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Smooth conjugacy classes of circle diffeomorphisms with irrational rotation number

by

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Abstract. We prove the C^1-density of every C^r-conjugacy class in the closed subset of diffeomorphisms of the circle with a given irrational rotation number.

1. Introduction. One knows from H. Poincaré's work that the dynamic of a homeomorphism f of the circle depends strongly on the rotation number $\rho(f)$: the existence of periodic orbits is equivalent to the rationality of $\rho(f)$. If, on the contrary, the rotation number is irrational then f is semiconjugate to the corresponding irrational rotation. The non-injectivity of the semiconjugacy consists in collapsing each wandering interval to a point. In the thirties, A. Denjoy exhibited examples of C^1-diffeomorphisms with irrational rotation number but having wandering intervals. He also proved that such a phenomenon cannot appear if f is assumed to be C^2: every C^2-diffeomorphism with irrational rotation number is topologically conjugate to the corresponding irrational rotation. Note that the conjugating homeomorphism (or semiconjugacy) is unique up to composition with a rotation.

However, for a C^2 or C^∞ or even analytic diffeomorphism with irrational rotation number, the conjugating homeomorphism is *in general* not differentiable. The expression *in general* here leads to important and deep results, in particular by V. Arnold [Ar], M. Herman [He] and J. C. Yoccoz [Yo]. Indeed, for rotation numbers satisfying a Diophantine condition, every smooth diffeomorphism is smoothly conjugate to a rotation. Later, different proofs and some generalizations were given by K. Khanin and Ya. Sinai [KS1], [KS2] and Y. Katznelson and D. Ornstein [KO1], [KO2].

In this paper, we consider C^1-diffeomorphisms. In this class of regularity, no arithmetic condition may ensure regularity of the conjugacy homeomorphism. Even if we have not found a precise reference for this statement, it is

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beyond doubt that every irrational rotation number corresponds to infinitely many C^1-conjugacy classes. Let us illustrate this by distinct behaviors of different conjugacy classes:

- The C^1-*centralizer* of a diffeomorphism f is the group of diffeomorphisms commuting with f. Any diffeomorphism g C^1-conjugate to f has a C^1-centralizer conjugate to the one of f (by the same diffeomorphism). Therefore, the isomorphism class of the centralizer is a C^1-invariant for a C^1-conjugacy class: in particular, if f is C^1-conjugate to a rotation then its C^1-centralizer is isomorphic to S^1. There are examples of diffeomorphisms for which the centralizer is trivial, or some dense proper subgroup of $\mathbb{R}$, or much larger than $\mathbb{R}$ if f admits wandering intervals.
- The asymptotic behavior of the iterates f^n leads also to an invariant of a C^1-conjugacy class: if a C^1-diffeomorphism is C^1-conjugate to a rotation, its derivatives df^n are uniformly bounded for $n \in \mathbb{Z}$. However [BCW, Theorem B] implies that for any rotation number there is a C^1-diffeomorphism for which the sequence $\sup\{df^n(x), df^{-n}(x)\}$, $n \in \mathbb{Z}$ is unbounded in any orbit.

All such properties are invariant under C^1-conjugacy, and they show a great variety of C^1-behaviors of C^1-diffeomorphisms with the same irrational rotation number.

In this paper we consider the space of diffeomorphisms having a given irrational rotation number $\alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$. In his thesis Herman denoted by $F_\alpha^r \subset \mathrm{Diff}^r(S^1)$ the closed subset of C^r-diffeomorphisms whose rotation number is α. He proved several results on F_α^r: it is connected, and F_α^s for $s > r$ is dense in F_α^r for the C^r-topology. As mentioned above, F_α^1 always contains many different C^1-behaviors. The aim of this paper is to show that these behaviors are indeed equidistributed in F_α^1, giving some homogeneity of this space. More precisely:

Given any $f \in \mathrm{Diff}^1(S^1)$ and $r \in \mathbb{N}$, we denote by $\mathcal{C}^r(f)$ the C^r-conjugacy class $\{hfh^{-1} : h \in \mathrm{Diff}^r(S^1)\}$; notice that all elements in $\mathcal{C}^1(f)$ share all the C^1-properties of f (same C^1-centralizer, same distortion properties, etc.). We prove:

Theorem 1.1. *Given any $\alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ and any $f \in F_\alpha^1$, the C^1-conjugacy class $\mathcal{C}^1(f)$ of f is dense in F_α^1 for the C^1-topology.*

Approaching the conjugation diffeomorphism h by some smooth diffeomorphism, one finds that $\mathcal{C}^r(f)$ is also C^1-dense, for every $r \in \mathbb{N}$.

The same kind of question can also be considered for the rational rotation number case. That case is closely related to the question of conjugacy classes of diffeomorphisms of $[0,1]$; this problem is solved in [Fa], which gives complete (and different) answers to two natural questions:

- Under what conditions the C^1-conjugacy class of a diffeomorphism f of $[0,1]$ contains g in its closure?
- Under what conditions does there exist a path h_t, $t \in [0,1)$, of diffeomorphisms such that $h_0 = \mathrm{id}$ and $h_t f h_t^{-1}$ tends to g as $t \to 1$?

This approach suggests a natural question in our setting:

QUESTION 1. *Given $f, g \in F_\alpha^1$, does there exist a path h_t, $t \in [0,1)$, of diffeomorphisms of S^1 such that $h_0 = \mathrm{id}$ and $h_t f h_t^{-1}$ tends to g as $t \to 1$?*

After our results and the ones of [Fa] were announced, A. Navas [Na] found a very simple, elementary and clever argument that partially answers this question. He showed

THEOREM 1.2 (Navas). *Given any irrational α and $f \in F_\alpha^1$, there is a path h_t, $t \in [0,1)$, of diffeomorphisms of S^1 such that $h_0 = \mathrm{id}$ and $h_t f h_t^{-1}$ tends to R_α as $t \to 1$.*

Navas' argument consists in building the derivatives of the conjugacy h_t as an approximate solution of a cohomological equation, the rotation R_α being characterized in F_α^1 by the vanishing logarithm of its derivative. This argument does not seem to be adaptable to going from f to g when g is not smoothly conjugate to a rotation.

Notice that a similar result had been proved by Herman [He] for C^2-diffeomorphisms: he proved in that setting that f can be conjugate arbitrarily close to the rotation in the $C^{1+\text{bounded variations}}$ topology.

Given $(f_0, g_0) \in \mathrm{Diff}^1(S^1) \times \mathrm{Diff}^1(S^1)$ and $r \in \mathbb{N}$, we denote by $\mathcal{C}^r(f_0, g_0)$ the C^r-conjugacy class $\{(f,g) : f = h f_0 h^{-1}$ and $g = h g_0 h^{-1}$ for some $h \in \mathrm{Diff}^r(S^1)\}$. One of our motivations for this paper is the same question for commuting diffeomorphisms.

QUESTION 2. *Given two irrational numbers α, β, we consider the space of commuting C^1-diffeomorphisms f, g with rotation numbers α and β, endowed with the C^1-topology.*

Are all the C^1-conjugacy classes dense in this space?

This problem is closely related to a famous old question of Rosenberg: does there exist a pair (f,g) such that the induced $\mathbb{Z}^2$ action is C^r-structurally stable? A positive answer to Question 2 would answer Rosenberg's question negatively for $r = 1$. In that direction, Navas [Na] proved recently that every C^1-conjugacy class contains a pair of rotations (R_α, R_β) in its closure.

Notice that, in higher differentiability, [KN] and [DKN] provide a generalization of the Denjoy theorem for $\mathbb{Z}^n$ actions on the circle by $C^{1+\theta}$-diffeomorphisms, where $\theta \in (0,1)$ depends on n. For smooth actions J. Moser

[Mo] posed the problem of smooth linearization of commuting circle diffeomorphisms. In this direction Fayad and Khanin [FK] proved that a finite number of commuting C^∞ diffeomorphisms with simultaneously Diophantine rotation numbers are smoothly conjugate to rotations.

1.1. Idea of the proof and organization of the paper. The idea of the proof is very simple. Given f and g with the same irrational rotation number, we want to build a conjugate hfh^{-1} of f arbitrarily C^1-close to g. For that, we consider long orbit segments $\{x, \dots, f^n(x)\}$ and $\{y, \dots, g^n(y)\}$ of the same length n. They are ordered in the same way on the circle.

Therefore, one may consider a homeomorphism H of the circle such that $H(f^i(x)) = g^i(y)$ for $0 \leq i \leq n$. Furthermore, we can choose H to be affine on each connected component of the complement of the orbit segment. If n is large enough, and if f and g have dense orbits, all connected components of the complement of each of these segments are arbitrarily small. Thus f and g are almost affine on each component, and the derivative of H on each component is almost the ratio between the component and its image.

Consider now the piecewise C^1-homeomorphism HfH^{-1}. It is also almost affine on each connected component of the complement of the orbit segment $\{y, \dots, g^n(y)\}$. Furthermore, up to the components starting at y or at $g^n(y)$ (i.e. the extremities of the orbit segment) the images of a component under g and under HfH^{-1} are the same. As a direct consequence, their derivatives are almost equal. We show that for the derivatives of HfH^{-1} and g to be everywhere almost equal (that is, even on the components adjacent to y and $g^n(y)$), it is sufficient that the ratios between the lengths of components adjacent to the extremal points x, $f^n(x)$ and y, $g^n(y)$ are the same for f and for g. These ratios of the lengths of the components adjacent to the initial point and end point of the orbit segment are called the *initial* and *final ratios* of f and g.

Then, the announced diffeomorphism h is obtained by smoothing H. This is not so easy because the derivative of H can be very different on the right and the left of a singular point, but Proposition 2.12 solves this difficulty.

Another difficulty comes from the fact that f or g may not have dense orbits, when we deal with C^1-diffeomorphisms. The argument can be adapted to that case, once one notices that one may perform a C^1-conjugacy of f so that the distortion on the wandering interval is arbitrarily small (see Proposition 2.16); thus the diffeomorphism is still almost affine on the complements of long orbit segments.

To conclude the proof it remains to show that one can perform a small perturbation of g so that its initial and final ratios will be equal to the ones of f.

To perform such a perturbation, we would like the components adjacent to the extremal points to be disjoint from their iterates during a long time, allowing their ratios to change slowly. This is not always the case. For that, we need to choose the length n of the orbit segments carefully. We build a sequence of times k_i called *characteristic times* with the property of having a long wandering time. Lemma 4.14 gives a bound of the ratio. This bound allows us to show in Proposition 5.1 that a small perturbation of g at the characteristic times enables us to get any possible initial and final ratios of f, ending the proof.

2. Geometry of orbit segments

2.1. Informal sketch. In this section we define the fundamental tools of the proof: for every diffeomorphism f with irrational rotation number α we consider orbit segments $\{x, \dots, f^n(x)\}$, forbidding some exceptional relative positions of the first and end points; we call them adapted segments.

For adapted segments we define the *initial* and *final ratios* which are the ratios of the lengths of the components adjacent to x and to $f^n(x)$.

We consider diffeomorphisms f and g with the same irrational rotation number and adapted segments $\{f^i(x)\}, \{g^i(y)\}$, $0 \leq i \leq n$, of the same length n. We assume that:

1. the distortion of f and g on each component of the complement of the respective orbit segment is small; this hypothesis is not so hard to get:
   - when f and g have dense orbit (for instance if they are C^2), it is enough to choose sufficiently long orbit segments;
   - if f has wandering intervals, we solve the difficulty in Section 2.6 by conjugating f to a diffeomorphism with small distortion on wandering intervals;
   - if g has wandering intervals, as we just want to approach g, one will perturb g in order to get a C^2-diffeomorphism.
2. The adapted segments $\{f^i(x)\}, \{g^i(y)\}$, $0 \leq i \leq n$, have the same initial and final ratios; achieving this is the hard part of this paper and will be the aim of Sections 3–5.

We consider the piecewise affine homeomorphism H defined by

- $H(f^i(x)) = g^i(y)$, for $0 \leq i \leq n$,
- H is affine on each component of $S^1 \setminus \{f^i(x)\}$.

We notice that our hypotheses imply that HfH^{-1} is a piecewise C^1-homeomorphism whose derivative at each point is close to the one of g. The aim of this section is to build a smooth conjugacy of f to some diffeomorphism close to g by smoothing the homeomorphism H (see Section 2.5).

2.2. Adapted segments, initial and final ratios, and conjugacy. In this paper, S^1 is the oriented circle $\mathbb{R}/\mathbb{Z}$. For $x, y \in S^1$, $[x, y]$ denotes the positively oriented segment joining x to y and (x, y) denotes its interior.

If $X \subset S^1$ is a finite set, then two points $x, y \in X$ are *adjacent* if (x, y) or (y, x) is a connected component of $S^1 \setminus X$. If (x, y) is a component of $S^1 \setminus X$, one says that y is the *first point to the right* of x and x is the *first point to the left* of y in X.

Two different components (x, y) and (y, z) of $S^1 \setminus X$ are called *adjacent*, (y, z) being to the right of (x, y).

Two sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of points of S^1 are *similarly ordered* on the circle if there is an orientation preserving homeomorphism $\varphi\colon S^1 \to S^1$ with $\varphi(x_i) = y_i$ for $i \in \{1, \ldots, n\}$.

