| \leq \mu$, $\mathrm{cf}(\delta) = \kappa$ and

$$(\forall i < \delta)\left(M_{\delta+1} \underset{M_i}{\overset{N_{\delta+1}}{\bigcup}} N_i \right),$$

then $M_{\delta+1} \underset{M_\delta}{\overset{N_{\delta+1}}{\bigcup}} N_\delta$.

Again we will mean the same thing by saying that $\mathbf{T}$ has continuous non-forking in (μ, κ).

Our next goal is to show that if $\mathbf{T}$ fails to possess these features for some $\mu < \lambda$ such that $\mu \geq \kappa + \mathrm{LS}(\mathcal{K})$, then $\mathbf{T}$ has many models in λ.

Let us recall in this context a further important result from [13, II, 3.10]:

Theorem 1.14. *Assume* $\mathbf{T}$ *is a* λ*-categorical theory, or just* $\mathcal{K}_{<\lambda}$ *has amalgamation and every* $N \in K_{<\lambda}$ *is nice.*

(1) *Let* $\mathrm{LS}(\mathbf{T}) < \mu \leq \lambda$ *and* $M \in K_\mu$. *Then the following are equivalent*:

   (A) M *is universal-homogeneous*: *if* $N \preceq_{\mathcal{F}} M$, $\|N\| < \mu$ *and* $N \preceq_{\mathcal{F}} N' \in K_{<\mu}$, *then there is an* $\mathcal{F}$*-elementary embedding* $g : N' \xrightarrow{\mathcal{F}} M$ *such that* $g{\restriction}N = \mathrm{id}_N$.

   (B) *If* $N \preceq_{\mathcal{F}} M$, $\|N\| < \mu$ *and* $p \in S(N)$, *then* p *is realized in* M, *i.e.*, N *is saturated.*

(2) M *as in* (A) *or* (B) *is unique for fixed* $\mathbf{T}$, μ.

(3) *Let* $\mathrm{LS}(\mathbf{T}) \leq \mu < \lambda$ *and* $\kappa \leq \mu$. *Any two* (μ,κ)*-saturated models are isomorphic* (see 1.4(7)).

(4) *Let* $\mathrm{LS}(\mathbf{T}) \leq \mu < \lambda$ *and* $\kappa \leq \mu$. *If* N_1, N_2 *are* (μ,κ)*-saturated over* M *then* N_1, N_2 *are isomorphic over* M.

Proof. (1), (2) See [13, II 3.10], or better presented [5, 0.19].

(3) Easy and proofs exist but we shall prove. Assume N_1, N_2 are (μ,κ)-saturated, hence for $l = 1,2$ there is a $\preceq_{\mathcal{F}}$-increasing continuous sequence $\langle M_{l,\alpha} : \alpha < \kappa\rangle$ in K_μ such that $M_{l,\kappa} = N_l$ and $M_{l,\alpha+1}$ is universal over $M_{l,\alpha}$. We now choose by induction on $\alpha \leq \kappa$ a triple $(f_l, M'_{1,\alpha}, M'_{2,\alpha})$ such that:

(a) for $l \in \{1,2\}$, $M'_{l,\alpha} \in K_\mu$ is $\preceq_{\mathcal{F}}$-increasing continuous with $\alpha < \kappa$,
(b) f_α is an isomorphism from $M'_{1,\alpha}$ onto $M'_{2,\alpha}$ increasing with α,
(c) if α is even then $M'_{1,\alpha} = M_{1,\alpha}$ and $M'_{2,\alpha} \preceq_{\mathcal{F}} M_{2,\alpha+1}$,
(d) if α is odd then $M'_{2,\alpha} = M_{2,\alpha}$ and $M'_{1,\alpha} \preceq_{\mathcal{F}} M_{1,\alpha+1}$,
(e) if α is a limit ordinal then $M'_{1,\alpha} = M_{1,\alpha}$ and $M'_{2,\alpha} = M_{2,\alpha}$.

Using the universality assumptions there is no problem to carry out the induction and f_κ is an isomorphism from $N_1 = M_{1,\kappa}$ onto $N_2 = N_2$.

(4) Similar to (3) (just let $M = M_{1,0} = M_{2,0}$, $f_0 = \mathrm{id}_M$). ∎

PROPOSITION 1.15. *Assume* $\mathbf{T}$ *is* λ*-categorical or just* $\mathcal{K}_{<\lambda}$ *has amalgamation.*

(1) *If* $\mathrm{LS}(\mathbf{T}) \leq \mu < \lambda$ *and* $N_0 \preceq_{\mathcal{F}} N_1$ *are in* K_μ, *then the following are equivalent*:

   (A) N_1 *is* (μ,μ)*-saturated over* N_0,
   (B) *there is a* $\preceq_{\mathcal{F}}$*-increasing continuous* $\langle M_i : i \leq \mu \times \mu\rangle$ *such that* $M_{\mu\times\mu} = N_1$, $M_0 = N$ *and every* $p \in S(M_i)$ *is realized in* M_{i+1}.

(2) *Also the following are equivalent for* $\kappa = \mathrm{cf}(\kappa) \leq \mu^+$:

   (A_κ) N_1 *is* (μ,κ)*-saturated over* N_0,
   (A_κ) *there is a* $\preceq_{\mathcal{F}}$*-increasing continuous* $\langle M_i : i \leq \mu \times \kappa\rangle$ *with* $M_{\mu\times\kappa} = N_1$, $M_0 = N$ *and every* $p \in S(M_i)$ *is realized in* M_{i+1}.

(3) *If* $\mathcal{K}$ *is stable in* μ, $\lambda > \mu \geq \mathrm{LS}(\mathcal{K})$, $\kappa = \mathrm{cf}(\kappa) \leq \mu^+$ *then there is a* (μ,κ)*-saturated model* (*in fact, over any given model in* K_μ).

Proof. (1) Follows from the proof of 1.14(1).
(2), (3) Straightforward. ∎

PROPOSITION 1.16. [$\mathbf{T}$ *categorical in* λ]

(1) *Any* $M \in K_\lambda$ *is saturated.*
(2) *Every* $N \in K_{<\lambda}$ *is nice.*
(3) $\mathcal{K}_{<\lambda}$ *has* $\preceq_{\mathcal{F}}$*-amalgamation.*

(4) *If* $\mu \in [\mathrm{LS}(\mathbf{T}), \lambda)$ *and* $M \in K_\mu$, *then there is* $N \in M_\mu$ *which is* μ*-universal over* M (see Definition 1.4).

(5) $\mathcal{K}$ *is stable in* μ *for* $\mu \in [\mathrm{LS}(\mathbf{T}), \lambda)$.

(6) *If* $\mu \in [\mathrm{LS}(\mathbf{T}), \lambda)$, $\kappa \leq \mu$ *and* $M \in K_\mu$, *then there is* $N \in K_\mu$ *which is* (μ, κ)*-saturated over* M.

Proof. (1) By the proof of [2, 5.4] (for λ regular easier).

(2) See [2, 5.4].
(3) See [2, 5.5].
(4) See [2, 3.7].
(5) Follows by the two previous parts.
(6) Follows by (3)+(5) and 1.15. ∎

Intermediate Corollary 1.17. (1) *Suppose that* $\mathbf{T}$ *is* λ*-categorical. If* $\mu < \lambda$, $\mu > \mathrm{LS}(\mathbf{T})$ *and* $\mathbf{T}$ *is not* μ*-categorical*, *then there is an unsaturated model* $M \in K_\mu$.

(2) *It now follows that if we show that the existence of an unsaturated model in* K_μ *implies that of an unsaturated model in* K_λ, *then* λ*-categoricity of* $\mathbf{T}$ *implies* μ*-categoricity of* $\mathbf{T}$.

Conclusion 1.18. [$\mathbf{T}$ *categorical in* λ] *If* I *is a linear order*, $I = I_1 + I_2$, $|I| < \lambda$ *and* $J = I_1 + \omega + I_2$ *then every* $p \in S(\mathrm{EM}(I))$ *is realized in* $\mathrm{EM}(J)$.

Proof. Clearly $\mathrm{EM}(I_1 + \lambda + I_2)$ is in K_λ, and hence is saturated, and so every $p \in S(\mathrm{EM}(I))$ is realized in it, say by a_p; for some finite $w_p \subseteq \lambda$ we have $a_p \in \mathrm{EM}(J_1 + w_p + I_2)$; now we use indiscernibility. ∎

Remark 1.19. By changing Φ we can replace "ω" by "1".

Conclusion 1.20. [$\mathbf{T}$ *categorical in* λ]

(1) *If* $J = \bigcup_{\alpha<\delta} I_\alpha$, δ *divisible by* μ, *with* $|J| = \mu \in [\mathrm{LS}(\mathbf{T}), \lambda)$ *or* $|J| = \mu = \lambda$ & $\mathrm{LS}(\mathbf{T}) \leq \chi = |I_\alpha| < \lambda$, I_α *increasing continuous*, *and if for each* α *some Dedekind cut of* I_α *is realized by infinitely many members of* $I_{\alpha+1} \setminus I_\alpha$ *then* $\mathrm{EM}(J)$ *is* $(\chi, \mathrm{cf}(\delta))$*-saturated over* $\mathrm{EM}(I_0)$.

(2) *If* Φ *is "corrected" as in* 1.19, $I_0 \subseteq J$, $|J \setminus I_0| = |J| = \mu$, $\mu \in [\mathrm{LS}(\mathbf{T}), \lambda)$, *or* $|J| = \mu = \lambda$ & $\mathrm{LS}(\mathbf{T}) \leq |I_0|$ & $|I_0|^+ = \lambda$, *then* $\mathrm{EM}(J)$ *is* $(\mathrm{cf}(\mu), |I_0|)$*-saturated over* $\mathrm{EM}(I_0)$; *moreover*, *for any* $\kappa = \mathrm{cf}(\kappa) \leq \mu$ *it is* $(|I_0|, \kappa)$*-saturated.*

(3) *If* $\langle M_i : i \leq \kappa \rangle$ *is* $\preceq_{\mathcal{F}}$*-increasing continuous*, $\kappa \leq \mu$, $M_i \in K_\mu$, *and* M_{i+1} *is universal over* M_i *then* M_κ *is* (μ, θ)*-saturated over* M_0 *for every* $\theta \leq \mu$, *even* $\theta \leq \mu^+$, *so* $N \in K_\mu$ *which is saturated over* $M \in M_\mu$ *is unique up to isomorphism over* M. *So if* $\mu > \mathrm{LS}(\mathbf{T})$ *then* M_κ *is saturated* (*also for* $\kappa = \mu^+$).

Proof. (1), (2) by 1.20+1.15(1).
(3) Follows. ∎

Proposition 1.21. (1) *Suppose $\langle N_i^l : i \le \alpha\rangle$ is $\preceq_{\mathrm{nice}}$-increasing continuous for $l = 1, 2$, $N_i^1 \preceq_{\mathcal{F}} N_i^2 \in K_{<\lambda}$ and*

$$N_i^2 \underset{N_i^1}{\overset{N_{i+1}^2}{\bigcup}} N_{i+1}^1 \quad \text{for each } i < \alpha.$$

Then

$$N_0^2 \underset{N_0^1}{\overset{N_\alpha^2}{\bigcup}} N_\alpha^1.$$

(2) (*Monotonicity properties of* $\bigcup$) *If* $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} M_2$ *and for some operation* Op *and models* M_1', M_2', M_3' *we have* $M_3 \preceq_{\mathcal{F}} M_3' \preceq \mathrm{Op}(M_3)$ *and* $M_0 \preceq_{\mathcal{F}} M_1' \preceq_{\mathcal{F}} M_1$ *and* $M_0 \preceq_{\mathcal{F}} M_2' \preceq_{\mathcal{F}} M_2$, *then* $M_1' \underset{M_0}{\overset{M_3'}{\bigcup}} M_2'$.

