Best constants for Lipschitz embeddings of metric spaces into $c_0$

by

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Abstract. We answer a question of Aharoni by showing that every separable metric space can be Lipschitz 2-embedded into $c_0$ and this result is sharp; this improves earlier estimates of Aharoni, Assouad and Pelant. We use our methods to examine the best constant for Lipschitz embeddings of the classical $\ell_p$-spaces into $c_0$ and give other applications. We prove that if a Banach space embeds almost isometrically into $c_0$, then it embeds linearly almost isometrically into $c_0$. We also study Lipschitz embeddings into $c_0^+$. 

1. Introduction. In 1974, Aharoni [1] proved that every separable metric space $(M,d)$ is Lipschitz isomorphic to a subset of the Banach space $c_0$. Thus, for some constant $K$, there is a map $f : M \to c_0$ which satisfies the inequality

$$d(x,y) \leq \|f(x) - f(y)\| \leq Kd(x,y), \quad x, y \in M.$$ 

Aharoni proved this result with $K = 6 + \varepsilon$ where $\varepsilon > 0$, so that every separable metric space $(6 + \varepsilon)$-embeds into $c_0$. He also noted that if one takes $M$ to be the Banach space $\ell_1$, one cannot have $K < 2$. In fact, the map defined by Aharoni took values in the positive cone $c_0^+$ of $c_0$. Later Assouad [3] refined Aharoni’s result by showing that every separable metric space $(3 + \varepsilon)$-embeds into $c_0^+$ (see [6, pp. 176 ff.]). A further improvement was obtained by Pelant in 1994 [16] who showed that every separable metric space 3-embeds into $c_0^+$ and that this result is sharp in the sense that $\ell_1$ cannot be $\lambda$-embedded into $c_0^+$ with $\lambda < 3$ (see also [2] for the lower bound).

These results leave open the question of the best constant for Lipschitz embeddings into $c_0$. Note that $c_0$ can only be 2-embedded into $c_0^+$. The main result of this paper is that every separable metric space 2-embeds into $c_0$, and this is sharp by Aharoni’s remark above. To prove this result, for $1 < \lambda \leq 2$ we establish a criterion $\Pi(\lambda)$ which is sufficient to imply

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that a separable metric space $\lambda$-embeds into $c_0$ (and the converse statement is almost true). This criterion enables us to establish sharp results concerning the embedding of $\ell_p$-spaces into $c_0$: thus $\ell_p 2^{1/p}$-embeds into $c_0$ if $1 \leq p < \infty$, and the constant is best possible. Using a previous work of the first author and D. Werner [12], we also show that a Banach space which embeds almost isometrically into $c_0$ embeds linearly almost isometrically into $c_0$.

The same techniques can be applied to embeddings into $c_0^+$. Here we show that $\ell_p (2^p + 1)^{1/p}$-embeds into $c_0^+$ and $\ell_p^+ 3^{1/p}$-embeds into $c_0^+$ and in each case the result is best possible.

We conclude the paper by proving that every separable ultrametric space embeds isometrically into $c_0^+$ and the infinite branching tree embeds isometrically into $c_0$.

2. Lipschitz embeddings into $c_0$. Let $(M, d)$ be a metric space and let $A$ and $B$ be non-empty subsets of $M$.

We define

$$\delta(A, B) = \inf_{a \in A, b \in B} d(a, b), \quad D(A, B) = \sup_{a \in A, b \in B} d(a, b).$$

In this paper all metric balls are closed with strictly positive radii.

If $f : (M_1, d_1) \to (M_2, d_2)$ is a Lipschitz map between metric spaces we write Lip($f$) for the Lipschitz constant of $f$, i.e. the least constant $K$ such that $d_2(f(x), f(y)) \leq Kd_1(x, y)$ for $x, y \in M_1$.

**Lemma 2.1.** Let $(M, d)$ be a metric space and suppose that $A, B$ and $C$ are non-empty subsets of $M$. Then for $\varepsilon > 0$, there exists a Lipschitz function $f : M \to \mathbb{R}$ with Lip($f$) \leq 1 such that

(i) $|f(x)| \leq \varepsilon, x \in C$,

(ii) $|f(x) - f(y)| = \theta = \min(\delta(A, B), \delta(A, C) + \delta(B, C) + 2\varepsilon), x \in A, y \in B$.