Let $\mathrm{Diff}^1(S^1)$ be the set of all C^1-diffeomorphisms of the circle, and $\mathrm{Diff}^1_+(S^1)$ the subset of orientation preserving ones. An *orbit segment of length* n of $f \in \mathrm{Diff}^1(S^1)$ is a sequence $\{x, f(x), \ldots, f^n(x)\}$. The point x is the *initial point* of the segment and $f^n(x)$ is its *final point*.

Given an orbit segment $\{x, f(x), \ldots, f^n(x)\}$, $n \geq 2$, we define its *initial* and *final basic intervals* to be the intervals $[a, b]$ and $[c, d]$, respectively, such that:

- a is the first point to the left of x in the orbit segment,
- b is the first point to the right of x,
- c is the first point to the left of $f^n(x)$,
- d is the first point to the right of $f^n(x)$.

In other words:

- $a, b, c, d \in \{x, f(x), \ldots, f^n(x)\}$,
- $x \in (a, b)$ and $\{x\} = (a, b) \cap \{x, f(x), \ldots, f^n(x)\}$,
- $f^n(x) \in (c, d)$ and $\{f^n(x)\} = (c, d) \cap \{x, f(x), \ldots, f^n(x)\}$.

Lemma 2.1. *Let $f \in \mathrm{Diff}^1_+(S^1)$ with irrational rotation number α. Let $[a, b]$ and $[c, d]$ be the initial and final basic intervals of an orbit segment $\{x, f(x), \ldots, f^n(x)\}$. Let $i, j \in \{1, \ldots, n-1\}$ be such that $c = f^i(x)$ and $d = f^j(x)$. Then*

$$a = f^{n-j}(x) \quad \textit{and} \quad b = f^{n-i}(x).$$

Proof. Any orbit segment $\{x, \ldots, f^n(x)\}$ is similarly ordered to the orbit segment $\{0, \alpha, \ldots, n\alpha\}$ of the rotation R_α.

Consider the symmetry $\sigma\colon t \mapsto n\alpha - t$ of S^1. The symmetry σ leaves the segment $\{0, \alpha, \ldots, n\alpha\}$ globally invariant, and $\sigma(k\alpha) = (n-k)\alpha$. In particular σ keeps the adjacent pairs but reverses right and left, leading to the conclusion. ∎

Definition 2.2. Let $f \in \mathrm{Diff}^1_+(S^1)$. One says that an orbit segment $\{x, \ldots, f^n(x)\}$ is *adapted* if:

- x and $f^n(x)$ are not adjacent,
- the image under f of the final basic segment is not the initial basic segment.

REMARK 2.3. Let $f \in \mathrm{Diff}^1_+(S^1)$ with irrational rotation number, and $\{x, \dots, f^n(x)\}$ be an orbit segment. Let $[a, b]$ and $[c, d]$ be the initial and final basic intervals of this segment, with $c = f^i(x)$ and $d = f^j(x)$. Then the orbit segment is adapted if and only if $i \neq 0$, $j \neq 0$, and $i + j \neq n - 1$.

Proof. ($i \neq 0$ and $j \neq 0$) is equivalent to x and $f^n(x)$ not being adjacent; and $i+j \neq n-1$ is equivalent to $f([f^i(x), f^j(x)]) \neq [f^{n-j}(x), f^{n-i}(x)]$, that is (according to Lemma 2.1), $f([c, d]) \neq [a, b]$. ∎

The next observation is fundamental to our argument:

LEMMA 2.4. *If $\{x, \dots, f^n(x)\}$ is an adapted orbit segment with initial and final basic intervals $[a, b]$ and $[c, d]$ then the open intervals $(f(c), f(d))$ and $(f^{-1}(a), f^{-1}(b))$ are disjoint from the orbit segment $\{x, \dots, f^n(x)\}$.*

The idea is very simple: if one considers the image under f, most of the points of the orbit segment remain in that segment. The point $f^n(x)$ is the only one which goes out of the orbit segment, and $f^{-1}(x)$ is the only one which enters. One deduces that the unique point of the orbit segment which may belong to $(f(c), f(d))$ is x; that means that $f([c, d]) = [a, b]$, which is forbidden by the definition of adapted interval. More precisely:

Proof. As i, j are different from n, by definition, $f(c)$ and $f(d)$ are points of the orbit segment $\{x, \dots, f^n(x)\}$.

Assume for contradiction that $(f(c), f(d)) \cap \{x, \dots, f^n(x)\} \neq \emptyset$. That is, there is $k \in \{0, \dots, n\}$ with $f^k(x) \in (f(c), f(d))$; then $f^{k-1}(x) \in (c, d)$.

If $k \neq 0$, then $f^{k-1}(x)$ belongs to the orbit segment and is different from $f^n(x)$; this contradicts the fact that $(c, d) \cap \{x, \dots, f^n(x)\} = f^n(x)$.

Therefore $k = 0$. In particular k is unique. This means that $(f(c), f(d)) \cap \{x, \dots, f^n(x)\} = x$. Thus $f([c, d]) = [a, b]$, contradicting the definition of adapted segment.

This proves that $(f(c), f(d)) \cap \{x, \dots, f^n(x)\} = \emptyset$.

The proof of $(f^{-1}(a), f^{-1}(b)) \cap \{x, \dots, f^n(x)\} = \emptyset$ is analogous. ∎

We first complete Lemma 2.4 by the following observation:

REMARK 2.5. Let $f \in \mathrm{Diff}^1_+(S^1)$. Let $\{x, \dots, f^n(x)\}$ be an adapted orbit segment of f, and C be a connected component of $S^1 \setminus \{x, \dots, f^n(x)\}$. Then:

- If C is neither $(f^{-1}(a), f^{-1}(b))$, $(c, f^n(x))$ nor $(f^n(x)), d)$, then $f(C)$ is a connected component of $S^1 \setminus \{x, \dots, f^n(x)\}$.
- If C is either $(c, f^n(x))$ or $(f^n(x)), d)$, then $f(C)$ lies in $(f(c), f(d))$, which is a connected component of $S^1 \setminus \{x, \dots, f^n(x)\}$.

- Finally, if $f([f^{-1}(a), f^{-1}(b)]) = [a, x] \cup [x, b]$, then the image of a component covers two components.

One gets the same statement for f^{-1} by replacing $(f^{-1}(a), f^{-1}(b))$, $(c, f^n(x))$ and $(f^n(x)), d)$ by $(f(c), f(d))$, $[a, x]$ and $[x, b]$.

For a given irrational rotation number, the orbit segments are all similarly ordered. Therefore:

Remark 2.6. Given a irrational rotation number α, whether or not an orbit segment $\{x, \dots, f^n(x)\}$ is adapted only depends on its length $n \geq 0$: more precisely, if $f, g \in F^0_\alpha$ and if $\{x, \dots, f^n(x)\}$ is an adapted orbit segment for f, then for every $y \in S^1$, $\{y, \dots, g^n(y)\}$ is an adapted orbit segment.

We are now ready for defining the initial and final ratios, which are fundamental notions used in our argument:

Definition 2.7. Given an adapted orbit segment $\{x, \dots, f^n(x)\}$, we define its *initial* and *final ratios* to be the quotients

$$R_0 = \frac{\ell([a, x])}{\ell([x, b])} \quad \text{and} \quad R_n = \frac{\ell([c, f^n(x)])}{\ell([f^n(x), d])}.$$

2.3. Distortion, initial and final ratios, and conjugacy

Definition 2.8. The *distortion* $\Delta(g, I)$ of $g \in \mathrm{Diff}^1(S^1)$ on some interval I is

$$\Delta(g, I) = \max_{x,y \in I} \log\left(\frac{dg(x)}{dg(y)}\right).$$

Remark 2.9. If I and J are two intervals such that $I \cap J \neq \emptyset$ then $I \cup J$ is an interval and $\Delta(g, I \cup J) \leq \Delta(g, I) + \Delta(g, J)$.

The aim of the next subsection is the proof of the following result, which is an important step in proving Theorem 1.1.

Theorem 2.1. *Let f and g be two diffeomorphisms with irrational rotation number α, and assume that g has dense orbits. Assume that, for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there are:*

- *a diffeomorphism $\tilde{g}$, ε-C^1-close to g,*
- *$n > N$,*
- *adapted orbit segments $\{x, \dots, f^n(x)\}$ and $\{y, \dots, \tilde{g}^n(y)\}$ similarly ordered, and*
- *$h \in \mathrm{Diff}^1_+(S^1)$*

such that the diffeomorphism $\tilde{f} = hfh^{-1}$ satisfies:

– *the (adapted) orbit segments*

$$\{x, \tilde{f}(x), \dots, \tilde{f}^n(x)\} \quad \textit{and} \quad \{y, \tilde{g}(y), \dots, \tilde{g}^n(y)\}$$

have the same initial and final ratios,

– *the distortion of* $\tilde{f}$ *on each connected component of the difference* $S^1 \setminus \{x, \tilde{f}(x), \ldots, \tilde{f}^n(x)\}$ *is less than* ε.

Then there is a sequence $\{h_m\}_{m\in\mathbb{N}}$ *of diffeomorphisms such that* $h_m f h_m^{-1} \to g$ *in the* C^1*-topology as* $m \to \infty$.

2.4. Piecewise linear conjugacy and the proof of Theorem 2.1

LEMMA 2.10. *Let* f, g *be diffeomorphisms, and* $\varepsilon > 0$. *Assume that* f *and* g *admit adapted orbit segments* $\{x, \ldots, f^n(x)\}$ *and* $\{y, \ldots, g^n(y)\}$ *similarly ordered and with the same initial and final ratios. Assume furthermore that the distortion of* f *and* g *on each connected component of the complement of the respective orbit segment is bounded by* ε.

Consider the piecewise affine homeomorphism H *defined as* $H(f^i(x)) = g^i(y)$ *for* $i \in \{0, \ldots, n\}$ *and* H *is affine on each connected component of the complement of the orbit segment of* f. *Then:*

(i) H *is differentiable at* x *and at* $f^n(x)$.
(ii) HfH^{-1} *is a piecewise* C^1*-homeomorphism.*
(iii) HfH^{-1} *is* C^1 *on* $S^1 \setminus \{y, \ldots, g^n(y)\}$; *the derivatives of* HfH^{-1} *and* $Hf^{-1}H^{-1}$ *are defined and continuous on the closure of each connected component of* $S^1 \setminus \{y, \ldots, g^{n-1}(y)\}$ *and* $S^1 \setminus \{g(y), \ldots, g^n(y)\}$, *respectively.*
(iv) *The right and left derivatives of* HfH^{-1} *are well defined at every point* $y \in S^1$ *and are close to the derivative of* g *at* y. *More precisely,*

$$\exp(-4\varepsilon) \leq \frac{d^{\pm}(HfH^{-1})(y)}{dg(y)} \leq \exp(4\varepsilon)$$

where d^- *and* d^+ *denote the left and right derivative respectively.*

Proof. The map HfH^{-1} and its inverse are piecewise C^1-homeomorphisms since they are the composition of a diffeomorphism with two piecewise affine homeomorphisms, proving (ii).

Let $[a(f), b(f)]$, $[c(f), d(f)]$ and $[a(g), b(g)]$, $[c(g), d(g]$ denote the initial and final basic segments of f and g, respectively.

Notice that H is affine from $[a(f), x]$ to $[a(g), y]$ and from $[x, b(f)]$ to $[y, b(g)]$. Furthermore the ratios $\frac{\ell([a(f),x])}{\ell([x,b(f)])}$ and $\frac{\ell([a(g),y])}{\ell([y,b(g)])}$ are equal. Therefore

$$\frac{\ell([a(g), y])}{\ell([a(f), x])} = \frac{\ell([y, b(g)])}{\ell([x, b(f)])}.$$

This implies that H has the same right and left derivatives at x, hence is affine in $[a(f), b(f)]$ (and so smooth at x).

The proof that H is affine on $[c(f), d(f)]$ (and so differentiable at $f^n(x)$) is analogous, using the final ratios of f and g. This proves (i).

We have shown that H and H^{-1} are affine on each connected component of $S^1 \setminus \{f(x), \ldots, f^{n-1}(x)\}$ and $S^1 \setminus \{g(y), \ldots, g^{n-1}(y)\}$ respectively. Using the fact that $H(f^i(x)) = g^i(x)$ for $i \in \{0, \ldots, n\}$ one deduces that a point z is non-singular for HfH^{-1} if $z \notin \{g(y), \ldots, g^{n-1}(y)\}$ and $f(H^{-1}(z)) \notin \{f(x), \ldots, f^{n-1}(x)\}$, that is, if $z \notin \{y, \ldots, g^{n-1}(y)\}$. One shows analogously that z is non-singular for $Hf^{-1}H^{-1}$ if $z \notin \{g(y), \ldots, g^n(y)\}$. This shows (iii).

It remains to compare the derivative of HfH^{-1} with the derivative of g. For that, notice that on each connected component C_g of the complement of $\{y, \ldots, g^n(y)\}$ the map is the composition of:

- $H^{-1}\colon C_g \to H^{-1}(C_g)$ which is affine; furthermore $C_f = H^{-1}(C_g)$ is a connected component of $S^1 \setminus \{x, \ldots, f^n(x)\}$,
- $f\colon C_f \to f(C_f)$ which has bounded distortion, by the assumption on f,
- $H\colon f(C_f) \to H(f(C_f))$.

Claim 1. *$H\colon f(C_f) \to H(f(C_f))$ is affine.*

Proof. By Remark 2.5, as the orbit segment $\{x, \ldots, f^n(x)\}$ is adapted, if C_f is neither $(c(f), f^n(x))$, $(f^n(x), d(f))$, nor $(f^{-1}(a(f)), f^{-1}(b(f)))$ then $f(C_f)$ is already a connected component of the complement of $\{x, \ldots, f^n(x)\}$. Thus, in that case, H is affine on $f(C_f)$.