(3) *If* $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} A$ *and* $M_0 \preceq_{\mathcal{F}} M_0' \preceq_{\mathcal{F}} M_1' \preceq_{\mathcal{F}} M_1 \preceq_{\mathcal{F}} M_3' \preceq_{\mathcal{F}} M_3''$ *and* $M_3 \preceq_{\mathcal{F}} M_3'' \prec \mathrm{Op}(M_3)$ *for some operation* Op *and* $A' \subseteq A$, *then* $M_1' \underset{M_0'}{\overset{M_3'}{\bigcup}} A'$.

(4) *Note that by definition if* $A_1 \underset{N_0}{\overset{N_3}{\bigcup}} A_2$ *and* $N_0 \subseteq N_0' \subseteq A_1$, *and* $N_0' \preceq_{\mathcal{F}} N_3$, *then* $A_1 \underset{N_0'}{\overset{N_3}{\bigcup}} A_2$ (*the same operation witnesses this*).

Proof. Use [2, 1.11], e.g.:

(1) For each $i < \alpha$ there is Op_i such that $N_{i+1}^1 \preceq_{\mathcal{F}} \mathrm{Op}_i(N_i^1)$ and $N_{i+1}^2 \preceq_{\mathcal{F}} \mathrm{Op}_i(N_i^2)$. We can find Op resulting from the iterated $\langle \mathrm{Op}_i : i < \alpha\rangle$. Let $N_1^* = \mathrm{Op}(N_0^1)$ and $N_2^* = \mathrm{Op}(N_0^2)$, so we can choose by induction on i a $\preceq_{\mathcal{F}}$-embedding f_i of N_i^2 into N_2^* mapping N_i^1 into N_1^*, increasing continuous with i, such that $f_i(N_i^2)$ is included in $\langle \mathrm{Op}_i : i < \alpha\rangle(N_0^2)$. ∎

Proposition 1.22. [**T** *is λ-categorical*] *If $M_0 \preceq_{\mathrm{nice}} M_1, M_2$ are in $K_{<\lambda}$ then we can find $M_4 \in K_{<\lambda}$, $M_0 \preceq_{\mathcal{F}} M_4$ and $\preceq_{\mathcal{F}}$-embeddings f_1, f_2 of M_1, M_2 respectively into M_4 such that*

(α) $f_1(M_1) \underset{M_0}{\overset{M_4}{\bigcup}} f_2(M_2)$,

(β) $f_2(M_2) \underset{M_0}{\overset{M_4}{\bigcup}} f_1(M_1)$.

REMARK 1.23. Note 1.7 deals only with models in $\bigcup\{K_\mu : \mu^+ < \lambda\}$, hence (β) is not totally redundant.

Proof of Proposition 1.2.2. If we want to get (α) only, use an operation Op such that $\mathrm{Op}(M_0)$ has cardinality $\geq \lambda$, and choose $N \preceq_{\mathcal{F}} \mathrm{Op}(M_0)$ with $\|N\| = \lambda$. Hence N is saturated and we can find a $\preceq_{\mathcal{F}}$-embedding $f_2 : M_2 \to N$. Let $N_1 = \mathrm{Op}(M_1)$, so $N \preceq_{\mathcal{F}} \mathrm{Op}(M_0) \preceq_{\mathcal{F}} \mathrm{Op}(M_1) = N_1$, and choose $M_4 \prec N_1$ with $M_4 \in K_\mu$, $\mu < \lambda$, such that $M_1 \cup \mathrm{Rang}\, f_2 \subseteq N$. So we have clause (α) and if $\mu^+ < \lambda$ we are done by 1.7; but as we need the case $\mu^+ = \lambda$ we have to restart the proof.

Since every $N \in K_\lambda$ is saturated, there are an operation Op and $N \in K_\lambda$ such that $M_0 \preceq_{\mathcal{F}} N \preceq_{\mathcal{F}} \mathrm{Op}(M_0)$. Hence there are M_0^+, M_1^+, M_2^+ in $K_{<\lambda}$ such that:

$(*)_0$ $(M_1^+, M_0^+) \preceq_{\mathcal{F}} \mathrm{Op}(M_1, M_0)$, $(M_2^+, M_0^+) \preceq_{\mathcal{F}} \mathrm{Op}(M_2, M_0)$ and M_0^+ has the form $\mathrm{EM}(I_0)$, I_0 a linear order with $|I_0|$ Dedekind cuts with cofinality (κ^*, κ^*). [Note that by 1.20(2) if $|I_0| = \lambda$ then $\mathrm{EM}(I_0)$ is saturated and N is saturated; clearly there is an I_0 as required.]

Clearly we can assume that the cardinality of I_0 is $< \lambda$. Hence we can find I_1, I_2, I_3 such that $I_0 := I \subseteq I_1 \subseteq I_3$, $I_0 \subseteq I_2 \subseteq I_3$, $I_1 \cap I_2 = I$, no $t_1 \in I_1 \backslash I_0$, $t_2 \in I_2 \backslash I_0$ realize the same Dedekind cut of I, and every $t \in I_3 \backslash I_0$ realizes a cut of I with cofinality (κ^*, κ^*) and $|I_1 \backslash I_0| = |I_2 \backslash I_0| = |I_0|$. Hence $I_0 \subseteq_{\mathrm{nice}} I_l$ $(l \leq 3)$, moreover $I_1 \underset{I_0}{\overset{I_3}{\biguplus}} I_2$ and $I_2 \underset{I_0}{\overset{I_3}{\biguplus}} I_1$. Hence

$$(*)_1 \qquad \mathrm{EM}(I_1) \underset{\mathrm{EM}(I_0)}{\overset{\mathrm{EM}(I_3)}{\biguplus}} \mathrm{EM}(I_2), \qquad \mathrm{EM}(I_2) \underset{\mathrm{EM}(I_0)}{\overset{\mathrm{EM}(I_3)}{\biguplus}} \mathrm{EM}(I_1).$$

Also by 1.20(2), without loss of generality $M_l^+ \preceq_{\mathcal{F}} \mathrm{EM}(I_l)$ $(l = 1, 2)$. So by 1.21(2),

$$(*)_2 \qquad M_1^+ \underset{M_0^+}{\overset{\mathrm{EM}(I_3)}{\biguplus}} M_2^+, \qquad M_2^+ \underset{M_0^+}{\overset{\mathrm{EM}(I_3)}{\biguplus}} M_1^+.$$

By $(*)_0 + (*)_2$ and 1.21(1) (for $\alpha = 2$) we get the conclusions. ∎

PROPOSITION 1.24. [$\mathbf{T}$ *is* λ-*categorical*]

(1) *If* $M_1^l \underset{M_0^l}{\overset{M_3^l}{\biguplus}} M_2^l$ *for* $l = 1, 2$, $M_3^l \in K_{<\lambda}$, $\|M_3^l\|^+ < \lambda$ *and* f_k *is an isomorphism from* M_k^1 *onto* M_k^2 *for* $k = 0, 1, 2$ *such that* $f_0 \subseteq f_1$, $f_0 \subseteq f_2$ *then there is* M *with* $M_3^2 \preceq_{\mathcal{F}} M \in K_{<\lambda}$, $\|M\| = \|M_3^1\| + \|M_3^2\|$ *and a* $\preceq_{\mathcal{F}}$-*embedding* f *of* M_3^1 *into* M_3^2 *extending* f_1 *and* f_2.

(2) *Assume* $M_1^l \underset{M_0^l}{\overset{M_3^l}{\bigcup}} A_2^l$ *for* $l = 1, 2$ *and* $A_2^l \subseteq M_2^l \preceq M_3^l$, *and* $M_3^l \in K_{<\lambda}$ *with* $\|M_3^l\|^+ < \lambda$, *and* f_k *is an isomorphism from* M_k^1 *onto* M_k^2 *for* $k = 0, 1, 2$ *such that* $f_0 \subseteq f_1$ *and* $f_0 \subseteq f_2$ *and* f_2 *maps* A_2^1 *onto* A_2^2. *Then there is* M *with* $M_2^3 \preceq_{\mathcal{F}} M \in K_{<\lambda}$ *such that* $\|M\| = \|M_3^1\| + \|M_3^2\|$ *and a* $\preceq_{\mathcal{F}}$*-embedding* f *of* M_3^1 *into* M_3^2 *extending* f_1 *and* $f_2 \restriction A_2^1$.

(3) *If for* $l = 1, 2$, $p_l \in S(N)$ *does not fork over* M (see Definition 1.8), $M \preceq_{\mathcal{F}} N \in K_\mu$, $\mu^+ < \lambda$ *and* $p_1 \restriction M = p_2 \restriction M$ *then* $p_1 = p_2$.

REMARK 1.25. (1) This is uniqueness of non-forking amalgamation.

(2) The requirement is $\|M_3^l\|^+ < \lambda$ rather than $\|M_3^l\| < \lambda$ only because of the use of symmetry, i.e., 1.7.

Proof of Proposition 1.2.4. (1) We can assume $f_0 = \mathrm{id}$, $M_0^1 = M_0^2$ (call it M_0) and $f_1 = \mathrm{id}_{M_1^1}$, $M_1^1 = M_1^2$ (call it M_1). By assumption for some operation Op_l we have $(M_3^l, M_2^l) \preceq_{\mathcal{F}} \mathrm{Op}_l(M_1^l, M_0^l)$. Let $\mathrm{Op} = \mathrm{Op}_1 \circ \mathrm{Op}_2$, so without loss of generality $M_3^l \preceq_{\mathcal{F}} \mathrm{Op}(M_1)$ and $M_2^l \preceq_{\mathcal{F}} \mathrm{Op}(M_0)$. We can assume $\|\mathrm{Op}(M_0)\| \geq \lambda$ and $\|\mathrm{Op}(M_1)\| \geq \lambda$, so there is N_0 with $\bigcup_{l=1}^2 M_2^l \subseteq N_0 \preceq_{\mathcal{F}} \mathrm{Op}(M_0)$ such that $\|N_0\| = \lambda$. Hence N_0 is saturated and so there is an automorphism g_0 of N_0 such that $g_0 \restriction M_2^1 = f_2$ (thus $g_0 \restriction M_0 = \mathrm{id}_{M_0}$). So there is N_2 such that $\bigcup_{l=1}^2 M_2^l \subseteq N_2 \preceq_{\mathcal{F}} N_0$, $\|N_2\|^+ < \lambda$, and N_2 is closed under g_0, g_0^{-1}. Now there is N_3 such that $N_0 \cup M_1 \subseteq N_3 \preceq_{\mathcal{F}} \mathrm{Op}(M_1)$, $N_3 \in K_\lambda$, hence N_3 is saturated. So $M_1 \underset{M_0}{\overset{N_3}{\bigcup}} N_2$ and hence $N_2 \underset{M_0}{\overset{N_3}{\bigcup}} M_1$ (by symmetry, i.e., 1.7). Hence for some N_3' we have $N_3' \preceq_{\mathcal{F}} N_3 \in K_{<\lambda}$ and some automorphism g_1 of N_3' extends $(g_0 \restriction N_2) \cup \mathrm{id}_{M_1}$. [Why? for some Op', we have $(N_3, M_1) \preceq_{\mathcal{F}} \mathrm{Op}'(N_1, M_0)$ and $\mathrm{Op}'(N_1)$, $\mathrm{Op}'(g_0 \restriction N_2)$ are as required except having too large cardinality, but this can be rectified.]

Clearly we are done.

(2), (3) Follow from part (1). ■

2. Various constructions. In this section we will attempt to describe some constructions of models of $\mathbf{T}$ relating to the situations in 1.12 and 1.13, i.e., we want to prove there are "many complicated" models of $\mathbf{T}$ when $\mathbf{T}$ is "on the unstable side" of Definition 1.12 or Definition 1.13; they will be used in the proofs in 3.2–3.5. May we suggest that on a first reading the reader be content with the perusal of 2.1 and 2.2, leaving the heavier work of 2.2.1 until after Section 3 which contains the model-theoretic fruits of the paper. The construction should be meaningful for the classification problem.

What we actually need are 2.2.1, 2.2.2, 2.2.3.