**Proof.** Let us augment $M$ by adding an extra point $0$; let $M^* = M \cup \{0\}$. We define

$$d^*(x, y) = \begin{cases} 
\min(d(x, y), d(x, C) + d(y, C) + 2\varepsilon), & x, y \in M, \\
(d(x, C) + \varepsilon, & x \in M, y = 0, \\
d(y, C) + \varepsilon, & x = 0, y \in M, \\
0, & x = y = 0.
\end{cases}$$

One can easily check that $d^*$ is a metric on $M^*$. We can pick $s, t$ in $\mathbb{R}$ such that

$$-(\delta(B, C) + \varepsilon) \leq s \leq 0 \leq t \leq \delta(A, C) + \varepsilon \quad \text{and} \quad t - s = \theta.$$
Then we define $g : A \cup B \cup \{0\} \to \mathbb{R}$ by $g = t$ on $A$, $g = s$ on $B$ and $g(0) = 0$. This function is 1-Lipschitz for $d^*$ and can be extended to a 1-Lipschitz function $f^*$ on $(M^*, d^*)$. Let $f$ be the restriction of $f^*$ to $M$. Then $f$ satisfies the conditions of the lemma.

For $\lambda > 1$, we say that a metric space $(M, d)$ has property $\Pi(\lambda)$ if given any $\mu > \lambda$ there exists $\nu > \mu$ such that if $B_1$ and $B_2$ are two metric balls of radii $r_1, r_2$ respectively then there are finitely many sets $(U_j)_{j=1}^N, (V_j)_{j=1}^N$ such that

$$\lambda \delta(U_j, V_j) \geq \nu(r_1 + r_2), \quad 1 \leq j \leq N,$$

and

$$\{(x, y) \in B_1 \times B_2 : d(x, y) > \mu(r_1 + r_2)\} \subset \bigcup_{j=1}^N (U_j \times V_j).$$

In this definition the sets $U_j, V_j$ are allowed to be repeated. It is clearly possible, without loss of generality, to assume they are closed. We can also (altering the value of $\nu$) assume that they are open.

**Lemma 2.2.** Every metric space has property $\Pi(2)$.

**Proof.** For $\mu > 2$, let

$$U = B_1 \cap \{x : \exists y \in B_2, d(x, y) > \mu(r_1 + r_2)\},$$

$$V = B_2 \cap \{y : \exists x \in B_1, d(x, y) > \mu(r_1 + r_2)\}.$$  

Then

$$\{(x, y) \in B_1 \times B_2 : d(x, y) > \mu(r_1 + r_2)\} \subset U \times V.$$ 

Suppose $x \in U$, $y \in V$. Assume, without loss of generality, that $r_1 \leq r_2$. Then there exists $x' \in U$ with $d(x', y) > \mu(r_1 + r_2)$. Hence

$$d(x, y) > \mu(r_1 + r_2) - d(x, x') \geq \mu(r_1 + r_2) - 2r_1 \geq (\mu - 1)(r_1 + r_2).$$

Therefore we can take $\nu = 2\mu - 2 > \mu$.

We say that a metric space is *locally compact* (respectively, *locally finite*) if all its metric balls are relatively compact (respectively, finite). Note that we do not use the terminology “locally compact” in the usual way. The metric spaces with relatively compact metric balls are often called *proper metric spaces*.

**Lemma 2.3.** Let $\lambda > 1$. Then every locally compact metric space has property $\Pi(\lambda)$.

**Proof.** Let $\mu > \lambda > 1$ and $B_1, B_2$ be two balls of a locally compact metric space $(M, d)$, with respective radii $r_1$ and $r_2$. Pick $\nu$ such that $\mu < \nu < \lambda \mu$. We define $\Delta = \{(x, y) \in B_1 \times B_2 : d(x, y) > \mu(r_1 + r_2)\}$. Let $\varepsilon > 0$. Since $M$
is locally compact, there are finitely many points \((x_j, y_j)\) in \(\Delta\) such that

\[
\Delta \subset \bigcup_{j=1}^{N} (U_j \times V_j), \quad \text{where} \quad U_j = B(x_j, \varepsilon) \quad \text{and} \quad V_j = B(y_j, \varepsilon).
\]