If $C_f = (c(f), f^n(x))$ or $(f^n(x), d(f))$ then $f(C_f)$ lies in $(f(c(f)), f(d(f)))$, which is a connected component of the complement of $\{x, \ldots, f^n(x)\}$, because this orbit segment is adapted. Thus H is affine on $[f(c(f)), f(d(f)]$, hence on $f(C_f)$.

If $f(C_f) = (a(f), b(f))$, then (i) shows that H is affine on $[a(f), b(f)]$, which concludes the proof of the claim. ∎

Summarizing, the restriction of HfH^{-1} to C_g is the composition of affine maps with the restriction of f to a connected component of the complement of $\{x, \ldots, f^n(x)\}$.

Composing with affine maps does not modify distortion. Therefore the distortion of HfH^{-1} on C_g is bounded by ε. By hypothesis on g, the distortion of g on C_g is also bounded by ε.

Claim 2. *If C_g is neither $(c(g), g^n(y))$, nor $(g^n(y), d(g))$, then*

$$HfH^{-1}(C_g) = g(C_g).$$

Proof. The hypothesis implies that C_f is neither $(c(f), f^n(x))$ nor $(f^n(x), d(f))$.

If $C_g = (g^{-1}(a(g)), g^{-1}(b(g)))$ then $f(C_f) = (a(f), b(f))$ and $H(f(C_f)) = (a(g), b(g)) = g(C_g)$. Otherwise, $f(C_f)$ is a connected component of $S^1 \setminus \{x, \ldots, f^n(x)\}$ and $g(C_g)$ is the corresponding connected component of $S^1 \setminus \{y, \ldots, g^n(y)\}$, so that $H(f(C_f)) = g(C_g)$. Hence $HfH^{-1}(C_g) = g(C_g)$. ∎

Thus, if C_g is neither $(c(g), g^n(y))$, nor $(g^n(g), d(g))$, there is at least one point in C_g where the derivatives of HfH^{-1} and g coincide. As a consequence, for every $y \in C_g$,

$$\exp(-2\varepsilon) \leq \frac{d(HfH^{-1})(y)}{dg(y)} \leq \exp(2\varepsilon).$$

We finish this proof by considering the points in $[c(g), d(g)] = [c(g), g^n(y)] \cup [g^n(y), d(g)]$. As g and HfH^{-1} have distortion bounded by ε on $[c(g), g^n(y)]$ and on $[g^n(y), d(g)]$, their distortion on $[c(g), d(g)]$ is bounded by 2ε. Furthermore

$$\begin{aligned} HfH^{-1}([c(g), d(g)]) &= [H(f(c(f)), H(f(d(f))] = [g(c(g)), g(d(g))] \\ &= g([c(g), d(g)]). \end{aligned}$$

The same argument as above now shows that, for every $y \in [c(g), d(g)]$,

$$\exp(-4\varepsilon) \leq \frac{d(HfH^{-1})(y)}{dg(y)} \leq \exp(4\varepsilon). \ \blacksquare$$

REMARK 2.11. Notice that the (right and left) derivatives of HfH^{-1} at every point are ε_0-close to the derivative of g, where $\varepsilon_0 = (\exp(4\varepsilon) - 1)M$ and $M = \sup_{x \in S^1} |dg(x)|$. In formula:

$$\|d^{\pm}(HfH^{-1})(y) - dg(y)\| < (\exp(4\varepsilon) - 1)M.$$

Notice that for small ε one has $\exp(4\varepsilon) - 1 < 5\varepsilon$. Thus, if the constant ε in Lemma 2.10 has been chosen small enough, one gets

$$\|d^{\pm}(HfH^{-1})(y) - dg(y)\| < 5\varepsilon M.$$

PROPOSITION 2.12. *Let f be a C^1-diffeomorphism of the circle, $\varepsilon > 0$ and $\{x, \ldots, f^n(x)\}$ be an adapted orbit segment. Let H be a piecewise affine homeomorphism, smooth off $\{f(x), \ldots, f^{n-1}(x)\}$, such that the right and left derivatives of HfH^{-1} are ε-close at each $y \in S^1$. Then there is a smooth diffeomorphism h arbitrarily C^0-close to H and such that the derivative of hfh^{-1} is 2ε-close to the right and left derivatives of HfH^{-1} at every point.*

We postpone the proof of Proposition 2.12 to the next section.

Proof of Theorem 2.1. Consider f, g satisfying the hypotheses of Theorem 2.1. In particular the orbits of g are assumed to be dense. Let $M > 0$ be such that the derivatives of g and of g^{-1} are bounded by $M/2$. Fix some $\varepsilon_0 > 0$. Let $0 < \varepsilon_1 < \varepsilon_0$ with $\varepsilon_1 + 5(\exp(4\varepsilon_1) - 1)M < \varepsilon_0$.

CLAIM 3. *There is an integer $N > 0$ such that, for every $y \in S^1$, the distortion of g on every connected component of $S^1 \setminus \{y, g(y), \ldots, g^N(y)\}$ is less than $\varepsilon_1/2$.*

Proof. The orbits of g are all dense, so that the length of any connected component of the complement of $\{y, \dots, g^N(y)\}$ tends uniformly to 0 as N tends to infinity.

One concludes by recalling that the logarithm of the derivative of g is uniformly continuous. ∎

From now on, we fix such an $N > 0$.

Claim 4. *There is $0 < \varepsilon < \varepsilon_1$ such that, for any $\tilde{g}$ which is ε-C^1-close to g, and any $y \in S^1$, the distortion of $\tilde{g}$ on any connected component of $S^1 \setminus \{y, \tilde{g}(y), \dots, \tilde{g}^N(y)\}$ is less than ε_1.*

Proof. Assume that the claim is false. Then there are g_i converging to g in the C^1-topology, and for every i an orbit segment $\{y_i, \dots, g_i^N(y_i)\}$ and a connected component C_i of $S^1 \setminus \{y_i, \dots, g_i^N(y_i)\}$ on which the distortion of g_i is larger than ε_1.

Up to taking a subsequence, one may assume that the y_i tend to a point y and the component C_i tends to a component of $S^1 \setminus \{y, \dots, g^N(y)\}$ on which the distortion of g is larger than or equal to ε_1. This contradicts our choice of N. ∎

We fix now some $0 < \varepsilon < \varepsilon_1$ given by Claim 4.

As f and g satisfy the hypotheses of Theorem 2.1, there are

- a diffeomorphism $\tilde{g}$, ε-C^1-close to g,
- $n > N$,
- $\varphi \in \mathrm{Diff}^1_+(S^1)$ and $\tilde{f} = \varphi f \varphi^{-1}$, and
- adapted orbit segments $\{x, \dots, \tilde{f}^n(x)\}$ and $\{y, \dots, \tilde{g}^n(y)\}$

such that

- $\{x, \dots, \tilde{f}^n(x)\}$ and $\{y, \dots, \tilde{g}^n(y)\}$ are similarly ordered, and have the same initial and final ratios,
- the distortion of $\tilde{f}$ on each connected component of $S^1 \setminus \{x, \dots, \tilde{f}^n(x)\}$ is less than ε (hence than ε_1).

Let H be the piecewise affine homeomorphism defined by $H(\tilde{f}^i(x)) = \tilde{g}^i(x)$ for $i \in \{0, \dots, n\}$, and affine on each connected component of $S^1 \setminus \{x, \dots, \tilde{f}^n(x)\}$.

Recall $\tilde{f}$ and $\tilde{g}$ have distortion bounded by ε_1 on each connected component of $S^1 \setminus \{x, \dots, \tilde{f}^n(x)\}$ and $S^1 \setminus \{y, \dots, \tilde{g}^n(y)\}$ respectively. Thus, according to Lemma 2.10, $H\tilde{f}H^{-1}$ is a piecewise C^1-homeomorphism whose left and right derivatives are $(\exp(4\varepsilon_1) - 1)M$-$C^1$-close to the one of $\tilde{g}$, at each point.

The triangular inequality implies that for every $y \in S^1$,

$$|d^+(H\tilde{f}H^{-1})(y) - d^-(H\tilde{f}H^{-1})(y)| < 2(\exp(4\varepsilon_1) - 1)M.$$

According to Proposition 2.12, there is $h \in \mathrm{Diff}^1_+(S^1)$ which is C^0-close to H and such that for every $y \in S^1$ one has

$$|d(h\tilde{f}h^{-1})(y) - d^+(H\tilde{f}H^{-1})(y)| < 4(\exp(4\varepsilon_1) - 1)M.$$

By the triangular inequality one gets

$$|d(h\tilde{f}h^{-1})(y) - d\tilde{g}(y)| < 5(\exp(4\varepsilon_1) - 1)M.$$

Finally, as $|d\tilde{g}(y) - dg(y)| < \varepsilon < \varepsilon_1$ one gets

$$|d(h\tilde{f}h^{-1})(y) - dg(y)| < \varepsilon_1 + 5(\exp(4\varepsilon_1) - 1)M < \varepsilon_0.$$

In other words, for any $\varepsilon_0 > 0$ we have built a diffeomorphism $h\varphi \in \mathrm{Diff}^1_+(S^1)$ so that $h\varphi f\varphi^{-1}h^{-1}$ is ε_0-close to g, concluding the proof. ∎

2.5. Smoothing a piecewise linear conjugacy: proof of Proposition 2.12. We start by linearizing the diffeomorphism f in a neighborhood of an adapted orbit segment:

Lemma 2.13. *Let f be a diffeomorphism of S^1 and $\{x, \dots, f^n(x)\}$ be an orbit segment of a non-periodic point. There is a family of diffeomorphisms $\varphi_t\colon S^1 \to S^1$, $t \in (0, t_0]$, with the following properties:*

- *$\varphi_t(f^i(x)) = f^i(x)$ for all t and $i \in \{0, \dots, n+1\}$,*
- *φ_t tends to $\mathrm{id}|_{S^1}$ in the C^1-topology as $t \to 0$,*
- *the support of φ_t is the union of disjoint intervals $I_{t,i}$ centered at $f^i(x)$, $i \in \{0, \dots, n+1\}$, and whose total length tends to 0 as $t \to 0$,*
- *the derivative of φ_t at $f^i(x)$, $i \in \{0, \dots, n+1\}$, is equal to 1,*
- *the restriction of $\varphi_t f \varphi_t^{-1}$ to each segment $[f^i(x) - t, f^i(x) + t]$ is the orientation preserving affine map*

$$A_i\colon [f^i(x) - t, f^i(x) + t] \to [f^{i+1}(x) - t \cdot df(f^i(x)), f^{i+1}(x) + t \cdot df(f^i(x))].$$

Notice that A_i does not depend on t: it is the affine map sending $f^i(x)$ to $f^{i+1}(x)$ and with derivative equal to $df(f^i(x))$.

Proof. The proof is easy; as it is somewhat technical, let us just give some indications.

If $n = 0$, then x and $f(x)$ are different points, and we only need to choose $\varphi_{t,0}$ to be the identity map on a small neighborhood $I_{t,0}$ of x containing $[x - t, x + t]$ and to coincide with $A_0 \circ f^{-1}$ on $f([x - t, x + t])$ (the interval $I_{t,1}$ contains $f([x - t, x + t])$). Therefore, for every $y \in [x - t, x + t]$ one gets

$$\varphi_{t,0} f \varphi_{t,0}^{-1}(y) = \varphi_{t,0}(f(y)) = (A_0 \circ f^{-1})f(y) = A_0(y).$$

The derivative of $A_0 \circ f^{-1}$ at $f(x)$ is 1 so that, by shrinking t, one may choose $\varphi_{t,0}$ C^1-close to the identity map and with support $I_{t,1}$ tending to $\{f(x)\}$.

One can now prove Lemma 2.13 by induction on n. We fix an orbit segment $\{x, \dots, f^{n+1}(x)\}$ associated to a non-periodic point x, and we assume that $\varphi_{t,n}$ has already been built and satisfies:

- the support of $\varphi_{t,n}$ is the union of disjoint intervals $I_{t,i}$ centered at $f^i(x)$, $i \in \{0, \dots, n+1\}$, and whose lengths tend to 0 as $t \to 0$,
- $\varphi_{t,n} f \varphi_{t,n}^{-1}$ coincides with A_i on $[f^i(x) - t, f^i(x) + t]$ for $i = 0, \dots, n$,
- the derivative of $\varphi_{t,n}$ at $f^i(x)$ is 1 and $\varphi_{t,n} \to \mathrm{id}$ as $t \to 0$.

Let $f_{t,n} = \varphi_{t,n}^{-1} f \varphi_{t,n}$; we build $\varphi_{t,n+1}$ as follows:

- $\varphi_{t,n+1}$ coincides with $\varphi_{t,n}$ on $\bigcup_{i=0}^{n+1} I_{t,i}$,
- $\varphi_{t,n+1}$ coincides with $A_{n+1} f_{t,n}^{-1}$ on $f_{t,n}([f^{n+1}(x) - t, f^{n+1}(x) + t])$,
- the support of $\varphi_{t,n+1}$ is the union of $\bigcup_{i=0}^{n+1} I_{t,i}$ and an interval $I_{t,n+2}$ centered at $f^{n+2}(x)$, containing $f_{t,n}([f^{n+1}(x) - t, f^{n+1}(x) + t])$, and whose length tends to 0 as $t \to 0$.