CONSTRUCTION 2.1. First try

Data 2.1.1. Suppose that $\langle M_i \in K_{\leq\mu} : i \leq \kappa+1\rangle$ is a continuous $\preceq_{\mathrm{nice}}$-chain of models of $\mathbf{T}$ with $\mu < \lambda$; T is a non-empty subset of $({}^{\kappa+1\geq}\mathrm{Ord})$ and

(i) T is closed under initial segments, i.e., if $\eta \in T$ and $\nu \triangleleft \eta$, then $\nu \in T$,
(ii) if $\eta \in T$ and $\lg(\eta) = \kappa$ then $\eta^\wedge\langle 0\rangle \in T$ and for all i, $\eta^\wedge\langle 1+i\rangle \notin T$.

Let $\lim_\kappa(T) = \{\eta : \lg(\eta) = \kappa$ and $\bigwedge_{i<\kappa}(\eta\restriction i \in T)\}$. Let $\{\eta_i : i < i^*\}$ be an enumeration of T such that if $\eta_i \triangleleft \eta_j$ (η_i is an initial segment of η_j), then $i < j$, and if $\eta_i = \nu^\wedge\langle\alpha\rangle$, $\eta_j = \nu^\wedge\langle\beta\rangle$, $\alpha < \beta$, then $i < j$. For simplicity i^* is a limit ordinal.

First Try 2.1.2. From the data of 2.1.1 we shall build a model N^* with Skolem functions, $N^*\restriction L \in K$, and for $\eta \in T$, $M^*_\eta \subseteq N^*$, $f_\eta : M_{\lg(\eta)} \overset{\text{into}}{\underset{\mathcal{F}}{\mapsto}} M^*_\eta\restriction L$ such that if $\eta_i \triangleleft \eta_j$, then $f_{\eta_i} \subseteq f_{\eta_j}$, and $M^*_{\eta_i} \preceq_{\mathcal{F}^{\mathrm{sk}}} M^*_{\eta_j}$, where $\mathcal{F}^{\mathrm{sk}} \supseteq \mathbf{T}^{\mathrm{sk}}$ is a fragment of $(L^{\mathrm{sk}})_{\kappa^*,\omega}$ (see below).

Let $M^*_i = \mathrm{Sk}(M_i)$ be a Skolemization of M_i for $\mathcal{F}$, increasing ($\subseteq$) with i, i.e., for every formula $(\exists y)(\varphi(y,\overline{x}) \in \mathcal{F})$ we choose a function $F^{M_i}_{\varphi(y,\overline{x})}$ from M_i to M_i with $\lg(\overline{x})$-places such that

$$M_i \models (\exists y)(\varphi(y,\overline{a}) \to \varphi(F^{M_i}_{\varphi(y,\overline{x})}(\overline{a}),\overline{a}))$$

and

$$j < i \ \Rightarrow\ F^{M_i}_{\varphi(y,\overline{y})}\restriction M_j = F^{M_j}_{\varphi(y,\overline{x})}.$$

Note: we do not require even $M^*_i \prec M^*_{i+1}$.

To achieve this, let us define N^*_i, $M^*_{\eta_i}$ and f_{η_i} by induction on $i < i^*$. Without loss of generality $\eta_0 = \langle\ \rangle$ and i limit implies $\lg(\eta_i)$ limit. Let $N^*_0 = M^*_{\eta_0} = \mathrm{Sk}(M_0)$, the Skolemization of M_0, and $f_{\langle\ \rangle} = \mathrm{id}_{M_0}$. If i is a limit ordinal, let $N^*_i = \bigcup_{j<i} N^*_j$. If i is a successor ordinal and $\lg(\eta_i) = \alpha+1$, then letting $\eta_j = \eta_i\restriction\alpha$, note that $\eta_j \triangleleft \eta_i$ so $j < i$ and so $M^*_{\eta_j}$ and f_{η_j} are defined. We are assuming $M_\alpha \preceq_{\mathrm{nice}} M_{\alpha+1}$, hence there is an operator $\mathrm{Op} = \mathrm{Op}_\alpha$ such that $M_{\alpha+1} \preceq_{\mathrm{nice}} \mathrm{Op}(M_\alpha)$. Let $N^*_i = \mathrm{Op}(N^*_{i-1})$, $\mathrm{Op}(N^*_{i-1}, M_\alpha, f_{\eta_j}) = (N^*_i, \mathrm{Op}(M_\alpha), \mathrm{Op}(f_{\eta_j}))$, $f_{\eta_i} = \mathrm{Op}(f_{\eta_j})\restriction M_{\lg(\eta_i)}$ and $M^*_{\eta_i}$ be the Skolem hull of $\mathrm{Rang}(f_{\eta_i})$. (We can replace N^*_{i+1} by any N' such that $N^*_i \cup M^*_{\eta_i} \subseteq N' \preceq_{\mathcal{F}} \mathrm{Sk}(N^*_{i+1})$ so preserving $|N^*_i| \leq \mu + |i|$.) Finally, let $N^* = \bigcup_{i<i^*} N^*_i$. We are left with the case of i a successor ordinal, $\lg(\eta_i)$ a limit ordinal; we then let $N^*_i = N^*_{i+1}$, $M^*_{\eta_i} = \bigcup_{\nu\triangleleft\eta_i} M^*_\nu$ and $f_{\eta_j} = \bigcup_{\nu\triangleleft\eta_j} f_\nu$.

Explanation. In order to use this construction to prove non-structure results, we intend to use the property: for $\eta \in \lim_\kappa T$, it is possible to extend $f_\eta = \bigcup_{\alpha<\kappa} f_{\eta\restriction\alpha}$ to an $\mathcal{F}$-elementary embedding f^* of $M_{\kappa+1}$ into N^* iff $\eta \in T$.

Let us remark that if for example χ is a strong limit cardinal of cofinality κ^* and $\chi^{<\kappa} \subseteq T \subseteq \chi^{\leq\kappa} \cap \{\eta^\wedge\langle 0\rangle : (\exists\alpha < \kappa)(\lg(\eta) = \alpha+1)\}$, then over $\bigcup_{\eta\in\chi^{<\kappa}} M^*_\eta$ for χ parameters there are 2^χ independent decisions. This is not only a reasonable result: it has been shown ([9, VIII, §1] for χ as above, [6, III, §5] more generally) that this result is sufficient to prove the existence of many models in every cardinality $\lambda > \mu + \mathrm{LS}(\mathbf{T})$.

But to use this construction we have to have some continuity of non-forking, which we have not proved. Hence we shall use another variant of the construction.

CONSTRUCTION 2.2. We modify the construction of 2.1 to suit our purposes.

Modified Data 2.2.1. Let $\langle M_i \in K_{\le_\mu} : i \le \kappa + 1\rangle$ be a continuous $\preceq_{\mathrm{nice}}$-chain of models of $\mathbf{T}$ with $\|M_{\kappa+1}\| = \mu < \lambda$. Let T be a subset of ${}^{\kappa+1\ge}(\mathrm{Ord})$ and $<_{\mathrm{lex}}$ be the lexicographic order on T, which is a linear order of T; suppose that T is $\triangleleft$-closed, i.e. $(\nu \triangleleft \eta \in T \Rightarrow \nu \in T)$, and if $\eta \in {}^{\kappa}(\mathrm{Ord}) \cap T$, then $\eta^\wedge\langle 0\rangle$ is the unique $<_{\mathrm{lex}}$-successor of η in T. For $S \subseteq T$ let $S^{\mathrm{se}} = \{\eta \in S : \lg(\eta) \text{ successor}\}$. Let Op_{i+1} witness $M_i \preceq_{\mathrm{nice}} M_{i+1}$.

We choose $\mathrm{Op}_\eta = \mathrm{Op}_{\lg(\eta)}$ for $\eta \in T^{\mathrm{se}}$. We can iterate the operation Op_η with respect to $(T^{\mathrm{se}}, <_{\mathrm{lex}})$. Also, for each $S \subseteq T$, we can iterate Op_η with respect to $(S^{\mathrm{se}}, <_{\mathrm{lex}})$. Let us denote the result of this iteration with respect to $(S, <_{\mathrm{lex}})$ by Op^S (see [2, 1.11]). Note that for any $M \in K$, if $S_1 \subseteq S_2 \subseteq T$, then $M \preceq_{\mathcal{F}} \mathrm{Op}^{S_1}(M) \preceq_{\mathcal{F}} \mathrm{Op}^{S_2}(M) \preceq_{\mathcal{F}} \mathrm{Op}^{T}(M)$ (by natural embeddings). More formally:

CLAIM 2.2.2. *There exist operations* Op^S *for* $S \subseteq T$ *such that*:

(1) *for every* $S \subseteq T$ *which is* $\triangleleft$*-closed* $M_S = \mathrm{Op}^S(M_0)$ *is defined, and whenever* $S_1 \subseteq S_2 \subseteq T$, *then* $M_{S_1} \preceq_{\mathcal{F}} M_{S_2}$; *let* $M_\eta = M_{\{\eta\restriction\alpha : \alpha \le \lg(\eta)\}}$;

(2) *for* $\eta \in T$, h_η *is a surjective* $\prec_{\mathcal{F}}$*-elementary embedding from* $M_{\lg(\eta)}$ *onto* $M_\eta^- \prec_{\mathcal{F}} M_\eta$, *and* $\langle h_\eta : \eta \in T\rangle$ *is a* $\triangleleft$*-increasing sequence*, *i.e.*, $h_\eta \subseteq h_\nu$ *whenever* $\eta \triangleleft \nu$; *moreover* $\eta \triangleleft \nu \in T \Rightarrow M_\eta \underset{M_\eta^-}{\overset{M_\nu}{\Cup}} M_\nu^-$;

(3) *for every* $x \in M_T$, *there exists a* $\triangleleft$*-closed* $S \subseteq T$ *with* $|S| \le \kappa$ *such that* $x \in M_S$; *in fact* S *is the union of finitely many branches*, *hence* $(S, <_{\mathrm{lex}})$ *is well ordered*;

(4) *for* $\eta \in T$, *letting* $T[\eta] = \{\nu \in T : \neg(\eta \triangleleft \nu)\}$, $T^{\le}[\eta] = \{\nu \in T[\eta] : \nu \le_{\mathrm{lex}} \eta\}$, $T^{\ge}[\eta] = \{\nu \in T[\eta] : \eta \le_{\mathrm{lex}} \nu\}$ (*so* $T[\eta] = T^{\le}[\eta] \cup T^{\ge}[\eta]$) *and* $\alpha < \lg(\eta)$ *we have* $M_{T^{\le}[\eta\restriction\alpha]} \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\Cup}} M_\eta$ (*so we can replace* M_T *by* $M_{T^{\le}[\eta]}$) *and* $M_{T^{\le}[\eta]} \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\Cup}} M_{T^{\ge}[\eta\restriction\alpha]}$ *for* $\alpha < \kappa$;

(5) *if* $\eta \in \lim_\kappa(T)$ *and* $\eta \notin T$, *then* $M_T = \bigcup_{\alpha<\kappa} M_{T[\eta\restriction\alpha]}$;

(6) $\|M_S\| \le |S| + \|M_{\kappa+1}\|^{\kappa^*} + \sup_{\eta\in S} \|M_{\lg(\eta)}\|$;

(7) *for* $\eta \in T \cup \lim_\kappa(T)$, $\langle M_{T[\eta\restriction\alpha]} : \alpha \le \lg(\eta)\rangle$ *is* $\preceq_{\mathcal{F}}$*-increasing continuous. Note:* $\langle T_{[\eta\restriction\alpha]} : \alpha \le \lg(\eta)\rangle$ *is increasing but generally not continuous, however* $\langle T^{\mathrm{se}}_{[\eta\restriction\alpha]} : \alpha \le \lg(\eta)\rangle$ *is.*

FACT 2.2.3. (1) *By clause* (4), *if we have the conclusion of* 1.7 *for models of cardinality* $\le \mu$ (*and* 1.21(1)) *then*

(∗) *if* $\|M_{\eta\restriction\alpha}\| \le \mu$, $M_{\eta\restriction\alpha} \prec_{\mathcal{F}} M' \prec_{\mathcal{F}} M_\eta$, $\|M'\| \le \mu$, $M_{\eta\restriction\alpha} \prec_{\mathcal{F}} M'' \prec_{\mathcal{F}} M_{T[\eta\restriction\alpha]}$ *and* $\|M''\| \le \mu$, *then* $M_\eta \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\bigcup\!\!\!\!\!\bigcup}} M_{T[\eta\restriction\alpha]}$ *and hence* $M' \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\bigcup\!\!\!\!\!\bigcup}} M''$

and (*recall* $M_{\eta\restriction\alpha} \underset{M^-_{\eta\restriction\alpha}}{\overset{M_\eta}{\bigcup\!\!\!\!\!\bigcup}} M^-_\eta$)

(∗∗) *if* $M^-_{\eta\restriction\alpha} \prec_{\mathcal{F}} M' \prec_{\mathcal{F}} M^-_\eta$ *and* $M^-_{\eta\restriction\alpha} \prec_{\mathcal{F}} M'' \prec_{\mathcal{F}} M_{T[\eta\restriction\alpha]}$ *and* $\|M''\| \le \mu$ *then* $M' \underset{M^-_{\eta\restriction\alpha}}{\overset{M_T}{\bigcup\!\!\!\!\!\bigcup}} M''$.

