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How many normal measures can $\aleph_{\omega+1}$ carry?

by

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Abstract. We show that assuming the consistency of a supercompact cardinal with a measurable cardinal above it, it is possible for $\aleph_{\omega+1}$ to be measurable and to carry exactly τ normal measures, where $\tau \geq \aleph_{\omega+2}$ is any regular cardinal. This contrasts with the fact that assuming AD + DC, $\aleph_{\omega+1}$ is measurable and carries exactly three normal measures. Our proof uses the methods of [6], along with a folklore technique and a new method due to James Cummings.

1. Introduction and preliminaries. It is a consequence of AD + DC that $\aleph_{\omega+1}$ is a measurable cardinal and carries exactly three normal measures. This follows since assuming AD + DC, there are only three regular cardinals (namely $\aleph_0$, $\aleph_1$, and $\aleph_2$) below $\aleph_{\omega+1}$, AD + DC implies that $\aleph_{\omega+1}$ satisfies the strong partition property $\aleph_{\omega+1} \to (\aleph_{\omega+1})^{\aleph_{\omega+1}}$, and if a successor cardinal κ satisfies the weak partition property $\forall\delta < \kappa[\kappa \to (\kappa)^{\delta}]$, then κ is measurable and carries exactly the same number of normal measures as regular cardinals below κ. (In fact, if a successor cardinal κ satisfies the weak partition property, then any normal measure κ carries must be of the form $\{x \subseteq \kappa \mid x$ contains a set which is δ club$\}$, where $\delta < \kappa$ is a regular cardinal.) The proofs of these last three facts can be found respectively in [14] (see also [13]), [10] (see also [11]), and [15].

When the Axiom of Determinacy is not assumed, however, the situation concerning the number of normal measures that $\aleph_{\omega+1}$ can carry if $\aleph_{\omega+1}$ is measurable is not so clear. In fact, in the articles [3], [4], [1], and [6], in which the measurability of $\aleph_{\omega+1}$ is forced from supercompactness hypotheses, the number of normal measures $\aleph_{\omega+1}$ possesses in the relevant models constructed is completely unclear.

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The purpose of this paper is to shed new light on the situation mentioned in the preceding paragraph and construct, via forcing, models in which $\aleph_{\omega+1}$ is measurable and carries exactly τ normal measures, where $\tau \geq \aleph_{\omega+2}$ is any regular cardinal. Specifically, we will prove the following two theorems.

THEOREM 1. *Let* $V^* \vDash$ *"*ZFC + GCH + $\kappa < \lambda$ *are such that* κ *is supercompact and* λ *is the least measurable cardinal above* κ + $\tau > \lambda^+$ *is a fixed but arbitrary regular cardinal". There is then a generic extension* V *of* V^*, *a partial ordering* $\mathbb{P} \in V$, *and a symmetric submodel* $N \subseteq V^{\mathbb{P}}$ *such that* $N \vDash$ *"*ZF + $\mathrm{DC}_{\aleph_\omega}$ + $\lambda = \aleph_{\omega+1}$ *is a measurable cardinal". In* N, *the cardinal and cofinality structure at and above* λ *is the same as in* V (*which has the same cardinal and cofinality structure at and above* λ *as* V^*), *and* $\aleph_{\omega+1}$ *carries exactly* τ *normal measures.*

THEOREM 2. *Let* $V^* \vDash$ *"*ZFC + GCH + $\kappa < \lambda$ *are such that* κ *is supercompact and* λ *is the least measurable cardinal above* κ*". There is then a generic extension* V *of* V^*, *a partial ordering* $\mathbb{P} \in V$, *and a symmetric submodel* $N \subseteq V^{\mathbb{P}}$ *such that* $N \vDash$ *"*ZF + $\mathrm{DC}_{\aleph_\omega}$ + $\lambda = \aleph_{\omega+1}$ *is a measurable cardinal". In* N, $\aleph_{\omega+2}$ *is regular, and* $\aleph_{\omega+1}$ *carries exactly* $\aleph_{\omega+2}$ *normal measures.*

Theorems 1 and 2 provide our desired results. Taken together, these theorems show that relative to the appropriate assumptions, it is consistent for $\aleph_{\omega+1}$ to be measurable and to carry τ normal measures, where $\tau \geq \aleph_{\omega+2}$ is any regular cardinal.