Then, for all \(1 \leq j \leq N\), \(\lambda \delta(U_j, V_j) > \lambda \mu(r_1 + r_2) - 2 \lambda \varepsilon > \nu(r_1 + r_2)\), if \(\varepsilon\) was chosen small enough, namely \(\varepsilon < (2 \lambda)^{-1}(\lambda \mu - \nu)(r_1 + r_2)\). □

**Proposition 2.4.** Let \(\lambda_0 \geq 1\). If a metric space \((M, d)\) \(\lambda_0\)-embeds into \(c_0\) then it has property \(\Pi(\lambda)\) for every \(\lambda > \lambda_0\).

**Proof.** Suppose \(\mu > \lambda\). Let \(B_1, B_2\) be metric balls of radii \(r_1, r_2\) and centers \(a_1, a_2\). Let \(\Delta = \{(x, y) \in B_1 \times B_2 : d(x, y) > \mu(r_1 + r_2)\}\). Let \(f : M \to c_0\) be an embedding such that

\[
d(x, y) \leq \|f(x) - f(y)\| \leq \lambda_0 d(x, y), \quad x, y \in M.
\]

Suppose \(f(x) = (f_i(x))_{i=1}^{\infty}\). Then there exists \(n\) so that

\[
|f_i(a_1) - f_i(a_2)| < (\mu - \lambda)(r_1 + r_2), \quad i \geq n + 1.
\]

Thus if \((x, y) \in \Delta\) we have

\[
|f_i(x) - f_i(y)| < (\mu - \lambda)(r_1 + r_2) + \lambda_0 r_1 + \lambda_0 r_2 < d(x, y), \quad i \geq n + 1.
\]

Hence

\[
d(x, y) \leq \max_{1 \leq i \leq n} |f_i(x) - f_i(y)|, \quad (x, y) \in \Delta.
\]

Choose \(\varepsilon > 0\) so that \(\lambda(\mu - \varepsilon) > \lambda_0 \mu\). By a compactness argument we can find coverings \((W_k)_{k=1}^{m}\) of \(B_1\) and \((W'_k)_{k=1}^{m'}\) of \(B_2\) such that

\[
|f_i(x) - f_i(x')| \leq \frac{1}{2} \varepsilon(r_1 + r_2), \quad x, x' \in W_k, \quad 1 \leq i \leq n, \quad 1 \leq k \leq m,
\]

and

\[
|f_i(x) - f_i(x')| \leq \frac{1}{2} \varepsilon(r_1 + r_2), \quad x, x' \in W'_k, \quad 1 \leq i \leq n, \quad 1 \leq k \leq m'.
\]

Let

\[
S = \{(k, k') : 1 \leq k \leq m, \quad 1 \leq k' \leq m', \quad W_k \times W'_{k'} \cap \Delta \neq \emptyset\}
\]

and then define \((U_j)_{j=1}^{N}, (V_j)_{j=1}^{N}\) in such a way that \((U_j \times V_j)_{j=1}^{N}\) is an enumeration of \((W_k \times W'_{k'})_{(k,k') \in S}\). Clearly, \(\Delta \subset \bigcup_{j=1}^{N} U_j \times V_j\). Now suppose \(x \in U_j, y \in V_j\). Then there exist \(x' \in U_j, y' \in V_j\) so that \(d(x', y') > \mu(r_1 + r_2)\).

Thus there exists \(i, 1 \leq i \leq n, \) so that \(|f_i(x') - f_i(y')| > \mu(r_1 + r_2)\). However,

\[
|f_i(x) - f_i(y)| > |f_i(x') - f_i(y')| - \varepsilon(r_1 + r_2) > (\mu - \varepsilon)(r_1 + r_2).
\]

Hence

\[
\delta(U_j, V_j) > \frac{\mu - \varepsilon}{\lambda_0} (r_1 + r_2).
\]

Thus we can take \(\nu = \lambda \lambda_0^{-1}(\mu - \varepsilon) > \mu\). □
We next observe that the definition of $\Pi(\lambda)$ implies a formally stronger conclusion.