When t is small, the interval $I_{t,n+2}$ is disjoint from $\bigcup_{i=0}^{n+1} I_{t,i}$. Then it is easy to check that $\varphi_{t,n+1} f \varphi_{t,n+1}^{-1}$ coincides with A_i on $[f^i(x) - t, f^i(x) + t]$ for $i = 0, \dots, n+1$.

One concludes by noticing that the derivative of $\varphi_{t,n+1}$ is 1 at $f^{n+2}(x)$, and $\varphi_{t,n+1}$ can be chosen C^1-close to the identity for small t. ∎

For every positive α, β, we denote by $h_{\alpha,\beta}\colon \mathbb{R} \to \mathbb{R}$ the map defined by

- $x \mapsto \alpha x$ for $x < -1$,
- $x \mapsto \frac{\beta-\alpha}{4} x^2 + \frac{\beta+\alpha}{2} x + \frac{\beta-\alpha}{4}$ for $x \in [-1, 1]$,
- $x \mapsto \beta x$ for $x > 1$.

An elementary calculation shows that

Lemma 2.14. *$h_{\alpha,\beta}$ is a C^1-diffeomorphism of $\mathbb{R}$ whose derivative at each point is in $[\alpha, \beta]$.*

Lemma 2.15. *Given $\alpha, \beta, \gamma, \delta > 0$ and $x \in \mathbb{R}$ one has*

$$\min\left\{\frac{\alpha}{\gamma}, \frac{\beta}{\delta}\right\} \le \frac{dh_{\alpha,\beta}(x)}{dh_{\gamma,\delta}(x)} \le \max\left\{\frac{\alpha}{\gamma}, \frac{\beta}{\delta}\right\}.$$

Proof. The proof is straightforward for $x \notin [-1, 1]$, since the maps $h_{\alpha,\beta}$ and $h_{\gamma,\delta}$ are linear with slope α and γ (if $x < -1$) or β and δ (if $x > 1$).

For $x \in [-1, 1]$ one has $dh_{\alpha,\beta}(x) = \frac{\beta-\alpha}{2} x + \frac{\beta+\alpha}{2} = \frac{1-x}{2}\alpha + \frac{1+x}{2}\beta$ and $h_{\gamma,\delta} = \frac{1-x}{2}\gamma + \frac{1+x}{2}\delta$, so that

$$\frac{dh_{\alpha,\beta}(x)}{dh_{\gamma,\delta}(x)} = \frac{\frac{1-x}{2}\alpha + \frac{1+x}{2}\beta}{\frac{1-x}{2}\gamma + \frac{1+x}{2}\delta}.$$

The stated inequality now follows immediately from the following (classical) claim:

Claim 5. *Let a, b, c, d be positive numbers. Then*

$$\inf\left\{\frac{a}{c}, \frac{b}{d}\right\} \leq \frac{a+b}{c+d} \leq \max\left\{\frac{a}{c}, \frac{b}{d}\right\}.$$

Proof of the claim. Assume $\frac{a}{c} \leq \frac{b}{d}$ (the converse case is similar). Then $a \leq \frac{cb}{d}$. Therefore

$$\frac{a+b}{c+d} \leq \frac{\frac{cb}{d}+b}{c+d} = \frac{\frac{(c+d)b}{d}}{c+d} = \frac{b}{d}.$$

This inequality, applied now to $\frac{c+d}{a+b}$, gives $\frac{c+d}{a+b} \leq \frac{c}{a}$, that is,

$$\frac{a}{c} \leq \frac{a+b}{c+d} \leq \frac{b}{d},$$

which is the desired inequality in that case. ■Claim 5 ■Lemma 2.15

Let H be a piecewise affine homeomorphism of S^1, and $x \in S^1$ a singular point. Let α, β be the right and left derivatives of H at x, and $\eta > 0$ be small enough so that H is affine on $[x-\eta, x]$ and on $[x, x+\eta]$.

We denote by $h_{\alpha,\beta,x,\eta} \colon [x-\eta, x+\eta] \to H([x-\eta, x+\eta])$ the diffeomorphism obtained as follows:

- Let A be the orientation preserving affine diffeomorphism that sends $[x-\eta, x+\eta]$ onto $[-1, 1]$.
- $h_{\alpha,\beta}$ induces a diffeomorphism of $[-1, 1]$ onto $[-\alpha, \beta]$,
- Let B be the orientation preserving affine diffeomorphism sending $H([x-\eta, x+\eta]) = [H(x) - \alpha\eta, H(x) + \beta\eta]$ onto $[-\alpha, \beta]$.

Then

$$h_{\alpha,\beta,x,\eta} = B^{-1} \circ h_{\alpha,\beta} \circ A \colon [x-\eta, x+\eta] \to H([x-\eta, x+\eta]).$$

Notice that:

1. The linear parts of A and B coincide (the derivative is $1/\eta$), therefore the derivative $dh_{\alpha,\beta,x,\eta}(z)$ is $dh_{\alpha,\beta}(A(z))$.
2. The derivatives of $h_{\alpha,\beta,x,\eta}$ and of H coincide at $x-\eta$ and at $x+\eta$.
3. If H is smooth at x, that is, $\alpha = \beta$, then $h_{\alpha,\beta,x,\eta}$ coincides with H.

We are now ready to prove Proposition 2.12:

Proof of Proposition 2.12. Up to replacing f by a conjugate $\varphi_t^{-1} f \varphi_t$ given by Lemma 2.13, one may assume that there is $t > 0$ such that f is affine on each interval $[f^i(x) - t, f^i(x) + t]$ for $i \in \{0, \dots, n\}$.

Notice that, for any $\eta > 0$ small enough, and $i \in \{0, \dots, n+1\}$, the interval $f^i([x-\eta, x+\eta])$ is contained in $[f^i(x) - t, f^i(x) + t]$ where f is affine, and $f^i([x-\eta, x+\eta]) = [f^i(x) - df^i(x) \cdot \eta, f^i(x) + df^i(x) \cdot \eta]$.

Let us denote for simplicity:

- $\eta_i = df^i(x) \cdot \eta$,

- α_i, β_i are the left and right derivatives of H at $f^i(x)$,
- $A_i\colon [f^i(x)-\eta_i, f^i(x)+\eta_i] \to [-1,1]$ and $B_i\colon [H(f^i x)-\alpha_i\eta_i, H(f^i(x))+\beta_i\eta_i] \to [-\alpha_i, \beta_i]$ are the orientation preserving affine maps.

We denote by h_η the diffeomorphism of S^1 defined as follows:

- h_η coincides with H outside $\bigcup_{i=1}^{n-1}[f^i(x)-\eta_i, f^i(x)+\eta_i]$,
- $h_\eta = h_{\alpha_i,\beta_i,f^i(x),\eta_i}$ on $[f^i(x)-\eta_i, f^i(x)+\eta_i]$.

Consider $h_\eta f h_\eta^{-1}$. For $x \notin H([f^i(x)-\eta_i, f^i(x)+\eta_i])$, $i \in \{0,\dots,n-1\}$, we have $h_\eta f h_\eta^{-1}(x) = HfH^{-1}$ so that there is nothing to prove.

Pick $y \in H([f^i(x)-\eta_i, f^i(x)+\eta_i])$. Then

$$h_\eta f h_\eta^{-1}(y) = h_{\alpha_{i+1},\beta_{i+1},f^{i+1}(x),\eta_{i+1}} \circ f \circ h^{-1}_{\alpha_i,\beta_i,f^i(x),\eta_i}(y).$$

Thus, if we set $z = h^{-1}_{\alpha_i,\beta_i,f^i(x),\eta_i}(y)$, then

$$\begin{aligned} d(h_\eta f h_\eta^{-1})(y) &= df(z)\cdot \frac{dh_{\alpha_{i+1},\beta_{i+1},f^{i+1}(x),\eta_{i+1}}(f(z))}{dh_{\alpha_i,\beta_i,f^i(x),\eta_i}(z)} \\ &= df(z)\cdot \frac{dh_{\alpha_{i+1},\beta_{i+1}}(A_{i+1}(f(z)))}{dh_{\alpha_i,\beta_i}(A_i(z))}. \end{aligned}$$

From the fact that f is affine and from the definition of A_i and A_{i+1} one easily checks that $A_{i+1}f = A_i$. This implies

$$d(h_\eta f h_\eta^{-1})(y) = df(z)\cdot \frac{dh_{\alpha_{i+1},\beta_{i+1}}(A_i(z))}{dh_{\alpha_i,\beta_i,}(A_i(z))}.$$

Since $z \in [f^i(x)-\eta_i, f^i(x)+\eta_i]$, one has

$$d(h_\eta f h_\eta^{-1})(y) = df(f^i(x))\cdot \frac{dh_{\alpha_{i+1},\beta_{i+1}}(A_i(z))}{dh_{\alpha_i,\beta_i,}(A_i(z))}.$$

From Lemma 2.15 one deduces that

$$\begin{aligned} df(f^i(x))\min\left\{\frac{\alpha_{i+1}}{\alpha_i}, \frac{\beta_{i+1}}{\beta_i}\right\} &\le d(h_\eta f h_\eta^{-1})(y) \\ &\le df(f^i(x))\max\left\{\frac{\alpha_{i+1}}{\alpha_i}, \frac{\beta_{i+1}}{\beta_i}\right\}. \end{aligned}$$

Recall that the derivative of HfH^{-1} is $\frac{\alpha_{i+1}}{\alpha_i}df(f^i(x))$ on $H([f^i(x)-\eta_i, f^i(x)]$ and is $\frac{\beta_{i+1}}{\beta_i}df(f^i(x))$ on $H([f^i(x), f^i(x)+\eta_i]$. Therefore, the hypothesis on H is that

$$\left|\frac{\alpha_{i+1}}{\alpha_i}df(f^i(x)) - \frac{\beta_{i+1}}{\beta_i}df(f^i(x))\right| < \varepsilon.$$

One deduces that $|d(HfH^{-1})(y) - d(h_\eta f h_\eta^{-1})(y)| < 2\varepsilon$, as announced. ∎

2.6. Distortion in wandering intervals for Denjoy counterexamples

2.6.1. *Turning down the distortion of Denjoy counterexamples: statements.* The aim of this section is to prove the following proposition which allows one to show that every $f \in \mathrm{Diff}^1_+(S^1)$ satisfies the distortion hypothesis of Theorem 2.1 (with no assumption on dense orbits).

If f is a C^1-diffeomorphism with irrational rotation number, we define a (*maximal*) *wandering interval* to be the closure of any connected component of the complement of the unique minimal set of f.

PROPOSITION 2.16. *Let $f\colon S^1 \to S^1$ be a diffeomorphism with irrational rotation number α. Then for any $\varepsilon > 0$ there is a diffeomorphism h such that the distortion of $g = hfh^{-1}$ on each wandering interval I is bounded by ε.*

REMARK 2.17. Let f be a diffeomorphism with irrational rotation number and $\eta > 0$. Then for any point x belonging to the minimal set (that is, x does not belong to any wandering interval) there is $n_1 > 0$ such that for every $n > n_1$, the closure I of every connected component of the complement of the orbit segment $\{x, \dots, f^n(x)\}$ has one of the following properties: either

- the length of I is smaller than η, or
- there is a wandering interval J contained in I such that the sum of the lengths of the two components of $I \setminus J$ is smaller than η.

COROLLARY 2.18. *Given $f \in \mathrm{Diff}^1_+(S^1)$ with irrational rotation number and $\varepsilon > 0$, there are $h \in \mathrm{Diff}^1_+(S^1)$ and $N > 0$ such that for all $n \geq N$ and $x \in S^1$, the distortion of $\tilde{f} = hfh^{-1}$ on any connected component of the complement of the orbit segment $\{x, \dots, \tilde{f}^n(x)\}$ is bounded by ε.*

Proof. By Proposition 2.16, there is a diffeomorphism h such that the distortion of $\tilde{f} = hfh^{-1}$ is smaller than $\varepsilon/10$ on each wandering interval.

Notice that, due to the uniform continuity of the derivative of $\tilde{f}$, its distortion on small intervals, is very small: there is $\delta > 0$ such that the distortion of $\tilde{f}$ on every interval shorter than δ is less than $\varepsilon/10$.

By Remark 2.17 there is $N > 0$ such that, for any $n \geq N$ and any x, each connected component C of $S^1 \setminus \{x, \dots, \tilde{f}^n(x)\}$ is the union of at most three intervals, two of them having length less than δ and (at most) one being a wandering interval. The distortion of $\tilde{f}$ on each of these components is bounded by $\varepsilon/10$ so that the distortion of $\tilde{f}$ on C is bounded by $\frac{3}{10}\varepsilon < \varepsilon$. ∎

The proof of Proposition 2.16 is divided into two main parts. We first perturb the derivative by conjugacy inside the orbits of wandering intervals in order to get small distortion. Then we extend the conjugacy onto the circle without changing the distortion inside the wandering intervals.