We digress now to provide some preliminary information. Essentially, our notation and terminology are standard, although exceptions to this will be noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in the usual interval notation. For x a set of ordinals, $\overline{x}$ is the order type of x.

When forcing, $q \geq p$ means that q is stronger than p. For κ a regular cardinal, the partial ordering $\mathbb{P}$ is κ-directed closed if every directed set of conditions of size less than κ has a common extension. For κ regular and λ any ordinal, $\mathrm{Add}(\kappa, \lambda)$ is the standard partial ordering for adding λ Cohen subsets to κ. We abuse notation somewhat and use both $V^{\mathbb{P}}$ and $V[G]$ to denote the generic extension by the partial ordering $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V.

For $\kappa < \lambda$ regular cardinals, $\mathrm{Coll}(\kappa, \lambda)$ is the standard Lévy partial ordering for collapsing λ to κ. $\mathrm{Coll}(\kappa, <\lambda)$ is the standard Lévy partial ordering for collapsing every cardinal $\delta \in (\kappa, \lambda)$ to κ. For such a δ and any $S \subseteq \mathrm{Coll}(\kappa, <\lambda)$, we define $S \restriction \delta = \{p \in S \mid \mathrm{dom}(p) \subseteq \kappa \times \delta\}$. It is well known that if G is V-generic over $\mathrm{Coll}(\kappa, <\lambda)$ and $\delta \in (\kappa, \lambda)$ is a cardinal, then $G \restriction \delta$ is V-generic over $\mathrm{Coll}(\kappa, <\lambda) \restriction \delta$.

Note that we are assuming familiarity with the large cardinal notions of measurability and supercompactness. Interested readers may consult [12] for further details.

We conclude Section 1 by mentioning that there are two results critical to the proofs of Theorems 1 and 2 which will be taken as "black boxes". For the convenience of readers, we provide a brief discussion of these facts here. The first concerns the folklore result that if $V \vDash$ "ZFC + κ is a measurable cardinal + $2^\kappa = \kappa^+$", then any reverse Easton iteration adding a single Cohen subset to every element of an unbounded normal measure 0 subset of κ (such as a set of successor cardinals) preserves the measurability of κ and increases the number of normal measures κ carries to $2^{2^\kappa} = 2^{\kappa^+}$. This is the essential content of Lemma 1.1 of [2].

The second is Hamkins' Gap Forcing Theorem of [8] and [9]. We state the version of this theorem we will use here, along with some associated terminology, quoting freely from [8] and [9]. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$ and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed". In Hamkins' terminology of [8] and [9], $\mathbb{P}$ admits a gap at δ. Also, as in the terminology of [8] and [9] (and elsewhere), an embedding $j : \overline{V} \to \overline{M}$ is amenable to $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The relevant form of the Gap Forcing Theorem is then the following.

Theorem 3 (Hamkins). *Suppose that $V[G]$ is a forcing extension obtained by forcing that admits a gap at some $\delta < \kappa$ and $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^\delta \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V.*

It immediately follows from Theorem 3 that any cardinal κ measurable in a generic extension obtained by forcing that admits a gap below κ must also be measurable in the ground model.

2. The Proofs of Theorems 1 and 2. We turn now to the proofs of Theorems 1 and 2. The proofs of these theorems are quite similar to one another, so we prove them in tandem, making the relevant distinctions when necessary.

Proof. Let $V^* \vDash$ "ZFC + GCH + $\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above κ". By Laver's result of [16], we assume that V^* has been generically extended via the partial ordering $\mathbb{L}$ to a model $\overline{V}$ such that $\overline{V} \vDash$ "κ is indestructibly supercompact", i.e., $\overline{V} \vDash$ "κ is supercompact and κ's supercompactness is indestructible under κ-directed closed forcing".

Both Theorems 1 and 2 will require that $\overline{V}$ first be generically extended to a model V in which λ remains the least measurable cardinal above κ and carries the appropriate number of normal measures. For Theorem 1, let $\tau > \lambda^+$ be a fixed but arbitrary regular cardinal in V^*. We show that $\overline{V}$ may be generically extended further to a model V such that $V \vDash$ "κ is supercompact + λ is the least measurable cardinal above κ + λ carries τ normal measures". To do this, since $\mathbb{L}$ may be defined so that $|\mathbb{L}| = \kappa$, standard arguments in tandem with the Lévy–Solovay results [17] allow us to assume in addition that $\overline{V} \vDash$ "GCH holds at and above κ + λ is the least measurable cardinal above κ + The cardinal and cofinality structure at and above κ is the same as in V^*". In particular, this means we may infer that $\overline{V} \vDash$ "$\tau > \lambda^+$ is a regular cardinal".