**Lemma 2.5.** Let $(M,d)$ be a metric space with property $\Pi(\lambda)$. Then for every $\mu > \lambda$ there is a constant $\nu > \mu$ so that if $B_1$ and $B_2$ are two metric balls of radii $r_1, r_2$ respectively then there are finitely many sets $(U_j)_{j=1}^N, (V_j)_{j=1}^N$ such that if $(x,y) \in B_1 \times B_2$ and $d(x,y) > \mu (r_1 + r_2)$ then there exists $1 \leq j \leq N$ so that $x \in U_j, y \in V_j$ and

$$\lambda \mu \delta(U_j, V_j) \geq \nu d(x,y).$$

**Proof.** By the definition of $\Pi(\lambda)$ there exists $\nu' > \lambda$ so that if $B_1$ and $B_2$ are two metric balls of radii $r_1, r_2$ respectively then there are finitely many sets $(U_j)_{j=1}^N, (V_j)_{j=1}^N$ such that

$$\lambda \delta(U_j, V_j) \geq \nu' (r_1 + r_2), \quad 1 \leq j \leq N,$$

and

$$\{(x,y) \in B_1 \times B_2 : d(x,y) > \mu (r_1 + r_2)\} \subset \bigcup_{j=1}^N (U_j \times V_j).$$

Suppose $\mu < \nu < \nu'$ and let $\varepsilon > 0$ be chosen so that $(1 + \varepsilon)\nu = \nu'$. Let $B_1, B_2$ be a pair of metric balls of radii $r_1, r_2 > 0$. Let $D = D(B_1, B_2)$ and let $m$ be the greatest integer such that $(1 + \varepsilon)^m \mu (r_1 + r_2) \leq D$. We define $B_1^{(k)}$ for $0 \leq k \leq m$ to be the ball with the same center as $B_1$ and radius $(1 + \varepsilon)^k r_1$; similarly, $B_2^{(k)}$ for $0 \leq k \leq m$ is the ball with the same center as $B_2$ and radius $(1 + \varepsilon)^k r_2$. For each $0 \leq k \leq m$ we may determine sets $U_{kl}, V_{kl}$ for $1 \leq l \leq N_k$ so that

$$\lambda \delta(U_{kl}, V_{kl}) \geq \nu' (1 + \varepsilon)^k (r_1 + r_2)$$

and

$$\{(x,y) \in B_1^{(k)} \times B_2^{(k)} : d(x,y) > \mu (1 + \varepsilon)^k (r_1 + r_2)\} \subset \bigcup_{l=1}^{N_k} (U_{kl} \times V_{kl}).$$

Now if $x \in B_1, y \in B_2$ with $d(x,y) > \mu (r_1 + r_2)$ we may choose $0 \leq k \leq m$ so that

$$(1 + \varepsilon)^k \mu (r_1 + r_2) < d(x,y) \leq (1 + \varepsilon)^{k+1} \mu (r_1 + r_2).$$

Then for a suitable $1 \leq l \leq N_k$ we have $x \in U_{kl}, y \in V_{kl}$ and

$$\lambda \mu \delta(U_{kl}, V_{kl}) \geq \nu' (1 + \varepsilon)^k \mu (r_1 + r_2) \geq \frac{\nu'}{1 + \varepsilon} d(x,y) = \nu d(x,y).$$

Relabeling the sets $(U_{kl}, V_{kl})_{l \leq N_k, 0 \leq k \leq m}$ gives the conclusion. \[
\]

**Lemma 2.6.** Suppose $(M,d)$ has property $\Pi(\lambda)$. Suppose $0 < \alpha < \beta$. Let $F, G$ be finite subsets of $M$ and let $\Delta(F, G, \alpha, \beta)$ be the set of $(x,y) \in M \times M$
such that
\[ \lambda(d(x, G) + d(y, G)) + \alpha \leq d(x, y) < \lambda(d(x, F) + d(y, F)) + \beta. \]

Then there is a finite set \( F = F(F, G, \alpha, \beta) \) of functions \( f : M \to \mathbb{R} \) with \( \text{Lip}(f) \leq \lambda \) such that
\[ |f(x)| \leq \lambda \beta, \quad x \in F, \]
and
\[ d(x, y) < \max_{f \in F} |f(x) - f(y)|, \quad (x, y) \in \Delta(F, G, \alpha, \beta). \]

**Proof.** Let \( R \) be the diameter of \( G \). Then for \( (x, y) \in \Delta(F, G, \alpha, \beta) \) we have
\[ \lambda(d(x, y) - R) + \alpha \leq d(x, y), \]
so that
\[ (\lambda - 1)d(x, y) < \lambda R. \]

Hence
\[ d(x, G) + d(y, G) < \frac{R}{\lambda - 1}. \]

We next let
\[ \mu = \lambda + \frac{(\lambda - 1)\alpha}{2R} \]
and choose \( \nu = \nu(\mu) \) according to the conclusion of Lemma 2.5.