(2) *Then in fact one can replace clause* (4) *above by the weaker condition*

$(4)^-$ $\mu \ge \kappa$ *and for every* $S \subseteq T$ *closed under initial segments*, *if* $|S| \le \mu$ *and* $(\forall \nu \in S)[\eta\restriction(\alpha+1) \trianglelefteq \nu \Rightarrow (\nu \trianglelefteq \eta \vee \eta \trianglelefteq \nu)]$ *and* $\{\eta\restriction i : i \le \alpha\} \subseteq S \subseteq T$, *then* $M_\eta \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\bigcup\!\!\!\!\!\bigcup}} M_S$.

Short proof of 2.2.2. As $\langle M_i : i \le \kappa+1\rangle$ is $\preceq_{\text{nice}}$-increasing continuous, by renaming there is $\langle M^*_i : i \le \kappa+1\rangle$ $\preceq_{\text{nice}}$-increasing continuous such that $M^*_0 = M_0$, $M^*_{i+1} = \mathrm{Op}_{i+1}(M^*_i)$, $M_i \preceq_{\mathcal{F}} M^*_i$ and $M^*_i \underset{M_i}{\overset{M^*_{i+1}}{\bigcup\!\!\!\!\!\bigcup}} M_{i+1}$ (for $i \le \kappa$). We can assume $\|M^*_i\| \le \|M_i\|^{\kappa^*}$. Let $\mathrm{Op}_\eta = (I_\eta, D_\eta, G_\eta)$ be a copy of $\mathrm{Op}_{\lg(\eta)}$ for $\eta \in T^{\mathrm{se}}$ with I_η's pairwise disjoint. Define $I = \prod\{I_\eta : \eta \in T^{\mathrm{se}}\}$, D, G as in the proof of [2, 1.11], so every equivalence relation $e \in G$ has a finite subset $w[e] = \{\eta^l_0 <_{\text{lex}} \ldots <_{\text{lex}} \eta^l_{n(l)-1}\} \subseteq T^{\mathrm{se}}$ and $\mathfrak{e}_l[e] \in G_{\eta^l_e}$ as there. We let $\mathrm{Op}_{T^{\mathrm{se}}} = (I, D, G)$, $M_{T^{\mathrm{se}}} = \mathrm{Op}_{T^{\mathrm{se}}}(M_0)$ and for $S \subseteq T^{\mathrm{se}}$ we let

$$M_S = \{x \in M_T : w[\mathrm{eq}(x)] \subseteq S\}.$$

This defines Op^S implicitly. Naturally there are canonical maps f^*_η from $M^*_{\lg(\eta)}$ onto $M_{\{\nu:\nu\triangleleft\eta\}}$ and let $M_\eta = f''_\eta(M^*_{\lg(\eta)})$ and $h_\eta = f_\eta\restriction M_{\lg(\eta)}$. ∎

Improvement in cardinality 2.2.1. We can replace $\|M_{\kappa+1}\|^{\kappa^*}$ by $\|M_{\kappa+1}\| + \mathrm{LS}(\mathbf{T})$ in part (6) of claim 2.2.2. After choosing $\langle M^*_i : i \le \kappa+1\rangle$, let M^+_0 be a Skolemization of $M_0 = M^*_0$, $M^+_{i+1} = \mathrm{Op}(M^+_i)$, $M^+_\delta = \bigcup_{i<\delta} M^+_i$. Of course

M_S^T ($S \subseteq T$ is $\lhd$-closed) are well defined similarly. Let N_i be the Skolem hull of M_i in M_i^*. For $\eta \in T$ let $N_\eta = f_\eta^*(N_{\lg(\eta)})$. Now for any $\lhd$-closed $S \subseteq T$ let

$$N_S = \text{Skolem hull in } M_S^+ \text{ of } \bigcup\{N_\eta : \eta \in S\}.$$

* * *

There are two different ways to carry on the construction (under Data 2.2.1). We will consider each in its turn.

Construction 2.3. Recall that it is possible to iterate the operation Op with respect to the linear order $(T, <_{\text{lex}})$ and this iteration can be defined as the direct limit of finite approximations. We shall use different approximations and take the direct limit to obtain the required operation.

Suppose that $w \subseteq T$ is closed with respect to $\lhd$ (i.e., initial segment) and is $<_{\text{lex}}$-well-ordered. For each approximation w of this kind, the iterated ultrapower $\mathrm{Op}^w(M_0)$ of M_0 with respect to w is defined as a limit ultrapower and there are natural elementary embeddings into this limit. The principal difference is that this limit is a little larger than a limit obtained using only finite approximations. For example, if $\langle \eta_n : n \le \omega \rangle$ is a $<_{\text{lex}}$-increasing sequence, then in $\mathrm{Op}^{\eta_\omega}(\ldots \mathrm{Op}^{\eta_n}(\ldots(\mathrm{Op}^{\eta_0}(M_0))))$, the last operation $\mathrm{Op}^{\eta_\omega}$ adds elements which are dispersed all over $\mathrm{Op}^{\eta_n}(\ldots \mathrm{Op}^{\eta_0}(M_0))$. (This is of more interest when the sequence has length κ^*.) Now it is easy to check the symmetry (for $\eta \in {}^\alpha\lambda$, $\alpha < \kappa$) between the $<_{\text{lex}}$-successors and $<_{\text{lex}}$-predecessors of η.

We define the embeddings h_η for $\eta \in T$ as follows. For $\eta = \langle\ \rangle$, $h_\eta = \mathrm{id}\restriction M_0$. If $\eta = \nu^\wedge\langle i \rangle$, then Op^η acts on $M_\nu = h_\nu[M_{\lg(\nu)}]$ and we use the commuting diagram:

$$\begin{tikzcd}
\mathrm{Op}^\eta(M_{\lg(\nu)}) \arrow[r, "\text{canonical}"'] & \mathrm{Op}^\eta(M_\nu) \\
M_{\lg(\eta)} \arrow[u] \arrow[r, "\text{canonical}"'] & M_\eta \arrow[u, "\text{canonical}"'] \\
M_{\lg(\nu)} \arrow[u] \arrow[r, "h_\nu"'] & M_\nu \arrow[u]
\end{tikzcd}$$

This completes the construction.

Construction 2.4. In this approach, we employ the generalized Ehrenfeucht–Mostowski models $\mathrm{EM}(I, \Phi)$ from Chapter VII in [9] or [14]. For this we need to specify the generators of the model and what the types are.

Let M_0^+ be the model obtained from M_0 by adding Skolem functions and individual constants for each element of M_0. We know that there is an operation Op such that $M_i \preceq_{\mathcal{F}} M_{i+1} \preceq_{\mathcal{F}} \mathrm{Op}(M_i)$ for $i \leq \kappa$. As in [2, 1.7.4] this means that there are I, D and G such that $\mathrm{Op}(M) = \mathrm{Op}(M, I, D, G)$ where I is a non-empty set, D is an ultrafilter on I, and G is a suitable set of equivalence relations on I, i.e.,

(i) if $e \in G$ and e' is an equivalence relation on I coarser than e, then $e' \in G$;

(ii) G is closed under finite intersections;

(iii) if $e \in G$, then

$$D/e = \Big\{ A \subset I/e : \bigcup_{x \in A} x \in D \Big\}$$

is a κ^*-complete ultrafilter on I/e.

For each $b \in M_{i+1} \backslash M_i$, let $\langle x_t^b : t \in I \rangle / D$ be the image of b in $\mathrm{Op}(M_i)$. We also write $\langle x_t^b : t \in I \rangle / D$ for the canonical image $d(b)$ of $b \in M_i$ in $\mathrm{Op}(M_i)$.

$$\begin{tikzcd} M_{i+1} \ni b \mapsto \langle x_t^b : t \in I \rangle / D \in \mathrm{Op}(M_i) & \\ M_i \arrow[u] \arrow[u] \end{tikzcd}$$

We define a model M^+ with $M_0^+ \preceq_{L_{\kappa^*,\omega}} M^+$ as follows. M^+ is generated by the set $\{x_\eta^b : b \in M_{i+1} \setminus M_i,\ \eta \in T,\ \lg(\eta) = i+1\}$. Note that this set does generate a model since M_0^+ is closed under Skolem functions. Since functions have finite arity, it is enough to specify, for each finite set of the x_η^b, what quantifier-free type it realizes. Since there is monotonicity, we shall obtain indiscernibility as in [9]. The type of a finite set $\langle x_{\eta_l}^{b_l} : l = 1, \dots, n \rangle$ depends on the set $\langle b_1, \dots, b_n \rangle$ and the atomic (i.e., quantifier-free) type of $\langle \eta_1, \dots, \eta_n \rangle$ in the model $\langle T, \triangleleft, <_{\mathrm{lex}}, \text{“}\eta{\restriction}i = \nu{\restriction}i\text{”} \rangle$. Now we can allow a finite sequence $\bar{b}$ instead of b for $\bar{b} \in M_{i+1} \setminus M_i$ and thus without loss of generality $\eta_1, \dots, \eta_n$ is repetition-free, so we can assume $\eta_1 <_{\mathrm{lex}} \dots <_{\mathrm{lex}} \eta_n$. Necessarily, the lexicographic order $<_{\mathrm{lex}}$ on $\{\eta_l{\restriction}\alpha : \alpha \leq \lg(\eta_l)$ and $l = 1, \dots, n\}$ is a well-order and the sequence $\langle \nu_\zeta : \zeta < \zeta(*) \rangle$ is $<_{\mathrm{lex}}$-increasing. We define $N_0 = M_0^+$, $N_{\zeta+1} = \mathrm{Op}(N_\zeta)$, $N_\zeta = \bigcup_{\xi<\zeta} N_\xi$ (for limit ζ). Next, we define $h_{\nu_\zeta} : M_{\lg(\nu_\zeta)} \to_{\mathcal{F}} N_{\zeta+1}$, $h_{\nu_\zeta{\restriction}\beta} \subseteq h_{\nu_\zeta}$. If $\lg(\nu)$ is a limit ordinal, then $\alpha < \lg(\nu) \Rightarrow h_{\nu{\restriction}\alpha}$ is defined and we let $h_\nu = \bigcup_{\alpha<\lg(\nu)} h_{\nu{\restriction}\alpha}$. If $\nu_\zeta = \nu_\xi{}^\wedge\langle\gamma\rangle$, $i = \langle u_\xi \rangle$, then $M_{\zeta+1} \prec_{\mathcal{F}} \mathrm{Op}(M_\zeta, I, D, G)$, identifying elements of M_ζ with their images in the ultrapower. Now define

$$h_{\nu_\zeta}(b) = \begin{cases} d(H_{\nu_\zeta}(b)) & \text{if } b \in M_i, \\ \langle h_{\nu_\zeta}(x_t^b) : t \in I \rangle / D & \text{if } b \in M_{i+1} \setminus M_i, \end{cases}$$

where $d(h_{\nu_\xi}(b))$ is the canonical image of $H_{\nu_\xi}(b)$ in the ultrapower. The type of $\langle x^{b_l}_{\eta_l} : l = 1,\dots,n\rangle$ is defined to be the type of $\langle h_{\eta_l}(b_l) : l = 1,\dots,n\rangle$ in N_ξ.