Let V be the generic extension obtained by forcing over $\overline{V}$ with the partial ordering $\mathrm{Add}(\lambda^+, \tau) * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is a term for the reverse Easton iteration of length λ which begins by adding a Cohen subset to κ^+ and then adds a Cohen subset to the successor of each inaccessible cardinal in the open interval (κ^+, λ). By its definition, $\mathrm{Add}(\lambda^+, \tau) * \dot{\mathbb{R}}$ is κ-directed closed, which means by indestructibility that $V \vDash$ "κ is supercompact". Further, since $\mathrm{Add}(\lambda^+, \tau)$ is λ^+-directed closed and GCH holds at and above κ in $\overline{V}$ (GCH holding at and above λ in $\overline{V}$ is sufficient for what follows), λ remains the least measurable cardinal above κ in $\overline{V}^{\mathrm{Add}(\lambda^+, \tau)}$, and $\overline{V}^{\mathrm{Add}(\lambda^+, \tau)} \vDash$ "$2^\lambda = \lambda^+$ and $2^{\lambda^+} = \tau$". Therefore, by Lemma 1.1 of [2], $V \vDash$ "λ is measurable and carries τ normal measures". However, by Theorem 3, any cardinal in the open interval (κ^+, λ) measurable in V had to have been measurable in $\overline{V}^{\mathrm{Add}(\lambda^+, \tau)}$. Since $\overline{V}^{\mathrm{Add}(\lambda^+, \tau)} \vDash$ "λ is the least measurable cardinal above κ", $V \vDash$ "λ is the least measurable cardinal above κ" as well. In addition, by our GCH assumptions, forcing over $\overline{V}$ with $\mathrm{Add}(\lambda^+, \tau) * \dot{\mathbb{R}}$ preserves the cardinality and cofinality structures at and above λ.

For Theorem 2, we need to show that $\overline{V}$ can be generically extended further to a model V such that $V \vDash$ "κ is supercompact + λ is the least measurable cardinal above κ + λ carries λ^+ normal measures". To do this, we use a new method due to James Cummings, which appears in [5] in a broader context. We isolate Cummings' techniques in the following lemma, which we state in a slightly generalized form.

Lemma 2.1. *Suppose $M \vDash$ "ZFC + δ is measurable + GCH holds at and above δ". Then for any $\gamma < \delta$, there is a γ-directed closed partial ordering $\mathbb{P}$ such that $M^{\mathbb{P}} \vDash$ "ZFC + δ is measurable + δ carries δ^+ normal measures".*

Proof. Let M be as in the hypotheses of Lemma 2.1. As above, if we first force over M with $\mathrm{Add}(\delta^+, \delta^{++}) * \dot{\mathbb{R}}$, where $\mathbb{R}$ is a term for the reverse Easton iteration of length δ which begins by adding a Cohen subset to γ^+ and then adds a Cohen subset to the successor of each inaccessible cardinal in the open interval (γ^+, δ), we obtain a model in which δ carries $2^{2^\delta} = 2^{\delta^+} = \delta^{++}$ normal measures. By its definition, this forcing is γ-directed closed. With a slight abuse of notation, we denote for the rest of Lemma 2.1 the model which results after the forcing also as M.

Working in M, let $\mathbb{Q} = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$, where $\mathbb{Q}_0 = \mathrm{Add}(\gamma^+, 1)$, and $\dot{\mathbb{Q}}_1$ is a term for $\mathrm{Coll}(\delta^+, \delta^{++})$. Since $|\mathbb{Q}_0| < \delta$, by the results of [17], $M^{\mathbb{Q}_0} \vDash$ "δ is measurable". Therefore, as $M^{\mathbb{Q}_0} \vDash$ "$\dot{\mathbb{Q}}_1$ is δ^+-directed closed" (which means that $M^{\mathbb{Q}_0}$ and $M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}$ contain the same subsets of δ), $M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1} \vDash$ "δ is measurable" as well. In particular, any normal measure over δ in $M^{\mathbb{Q}_0}$ remains a normal measure over δ in $M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}$.