2.6.2. *Turning down the distortion on wandering intervals.* The aim of this section is to prove

LEMMA 2.19. *Let f be a C^1-diffeomorphism of S^1 with irrational rotation number, and $\varepsilon > 0$. Let $[a,b]$ be a maximal wandering interval. Then there is a family of diffeomorphisms $h_i\colon f^i([a,b]) \to f^i([a,b])$, $i \in \mathbb{Z}$, such that:*

- *there is n_0 such that $h_i = \mathrm{id}|_{f^i([a,b])}$ for $|i| \geq n_0$,*
- *the distortion of $h_{i+1} \circ f \circ h_i^{-1}\colon f^i([a,b]) \to f^{i+1}([a,b])$ is bounded by ε for every $i \in \mathbb{Z}$.*

To prove Lemma 2.19 we will use

LEMMA 2.20. *Let $\{f_i\}_{i\in\mathbb{Z}}$ be a sequence of diffeomorphisms of $[0,1]$ such that $f_i \to \mathrm{id}$ in the C^1-topology as $i \to \pm\infty$, and let $\varepsilon > 0$. Then there is a sequence $\{g_i\}_{i\in\mathbb{Z}}$ of diffeomorphisms of $[0,1]$, ε-C^1-close to the identity map, and n_0 such that:*

- *$g_i = f_i$ for $|i| \geq n_0$,*
- *$g_{n_0} \circ g_{n_0-1} \circ \cdots \circ g_{-n_0+1} \circ g_{-n_0} = f_{n_0} \circ f_{n_0-1} \circ \cdots \circ f_{-n_0+1} \circ f_{-n_0}$.*

Proof. Let $n_1 > 1$ be such that f_n and $f_{n+1}f_n$ are $\varepsilon/2$-close to the identity for $|n| \geq n_1$.

We fix $g_i = f_i$ for $i < -n_1$.

Consider the diffeomorphism $F = f_{n_1} \circ \cdots \circ f_{-n_1}$. The fragmentation lemma (which is elementary for diffeomorphisms of the interval) asserts that any orientation preserving diffeomorphism of $[0,1]$ is the product of finitely many diffeomorphisms arbitrarily close to the identity. Therefore there is $m > 0$ and diffeomorphisms g_i, $i = -n_1, \ldots, m$, such that

- every g_i, $i = -n_1, \ldots, m$, is ε-C^1-close to the identity,
- $F = g_m \circ \cdots \circ g_{-n_1}$.

If $m \leq n_1$ one fixes $g_i = \mathrm{id}$ for $m \leq i \leq n_1$. Therefore one may always assume that $m > n_1$. Let us write $m = n_1 + k$ with $k > 0$. Then we define:

- $g_{m+i} = f_{n_1+2i}f_{n_1+2i-1}$ for $i = 1, \ldots, k$,
- $g_i = f_i$ for $i > m + k = n_1 + 2k$,
- $n_0 > n_1 + 2k$.

Thus g_i is ε-close to the identity for every i and

$$g_{m+k_1} \circ \cdots \circ g_m \circ \cdots \circ g_{-n_1} = f_{n_1+2k_1} \circ \cdots \circ f_{n_1+1} \circ F = f_{n_1+2k_1} \circ \cdots \circ f_{-n_1}.$$

As a direct consequence, $g_{n_0} \circ g_{n_0-1} \circ \cdots \circ g_{-n_0+1} \circ g_{-n_0} = f_{n_0} \circ f_{n_0-1} \circ \cdots \circ f_{-n_0+1} \circ f_{-n_0}$, concluding the proof of the lemma. ∎

Proof of Lemma 2.19. Let $\varphi_i\colon f^i([a,b]) \to [0,1]$ be the unique orientation preserving affine diffeomorphism. We write

$$f_i = \varphi_{i+1} f \varphi_i^{-1}\colon [0,1] \to [0,1].$$

Notice that the distortion of f_i on $[0,1]$ is equal to the distortion of the restriction of f to $f^i([a,b])$.

Notice that for i large enough, the length of $f^i([a,b])$ is arbitrarily small, so the distortion of f on $f^i([a,b])$ tends to 0 as $i \to \pm\infty$. Therefore the distortion of f_i on $[0,1]$ tends to 0 as $i \to \pm\infty$. Since the f_i are orientation preserving diffeomorphisms of $[0,1]$, they each have at least one point at which the derivative is 1. Therefore, the fact that the distortion of the f_i tends to 0 implies:

Claim 6. *The C^1-distance from f_i to the identity map tends to* 0 *as* $i \to \pm\infty$.

Therefore, $\{f_i\}_{i\in\mathbb{Z}}$ satisfies the hypothesis of Lemma 2.20. Consider $n_0 > 0$ and the sequence $\{g_i\}_{i\in\mathbb{Z}}$ of diffeomorphisms given by Lemma 2.20 associated to the f_i and the constant $\frac{\varepsilon}{4}$ (i.e. the g_i are $\frac{\varepsilon}{4}$-C^1-close to identity).

In particular $f_i = g_i$ for $i < -n_0$. We set

- $h_i = \mathrm{id}$ for $i < -n_0 - 1$,
- $h_i = \varphi_i^{-1} \circ g_{i-1} \circ \cdots \circ g_{-n_0-1} \circ f_{-n_0-1}^{-1} \circ \cdots \circ f_{i-1}^{-1} \circ \varphi_i$.

By definition of the g_i, one has $g_{n_0} \circ \cdots \circ g_{-n_0} = f_{n_0} \circ \cdots \circ f_{-n_0}$ and $g_i = f_i$ for $|i| > n_0$; one deduces that $h_i = \mathrm{id}$ for $i > n_0$. Furthermore

$$\begin{aligned} h_{i+1} f h_i^{-1} &= \varphi_{i+1}^{-1} \\ &\qquad \circ g_i \circ \cdots \circ g_{-n_0-1} \circ f_{-n_0-1}^{-1} \circ \cdots \circ f_i^{-1} \\ &\qquad \circ \varphi_{i+1} \circ f \circ \varphi_i^{-1} \\ &\qquad \circ f_{i-1} \circ \cdots \circ f_{-n_0-1} \circ g_{-n_0-1}^{-1} \circ \cdots \circ g_{i-1}^{-1} \\ &\qquad \circ \varphi_i \\ &= \varphi_{i+1}^{-1} \\ &\qquad \circ g_i \circ \cdots \circ g_{-n_0-1} \circ f_{-n_0-1}^{-1} \circ \cdots \circ f_i^{-1} \\ &\qquad \circ f_i \\ &\qquad \circ f_{i-1} \circ \cdots \circ f_{-n_0-1} \circ g_{-n_0-1}^{-1} \circ \cdots \circ g_{i-1}^{-1} \\ &\qquad \circ \varphi_i \\ &= \varphi_{i+1}^{-1} \circ g_i \circ \varphi_i. \end{aligned}$$

Note that as g_i is $\frac{\varepsilon}{4}$-C^1-close to the identity map, one has

$$\left|\frac{dg_i(x)}{dg_i(y)} - 1\right| = \left|\frac{dg_i(x) - dg_i(y)}{dg_i(y)}\right| \le \frac{2\varepsilon/4}{1-\varepsilon/4} < \frac{2\varepsilon}{3} \quad \text{for } \varepsilon < 1.$$

As φ_i and φ_{i+1} are affine one concludes that the distortion of $h_{i+1} f h_i^{-1}$ on $f^i([a,b])$ is bounded by $\log(1+2\varepsilon/3)$, therefore it is smaller than ε. ∎

2.6.3. *Extension of the conjugacy onto the whole circle: proof of Proposition 2.16.* The aim of this section is to show that one can extend the conjugacy, defined inside the wandering interval by Lemma 2.19, onto the whole circle without changing the distortion.

First notice that, due to the uniform continuity of the derivative of f, its distortion is smaller than $\varepsilon/2$ on every small enough interval. This shows:

LEMMA 2.21. *There are $N > 0$, $k \geq 0$ and, for every $0 < i \leq k$, a maximal wandering interval $[a_i, b_i]$ such that:*

- *the orbits of $[a_i, b_i]$ are pairwise distinct,*
- *for any n with $|n| > N$, the distortion of f on $f^n([a_i, b_i])$ is smaller than $\varepsilon/2$,*
- *for any wandering interval $[a,b]$ whose orbit is distinct from the orbits of the $[a_i, b_i]$, the distortion is bounded by $\varepsilon/2$ on each $f^n([a,b])$, $n \in \mathbb{Z}$.*

For every $i \geq 0$, we fix $n_{0,i} > 0$ and a sequence of diffeomorphisms $h_{i,t}\colon [f^t(a_i), f^t(b_i)] \to [f^t(a_i), f^t(b_i)]$, $t \in \mathbb{Z}$, associated by Lemma 2.19 to f, the wandering interval $[a_i, b_i]$ and the constant $\varepsilon > 0$. We denote $n_0 = \max\{n_{0,i}\}$, $0 < i \leq k$. By definition, for every $0 < i \leq k$, one has:

- $h_{i,t} = \mathrm{id}|_{f^t([a_i,b_i])}$ for $|t| \geq n_0$,
- the distortion of $h_{i,t+1} \circ f \circ h_{i,t}^{-1}\colon f^t([a_i,b_i]) \to f^{t+1}([a_i,b_i])$ is bounded by ε.

Notice that the $[f^t(a_i), f^t(b_i)]$, $i \in \{1,\dots,k\}$, $t \in \{-n_0,\dots,n_0\}$, are finitely many compact disjoint segments.

We are now ready to state the main result of this section:

LEMMA 2.22. *There is a diffeomorphism h of S^1 such that h coincides with the $h_{i,t}$ for $i \in \{1,\dots,k\}$, $t \in \{-n_0,\dots,n_0\}$, and the derivative of h is constant in every wandering interval distinct from the $[f^t(a_i), f^t(b_i)]$, $i \in \{1,\dots,k\}$, $t \in \{-n_0,\dots,n_0\}$.*

To construct h we will build its derivative dh. This derivative is a classical *Lebesgue devil staircase* as it is continuous, and constant on each connected component of the complement of a Cantor set. We will use the following result:

LEMMA 2.23. *Let $\mathcal{C} \subset \mathbb{R}$ be a Cantor set, and $\alpha, \beta, \delta > 0$. Let $I = [a,b]$ be the convex hull of $\mathcal{C}$. Then there is a continuous function $\varphi\colon [a,b] \to \mathbb{R}$ such that:*

- *$\varphi(t) > 0$ for every $t \in I$,*
- *$\varphi(a) = \alpha$ and $\varphi(b) = \beta$,*

- *φ is constant on each connected component of $I \setminus \mathcal{C}$,*
- *$\int_a^b \varphi\, dt = \delta$.*

Proof. The unique point which is not classical is the condition on the integral.

We begin by constructing a continuous map $\varphi_-\colon [a,b] \to [0,\infty)$, constant on each connected component of $[a,b]\setminus\mathcal{C}$, such that $\varphi_-(a) = \alpha$, $\varphi_-(b) = \beta$ and $\int_a^b \varphi\, dt < \delta$; for that, it is enough to build φ_- bounded by $\max\{\alpha,\beta\}$ and equal to 0 on a segment $[c,d]$ with $c,d \in \mathcal{C}$ and $(c-a)+(b-d) < \frac{b-a}{\max\{\alpha,\beta\}}$.

Then we construct a continuous map $\varphi_0\colon [a,b] \to [0,\infty)$, constant on each connected component of $[a,b]\setminus\mathcal{C}$, and such that $\varphi_0(a) = \varphi_0(b) = 0$ and $\varphi_0(x) > 0$ for $a < x < b$. In particular $\int_a^b \varphi_0(t)\, dt > 0$.

Then the map φ defined by

$$\varphi(t) = \varphi_-(t) + \left(\frac{\delta - \int_a^b \varphi_-(s)\, ds}{\int_a^b \varphi_0(s)\, ds}\right)\varphi_0(t)$$

has all the announced properties. ∎

Corollary 2.24. *Let $\mathcal{C} \subset \mathbb{R}$ be a Cantor set, and $\alpha, \beta > 0$. Let $I = [a,b]$ be the convex hull of $\mathcal{C}$. There is a C^1-diffeomorphism $h\colon I \to I$, affine on each connected component of $I\setminus\mathcal{C}$ and such that $dh(a) = \alpha$ and $dh(b) = \beta$.*

Proof. Let $\varphi\colon I \to (0,\infty)$ be the map associated by Lemma 2.23 to $\mathcal{C}$, $[a,b]$, α, β and $\delta = b-a$. Then $h\colon [a,b] \to \mathbb{R}$ defined by $h(t) = a + \int_a^t \varphi(s)\, ds$ induces a C^1-diffeomorphism of $[a,b]$ which is affine on each connected component of $I\setminus\mathcal{C}$. ∎

Proof of Lemma 2.22. Recall that $f \in \mathrm{Diff}^1_+(S^1)$ is a C^1-diffeomorphism with irrational rotation number, whose orbits are not dense (otherwise there is nothing to prove). Thus f has a unique minimal set which is a Cantor set, $\mathcal{C}$.

Consider the closure $I = [f^{t_1}(b_i), f^{t_2}(a_j)]$ of a connected component of $S^1 \setminus \bigcup_{i\in\{1,\dots,k\},\, t\in\{-n_0,\dots,n_0\}} [f^t(a_i), f^t(b_i)]$. Notice that the interior of the last union is disjoint from $\mathcal{C}$. Consequently, the extremities of I are not isolated points of $I\cap\mathcal{C}$. Hence $I\cap\mathcal{C}$ is a Cantor set.

According to Corollary 2.24 there is an orientation preserving C^1-diffeomorphism $h_I\colon I \to I$ such that:

- $dh_I(f^{t_1}(b_i)) = dh_{i,t_1}(f^{t_1}(b_i))$,
- $dh_I(f^{t_2}(a_j)) = dh_{j,t_2}(f^{t_2}(a_j))$,
- dh_I is constant on each connected component of $I\setminus\mathcal{C}$.