REMARK 2.4.1. It is possible to split the construction into two steps. For $i \le j \le \kappa+1$, there is an operation $\mathrm{Op}^{i,j}$ with $M_i \preceq M_j \preceq \mathrm{Op}^{i,j}(M_i)$, moving b to $\langle {}^{i,j}a^b_t : t \in I\rangle$, $b \in M_j$, ${}^{i,j}a^b_t \in M_i$, with the obvious commutativity and continuity properties. Now the construction is done on a finite tree $\langle \eta_l : l = 1,\dots,n\rangle$, $\langle \eta_l \cap \eta_m : l,m < \omega\rangle$. We omit the details of monotonicity.

Notation 2.4.2. Let $M_T = M$ be the Skolem closure. If $S \subseteq T$ is closed with respect to initial segments, let $M_S = \mathrm{Sk}_{M_T}(x^b_\eta : \eta \in S,\ b \in M_{\lg(\eta)})$ and $M^*_\eta = M_{\{\eta\restriction\alpha : \alpha\le\lg(\eta)\}}$. Define $h_\eta : M_{\lg(\eta)} \to M^*_\eta$ by $h_\eta(b) = x^b_{\eta\restriction\tau(\mathbf{T})}$ and $N_\eta = h_\eta[M_\eta]$.

REMARK 2.4.3. The construction can be used to get many fairly saturated models. We list the principal properties below.

FACT 2.4.4. *Suppose that $S_l \subseteq T$ is closed with respect to initial segments*, $S_0 = S_1 \cap S_2$ *and*

$$\eta \in S_1 \ \&\ \nu \in S_2 \setminus S_1 \ \Rightarrow\ \eta <_{\mathrm{lex}} \nu.$$

Then

$$M_{S_1} \underset{M_{S_0}}{\overset{M_T}{\Cup}} M_{S_2}.$$

Proof. We can assume S_l is closed, $M_{\mathrm{cl}(S_l)} = M_{S_l}$. Let $S_2 \setminus S_0 = \{\nu_\zeta : \zeta < \zeta(*)\}$ be a list such that $\nu_\zeta < \zeta_\xi \ \Rightarrow\ \zeta < \xi$; let $S^\xi_2 = S_0 \cup \{\nu_\zeta : \zeta < \zeta(*)\}$. Then:

(1) $\langle M_{S^\xi_2} : \xi \le \xi(*)\rangle$ is continuous increasing;
(2) $\langle M_{S^\xi_2 \cap S_1} : \xi \le \xi(*)\rangle$ is continuous increasing.

Hence one has

(3) $$M_{S^\xi_2 \cup S_1} \underset{M_{S^\xi_2}}{\overset{M_{S^{\xi+1}_2 \cup S_1}}{\Cup}} M_{S^{\xi+1}_2}.$$

This is immediate from the definitions, because $M_{S^{\xi+1}_2 \cup S_1}$ is the Skolem closure of $M_{S^2_\xi \cup S_1} \cup N_{\nu_\xi}$, and so elements of N_{ν_ξ} can be represented as averages. ■

3. Categoricity in μ when $\mathrm{LS}(\mathbf{T}) \le \mu < \lambda$

HYPOTHESIS 3.1. Every $M \in K_{<\lambda}$ is nice hence has a $\prec_{\mathcal{F}}$-extension of cardinality λ which is saturated and $\mathcal{K}_{<\lambda}$ has amalgamation.

This section contains the principal theorems of the paper: if $\mathbf{T}$ is λ-categorical and $\mathrm{LS}(\mathbf{T}) \le \mu < \lambda$, then (by [2, 5.4, 5.5], 3.1 holds and):

(i) $\boldsymbol{\kappa}_\mu(\mathbf{T}) = \emptyset$ when $\mu \in [\mathrm{LS}(\mathbf{T}), \lambda)$ (see Def. 1.10),

(ii) when $\mathrm{LS}(\mathbf{T}) \le \chi = \mathrm{cf}(\chi) < \lambda$, $\mathbf{T}$ is χ-based (and $\mathcal{K}$ does not have (μ,κ)-continuous non-forking when $\mu \in [\mathrm{LS}(\mathbf{T}), \lambda)$, $\kappa \le \mu$) (see Defs. 1.12, 1.13),

(iii) there is a saturated model in $\mathcal{K}_\mu = \langle K_\mu, \preceq_{\mathcal{F}} \rangle$,

(iv) $\mathbf{T}$ is categorical in every large enough $\mu < \lambda$.

We first deal with some preliminary results, quoting [6] for "black boxes" which do for us much of the combinatorial work for "there are many non-isomorphic models" extensively.

Theorem 3.2. *Assume the conclusion of* 1.7 *for* $\boldsymbol{\kappa}_{\le\mu}$ (*e.g.*, $\mu^+ < \lambda$) *and* $\kappa \le \mu^+$. *Suppose that the tree* T *is as in Claim* 2.2.2 *and suppose further* $\langle M_i \in K_{\le\mu} : i \le \kappa+1 \rangle$ *is a* $\preceq_{\mathrm{nice}}$-*increasing continuous sequence of members of* $K_{\le\mu}$ *such that* $\|M_{\kappa+1}\| = \|M_\kappa\|$ *and*

(∗) *there is no* $\preceq_{\mathcal{F}}$-*increasing continuous sequence* $\langle N_i \in K_{\le\mu} : i \le \kappa \rangle$ *such that*:

(i) $M_i \preceq_{\mathcal{F}} N_i$,

(ii) $M_{\kappa+1} \preceq_{\mathcal{F}} N_\kappa$,

(iii) *if* $i < j \le \kappa$ *and* $\|N_j\| < \mu^*$, *then* $N_i \underset{M_i}{\overset{N_j}{\biguplus}} M_j$.

Assume that T, M_T, M_ν, M_ν^-, h_ν (*for* $\nu \in T$) *are as in Section* 2. *Then the following are equivalent for* $\eta \in \lim_\kappa(T) := \{\eta \in {}^\kappa(\mathrm{Ord}) : \bigwedge_{i<\kappa}(\eta{\restriction}(i+1) \in T)\}$:

(α) *There is an* $\mathcal{F}$-*elementary embedding* h *from* $M_{\kappa+1}$ *into* M_T *such that*

$$\bigcup_{i<\kappa} h_{\eta\restriction i+1} \subseteq h.$$

(β) $\eta^\wedge\langle 0\rangle \in T$ (*equivalently*, $\eta \in T$, *see* 2.2.1).

Proof. For (β)⇒(α), assume $\eta \in T$ and consider the $\mathcal{F}$-elementary embedding $h_{\eta^\wedge\langle 0\rangle}$. Check that $h_{\eta^\wedge\langle 0\rangle}$ is as required in (α). The other direction follows by 2.2.3(1) and (∗). That is, we assume that h exemplifies clause (α) but $\eta^\wedge\langle 0\rangle \notin T$, equivalently $\eta \notin T$ and we shall get a contradiction. We let $\eta_\alpha = \eta{\restriction}\alpha$ for $\alpha \le \kappa$, and let $T_\alpha = T[\eta_\alpha]$ so $T_\kappa = T$. Hence $\langle M_{T_\alpha} : \alpha \le \kappa \rangle$ is $\preceq_{\mathcal{F}}$- increasing continuous (see 2.2.2(7)). By induction on $\alpha \le \kappa$, we can choose a model $N_\alpha \preceq_{\mathcal{F}} M_{T_\alpha}$ such that $\|N_\alpha\| \le \|M_\alpha\| + \mathrm{LS}(\mathbf{T}) \le \mu$, $M^-_{\eta\restriction\alpha} \subseteq N_\alpha$ and $N = \bigcup_{\alpha<\kappa} N_\alpha$ includes $h(M_{\kappa+1})$. By 2.2.3 we get $N_i \underset{M_i^-}{\overset{N_j}{\biguplus}} M_j^-$ if

$i < j < \kappa$ as $\|M_j^-\| \le \mu$, hence by 1.21 we may allow $j = \kappa$, so we have contradicted $(*)$. ■

PROPOSITION 3.3. *Suppose the conclusion of* 1.7 *for* μ, *and* $\kappa \le \mu^+$ *and an* $\preceq_{\mathcal{F}}$*-increasing sequence* $\overline{M} = \langle M_i : i \le \kappa + 1\rangle$ *is given with* $M_i \in K_{\le\mu}$ *when* $i < \kappa$, $M_j \in K_{\le \mu+\kappa}$ *if* $j \le \kappa + 1$. *Then* $\overline{M}$ *satisfies* $(*)$ *of* 3.2 *if one of the following holds*:

(α) *there is* $a \in M_{\kappa+1}$ *such that* $i < \kappa \Rightarrow M_\kappa \underset{M_{i+1}}{\overset{M_{\kappa+1}}{\uplus}} a$, *or*

(β) $\kappa = \mathrm{cf}(\kappa) = \mu > \mathrm{LS}(\mathbf{T})$ *and* $\kappa < \lambda$ *and* $i < \kappa \Rightarrow \|M_i\| < \kappa$, *and there is a continuous* $\prec_{\mathcal{F}}$*-chain* $\langle N_i : i \le \kappa\rangle$ *such that* $M_{\kappa+1} = \bigcup_{i\le\kappa} N_i$, $\bigwedge_{i<\kappa}(N_i \in K_{<\kappa})$, *and* $E = \{i < \kappa : M_{i+1} \underset{M_i}{\overset{N_\kappa}{\uplus}} N_i\}$ *is a stationary subset of* κ.

Proof. Straightforward from 3.2, and the monotonicity of $\biguplus$, that is, 1.21(3). ■

REMARK 3.4. Clause (β) can also be proved using niceness as in the proof of 3.8. This works for any $\kappa < \lambda$. Also we can imitate 2.2.2 but no need arises.

COROLLARY 3.5. *If* $\mathbf{T}$ *is a* λ*-categorical theory* ([1]), *then*

(1) $\mathbf{T}$ *is* χ*-based if* $\chi^+ < \lambda$ *and* $\chi \ge \mathrm{LS}(\mathbf{T})$; *also it is* $({<}\mu)$*-based if* $\mu = \mathrm{cf}(\mu)$, $\mathrm{LS}(\mathbf{T}) < \mu$, $\mu < \lambda$;

(2) $\boldsymbol{\kappa}_\mu(\mathbf{T}) = \emptyset$ *for every* μ *such that* $\mu^+ < \lambda$ *and* $\mu \ge \mathrm{LS}(\mathbf{T})$.

Proof. (1), (2) We use 3.2, 3.3 to contradict λ-categoricity. In the first phrase of (1) let $\mu = \chi$, $\kappa = \chi^+$, in the second let us repeat the proofs (i.e., prove the appropriate variants of 3.2, 3.3); in the proof of part (2) let $\kappa \le \mu$ be regular, $\kappa \in \boldsymbol{\kappa}_\mu(\mathbf{T})$; so $\kappa = \mathrm{cf}(\kappa)$ and $\kappa^+ < \lambda$.

CASE 1: $\lambda^\mu = \lambda$. By [13, III, 5.1] = [6, IV, 2.1], using 3.2.

CASE 2: λ is regular. We can find a stationary $W^* \in I[\lambda]$ with $W^* \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ (by [15, §1]). Hence, possibly replacing W^* by its intersection with some club of λ, there is W^+ with $W^* \subseteq W^+$ and $\langle a_\alpha : \alpha \in W^+\rangle$ such that $a_\alpha \subseteq \alpha$, $\alpha \in a_\beta$ (so $\beta \in W^+$) implies $\alpha \in W^+$, $a_\alpha = a_\beta \cap a_\alpha$ and $\mathrm{otp}(a_\alpha) \le \kappa$ and

$$\alpha = \sup a_\alpha \Leftrightarrow \mathrm{cf}(\alpha) = \kappa \Leftrightarrow \alpha \in W^*.$$

Now let η_α enumerate a_α in increasing order (for $\alpha \in W^+$), and for any $W \subseteq W^*$ let

$$T_W = \{\eta_\alpha : \alpha \in W^+ \text{ but } \alpha \notin W^* \setminus W\} \cup \{\eta_\alpha{}^\wedge\langle 0\rangle : \alpha \in W\}.$$

([1]) Or just has $< 2^\lambda$ non-isomorphic models in λ.