Let $M^* = M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}$. By the preceding paragraph, let $\mathcal{U}^* \in M^*$ be a normal measure over δ, with $j^* : M^* \to N^*$ the associated ultrapower embedding. Note that $N^* = N^{j^*(\mathbb{Q}_0 * \dot{\mathbb{Q}}_1)}$ for the appropriate model N. In addition, N^* has the properties that $N^* \subseteq M^*$ and $(N^*)^\delta \subseteq N^*$ (so in particular, for any $\eta < \delta$, $(N^*)^\eta \subseteq N^*$). Since $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$ is such that $|\mathbb{Q}_0| = |[\gamma^+]^\gamma| < \delta$ and $\Vdash_{\mathbb{Q}_0}$ "$\dot{\mathbb{Q}}_1$ is $|\mathbb{Q}_0|^{++}$-directed closed", by Theorem 3, j^* must lift an elementary embedding $j : M \to N$ such that $j \restriction A \in M$ for any $A \in M$. Hence, for $\mathcal{U} = \{x \subseteq \delta \mid \delta \in j(x)\}$, $\mathcal{U} \in M$, $\mathcal{U}$ is a normal measure over δ, and $\mathcal{U} \subseteq \mathcal{U}^*$.

By the results of [17], $\mathcal{U}' = \{x \subseteq \delta \mid \exists y \subseteq x [y \in \mathcal{U}]\}$ is in $M^{\mathbb{Q}_0}$ a normal measure over δ. As was mentioned above, $\mathcal{U}'$ is a normal measure over δ in $M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}$ as well. However, by their definitions, it must be the case that $\mathcal{U}' = \mathcal{U}^*$, since otherwise, if $x \in \mathcal{U}^*$ but $x \notin \mathcal{U}'$, then $\delta - x \in \mathcal{U}'$. This means that x is disjoint from a set in $\mathcal{U}$, which is absurd since $\mathcal{U} \subseteq \mathcal{U}^*$. Thus, it is actually the case that $\mathcal{U}^* \in M^{\mathbb{Q}_0}$, i.e., any normal measure over δ in $M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}$ is actually an element of $M^{\mathbb{Q}_0}$. However, again by the results of [17], there are the same number of normal measures over δ in $M^{\mathbb{Q}_0}$ as there are in M, i.e., there are $(\delta^{++})^M = (\delta^{++})^{M^{\mathbb{Q}_0}}$ normal measures over δ in $M^{\mathbb{Q}_0}$. Consequently, for $\zeta = (\delta^+)^M = (\delta^+)^{M^{\mathbb{Q}_0}} = (\delta^+)^{M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}}$, as $M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1} \vDash$ "$|(\delta^{++})^{M^{\mathbb{Q}_0}}| = \zeta$", δ carries δ^+ normal measures in $M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}$. Since $\mathrm{Add}(\delta^+, \delta^{++}) * \dot{\mathbb{R}} * \dot{\mathrm{Add}}(\gamma^+, 1) * \dot{\mathrm{Coll}}(\delta^+, \delta^{++})$ is γ-directed closed over our ground model, this completes the proof of Lemma 2.1. ■

Returning to the construction of the model V used in the proof of Theorem 2, let $V^* \vDash$ "ZFC + GCH + $\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above κ". As in the proof of Theorem 1, again using indestructibility, we may assume that V^* has been generically extended to a model $\overline{V}$ such that $\overline{V} \models$ "κ is indestructibly supercompact + GCH holds at and above κ + λ is the least measurable cardinal above κ + The cardinal and cofinality structure at and above κ is the same as in V^*". We then force over $\overline{V}$ with $\mathrm{Add}(\lambda^+, \lambda^{++}) * \dot{\mathbb{R}} * \dot{\mathrm{Add}}(\kappa^+, 1) * \dot{\mathrm{Coll}}(\lambda^+, \lambda^{++})$, where $\dot{\mathbb{R}}$ is a term for the reverse Easton iteration of length λ which begins by adding a Cohen subset to κ^+ and then adds a Cohen subset to the successor of each inaccessible cardinal in the open interval (κ^+, λ). Call the resulting model V. Since this partial ordering by its definition is κ-directed closed, $V \models$ "κ is supercompact". By Lemma 2.1, $V \models$ "λ is measurable and carries λ^+ normal measures", and by the remarks immediately prior to the proof of Lemma 2.1, $\overline{V}^{\mathrm{Add}(\lambda^+, \lambda^{++}) * \dot{\mathbb{R}}} \models$ "λ is the least measurable cardinal above κ". Since $\overline{V}^{\mathrm{Add}(\lambda^+, \lambda^{++}) * \dot{\mathbb{R}}} \models$ "$|\mathrm{Add}(\kappa^+, 1)| < \lambda$", by the results of [17], $\overline{V}^{\mathrm{Add}(\lambda^+, \lambda^{++}) * \dot{\mathbb{R}} * \dot{\mathrm{Add}}(\kappa^+, 1)} \models$ "λ is the least measurable cardinal above κ". Therefore, since $\overline{V}^{\mathrm{Add}(\lambda^+, \lambda^{++}) * \dot{\mathbb{R}} * \dot{\mathrm{Add}}(\kappa^+, 1)} \models$ "$\mathrm{Coll}(\lambda^+, \lambda^{++})$ is λ^+-directed closed", $\overline{V}^{\mathrm{Add}(\lambda^+, \lambda^{++}) * \dot{\mathbb{R}} * \dot{\mathrm{Add}}(\kappa^+, 1) * \dot{\mathrm{Coll}}(\lambda^+, \lambda^{++})} = V \models$ "λ is the least measurable cardinal above κ" as well.