Now, one defines $h\colon S^1 \to S^1$ as $h|_I = h_I$ and $h|_{[f^t(a_i), f^t(b_i)]} = h_{i,t}$ for $t \in \{-n_0,\dots,n_0\}$. ∎

Proof of Proposition 2.16. Consider the diffeomorphism h given by Lemma 2.22. Then hfh^{-1} coincides with $h_{i,t+1}fh_{i,t}$ on the $f^t([a_i,b_i])$ for $|t|<n_0$; therefore, the distortion is bounded by ε. On other wandering intervals, h is affine, so that the conjugacy does not affect the distortion, which was bounded by $\varepsilon/2$ by definition of the $[a_i,b_i]$ and n_0. ∎

3. Proof of the main result. The aim of this section is to prove our main result (Theorem 1.1) assuming Theorem 3.1 which explains that one can change the initial and final ratios by arbitrarily small perturbations if one choose an adapted segment of a specific length.

3.1. Perturbing the initial and final ratios at *characteristic times*. Our main technical result is

THEOREM 3.1. *Given any $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ there is a (strictly increasing) sequence $\{k_i\}\subset\mathbb{N}$ with the following properties: Let f, g be C^1-diffeomorphisms with rotation number α, let $x,y\in S^1$, and let $\varepsilon>0$. Then:*

- *for any i the orbit segments $\{x,\dots,f^{k_i}(x)\}$ and $\{y,\dots,g^{k_i}(y)\}$ are adapted,*
- *there is i_0 such that for every $i\geq i_0$ there is a C^1-diffeomorphism g_i such that:*
  - *g_i is ε-C^1-close to g,*
  - *$\{y,\dots,g_i^{k_i}(y)\}$ is an adapted segment of g_i ordered on S^1 in the same way as $\{x,\dots,f^{k_i}(x)\}$ and $\{y,\dots,g^{k_i}(y)\}$,*
  - *the initial and final ratios of g_i on $\{y,\dots,g_i^{k_i}(y)\}$ are the same as the ones of f on $\{x,\dots,f^{k_i}(x)\}$.*

In Section 4.2 we will build the sequence $\{k_i\}$, called the *characteristic times*, and Section 5 will be dedicated to the proof of Theorem 3.1.

The aim of this section is to show that Theorem 3.1 together with Theorem 2.1, Proposition 2.16, Lemma 2.10 and Proposition 2.12 imply Theorem 1.1.

3.2. Proof of Theorem 1.1. Let α be an irrational number, $f,g\in\mathrm{Diff}^1_+(S^1)$ with rotation number α, and $\varepsilon>0$. We have to prove that there is a diffeomorphism h of S^1 such that hfh^{-1} is ε-close to g.

Recall that F^r_α denotes the space of C^r-diffeomorphisms with rotation number α. According to [He, Proposition 4.4.2], F^r_α is C^s-dense in F^s_α for any $s\leq r$. In particular, F^2_α is C^1-dense in F^1_α. Thus there is a C^2-diffeomorphism g_0 with rotation number α and $\varepsilon/2$-C^2-close to g. In other words, up to replacing ε by $\varepsilon/2$ and g by g_0, we may (and will) assume that g is C^2.

According to Proposition 2.16, f is C^1-conjugate to $f_0=h_0fh_0^{-1}$ such that the distortion of f_0 on each wandering interval is bounded by $\varepsilon/(150M)$, where M is an upper bound for dg. Therefore, by Corollary 2.18, for any sufficiently large orbit segment of a point x in the minimal set of f_0, the distortion of f_0 on each connected component of the complement of that orbit segment will be bounded by $\varepsilon/(50M)$.

Thus, we choose x in the minimal set of f; then $x_0 = h_0(x)$ is in the minimal set of f_0. We choose a sufficiently large characteristic time k_i so that, according to Theorem 3.1, g admits an $\frac{\varepsilon}{200}$-C^1-perturbation g_1 for which:

- the orbit segment $\{0, \ldots, g_1^{k_i}(0)\}$ is adapted, and is ordered in the same way as $\{0, \ldots, g^{k_i}(0)\}$ and $\{x_0, \ldots, f_0^{k_i}(x_0)\}$,
- the initial and final ratios associated to $\{0, \ldots, g_1^{k_i}(0)\}$ are the same as the ones of f_0 on $\{x_0, \ldots, f_0^{k_i}(x_0)\}$.
- the distortion of g_1 on each connected component of the complement of $\{0, \ldots, g_1^{k_i}(0)\}$ is bounded by $\varepsilon/(50M)$, since g_1 was chosen C^1-close to g.

Now Lemma 2.10 yields a piecewise linear conjugacy H such that Hf_0H^{-1} satisfies $|d(Hf_0H^{-1}) - dg| < \varepsilon/8$ (see Remark 2.11).

Finally, Proposition 2.12 ensures the existence of a diffeomorphism h for which $|d(Hf_0H^{-1}) - d(hf_0h^{-1})| < \varepsilon/2$. One infers that $hh_0f(hh_0)^{-1}$ is ε-C^1-close to g, concluding the proof. ■

It now remains to prove Theorem 3.1.

4. Characteristic times

4.1. Informal sketch. To end the proof of Theorem 1.1, it remains to prove Theorem 3.1, that is, to control and modify the initial and final ratios of (well chosen) adapted segments.

In this section, we will choose specific adapted segments that we will call *characteristic segments*. They will be chosen for the rotation R_α, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and we will then control the ratios of characteristic segments for the diffeomorphisms f, g in F^r_α, $r = 1, 2$, and for their C^1-perturbations.

Let us roughly explain how we choose these characteristic segments:

- First we consider the sequence of closest returns to 0 of the $n\alpha$; they are the extremities of an interval that contains 0. We consider the time $n-1$ such that $R_\alpha(R_\alpha^{n-1}(0))$ belongs to the interval.
- We will see that we can extract a subsequence of such times for which the initial and final ratios (of the corresponding segments for the rotation R_α) are uniformly bounded, between $1/2$ and 2.
- Then we will extract a new subsequence (called *characteristic times*) for which the union of the two segments adjacent to 0 will have a large number of disjoint successive positive iterates, which are also

disjoint from the negative iterates of the two segments adjacent to $n\alpha$. This number of iterates will be called the *wandering time*. This long wandering time will allow us to modify these ratios as we want by a C^1-perturbation.

As we need to control the complete geometry of the orbit segment until the closest return, we will first reconstruct the sequence of these closest return times, paying attention to the wandering time of the union of the segments adjacent to 0.

4.2. Ordering the orbit segments of rotations. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number and R_α the rotation $x \mapsto x + \alpha$ on $S^1 = \mathbb{R}/\mathbb{Z}$.

Every orbit segment $\{x, \dots, R_\alpha^n(x)\}$ of length n is the image under the isometry R_x of the corresponding orbit segment starting at 0. Therefore we consider the orbit segments $\{0, \alpha, \dots, n\alpha\}$.

We consider the points $-1/2 < -a_n < 0 < b_n < 1/2$ which are adjacent to 0 in this orbit segment. We define $r_n, s_n \in \{1, \dots, n\}$ by $-a_n = r_n\alpha$ and $b_n = s_n\alpha$. According to Lemma 2.1, $(n - s_n)\alpha < n\alpha < (n - r_n)\alpha$ are the points adjacent to $n\alpha$. Note that:

- $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$, and hence
- $r_n \to \infty$ and $s_n \to \infty$ as $n \to \infty$.

The following lemma is elementary and classical (and can be deduced from Lemma 2.1 and Remark 2.3):

Lemma 4.1.

(i) *The length of each connected component of the complement of the orbit segment belongs to* $\{a_n, b_n, a_n + b_n\}$.

(ii) *We have*

$$r_n + s_n \neq n + 1 \;\Leftrightarrow\; R_\alpha(n\alpha) = (n+1)\alpha \notin (-a_n, b_n).$$

In that case:

- *the points* $(s_n - 1)\alpha$ *and* $(r_n - 1)\alpha$ *are adjacent, and the length of the component* $((s_n - 1)\alpha, (r_n - 1)\alpha)$ *is* $a_n + b_n$,
- $a_{n+1} = a_n$, $b_{n+1} = b_n$, $r_{n+1} = r_n$, *and* $s_{n+1} = s_n$.

(iii) *If* $r_n + s_n = n + 1$ *then the image of* $[(n - s_n)\alpha, (n - r_n)\alpha]$ *under* R_α *is the segment* $[r_n\alpha, s_n\alpha] = [-a_n, b_n]$. *Hence the length of each connected component of the complement of the orbit segment belongs to* $\{a_n, b_n\}$. *Furthermore:*

(a) *If $a_n > b_n$ then $(n+1)\alpha \in (-a_n, 0)$ and*

$$\begin{cases} a_{n+1} = a_n - b_n, \\ b_{n+1} = b_n, \\ r_{n+1} = n+1 = r_n + s_n, \\ s_{n+1} = s_n. \end{cases}$$

(b) *If $b_n > a_n$ then $(n+1)\alpha \in (0, b_n)$ and*

$$\begin{cases} a_{n+1} = a_n, \\ b_{n+1} = b_n - a_n, \\ r_{n+1} = r_n, \\ s_{n+1} = n+1 = r_n + s_n. \end{cases}$$

Let n_i denote the sequence of numbers such that $(n_i+1)\alpha \in [-a_{n_i}, b_{n_i}]$. Lemma 4.1 asserts that:

- if $a_{n_i} > b_{n_i}$ then $a_{n_{i+1}} = a_{n_i+1} = a_{n_i} - b_{n_i}$ and $b_{n_{i+1}} = b_{n_i+1} = b_{n_i}$,
- if $a_{n_i} < b_{n_i}$ then $a_{n_{i+1}} = a_{n_i+1} = a_{n_i}$ and $b_{n_{i+1}} = b_{n_i+1} = b_{n_i} - a_{n_i}$.

One deduces:

LEMMA 4.2. *There is a subsequence $\{n_{i_j}\}_{j\in\mathbb{Z}}$ of n_i such that*

$$a_{n_{i_j}}/b_{n_{i_j}} \in [1/2, 2].$$

Proof. Assume $a_{n_i} > b_{n_i}$. Then $a_{n_{i+1}} = a_{n_i} - b_{n_i}$ and $b_{n_{i+1}} = b_{n_i}$. If $a_{n_{i+1}} < b_{n_{i+1}}$ this means $b_{n_i} < a_{n_i} < 2b_{n_i}$ so that n_i belongs to the announced sequence.

Otherwise, $a_{n_{i+1}} > b_{n_{i+1}}$ and $a_{n_{i+2}} = a_{n_{i+1}} - b_{n_{i+1}} < a_{n_i}$; if $a_{n_{i+1}} - b_{n_{i+1}} < b_{n_i} = b_{n_i+1} = b_{n_i+2}$ we are done; otherwise we continue until there is k such that $a_{n_{i+k}} > b_{n_{i+k}} = b_{n_i}$ but $a_{n_{i+k+1}} < b_{n_{i+k+1}}$; then n_{i+k} belongs to the announced sequence.

The case $a_{n_i} < b_{n_i}$ is analogous. Thus we have shown that the announced sequence contains numbers greater than any of the n_i, allowing us to define the n_{i_j} by induction. ∎

REMARK 4.3. $\{(n_{i_j}+1)\alpha\}_{j\in\mathbb{N}}$ is the sequence of *closest returns to* 0 of the orbit of 0 under the rotation R_α.

4.2.1. *Wandering time.* Consider an irrational number α, $n > 0$, and the orbit segment $\{0, \dots, n\alpha\}$. The numbers $r_n\alpha, s_n\alpha$ have been defined so that 0 is the unique point of the orbit segment in the open interval $I_n = (-a_n, b_n) = (r_n\alpha, s_n\alpha)$; as a consequence, n is the unique point of the segment in the open interval $J_n = ((n-s_n)\alpha, (n-r_n)\alpha)$.

Definition 4.4. With the notation above, we define the *wandering time* $w(n)$ to be the largest integer w such that the $2(w+1)$ intervals

$$I_n, R_\alpha(I_n), \dots, R_\alpha^w(I_n) \quad \text{and} \quad R_\alpha^{-w}(J_n), \dots, R_\alpha^{-1}(J_n), J_n$$

are pairwise disjoint.

Lemma 4.5. *For every $n > 0$,*

$$w(n) = \inf\{[n - r_n - 1/2], [n - s_n - 1/2]\},$$

where $[\cdot]$ denotes the integer part.

Proof. Notice that $f^{n-r_n}(I_n) \cap J_n$ contains $[n\alpha, (n-r_n)\alpha]$ and in particular is non-empty. This implies that $2w(n) < n - r_n$. One shows analogously that $2w(n) < n - s_n$.

On the other hand the intervals

$$(r_n\alpha, 0), \dots, (n\alpha, (n-r_n)\alpha) \quad \text{and} \quad (0, s_n\alpha), \dots, ((n-s_n)\alpha, n\alpha)$$

are pairwise disjoint.

One deduces that, for every w satisfying $2w < \min\{n - s_n, n - r_n\}$, the intervals $I_n, R_\alpha(I_n), \dots, R_\alpha^w(I_n)$ and $R_\alpha^{-w}(J_n), \dots, R_\alpha^{-1}(J_n), J_n$ are pairwise disjoint. ∎

Corollary 4.6. *For any j,*

$$w(n_{i_j}) = \inf\{[s_{n_{i_j}}/2] - 1, [r_{n_{i_j}}/2] - 1\}.$$

Proof. Recall that by definition $r_{n_{i_j}} + s_{n_{i_j}} = n_{i_j} + 1$. ∎

4.3. Characteristic times. Corollary 4.6 provides a lower bound of $w(n_{i_j})$ as the min of two quantities. Lemma 4.8 and Corollary 4.9 below choose a subsequence N_i of the n_{i_j} for which we have a simpler lower bound for the wandering times $w(N_i)$.