Now if $W_1, W_2 \subseteq W$ and $W_1 \setminus W_2$ is stationary, then $M_{T_{W_1}}$ cannot be $\preceq_{\mathcal{F}}$-embedded into $M_{T_{W_2}}$(again by [13, III, §5] = [6, IV, §2]).

Case 3: λ singular. Choose λ' with $\lambda > \lambda' = \mathrm{cf}(\lambda') > \mu^+$ and act as in case 2 using λ' instead of λ except that we add to T_W the set $\{\langle i\rangle : i < \lambda\}$ (to get 2^λ we need more, see in [6, IV, VI] on pairwise non-isomorphic models). ∎

Hypothesis 3.6. The conclusion of 3.5 (in addition to 3.1 of course).

Conclusion 3.7. *Suppose* $\mu \geq \mathrm{LS}(\mathbf{T})$, $\mu^+ < \lambda$, *and* $M \in K_\mu$.

(1) *If* $p \in S(M)$ *then* p *is determined by*

$$\{p{\restriction}N : N \preceq_{\mathcal{F}} M \text{ and } \|N\| = \mathrm{LS}(\mathbf{T})\}.$$

(2) *Assume further*

$(*)^M_{\{N_t:t\in I\}}$ (a) *I is a directed partial order*,

(b) $N_t \preceq_{\mathcal{F}} M$,

(c) $I \models t \leq s$ *implies* $N_t \subseteq N_s$ (*hence* $N_t \preceq_{\mathcal{F}} N_s$ *by* (b)),

(d) $\bigcup_{t\in I} N_t = M$.

Then:

(α) *every* $p \in S(M)$ *is determined by* $\{p{\restriction}N_t : t \in I\}$, *which means just that if* $q \in S(M)$ *and for every* $t \in I$ *we have* $p{\restriction}N_t = q{\restriction}N_t$ *then* $p = q$,

(β) *for some* $t \in I$, p *does not fork over* N_t.

Proof. (1) Follows from part (2): We can find $\overline{N} = \langle N_t : t \in I\rangle$ such that $(*)^M_{\{N_t:t\in I\}}$ holds, $\|N_t\| \leq \mathrm{LS}(\mathbf{T})$ and on it use part (2). Why does $\overline{N}$ exist? E.g., as in the proof of part (2) with $I = \{\emptyset\}$, $N_\emptyset = M$ and use $\langle N^*_u : u \in I^*\rangle$ for $I^* = ([M]^{<\aleph_0}, \subseteq)$. Now apply part (2).

(2) Clearly (and as in [12, §1]),

($\otimes$) by induction on $n < \omega$ for every $u \in [M]^n$ we can choose $t[u] \in I$ and N^*_u such that $u \subseteq N^*_u$, $N^*_u \preceq_{\mathcal{F}} N_{t[u]}$, $\|N^*_u\| \leq \mathrm{LS}(\mathbf{T})$ and

$$u \subseteq v \in [|M|]^{<\aleph_0} \quad \text{implies} \quad N^*_u \prec N^*_v \text{ and } t[u] \leq_I t[v].$$

For $U \subseteq |M|$ let $N^*_U := \bigcup\{N^*_u : u \subseteq U \text{ is finite}\}$ (the definitions are compatible). Clearly $U_1 \subseteq U_2 \subseteq |M| \Rightarrow N^*_{U_1} \preceq_{\mathcal{F}} N^*_{U_2} \preceq_{\mathcal{F}} M$. Now we prove by induction on $\theta \leq \|M\|$ that:

($**$) if $U \subseteq \|M\|$, $|U| = \theta$ and $p \in S(N^*_U)$ then for some $u \in [U]^{<\aleph_0}$, p does not fork over N^*_u.

This is enough for clause (β), as by monotonicity p also does not fork over $N_{t[u]}$. For θ finite this is trivial, and for θ infinite $\mathrm{cf}(\mu) \notin \boldsymbol{\kappa}_{\theta+\mathrm{LS}(\mathbf{T})}(\mathbf{T})$ (by 3.5(2)) so ($**$) holds. So we have proved clause (β) and clause (α) follows by 1.24(3), and we are done. ∎

THEOREM 3.8. *Suppose that* $\mathrm{cf}(\kappa) = \kappa \leq \mu < \lambda$ *and* $\mathrm{LS}(\mathbf{T}) < \mu$. *Then*:

(1) *The* (μ, κ)*-saturated model* M *is saturated* (*i.e.*, $N \preceq_{\mathcal{F}} M$, $\|N\| < \|M\|$, $p \in S(N) \Rightarrow p$ *realized in* M, *and hence unique*). *Hence there is a saturated model in* K_μ.

(2) *The union of a continuous* $\preceq_{\mathcal{F}}$*-chain of length* κ *of saturated models from* K_μ *is saturated.*

(3) *In part* (1) *we can replace saturated by* (μ, μ)*-saturated if* $\mu = \mathrm{LS}(\mathbf{T})$. *In part* (2) *we can replace saturated by* μ*-saturated if* $\mu > \mathrm{LS}(\mathbf{T})$.

Proof. (1), (2) Suppose that $M = M_\kappa$ and $\langle M_i : i \leq \kappa\rangle$ is a continuous $\preceq_{\mathcal{F}}$-chain of members of K_μ such that for the proof of (1), M_{i+1} is a universal extension of M_i, and for the proof of (2), M_{i+1} is saturated. Let $i \leq j \leq \kappa$. Then $M_i \preceq_{\mathrm{nice}} M_j$ (by [2, 5.4], or more exactly by Hypothesis 3.1). So there is an operation $\mathrm{Op}_{i,j}$ such that $M_i \preceq_{\mathcal{F}} M_j \preceq_{\mathcal{F}} \mathrm{Op}_{i,j}(M_i)$. It follows that there is an expansion $M^+_{i,j}$ of M_j by at most $\mathrm{LS}(\mathbf{T})$ Skolem functions such that if N is a submodel of $M^+_{i,j}$, then

$$\begin{array}{ccc} & M_j & \\ M_i & \bigcup\!\!\!\!\!\bigcup & N\restriction M_j. \\ & (N \cap M_j \restriction M_i) & \end{array}$$

[Why? as we use operations coming from equivalence relations with $\leq \kappa^*$ classes and $\mathrm{LS}(\mathbf{T}) \geq \kappa^*$ by its definition. More fully, letting $\mathrm{Op}_{i,j}(N) = N^I_D/G$, every element $b \in M_j$ being in $\mathrm{Op}_{i,j}(M_i)$ has a representation as the equivalence class of $\langle x^b_t : t \in I\rangle/D$ under $\mathrm{Op}_{i,j}$, with $x^b_t \in M_i$ and $|\{x^b_t : t \in I\}| \leq \kappa^*$. The functions of $M^+_{i,j}$ are the Skolem functions of M_j and M_i and functions F_ζ ($\zeta < \kappa^*$) such that $\{F_\zeta(b) : \zeta < \kappa^*\} \supseteq \{x^b_t : t \in I\}$.]

If $\kappa = \mu$, the theorem is immediate as κ is regular and $\mu > \mathrm{LS}(\mathbf{T})$. So we will suppose that $\kappa < \mu$. Suppose $N \preceq M = M_\kappa$, $\|N\| < \mu$ and $p \in S(N)$. Let $\chi := \|N\| + \kappa + \mathrm{LS}(\mathbf{T})$ so $\chi < \mu$ hence $\chi^+ < \lambda$. Without loss of generality there is no N_1 with $N \preceq_{\mathcal{F}} N_1 \prec M_\kappa$, $\|N_1\| \leq \chi$ and p_1 with $p \subseteq p_1 \in S(N_1)$ such that p_1 forks over N (by 3.3 but not used). If there is $i < \kappa$ such that $N \subseteq M_i$, then p is realized in M_{i+1}. By the choice of the models $M^+_{i,j}$, it is easy to find N' such that $N \preceq N' \preceq M_\kappa$, $\|N'\| = \chi := \|N\| + \kappa + \mathrm{LS}(\mathbf{T})$ and, for every $i \leq \kappa$,

$$\begin{array}{ccc} & M_\kappa & \\ M_i & \bigcup\!\!\!\!\!\bigcup & N'. \\ & M_i \cap N' & \end{array}$$

Now let $N_i = N' \cap M_i$ and note that $N_\kappa = N'$. The sequence $\langle N_i : i \leq \kappa\rangle$ is continuous increasing and there is an extension p' of p in $S(N_\kappa) = S(N')$. Hence there exists $i < \kappa$ such that $(i \leq j < \kappa) \Rightarrow (p'$ does not fork over $N_j)$. If we are proving part (2), then M_{i+1} is saturated but $\|M_i\| = \mu > \kappa = \|N_{i+1}\|$ and hence there is $a \in M_{i+1}$ realizing $p'\restriction N_{i+1}$. But by the non-forking relation above, $\mathrm{tp}(a, N', M_\kappa)$ does not fork over N_{i+1}, hence is p' as $\chi^+ < \lambda$, by 1.24(3), so a is as required. If we are proving part (1), M_{i+1} is universal over M_i hence we can find a saturated model $N^* \preceq_{\mathcal{F}} M_{i+1}$ which contains $M_i \cap N'$. Hence we can find $\langle N^*_\varepsilon : \varepsilon < \chi^+\rangle$ which is $\preceq_{\mathcal{F}}$-increasing continuous such that $N_i \preceq_{\mathcal{F}} N^*_\varepsilon \preceq_{\mathcal{F}} M_{i+1}$, $N^*_{\varepsilon+1}$ is a χ-universal extension of N^*_ε and $N^*_0 = M_i \cap N'$, and let $a_\varepsilon \in N^*_{\varepsilon+1}$ be such that $\mathrm{tp}(a_\varepsilon, N^*_\varepsilon, N^*_{\varepsilon+1})$ does not fork over $M_i \cap N'$ and extends $p' \restriction (M_i \cap N')$. By 3.5(1), for some $\varepsilon < \chi^+$ there is N'_ε such that $N' \cup N^*_\varepsilon \subseteq N'_\varepsilon$ and $N^*_\varepsilon \underset{N' \cap M_i}{\overset{N'_\varepsilon}{\bigcup\!\!\!\!\bigcup}} N'$, so a_ε realizes p'. (Recall symmetry and uniqueness of extensions.)

(3) Similar proof for the second sentence. For the first sentence, note that by 3.7 we can find a template Φ such that if $I \subset J$ are linear orders of cardinality $< \lambda$, then $\mathrm{EM}(I, \Phi) \prec_{\mathcal{F}} \mathrm{EM}(J, \Phi) \in K_{<\lambda}$, and every $p \in S(\mathrm{EM}(I, \Phi))$ is realized in $\mathrm{EM}(J, \Phi)$. Now, if $\langle M_i : i \le \kappa\rangle$ is as above, then we can prove that M_κ is isomorphic to $\mathrm{EM}(I_\kappa, \Phi)$ when $\langle I_i : i \le \kappa\rangle$ is an increasing continuous sequence of linear orders each of cardinality μ with $|I_{i+1} \setminus I_i| = \mu$. Hence the isomorphism type of $\mathrm{EM}(I_\kappa, \Phi)$ does not depend on I_κ as long as $|I_\kappa| = \mu$, so it does not depend on κ. Note that we can even do it over M_0 (i.e., expand by adding individual constants for each member of M_0). ■

Remark. Using categoricity we can prove 3.8 also by 1.20(2) (and uniqueness).