We continue with a unified proof of Theorems 1 and 2. We summarize where we are at this point. For both of these theorems, we have that $V \models$ "ZFC + $\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above κ". For Theorem 1, for τ as in the statement of that theorem, we have that in addition, $V \models$ "λ carries τ normal measures". For Theorem 2, we have that in addition, $V \models$ "λ carries λ^+ normal measures".

We outline now the construction of the model N witnessing the conclusions of the Theorem of [6], since this model (built within $V[G]$) will witness the desired conclusions of our theorems. We quote freely from [6], using portions verbatim as necessary. As in [6], the fact that κ is 2^λ supercompact for $\lambda > \kappa$ the least measurable cardinal implies there is a supercompact ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$ with the Menas partition property [18] such that $C_0 = \{p \in P_\kappa(\lambda) \mid p \cap \kappa$ is a measurable cardinal and $\overline{p}$ is the least measurable cardinal greater than $p \cap \kappa\} \in \mathcal{U}$.

The forcing conditions $\mathbb{P}$ used in the proof of Theorems 1 and 2 are the set of all finite sequences of the form $\langle p_1, \ldots, p_n, f_0, \ldots, f_n, A, F \rangle$ satisfying the following properties:

1. Each p_i for $1 \le i \le n$ is an element of C_0, and for $1 \le i < j \le n$, $p_i \subsetneqq p_j$, where as in [6], $p_i \subsetneqq p_j$ means $p_i \subseteq p_j$ and $\overline{p_i} < p_j \cap \kappa$.
2. $f_0 \in \mathrm{Coll}(\omega_1, {<}\overline{p_1})$, for $1 \le i < n$, $f_i \in \mathrm{Coll}(\overline{p}_i^+, {<}\overline{p_{i+1}})$, and $f_n \in \mathrm{Coll}(\overline{p}_n^+, {<}\lambda)$.
3. $A \subseteq C_0$, $A \in \mathcal{U}$, and for every $q \in A$, $p_n \subsetneqq q$ and the range and domain of f_n are both subsets of q, meaning that if $\langle\langle \alpha, \beta \rangle, \gamma\rangle \in f_n$, then $\alpha, \beta, \gamma \in q$.

4. F is a function defined on A such that for $p \in A$, $F(p) \in \mathrm{Coll}(\overline{p}^+, {<}\lambda)$, and if $q \in A$, $p \precsim q$, then the range and domain of $F(p)$ are both subsets of q.

Before we can define the ordering on $\mathbb{P}$, we need to define, for $p, q \in A$ with $p \precsim q$ and $f \in \mathrm{Coll}(\overline{p}^+, {<}\lambda)$ such that the range and domain of f are subsets of q, the collapse of f in q, denoted f^*_q. Let $h : q \to \overline{q}$ be the unique order isomorphism between q and $\overline{q}$. Then $f^*_q : \overline{p}^+ \times \overline{q} \to \overline{q}$ is defined as $f^*_q(\langle \alpha, h^{-1}(\beta)\rangle) = h(f(\langle \alpha, h^{-1}(\beta)\rangle))$ if $h^{-1}(\beta) \in q$. In other words, to define f^*_q given f, we transform using h^{-1} the appropriate $\langle \alpha, \beta\rangle \in \overline{p}^+ \times \overline{q}$ into an element of $\overline{p}^+ \times \lambda$, apply f to it, and collapse the result using h. It is easily checked $f^*_q \in \mathrm{Coll}(\overline{p}^+, {<}\overline{q})$.