Remark 4.7. To ease notation we will write sometimes $r(k) = r_k$, $a(k) = a_k$ etc., in particular when k is a number given by some formula; for instance $s(n_{i_{j-1}} + 1)$ means $s_{n_{i_{j-1}}+1}$.

Lemma 4.8. *There is a strictly increasing sequence $\{N_i\}_{i\in\mathbb{N}}$ of integers, which is a subsequence of $\{n_{i_j}\}_{j\in\mathbb{N}}$, for which we have, for every i, either*

- *$a_{N_i} < b_{N_i}$ and $r(N_i) \le 2s(N_i)$, or*
- *$a_{N_i} > b_{N_i}$ and $s(N_i) \le 2r(N_i)$.*

As a direct consequence of Lemma 4.8 and Corollary 4.6 one gets:

Corollary 4.9. *With the notations of Lemma 4.8, for every i, either*

- *$a_{N_i} < b_{N_i}$ and $[r(N_i)/4] - 1 \le w(N_i)$, or*
- *$a_{N_i} > b_{N_i}$ and $[s(N_i)/4] - 1 \le w(N_i)$.*

Proof of Lemma 4.8. Denote by $\mathcal{N}(\alpha)$ the subset of $\{n_{i_j}\}_{j\in\mathbb{N}}$ satisfying: either $a_{n_{i_j}} < b_{n_{i_j}}$ and $r_{n_{i_j}} \le 2s_{n_{i_j}}$, or $a_{n_{i_j}} > b_{n_{i_j}}$ and $s_{n_{i_j}} \le 2r_{n_{i_j}}$. We have to prove that $\mathcal{N}(\alpha)$ is infinite for every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.

Most of the α are solved by the following claim:

Claim 7. *If $i_j - i_{j-1} \ge 2$ then either*

- *$a(n_{i_j}) < b(n_{i_j})$ and $r(n_{i_j}) < s(n_{i_j})$, or*
- *$a(n_{i_j}) > b(n_{i_j})$ and $r(n_{i_j}) > s(n_{i_j})$,*

so that in both cases $i_j \in \mathcal{N}(\alpha)$.

Proof. Assume for instance $a(n_{i_j}) < b(n_{i_j})$; the other case is identical.

By the choice of the n_{i_j}, one has $a(n) < b(n)$ for every $n_{i_{j-1}} < n \le n_{i_j}$, and $a(n_{i_{j-1}}) > b(n_{i_{j-1}})$.

According to Lemma 4.1,

$$\begin{aligned} r(n_{i_{j-1}}+1) &= r(n_{i_{j-1}}) + s(n_{i_{j-1}}),\\ s(n_{i_{j-1}}+1) &= s(n_{i_{j-1}}). \end{aligned}$$

Furthermore, $r(n_{i_{j-1}+1}) = r(n_{i_{j-1}}+1)$ and $s(n_{i_{j-1}+1}) = s(n_{i_{j-1}}+1)$.

Then by Lemma 4.1, for every $0 \le k \le i_j - i_{j-1}$,

$$\begin{aligned} r(n_{i_{j-1}+k}) &= r(n_{i_{j-1}}+1),\\ s(n_{i_{j-1}+k}) &= s(n_{i_{j-1}}+1) + (k-1)r(n_{i_{j-1}}+1). \end{aligned}$$

In particular,

$$\begin{aligned} r(n_{i_j}) &= r(n_{i_{j-1}}+1),\\ s(n_{i_j}) &= s(n_{i_{j-1}}+1) + (i_j - i_{j-1} - 1)r(n_{i_{j-1}}+1), \end{aligned}$$

so that $r(n_{i_j}) < s(n_{i_j})$ as announced. ∎

Consider now α such that there is j_0 such that no $j \ge j_0$ satisfies the conclusion (and hence the hypothesis) of the claim. This implies that $i_{j_0+k} = i_{j_0} + k$ for every positive k.

Assume for instance $a(n_{i_{j_0}}) < b(n_{i_{j_0}})$. Then, for every $k > 0$,

$$\begin{aligned} r(n_{i_{j_0}}+2k) &= r(n_{i_{j_0}}+2k-1) + s(n_{i_{j_0}}+2k-1),\\ s(n_{i_{j_0}}+2k) &= s(n_{i_{j_0}}+2k-1),\\ r(n_{i_{j_0}}+2k+1) &= r(n_{i_{j_0}}+2k),\\ s(n_{i_{j_0}}+2k+1) &= r(n_{i_{j_0}}+2k) + s(n_{i_{j_0}}+2k). \end{aligned}$$

In particular $r(n_{i_{j_0+2}}) = r(n_{i_{j_0+1}}) + s(n_{i_{j_0+1}}) = 2r(n_{i_{j_0}}) + s(n_{i_{j_0}})$ and $s(n_{i_{j_0+2}}) = s(n_{i_{j_0+1}}) = r(n_{i_{j_0}}) + s(n_{i_{j_0}})$, so that

$$r(n_{i_{j_0+2}}) < 2s(n_{i_{j_0+2}}).$$

This proves that $n_{i_{j_0+2}} \in \mathcal{N}(\alpha)$, and ends the proof of the lemma. ∎

The N_i are almost the announced characteristic times. The unique defect is that the orbit segments $\{x, \dots, R_\alpha^{N_i}(x)\}$ are not adapted, because, as $\{N_i\}$ is a subsequence of $\{n_{i_j}\}$, one has $r(N_i) + s(N_i) = N_i + 1$.

DEFINITION 4.10. With the notations above, we define $k_i = N_i - 1$ and we call k_i the *characteristic times* of α. If f is a diffeomorphism with rotation number α then for every x and $i \in \mathbb{N}$ the orbit segments $\{0, \dots, f^{k_i}(0)\}$ will be the *characteristic segments* of f.

We denote $w_i = w(k_i)$, the wandering time of the characteristic segment. Then, summarizing the results of this section:

LEMMA 4.11.

- *The orbit segments* $\{x, , \dots, R_\alpha^{k_i}(x)\}$ *are adapted segments.*
- *The initial and final ratios of the rotation* R_α *on this orbit segment belong to* $[1/2, 2]$.
- *The wandering time* w_i *is bounded as follows: either*
  - $a_{k_i} < b_{k_i}$ *and* $[r(k_i)/4] - 2 \le w(k_i)$, *or*
  - $a_{k_i} > b_{k_i}$ *and* $[s(k_i)/4] - 2 \le w(k_i)$.

Proof. One deduces from Lemma 4.1 that $r(k_i) = r(N_i)$ and $s(k_i) = s(N_i)$. As a consequence, the orbit segment $\{x, \dots, R_\alpha^{k_i}(x)\}$ is adapted. Furthermore the initial and final ratios are the same as those associated to N_i and belong to $[1/2, 2)$ (see Lemma 4.2).

The last item is now a consequence of the fact that $w(k_i) = w(N_i)$ or $w(k_i) = w(N_i) - 1$ (see Lemma 4.5) applied to Corollary 4.9. ■

4.4. Geometry of the characteristic segment for $f \in F_\alpha^1$. Let f be a C^1-diffeomorphism with an irrational rotation number α. Classical results assert that f is *uniquely ergodic*, that is, f admits a unique invariant measure. The Lyapunov exponent of this measure is zero. This implies:

LEMMA 4.12. *For any* $\lambda > 1$ *there is* $n_\lambda > 0$ *such that for any* $n > n_\lambda$ *and any* $x \in S^1$,
$$df^n(x) \in [\lambda^{-n}, \lambda^n].$$

One checks easily:

COROLLARY 4.13. *Let* $x, y \in S^1$. *Assume that there is* n *with* $|n| > n_\lambda$ *and* $x < f^n(x) < f^{2n}(x) < y < f^{3n}(x)$. *Then*
$$\frac{|x - f^n(x)|}{\lambda^{|n|}} < |f^n(x) - y| < (\lambda^{|n|} + \lambda^{|2n|})|x - f^n(x)|.$$

Proof. Applying f^{-n} to the segment $[f^n(x), y]$, one gets
$$|f^n(x) - y| \ge \frac{|f^{-n}(f^n(x)) - f^{-n}(y)|}{\lambda^{|n|}} > \frac{|x - f^n(x)|}{\lambda^{|n|}},$$

proving the left inequality. Moreover,

$$|f^n(x) - y| < |f^n(x) - f^{2n}(x)| + |f^{2n}(x) - f^{3n}(x)| < (\lambda^{|n|} + \lambda^{|2n|})|x - f^n(x)|,$$

ending the proof. ∎

LEMMA 4.14. *Let $f \in F^1_\alpha$. Let $\{k_i\}_{i\in\mathbb{N}}$ be the sequence of characteristic times associated to α (see Definition 4.10). Then, given any $\lambda > 1$, there is $i(\lambda)$ such that, for all $i \geq i(\lambda)$ and $x \in S^1$, the initial and final ratios of the characteristic segments $\{x, \ldots, f^{k_i}(x)\}$ belong to $[\lambda^{-w_i}, \lambda^{w_i}]$.*

Proof. Consider the corresponding characteristic segment $\{0, \ldots, k_i\alpha\}$ for the rotation R_α. Assume for instance that $a(k_i) < b(k_i)$. By Lemma 4.11, we also have $b(k_i) < 2a(k_i)$. This means

$$R_\alpha^{r(k_i)}(0) < 0 < R_\alpha^{-r(k_i)}(0) < R_\alpha^{s(k_i)}(0) < R_\alpha^{-2r(k_i)}(0).$$

As the orbit segments of f are similarly ordered to the ones of R_α, one gets

$$f^{r(k_i)}(x) < x < f^{-r(k_i)}(x) < f^{s(k_i)}(x) < f^{-2r(k_i)}(x).$$

Given any $\lambda_1 > 1$ and i such that $r(k_i) > n_{\lambda_1}$, one deduces the following bounds from Corollary 4.13 applied to the point $f^{r(k_i)}(x)$ and $n = -r(k_i)$:

$$\frac{|x - f^{r(k_i)}(x)|}{\lambda_1^{|r(k_i)|}} < |f^{s(k_i)}(x) - x| < (\lambda_1^{|r(k_i)|} + \lambda_1^{|2r(k_i)|})|x - f^{r(k_i)}(x)|.$$

Thus the initial ratio belongs to $[\lambda_1^{-r(k_i)}, \lambda_1^{|r(k_i)|} + \lambda_1^{|2r(k_i)|}]$.

Recall that for the characteristic time k_i for which $a(k_i) < b(k_i)$ one has

$$[r(k_i)/4] - 1 \leq w(k_i).$$

Thus, there is λ_1 such that

$$\lambda_1^{|r(k_i)|} + \lambda_1^{|2r(k_i)|} < \lambda^{w(k_i)}$$

for every large enough $k_i > n_{\lambda_1}$. This gives the announced bound for the initial ratio; the bound of the final ratio is obtained similarly. ∎

We can restate Lemma 4.14 as follows:

REMARK 4.15. There is a sequence $\lambda_i > 1$ tending to 1 as $n \to \infty$ such that for all i and $x \in S^1$ the initial and final ratios of the characteristic segment $\{x, \ldots, f^{k_i}(x)\}$ belong to $[\lambda_i^{-w_i}, \lambda_i^{w_i}]$.

5. Perturbations. The aim of this section is to prove Theorem 3.1 with the characteristic times $\{k_i\}$ as the announced sequence. Let w_i be the corresponding wandering times. In the statement of Theorem 3.1, the diffeomorphism f appears only via its initial and final ratios. Let us recall that, according to Lemma 4.14 and Remark 4.15, these ratios are bounded: they are in an interval $[\lambda_i^{-w_i}, \lambda_i^{w_i}]$ where the sequence $\lambda_i > 1$ tends to 1; this sequence depends on f.

Theorem 3.1 states that there exists an ε-C^1-small perturbation of g whose initial and final ratios coincide with those of f.

As we already noticed, up to shrinking ε if necessary, one may assume that g is a C^2-diffeomorphism.

Having in mind these comments, Theorem 3.1 is a direct consequence of Proposition 5.1 below:

Proposition 5.1. *Given*

- *any* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, *the corresponding characteristic times* k_i *and wandering times* w_i, $i \in \mathbb{N}$,
- *any sequence* $\lambda_i > 1$ *tending to* 1,
- *any sequences* $\rho_i^-, \rho_i^+ \in [\lambda_i^{-w_i}, \lambda_i^{w_i}]$,
- *any* C^2-*diffeomorphism* g *with rotation number* α,
- *any* $y \in S^1$,
- *any* $\varepsilon > 0$,

there is i_0 *such that, for every* $i \geq i_0$, *there is a* C^1-*diffeomorphism* g_i *and a point* y_i *with the following properties:*

- g_i *is* ε-C^1-*close to* g,
- $\{y_i, \dots, g_i^{k_i}(y_i)\}$ *is an adapted segment of* g_i *similarly ordered on* S^1 *to* $\{y, \dots, g^{k_i}(y)\}$,
- *the initial and final ratios of* g_i *on* $\{y_i, \dots, g_i^{k_i}(y_i)\}$ *are* ρ_i^- *and* ρ_i^+, *respectively,*
- $y_i \to y$ *as* $i \to \infty$.