Conclusion 3.9. *Assume* $\mathrm{LS}(\mathbf{T}) \le \kappa < \mu < \lambda$ *and* $M \in K_\mu$ *is not* κ^+*-saturated; let* $\langle N^*_u : u \in [|M|]^{<\aleph_0}\rangle$ *and* N^*_U *(for* $U \subseteq |M|$*) be as in the proof of* 3.7(2) *(for* $I = \{\emptyset\}$, $N_\emptyset = M$*). Then there is* $U \subseteq |M|$, $|U| \le \kappa$ *and* $p \in S(N^*_U)$*, i.e., there are* N^+ *with* $N^*_U \preceq_{\mathcal{F}} N^+ \in K_\kappa$ *and* $a^+ \in N^+$ *satisfying* $(a^+, N^+)/E_{N^*_U} = p$*, such that for no* $a \in M$ *do we have*

$$u \in [U]^{<\aleph_0} \;\Rightarrow\; \mathrm{tp}(a, N^*_u, M) = \mathrm{tp}(a^+, N^*_u, N^+).$$

Equivalently: without loss of generality $N^+ \cap M = N^*_U$ *and we can define* N^+_u *for* $u \in [|N^+|]^{<\aleph_0}$ *such that* $\langle N^+_u : u \in [|N^+|]^{<\aleph_0}\rangle$ *is as in the proof of* 3.7(2), *and* $u \in [U]^{<\aleph_0} \Rightarrow N^+_u = N^*_u$ *and for no* $u_0 \in [|M|]^{<\aleph_0}$, $v_0 \in [|N^+|]^{<\aleph_0}$, $a^+ \in N^*_{v_0}$, *and* $a \in N^*_{u_0}$ *do we have*

$$\bigwedge_{u \in [U]^{<\aleph_0}} \mathrm{tp}(a, N^*_u, N^*_{u \cup u_0}) = \mathrm{tp}(a^+, N^+_u, N^+_{u \cup v_0}).$$

Proof. By 3.7. ■

Corollary 3.10. (1) *If* $\mathbf{T}$ *is* λ*-categorical and* $\mathrm{LS}(\mathbf{T}) < \mu < \lambda$, $\mathrm{LS}(\mathbf{T}) \le \chi$, $\delta(*) = (2^{\mathrm{LS}(\mathbf{T})})^+$ *and* $\beth_{\delta(*)}(\chi) \le \mu$ *then every* $M \in K_\mu$ *is* χ^+*-saturated. In fact for some* $\delta < \delta(*)$ *we can replace* $\delta(*)$ *by* δ.

(2) *If* $\mu = \beth_{(2^\chi)^+ \times \delta}$, δ *a limit ordinal then* $\mathbf{T}$ *is* μ*-categorical.*

Proof. By 3.9 (and 1.17(2), that is, 1.17(1)+ 1.14(1)) this problem is translated to an omitting type argument + cardinality of a predicate which holds (see [14, VIII, §4], [14, VII, §5] for a parallel result for first order logic, pseudo elementary classes, done independently in 1968 by G. Chudnovskii, J. Keisler and S. Shelah). See more on this in [12] and better [16]. The translated problem is: for $(\kappa, \lambda_1, \lambda_2)$ consider the statement:

$Q(\kappa, \lambda_1, \lambda_2)$ For a vocabulary L^* of cardinality $\leq \kappa$ and set Γ of 1-types (or $< \omega$-types, it does not matter), and a unary predicate P, the existence of an L-model M_1 omitting every $p \in \Gamma$ satisfying $\|M_1\| = \lambda_1 > |P_1^M| \geq \kappa$ implies the existence of an L-model M_2 omitting every $p \in \Gamma$ and satisfying $\|M_2\| = \lambda_2 > |P_2^M| \geq \kappa$.

So by 3.9 we see that $Q(\mathrm{LS}(\mathbf{T}), \lambda_1, \lambda_2)$, $\mathbf{T}$ categorical in $\lambda = \lambda_1 > \mathrm{LS}(\mathbf{T})$ and $\lambda_2 < \lambda_1$ implies $\mathbf{T}$ is categorical in λ_2 (the need for $\lambda_2 < \lambda_1$ is as only over models in $K_{<\lambda}$ do we somewhat understand types). ■

Proposition 3.11. [$\mathbf{T}$ *categorical in* λ]

(1) *If* $\langle M_i : i \leq \delta \rangle$ *is* $\preceq_{\mathcal{F}}$*-increasing continuous*, $M_i \in K_{<\lambda}$, $p \in S(M_\delta)$ *then for some* $i < \delta$, p *does not fork over* M_i.

(2) *If* $N \in K_{<\lambda}$ *and* $p, q \in S(N)$ *does not fork over* M, $M \preceq_{\mathcal{F}} N \in K_{<\lambda}$ *then* $p = q \Leftrightarrow p{\restriction}M = q{\restriction}M$. *Moreover, if* $M \preceq_{\mathcal{F}} N \preceq_{\mathcal{F}} N^+$ *and* $a \in N^+$ *then*

$$N \underset{M}{\overset{N^+}{\bigcup\!\!\!\!\!\bigcup}} a \ \Leftrightarrow\ a \underset{M}{\overset{N^+}{\bigcup\!\!\!\!\!\bigcup}} N.$$

(3) *If* $M \preceq_{\mathcal{F}} N \in K_{<\lambda}$ *and* $p \in S(M)$ *then there is* $q \in S(N)$ *extending* p *not forking over* M.

(4) *If* $M_0 \preceq_{\mathcal{F}} M_1 \preceq_{\mathcal{F}} M_2 \in K_{<\lambda}$, $p \in S(M_2)$, *and* $p{\restriction}M_{l+1}$ *does not fork over* M_l *for* $l = 0, 1$ *then* p *does not fork over* M_0.

(5) *If* $\mu, \delta < \lambda$, $M_i \in K_{\leq\mu}$ *for* $i < \delta$ *is* $\preceq_{\mathcal{F}}$*-increasing continuous*, $p_i \in S(M_i)$, $[j < i \Rightarrow p_j \subseteq p_i]$, *then there is* $p \in S(M_\delta)$ *such that* $i < \delta \Rightarrow p_i \subseteq p_\delta$.

Proof. (1) Otherwise we can find N with $M_\delta \preceq_{\mathcal{F}} N \preceq_{\mathcal{F}} \mathrm{Op}(M_\delta)$, $N \in K_\lambda$ such that $N \preceq_{\mathcal{F}} N^* := \bigcup_{i<\delta} \mathrm{Op}(M_i)$. As $N \in K_\lambda$, e.g. by 1.16(1), N is saturated so let $a \in N$ realize p; so for some i, $a \in \mathrm{Op}(M_i)$ and let $N_i' \preceq_{\mathcal{F}} \mathrm{Op}(M_i)$ be such that $M_i \cup \{a\} \subseteq N_i'$; clearly $M_\delta \underset{M_i}{\overset{N^*}{\bigcup\!\!\!\!\!\bigcup}} N_i'$. Hence $M_\delta \underset{M_i}{\overset{N^*}{\bigcup\!\!\!\!\!\bigcup}} a$, and hence, by part (2), $\mathrm{tp}(a, M_\delta, N^*)$ does not fork over M_i, so it is $\neq p$.

(2) The first sentence follows from the second as in the proof of 1.16(3). If the second fails then we can contradict stability in $\|N\|$ (holds by 1.16(5)), by a proof just as in 1.6(2).

(3) We can find an operation Op with $\|\mathrm{Op}(M)\| \geq \lambda$, so by 1.16(2) in $\mathrm{Op}(M)$ some $\overline{a}$ realizes p so $q = \mathrm{tp}(\overline{a}, N, \mathrm{Op}(N))$ is as required (actually done e.g. in 1.22).

(4) By part (3) there is $q \in S(M_2)$ such that $q \restriction M_0 = p \restriction M_0$ and q does not fork over M_0. Now by 1.21(3) usually and part (2) of the present proposition in general the type $q \restriction M_1$ does not fork over M_0; hence by 1.24(3), $q \restriction M_1 = p \restriction M_1$, and hence by the same argument $q = p$.

(5) Case 1: $\mathrm{cf}(\delta) > \aleph_0$. For every limit $\alpha < \delta$ for some $i < \alpha$, p_α does not fork over M_i. By Fodor's lemma, for some $i < \delta$ and stationary $S \subseteq \delta$ we have

$$j \in S \;\Rightarrow\; p_j \text{ does not fork over } M_i.$$

So the stationarization of p_i in $S(M_\delta)$ (which exists by 1.22 or use part (3)) is as required.

Case 2: $\mathrm{cf}(\delta) = \aleph_0$. So we can assume $\delta = \omega$. Here chasing arrows (using amalgamation) suffices. ∎

Lemma 3.12. *In $K_{<\lambda}$ we can define* $\mathrm{rk}(\mathrm{tp}(a, M, N))$ *with the right properties. That is*:

(A) *If $M \prec_{\mathcal{F}} N \in K_{<\lambda}$, $\overline{a} \subseteq N$, $M \in \bigcup_{\mu^+ < \lambda} K_\mu$, $p = \mathrm{tp}(\overline{a}, M, N)$ then*

$\mathrm{rk}(p) \geq \alpha$ *iff* *for every $\beta < \alpha$ there are p', M' such that $M \prec_{\mathcal{F}} M' \in \bigcup_{\mu^+ < \lambda} K_\mu$, $p' \in S(M')$, $p' \restriction M = p$, $\mathrm{rk}(p') \geq \beta$ and p' forks over M.*

(B) *For all M, N, $\overline{a}$, p as above*, $\mathrm{rk}(p)$ *is an ordinal.*

(C) *If $M_1 \prec_{\mathcal{F}} M_2 \in \bigcup_{\mu^+ < \lambda} K_\mu$ and $p_2 \in S(M_2)$, then $\mathrm{rk}(p_2 \restriction M_1) \geq \mathrm{rk}(p_2)$ and equality holds iff p_2 does not fork over M_1 and then $p_2 \restriction M_1$ (and M_2) determines p_2.*

(D) *If $\langle M_i : i \leq \delta \rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous, $M_i \in \bigcup_{\mu^+ < \lambda} K_\mu$ and $p_\delta \in S(M_\delta)$ then for some $i < \delta$ we have*

$$j \in [i, \delta] \;\Rightarrow\; \mathrm{rk}(p_\delta) = \mathrm{rk}(p_\delta \restriction M_j).$$

Proof. Straightforward; in fact by 3.11 we can use $K_{<\lambda}$ instead of $\bigcup_{\mu^+ < \lambda} K_\mu$. ∎

Lemma 3.13. *Assume $\mu \geq \mathrm{LS}(\mathbf{T})$ and $\mu^+ < \lambda$. If $M \in K_\mu$ is saturated (for $\mu = \mathrm{LS}(\mathbf{T})$ means (μ, μ)-saturated) and $p \in S(M)$ then there are N and a such that $N \in K_\mu$ is saturated, $a \in N$, $\mathrm{tp}(a, M, N) = p$ and N is isolated over $M \cup \{a\}$ (where we say that N is isolated over $M \cup \{a\}$ when $M \preceq_{\mathcal{F}} N$, $a \in N \in K_{<\lambda}$ and: if $N \preceq_{\mathcal{F}} N^+ \in K_{<\lambda}$ and $M \preceq_{\mathcal{F}} M^* \preceq_{\mathcal{F}} N^+$, and $\mathrm{tp}(a, M, N^+)$ does not fork over M then $M^* \underset{M}{\overset{N^+}{\bigcup}} N$).*

REMARK. As in [7, Ch. V] (or Makkai and Shelah [3, 4.22]) because we have 3.5(1) (by 3.6).

Proof of Lemma 3.13. We can find $\langle M'_n : n < \omega\rangle$ such that $M'_n \in K_\mu$ is saturated, M_{n+1} is saturated over M'_n, hence by definition $\bigcup_{n<\omega} M'_n$ is $(\mu, \aleph_0)$-saturated over M'_n and so is saturated; therefore by 3.8 we can assume it is M, so by 3.11(1), p does not fork over M'_n for some n, by renaming p does not fork over M'_0; note also that by 3.8, M is saturated over M'_0. We try to choose, by induction on $\alpha < \mu^+$, (M_α, N_α) such that

(a) $M_\alpha \in K_\mu$ is $\preceq_{\mathcal{F}}$-increasing continuous,
(b) $N_\alpha \in K_\mu$ is $\preceq_{\mathcal{F}}$-increasing continuous,
(c) M_α, N_α are saturated, $M_\alpha \preceq_{\mathcal{F}} N_\alpha$,
(d) $M_0 = M$, $a \in N_0$, $\mathrm{tp}(a, M_0, N_0)$ is p,
(e) if $\alpha = \beta+1$ with β successor, then $M_{\beta+1}$ is $(\lambda, \aleph_0)$-saturated over M_β,
(f) if $\alpha = \beta+1$ with β successor, then $N_{\beta+1}$ is $(\lambda, \aleph_0)$-saturated over N_β,
(g) $\mathrm{tp}(a, M_\alpha, N_\alpha)$ does not fork over M_0,
(h) $M_{\alpha+1} \underset{M_\alpha}{\overset{N_{\alpha+1}}{\uplus}} N_\alpha$ if α is a limit ordinal.