We are now able to define the ordering on $\mathbb{P}$. If $\pi_0 = \langle p_1, \ldots, p_n, f_0, \ldots, f_n, A, F\rangle$ and $\pi_1 = \langle q_1, \ldots, q_m, g_0, \ldots, g_m, B, H\rangle$, then $\pi_1 \geq \pi_0$ iff the following conditions hold:

1. $n \leq m$, $p_i = q_i$ for $1 \leq i \leq n$, and $q_i \in A$ for $n+1 \leq i \leq m$.
2. $f_i \subseteq g_i$ for $0 \leq i < n$, and $(f_n)^*_{q_{n+1}} \subseteq g_n$. If $n = m$, then $f_n \subseteq g_n$.
3. $(F(q_i))^*_{q_{i+1}} \subseteq g_i$ for $n+1 \leq i < m$, and $F(q_m) \subseteq g_m$.
4. $B \subseteq A$.
5. For every $p \in B$, $F(p) \subseteq H(p)$.

Let G be V-generic over $\mathbb{P}$. As in [6], we can define sequences $r = \langle p_i \mid i \in \omega - \{0\}\rangle$ and $g = \langle G_i \mid i < \omega\rangle$, where $p_i \in r$ iff $\exists \pi \in G[p_i \in \pi]$ and $G_i = \bigcup\{f_i \mid \exists \pi \in G[\pi = \langle p_1, \ldots, p_n, f_0, \ldots, f_i, \ldots, f_n, A, F\rangle]\}$. These sequences will be well-defined by the genericity of G.

We are now in a position to describe the inner model $N \subseteq V[G]$ which, when appropriately constructed, will witness either the conclusions of Theorem 1 or the conclusions of Theorem 2. For $\delta \in [\kappa, \lambda)$, δ inaccessible, let $r{\restriction}\delta = \langle p_i \cap \delta \mid i \in \omega - \{0\}\rangle$, and let $g{\restriction}\delta = \langle G^\delta_i \mid i < \omega\rangle$, where $G^\delta_i = G_i{\restriction}\overline{p_{i+1} \cap \delta}$. Intuitively, N is the least model of ZF extending V which contains, for each inaccessible $\delta \in [\kappa, \lambda)$, the sequences $r{\restriction}\delta$ and $g{\restriction}\delta$. More formally, let $\mathcal{L}_1$ be the sublanguage of the forcing language $\mathcal{L}$ with respect to $\mathbb{P}$ which contains symbols $\check{v}$ for each $v \in V$, a unary predicate symbol $\check{V}$ (to be interpreted $\check{V}(\check{v})$ iff $v \in V$), and for $\delta \in [\kappa, \lambda)$, δ inaccessible, symbols $\dot{r}{\restriction}\delta$ for $r{\restriction}\delta$ and $\dot{g}{\restriction}\delta$ for $g{\restriction}\delta$. Then N can be defined inside $V[G]$ as follows:

$$N_0 = \emptyset,$$
$$N_\lambda = \textstyle\bigcup_{\alpha<\lambda} N_\alpha \quad \text{if } \lambda \text{ is a limit ordinal,}$$
$$N_{\alpha+1} = \left\{ x \subseteq N_\alpha \;\middle|\; \begin{array}{l} x \text{ is definable over the model } \langle N_\alpha, \in, c\rangle_{c \in N_\alpha} \\ \text{via a term } \tau \in \mathcal{L}_1 \text{ of rank} \leq \alpha \end{array} \right\},$$
$$N = \textstyle\bigcup_{\alpha \in \mathrm{Ord}^V} N_\alpha.$$

The standard arguments show $N \vDash$ ZF. By Lemmas 1–7 and the intervening remarks of [6], $N \vDash$ "$\kappa = \aleph_\omega$ + $\lambda = \kappa^+ = \aleph_{\omega+1}$ + For any normal measure $\mathcal{U} \in V$ over λ, $\mathcal{U}^* = \{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$ is a normal measure over λ + $\mathrm{DC}_{\aleph_\omega}$". Further, Lemmas 3 and 4 of [6] and their proofs tell us that for $\delta < \kappa$ inaccessible, any formula mentioning only (terms for ground model sets and) $\dot{r}\restriction\delta$ and $\dot{g}\restriction\delta$ may be decided in $V[r\restriction\delta, g\restriction\delta]$ the same way as in $V[G]$, and that $V[r\restriction\delta, g\restriction\delta]$ is obtained by forcing with a partial ordering having size less than λ. In particular, any set of ordinals in N is actually a member of $V[r\restriction\delta, g\restriction\delta]$ for the appropriate $\delta < \kappa$. These facts will be critical in the proof of Theorems 1 and 2 and the following two lemmas.