5.1. Proof of Proposition 5.1. Let $I_i = I_i(g)$ and $J_i = J_i(g)$ denote the intervals $(g^{r(k_i)}(y), g^{s(k_i)}(y))$ and $(g^{k_i - s(k_i)}(y), g^{k_i - r(k_i)}(y))$, containing y and $g^{k_i}(y)$, respectively (in the terminology of Section 2, they are the interiors of the initial and final basic intervals, respectively). By definition of the wandering times, the intervals $I_i(g), g(I_i), \dots, g^{w_i}(I_i), g^{-w_i}(J_i), \dots, J_i$ are pairwise disjoint.

We will achieve the final ratio equal to ρ_i^+ by performing a perturbation of g with support in $g^{-w_i}(J_i), \dots, J_i$, and the initial ratio equal to ρ_i^- by a perturbation of g with support in $I_i, g(I_i), \dots, g^{w_i}(I_i)$. These supports are disjoint so that the construction can be performed independently. Furthermore, they are analogous. We will only present the construction of ρ_i^-.

5.1.1. *Distortion control and initial and final ratios for* g. According to Remark 4.15, there are $\tilde{\lambda}_i > 1$ tending to 1 as $i \to \infty$ such that the initial and final ratios of g belong to $[\tilde{\lambda}_i^{-w_i}, \tilde{\lambda}_i^{w_i}]$.

The intervals $g^j(I_i)$, $j \in \{0, \dots, w_i\}$, are pairwise disjoint in S^1, so the sum of their lengths is bounded by 1. As g is assumed to be C^2, a classical argument implies the following distortion control:

Lemma 5.2. *There is a constant $C > 1$ such that for all i and $j \in \{0, 1, \dots, w_i\}$, the distortion of g^j on I_i is bounded by $\log C$.*

As a consequence, for every i one has

$$\frac{|g^{r(k_i)+j}(y) - g^j(y)|}{|g^{s(k_i)+j}(y) - g^j(y)|} \in [C^{-2}\tilde{\lambda}_i^{-w_i}, C^2\tilde{\lambda}_i^{w_i}].$$

We denote

$$\mu_i = \sup\{\lambda_i, C^{2/w_i}\tilde{\lambda}_i\}.$$

Then

Lemma 5.3. *With the notations above:*

- $\mu_i \to 1$ *as* $i \to \infty$,
- $\rho_i^- \in [\mu_i^{-w_i}, \mu_i^{w_i}]$ *for every* i,
- $\dfrac{|g^{r(k_i)+j}(y) - g^j(y)|}{|g^{s(k_i)+j}(y) - g^j(y)|} \in [\mu_i^{-w_i}, \mu_i^{w_i}]$ *for all* i *and* $j \in \{0, \dots, w_i\}$.

5.1.2. *Rescaling the statement of Proposition 5.1 on* $[0, 1]$. For any j the restriction $g|_{g^j(I_i)}$ maps $g^j(I_i)$ to $g^{j+1}(I_i)$. It will be more comfortable to deal with diffeomorphisms of the same interval. Therefore we will rescale the intervals $g^j(I_i)$ by affine maps to $[0, 1]$. As this rescaling is affine, it will not affect the distortion of $g|_{g^j(I_i)}$, and small C^1-pertubations of the rescaled map will induce C^1-small perturbations of g with proportional C^1-size. More precisely:

Let $\varphi_{i,j} : g^j(I_i(g)) \to [0, 1]$, $j \in \{0, \dots, w_i\}$, be the affine orientation preserving maps. We denote by $G_{i,j} : [0, 1] \to [0, 1]$, $j \in \{0, \dots, w_i - 1\}$, the diffeomorphism

$$\varphi_{i,j+1} \circ g|_{g^j(I_i)} \circ \varphi_{i,j}^{-1}.$$

As g is assumed to be C^2, the orbits are all dense. Thus the length of $g^j(I_i)$ tends uniformly to 0 as $i \to \infty$. Consequently, the distortion of g tends to 0 on $g^j(I_i)$. As a direct consequence, one gets

Lemma 5.4. *The diffeomorphisms $G_{i,j}$, $i \in \mathbb{N}$, $j \in \{0, \dots, w_i\}$, tend uniformly to the identity map in the C^1-topology as $i \to \infty$.*

Our main lemma is

Lemma 5.5. *Let w_i be a sequence tending to infinity. Let $G_{i,j}$, $j \in \{0, \dots, w_i - 1\}$, be families of diffeomorphisms of $[0, 1]$ tending uniformly to the identity in the C^1-topology as $i \to \infty$. Given any points $t_{i,1}$, $t_{i,2}$ satisfying $\frac{t_{i,\eta}}{1-t_{i,\eta}} \in [\mu_i^{-w_i}, \mu_i^{w_i}]$, $\eta \in \{1, 2\}$, and any $\varepsilon > 0$, there is i_0 such that for any $i \geq i_0$ there are families $H_{i,j}$ such that:*

- $H_{i,j}$ *are* ε-C^1*-close to* $G_{i,j}$,
- $H_{i,j}$ *coincides with* $G_{i,j}$ *in neighborhoods of* 0 *and* 1,

- $H_{i,w_i-1} \circ \cdots \circ H_{i,0}(t_{i,1}) = t_{i,2}$.

We postpone the proof of Lemma 5.5 to the next section and we now conclude the proof of the proposition:

Proof of Proposition 5.1. Defined $g_i = g$ off the union of the intervals $g^j(I_i)$, $j \in \{0, \dots, w_i - 1\}$, and $g_i = \varphi_{i,j+1}^{-1} \circ H_{i,j} \circ \varphi_{i,j}$ on $g^j(I_i)$, where $H_{i,j}$ is given by Lemma 5.5 for the constants:

- ε/M where M is a bound for dg,
- $t_2 = \varphi_{i,w_i}(g^{w_i}(y))$,
- $t_1 =$ the point such that $t_1/(1 - t_1)$ is the initial ratio ρ_i^-.

Using the fact that $g = \varphi_{i,j+1}^{-1} \circ G_{i,j} \circ \varphi_{i,j}$ on $g^j(I_i)$ and $H_{i,j} = G_{i,j}$ in a neighborhood of 0 and 1, one easily checks that g_i is a diffeomorphism.

Furthermore, for i large enough, and every $j \in \{0, \dots, w_i\}$, the diffeomorphism $H_{i,j}$ is ε/M-close to $G_{i,j}$. As a consequence, g_i is ε-close to g in the C^1-topology.

Finally, let

$$y_i = \varphi_{i,0}^{-1}(t_1).$$

The orbit segment of length k_i through y_i satisfies:

- $g_i^{w_i}(y_i) = g^{w_i}(y)$,
- $g_i^j(y_i) = g^j(y)$ for any $j \in \{w_i, \dots, k_i\}$,
- $\{y_i, \dots, g_i^{k_i}(y_i)\}$ is an adapted segment for g_i ordered similarly to the adapted segment $\{y, \dots, g^{k_i}(y)\}$.

Hence the initial ratio of the orbit segment $\{y_i, \dots, g_i^{k_i}(y_i)\}$ is ρ_i^-, as announced. ∎

5.2. Proof of Lemma 5.5. Notice that, as the $G_{i,j}$ are assumed to tend uniformly to the identity, the condition that the $H_{i,j}$ are ε-close to the $G_{i,j}$ can be replaced by their being ε-C^1-close to the identity (up to shrinking ε slightly).

Furthermore, the condition that $H_{i,j}$ and $G_{i,j}$ coincide in an (arbitrarily small) neighborhood of 0 and 1 can be obtained by the use of a bump function, without introducing derivatives larger than $1 + 2\varepsilon$.

Therefore, up to replacing ε by $\varepsilon/2$, Lemma 5.5 is a direct consequence of the following lemma:

LEMMA 5.6. *Let w_i be a sequence tending to infinity and μ_i be a sequence tending to 1. Then given any points $t_{i,1}$, $t_{i,2}$ satisfying $\frac{t_{i,\eta}}{1-t_{i,\eta}} \in [\mu_i^{-w_i}, \mu_i^{w_i}]$, $\eta \in \{1, 2\}$, and given any $\varepsilon > 0$, there is i_0 such that for any $i \geq i_0$, there are families $H_{i,j}$ such that:*

- *$H_{i,j}$ are ε-C^1-close to* id,
- *$H_{i,w_i-1} \circ \cdots \circ H_{i,0}(t_{i,1}) = t_{i,2}$.*

The main step in the proof is the following elementary observation:

LEMMA 5.7. *Given* $\varepsilon > 0$ *small enough,* $t \in [-1, 1]$ *and* $y \in [0, 1]$*, there is a diffeomorphism* φ *of* $[0, 1]$ *which is equal to the identity in a neighborhood of* 0 *and* 1*,* 2ε*-*C^1*-close to the identity, and such that*

$$\frac{|\varphi(y)|}{|\varphi(y) - \varphi(1)|} = (1 + t\varepsilon)\frac{|y|}{|1 - y|}.$$

Proof. Let $y_1 \in [0, 1]$ be such that $\frac{y_1}{1-y_1} = (1 + t\varepsilon)\frac{|y|}{|1-y|}$. An easy calculation shows that $y_1 = \frac{(1+t\varepsilon)y}{1+t\varepsilon y}$.

The map φ is obtained by just smoothing the piecewise affine homeomorphism, affine from $[0, y]$ to $[0, y_1]$ and from $[y, 1]$ to $[y_1, 1]$. Notice that the slopes of the affine segments are

- $\frac{y_1}{y} = \frac{1+t\varepsilon}{1+t\varepsilon y} = 1 + t\varepsilon\frac{1-y}{1+t\varepsilon y} \in (1 - \varepsilon, 1 + \varepsilon)$ and
- $\frac{1-y_1}{1-y} = \frac{1}{1+t\varepsilon y} = 1 - \frac{ty}{1+t\varepsilon y}\varepsilon \in (1 - 2\varepsilon, 1 + 2\varepsilon)$ for $\varepsilon < 1/2$. ∎

Proof of Lemma 5.6. Applying Lemma 5.7 w_i times to $\varepsilon/2$, it is enough to show that

$$\frac{t_{i,2}}{1 - t_{i,2}} = \left(1 + t\frac{\varepsilon}{2}\right)^{w_i} \frac{t_{i,1}}{1 - t_{i,1}} \qquad \text{for some } t \in [0, 1].$$

By assumption $\frac{t_{i,1}}{1-t_{i,1}}, \frac{t_{i,2}}{1-t_{i,2}} \in [\mu_i^{-w_i}, \mu_i^{w_i}]$, that is, $\frac{t_{i,2}}{1-t_{i,2}}\frac{1-t_{i,1}}{t_{i,1}} \in [\mu_i^{-2w_i}, \mu_i^{2w_i}]$. Therefore, one can find t if $\mu_i^2 < 1+\varepsilon/2$. As $\mu_i \to 1$ when $i \to \infty$, it is enough to choose i large enough, ending the proof. ∎

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References

[Ar] V. Arnold, *Small denominators I. On the mapping of the circle into itself*, Izv. Akad. Nauk Ser. Mat. 25 (1961), 21–86; English transl.: Amer. Math. Soc. (2) 46 (1965), 213–284.

[BCW] C. Bonatti, S. Crovisier and A. Wilkinson, *The* C^1*-generic diffeomorphism has trivial centralizer*, Publ. Math. I.H.E.S. 109 (2009), 185–244.

[DKN] B. Deroin, V. Kleptsyn et A. Navas, *Sur la dynamique unidimensionnelle et régularité intermédiaire*, Acta Math. 199 (2007), 199–262.

[Fa] E. Farinelli, *Classes de conjugaison des difféomorphismes de l'intervalle en régularité* C^1, preprint, 2012.

[FK] B. Fayad and K. Khanin, *Smooth linearization of commuting circle diffeomorphisms*, Ann. of Math. 170 (2009), 961–980.

[He] M.-R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. I.H.E.S. 49 (1979), 5–233.

[KO1] Y. Katznelson and D. Ornstein, *The differentiability of the conjugation of certain diffeomorphisms of the circle*, Ergodic Theory Dynam. Systems 9 (1989), 643–680.

[KO2] Y. Katznelson and D. Ornstein, *The absolute continuity of the conjugation of certain diffeomorphisms of the circle*, Ergodic Theory Dynam. Systems 9 (1989), 681–690.

[KS1] K. Khanin and Y. Sinai, *A new proof of Herman theorem*, Comm. Math. Phys. 112 (1987), 89–101.

[KS2] K. Khanin and Y. Sinai, *Smoothness of conjugacies of diffeomorphisms of the circle with rotations*, Russian Math. Surveys 44 (1989), 69–99.

[KN] V. Kleptsyn and A. Navas, *A Denjoy type theorem for commuting circle diffeomorphisms with derivatives having different Hölder differentiability classes*, Moscow Math. J. 8 (2008), 477–492.

[Mo] J. Moser, *On commuting circle mappings and simultaneous Diophantine approximations*, Math. Z. 205 (1990), 105–121.

[Na] A. Navas, *Sur les rapprochements par conjugaison en dimension 1 et classe* C^1, Compos. Math. 150 (2014), 1183–1195.

[Yo] J. C. Yoccoz, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation satisfait à une condition diophantienne*, Ann. Sci. École Norm. Sup. 17 (1984), 333–359.

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