For $\alpha = 0$ just choose (M_0, N_0) to satisfy (c) for $\alpha = 0$ and (d); and let, e.g., $(M_1, N_1) = (M_0, N_0)$. For $\alpha = \beta + 2$ just satisfy (e)+(f) (and $M_\alpha \preceq_{\mathcal{F}} N_\alpha$ in K_μ), possible by 1.22 + 1.16(6). For α limit take unions (the results are saturated by definition, and (g) holds by 3.5(2)). Lastly for $\alpha = \beta + 1$ with β limit, if there are no such M_α, N_α then N_β is isolated over $M_\beta \cup \{a\}$.

Now both M_β and $M = M_0 = \bigcup_{n<\omega} M'_n$ are saturated over M'_0, and hence there is an isomorphism f from M_β onto M which is the identity over M'_0. By uniqueness of non-forking extensions, f maps $\mathrm{tp}(a, M_\beta, N_\beta)$ to p. Renaming we find that f is the identity and letting $N = N_\beta$ we get the desired conclusion. But if we succeed in carrying out the induction we get a contradiction to 3.6; so we are done.

Note that for a limit ordinal β, the model M_β is $(\mu, \mathrm{cf}(\mu))$-saturated over M_γ for any $\gamma < \beta$ and N_β is $(\mu, \mathrm{cf}(\mu))$-saturated over N_γ for any $\gamma < \beta$. ■

PROPOSITION 3.14. *If $M \preceq_{\mathcal{F}} N$ are in K_μ, $\mu \geq \mathrm{LS}(\mathbf{T})$, $\mu^+ < \lambda$, and $a \in N \setminus M$, then we can find saturated $M', N' \in K_\mu$ such that $M \preceq_{\mathcal{F}} M' \preceq_{\mathcal{F}} N'$, $N \preceq_{\mathcal{F}} N'$, $\mathrm{tp}(a, M', N')$ does not fork over M'; and N' is isolated over $M' \cup \{a\}$, M' is saturated over M, and N' is saturated over N.*

Proof. Contained in the proof of 3.13. ■

PROPOSITION 3.15. *If $\mu \in [\mathrm{LS}(\mathbf{T}), \lambda)$, $M \in K_\mu$ is saturated and $p \in S(M)$ then for some saturated $N \in K_\mu$, $M \preceq_{\mathcal{F}} N$, and $a \in N$, we have $\mathrm{tp}(\overline{a}, M, N)) = p$ and N is locally isolated over $M \cup \{a\}$, which means*:

($\boxtimes$) *$M \preceq_{\mathcal{F}} N \in K_{<\lambda}$, $a \in N$ and if $N \preceq_{\mathcal{F}} N^+ \in K_\lambda$, $M \preceq_{\mathcal{F}} M^* \preceq_{\mathcal{F}} N^+$, $M^* \in K_{<\lambda}$ and $\operatorname{tp}(a, M^*, N^+)$ does not fork over M ($\preceq_{\mathcal{F}} M^*$) and $A \subseteq M^*$ is finite, then $A \underset{M}{\overset{N^+}{\bigcup}} N$.*

Proof. Usually we can use 3.14. A problem arises only if $\mu^+ = \lambda$. We can find $\langle M'_i : i \leq \mu \rangle$ which is $\preceq_{\mathcal{F}}$-increasing continuous, $\|M'_i\| = |i| + \mathrm{LS}(\mathbf{T})$, $M'_\mu = M$, M'_i is saturated, M'_{i+1} is universal over M'_i and p does not fork over M_0 (recall 3.11(1)).

Now choose, by induction on $i \leq \mu$, (M_i, N_i, a) such that:

(a) $M_0 = M'_0$,
(b) $\|M_i\| = \|N_i\| = |i| + \mathrm{LS}(\mathbf{T})$,
(c) for i non-limit, (M_i, N_i, a) is as in 3.13 (with $|i| + \mathrm{LS}(\mathbf{T})$ instead of μ), that is, N_i is isolated over $M_i \cup \{a\}$,
(d) $\operatorname{tp}(a, M_0, N_0) = p \restriction M'_0$,
(e) $\langle M_i : i \leq \mu \rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous,
(f) $\langle N_i : i \leq \mu \rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous,
(g) $\operatorname{tp}(a, M_{i+1}, N_{i+1})$ does not fork over M_i (hence is the stationarization of $\operatorname{tp}(a, M_0, N_0) = p \restriction M'_0$, that is, does not fork over $M'_0 = M_0$),
(h) M_{i+1} is saturated over M_i and N_{i+1} is saturated over N_i,
(i) $M_i \preceq_{\mathcal{F}} N_i$.

There is no problem, so as M_μ is saturated and in K_μ, $M_0 = M'_0$ has cardinality $< \mu$ and uniqueness of non-forking extensions (3.11), we can assume $M_\mu = M$ and let $N_\mu = N$. For any candidates N^+, A, M^*, as in the definition of "N is locally isolated over $M \cup \{a\}$" assume toward contradiction that $N \underset{M}{\overset{N^+}{\biguplus}} A$; as A is finite, by 3.11(1), for some $i < \mu$, the type $\operatorname{tp}(A, M, N^+)$ does not fork over M_i, and for some $j < \mu$ the type $\operatorname{tp}(A, N, N^+)$ does not fork over N_j. We can assume $i = j$ is a successor ordinal and $\operatorname{tp}(A \cup \{a\}, M)$ does not fork over M_{i-1}. So as $N \underset{M}{\overset{N^+}{\biguplus}} A$, necessarily $\operatorname{tp}(A, N_i, N^+)$ forks over M_i, hence (by clause (c) above), $a \underset{M_i}{\overset{N^+}{\biguplus}} A$. But by construction M and M_i are saturated over M_{i-1}, and hence there is an isomorphism f from M_i onto M which is the identity over M_{i-1}. So by using uniqueness of non-forking extensions, it maps $\operatorname{tp}(A \cup \{a\}, M_i, N^+)$ to $\operatorname{tp}(A \cup \{a\}, M, N^+)$ and hence $a \underset{M}{\overset{N^+}{\biguplus}} A$ (by 1.21(4)). Thus we get $a \underset{M}{\overset{M^*}{\biguplus}} M^*$, contradiction to the choice of N^+, A, M^*.

Alternatively repeat the proof of 3.13 using 3.11(2)'s second sentence. ∎

THEOREM 3.16. *Assume λ is a successor cardinal, i.e., $\lambda = \lambda_0^+$. Then $\mathbf{T}$ is categorical in every $\mu \in [\beth_{(2^{\mathrm{LS}(\mathbf{T})})^+}, \lambda)$ (really for some $\mu_0 < \beth_{(2^{\mathrm{LS}(\mathbf{T})})^+}$, $\mu \in [\mu_0, \lambda)$ suffices).*

Proof. As in [3]. By 3.10, for some $\mu_1 < \beth_{(2^{\mathrm{LS}(\mathbf{T})})^+}$ every $M \in K_{[\mu_1,\lambda]}$ is $\mathrm{LS}(\mathbf{T})^+$-saturated. Let $\mu \in [\mu_1, \lambda)$, and assume $M \in K_\mu$ is not saturated, so for some $\kappa \in (\mathrm{LS}(\mathbf{T}), \mu)$ the model M is κ-saturated but not κ^+-saturated. Let p, $\langle N_u^* : u \in [\|M\|]^{<\aleph_0}\rangle$, U, N^+, $\langle N_u^+ : u \in [\|N^+\|]^{\aleph_0}\rangle$ be as in 3.9. Let $U_0 = U$. We can assume $N_{U_0}^*$ is saturated, p does not fork over $N_{u^*}^*$ with $u^* \in [U]^{<\aleph_0}$ finite, and $\mathrm{rk}(p)$ is minimal under the circumstances. Now let $b \in M \setminus N_{U_0}^*$, so there are M^+ satisfying $M \prec_{\mathcal{F}} M^+ \in K_\mu$ and $N_1 \preceq_{\mathcal{F}} M^+$ which is μ-isolated over $N_{U_0}^* \cup \{b\}$. By defining more N_u^* we can assume $N_1 = N_{U_1}^*$. So $\mathrm{tp}(b, N_{U_0}^*, M)$ and p are orthogonal (see [7, Ch. V]). Now we deal with orthogonal types and we continue as in [3]: define a $\prec_{\mathcal{F}}$-chain M_i^* $(i < \lambda)$ of saturated models of cardinality λ_0 all omitting some fixed $p \in S(M_0^*)$. ∎

DISCUSSION 3.17. (1) *Below* $\beth_{(2^{\mathrm{LS}(\mathbf{T})})^+}$: The problem is what occurs in $[\mathrm{LS}(\mathbf{T}), \beth_{(2^{\mathrm{LS}(\mathbf{T})})^+})$. As $\mathbf{T}$ is not necessarily complete, for any ψ and $\mathbf{T}$ we can consider $\mathbf{T}' := \{\psi \to \varphi : \varphi \in \mathbf{T}\}$ if $\neg\psi$ has a model in μ iff $\mu < \mu^*$, we get such examples where categoricity can start "late". So we may consider $\mathbf{T}$ complete in $L_{\kappa^*,\omega}$. Hart and Shelah [1] bound our possible improvement but we may want larger gaps, a worthwhile direction.

If $|\mathbf{T}| < \kappa^*$ we may look at what occurs in large enough $\mu < \kappa^*$.

(2) *Below* λ: If λ is a limit cardinal we get only 3.11; this is a more serious issue. The problem is that we can get a μ-saturated but not saturated model in K_{μ^+}, so we get, for $M \in K_\mu$ saturated, two orthogonal types $p, q \in S(M)$ (not realized in M). We want to build a prime model over $M \cup$ (a large indiscernible set for p). Clearly $\mathcal{P}^-(n)$-diagrams are called for.

(3) *Above* λ: In some sense we know every model is saturated: if $M \in K_{>\lambda}$, $N \preceq_{\mathcal{F}} M$, $\|N\| < \lambda$, $p \in S(N)$ then $\dim(p, N, M) = \|M\|$, i.e., if $N \preceq_{\mathcal{F}} N^+ \preceq_{\mathcal{F}} M$ and $\|N^+\| < \|M\|$ when λ is successor, or $\beth_{(2^{\mathrm{LS}(\mathbf{T})})^+}(\|N^+\|)$ when λ is a limit cardinal.

Another way to say it: the stationarization of p over N^+ is realized. But is every $q \in S(N^+)$ a stationarization of some $p \in S(N')$, $N' \preceq_{\mathcal{F}} N^+$, $\|N'\| \le \mathrm{LS}(\mathbf{T})$? We can find $N_0 \preceq_{\mathcal{F}} N^+$ with $\|N_0\| \subseteq |\mathbf{T}|$ such that $[N_0 \preceq_{\mathcal{F}} N_1 \le N^+ \ \&\ \|N_1\| \le \mathrm{LS}(\mathbf{T}) \ \Rightarrow\ q{\restriction}N_1$ does not fork over $N_0]$, we can get it for $\|N_1\| < \mu$, but does it hold for $N_1 = N^+$? A central point is

$(*)$ Does K satisfy amalgamation?

Again it seems that $\mathcal{P}^-(n)$-systems are called for. See more in [5], [4].

(4) If $|\mathbf{T}| < \kappa^*$ we can do better, as $\mathrm{Op}(\mathrm{EM}(I,\Phi)) = \mathrm{EM}(\mathrm{Op}(I),\Phi)$; will be discussed elsewhere.

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