LEMMA 2.2. *Suppose $\mathcal{U}^* \in N$ is a normal measure over λ. Then for some normal measure $\mathcal{U} \in V$ over λ, $\mathcal{U}^* = \{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$.*

Proof. We use ideas found in the proof of Theorem 2.3(e) of [7]. Let τ be a term for $\mathcal{U}^*$. Since $\mathcal{U}^* \in N$, we may choose $\delta < \kappa$, δ inaccessible, such that τ mentions only $\dot{r}\restriction\delta$ and $\dot{g}\restriction\delta$. By our remarks in the paragraph immediately preceding the statement of Lemma 2.2, the set $\mathcal{U}^*\restriction\delta = \mathcal{U}^* \cap V[r\restriction\delta, g\restriction\delta] \in V[r\restriction\delta, g\restriction\delta]$, which immediately implies that $\mathcal{U}^*\restriction\delta$ is in $V[r\restriction\delta, g\restriction\delta]$ a normal measure over λ. Again by our remarks in the paragraph immediately preceding the statement of Lemma 2.2 and by the results of [17], it must consequently be the case that for some $\mathcal{U} \in V$ a normal measure over λ, $\mathcal{U}^*\restriction\delta$ is definable in $V[r\restriction\delta, g\restriction\delta]$ as $\{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$. Therefore, since in N, $\mathcal{U}^*$ is a normal measure over λ, by the same argument as found in the last paragraph of the proof of Lemma 2.1, for $\mathcal{U}'$ defined in N as $\{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$, $\mathcal{U}' = \mathcal{U}^*$. This completes the proof of Lemma 2.2. ∎

LEMMA 2.3. *In N, the cardinal and cofinality structure above λ is the same as in V.*

Proof. Let β and γ be arbitrary ordinals, and suppose $N \vDash$ "$f : \beta \to \gamma$ is a function". Since f may be coded by a set of ordinals, by our remarks in the paragraph immediately preceding the statement of Lemma 2.2, $f \in V[r\restriction\delta, g\restriction\delta]$ for some $\delta < \kappa$. Since $V[r\restriction\delta, g\restriction\delta]$ is obtained by forcing with a partial ordering having size less than λ, f cannot witness that any V-cardinal greater than or equal to λ has a different cardinality or cofinality. This contradiction completes the proof of Lemma 2.3. ∎

By Lemmas 2.2 and 2.3 and our earlier work, if V^* is as in Theorem 1, then N witnesses the conclusions of Theorem 1. Similarly, Lemmas 2.2 and 2.3 and our earlier work imply that if V^* is as in Theorem 2, then N witnesses the conclusions of Theorem 2. This completes the proofs of Theorems 1 and 2. ∎

Suppose V is an inner model (e.g., as given in [19]) with $V \vDash$ "$\kappa < \lambda$ are such that κ is regular and λ is measurable + For some cardinal τ which is either less than or equal to κ or is one of the cardinals λ, λ^+, or λ^{++}, λ carries τ normal measures". We observe that a simplified version of the proof of Theorem 3.1 of [7] shows the existence of a partial ordering $\mathbb{P}$ and a symmetric inner model $N \subseteq V^{\mathbb{P}}$ such that $N \vDash$ "κ is regular + $\lambda = \kappa^+ + \tau$ is a cardinal + λ is measurable and carries τ normal measures". In addition, suppose we start with a model $V^* \vDash$ "ZFC + GCH holds at and above λ + $\kappa < \lambda$ are such that κ is regular and λ is measurable + $\tau > \lambda^+$ is a regular cardinal" and then force with the partial ordering $\mathrm{Add}(\lambda^+, \tau) * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is a term for the reverse Easton iteration of length λ which begins by adding a Cohen subset to κ^+ and then adds a Cohen subset to the successor of each inaccessible cardinal in the open interval (κ^+, λ). If we denote the resulting generic extension by V, then by standard arguments, κ remains regular in V. In addition, by our earlier remarks, $V \vDash$ "τ is a regular cardinal + λ is measurable and carries τ normal measures". Once again, a simplified version of the proof of Theorem 3.1 of [7] shows the existence of a partial ordering $\mathbb{P}$ and a symmetric inner model $N \subseteq V^{\mathbb{P}}$ such that $N \vDash$ "κ is regular + $\lambda = \kappa^+ + \tau$ is a regular cardinal + λ is measurable and carries τ normal measures". Note that in both cases mentioned above, $\mathbb{P} = \mathrm{Coll}(\kappa, {<}\lambda)$, and if G is V-generic over $\mathbb{P}$, N may intuitively be described as the least model of ZF extending V which contains, for each inaccessible cardinal δ in the open interval (κ, λ), the set $G \restriction \delta$.

It is thus true that because of the existence of the relevant inner models, it is relatively consistent for the successor of a regular cardinal to be measurable and to carry essentially any desired (regular) cardinality of normal measures. Due to the current state of knowledge, however, the existence of a model in which $\aleph_{\omega+1}$ carries, say, exactly four normal measures remains open. We therefore conclude this paper by reiterating and expanding upon the title question, i.e., by asking how many normal measures $\aleph_{\omega+1}$, or indeed, the successor of any singular cardinal, can carry. More specifically, is it relatively consistent for $\aleph_{\omega+1}$ to carry exactly τ normal measures, where τ is a cardinal and either $\tau = 1$, $\tau = 2$, or $4 \le \tau \le \aleph_{\omega+1}$?

References

[1] A. Apter, *On the class of measurable cardinals without the Axiom of Choice*, Israel J. Math. 79 (1992), 367–379.

[2] —, *Some remarks on normal measures and measurable cardinals*, Math. Logic Quart. 47 (2001), 35–44.

[3] —, *Successors of singular cardinals and measurability*, Adv. Math. 55 (1985), 228–241.

[4] —, *Successors of singular cardinals and measurability revisited*, J. Symbolic Logic 55 (1990), 492–501.

[5] A. Apter, J. Cummings and J. D. Hamkins, *Large cardinals with few measures*, submitted to Proc. Amer. Math. Soc.
[6] A. Apter and M. Magidor, *Instances of dependent choice and the measurability of* $\aleph_{\omega+1}$, Ann. Pure Appl. Logic 74 (1995), 203–219.
[7] E. Bull and E. M. Kleinberg, *A consistent consequence of AD*, Trans. Amer. Math. Soc. 247 (1979), 211–226.
[8] J. D. Hamkins, *Gap forcing*, Israel J. Math. 125 (2001), 237–252.
[9] —, *Gap forcing: Generalizing the Lévy–Solovay theorem*, Bull. Symbolic Logic 5 (1999), 264–272.
[10] S. C. Jackson, *A computation of* δ^1_5, Mem. Amer. Math. Soc. 140 (1999), no. 670.
[11] —, *AD and the very fine structure of* $L(\mathbb{R})$, Bull. Amer. Math. Soc. 21 (1989), 77–81.
[12] A. Kanamori, *The Higher Infinite*, Springer, Berlin, 1994.
[13] E. M. Kleinberg, *AD* $\vdash$ *"The* $\aleph_n$ *are Jonsson cardinals and* $\aleph_\omega$ *is a Rowbottom cardinal"*, Ann. Math. Logic 12 (1977), 229–248.
[14] —, *Infinitary Combinatorics and the Axiom of Determinateness*, Lecture Notes in Math. 612, Springer, Berlin, 1977.
[15] —, *Strong partition properties for infinite cardinals*, J. Symbolic Logic 35 (1970), 410–428.
[16] R. Laver, *Making the supercompactness of* κ *indestructible under* κ*-directed closed forcing*, Israel J. Math. 29 (1978), 385–388.
[17] A. Lévy and R. Solovay, *Measurable cardinals and the continuum hypothesis*, ibid. 5 (1967), 234–248.
[18] T. Menas, *A combinatorial property of* $P_\kappa(\lambda)$, J. Symbolic Logic 41 (1976), 225–234.
[19] W. Mitchell, *Sets constructible from sequences of ultrafilters*, ibid. 39 (1974), 57–66.

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