We now fix \( \varepsilon > 0 \) so that \( 4\mu\varepsilon < \alpha \). Let \( E = \{ x : d(x, G) < (\lambda - 1)^{-1}R \} \). Since \( E \) is metrically bounded and \( F \cup G \) is finite we can partition \( E \) into finitely many subsets \( (E_1, \ldots, E_m) \) so that for each \( z \in F \cup G \) we have
\[ |d(x, z) - d(x', z)| \leq \varepsilon, \quad x, x' \in E_j, 1 \leq j \leq m. \]

Since \( G \) is finite, for each \( j \) there exist \( z_j \in G \) and \( r_j \geq 0 \) so that
\[ \inf_{x \in E_j} d(x, z_j) = \inf_{x \in E_j} d(x, G) = r_j. \]

Thus \( E_j \) is contained in a ball \( B_j \) centered at \( z_j \) with radius \( r_j + \varepsilon \).

For each pair \( (j, k) \) we can find finitely many pairs of sets \( (U_{jkl}, V_{jkl})_{l=1}^{N_{jk}} \) such that for every \( (x, y) \in E_j \times E_k \) with \( d(x, y) > \mu(r_j + r_k + 2\varepsilon) \) there exists \( 1 \leq l \leq N_{jk} \) with \( x \in U_{jkl}, y \in V_{jkl} \) and
\[ \lambda \mu \delta(U_{jkl}, V_{jkl}) \geq \nu d(x, y). \]

We may as well assume that \( U_{jkl} \subset E_j \) and \( V_{jkl} \subset E_k \).

Then we apply Lemma 2.1 to construct Lipschitz functions \( f_{jkl} : M \to \mathbb{R} \) where \( 1 \leq j, k \leq m, 1 \leq l \leq N_{jk} \) such that \( \text{Lip}(f_{jkl}) \leq \lambda \),
\[ |f_{jkl}(x)| \leq \lambda \beta, \quad x \in F, \]
and
\[ |f_{jkl}(x) - f_{jkl}(y)| \geq \lambda \theta_{jkl}, \quad x \in U_{jkl}, y \in V_{jkl}, \]
where
\[ \theta_{jkl} = \min (\delta(U_{jkl}, V_{jkl}), \delta(U_{jkl}, F) + \delta(V_{jkl}, F) + 2\beta). \]

Now let us suppose \((x, y) \in \Delta(F, G, \alpha, \beta)\). Then there exists \((j, k)\) so that \(x \in E_j, y \in E_k\). Note that
\[
d(x, y) \geq \lambda(d(x, G) + d(y, G)) + \alpha \geq \lambda(r_j + r_k) + \alpha \\
= \mu(r_j + r_k + 2\varepsilon) + \alpha - 2\mu\varepsilon - (\mu - \lambda)(r_j + r_k) \\
\geq \mu(r_j + r_k + 2\varepsilon) + \alpha - 2\mu\varepsilon - (\mu - \lambda)(\lambda - 1)^{-1} R \\
> \mu(r_j + r_k + 2\varepsilon).
\]
Thus there exists \(1 \leq l \leq N_{jk}\) so that \(x \in U_{jkl}, y \in V_{jkl}\) and
\[
\lambda\delta(U_{jkl}, V_{jkl}) \geq \frac{\nu}{\mu} d(x, y) > d(x, y).
\]
On the other hand, \(\varepsilon < \alpha/2 < \beta/2\). So
\[
\lambda(\delta(U_{jkl}, F) + \delta(V_{jkl}, F) + 2\beta) \geq \lambda(d(x, F) + d(y, F) + 2\beta - 2\varepsilon) \\
> \lambda(d(x, F) + d(y, F) + \beta) \\
> d(x, y) + (\lambda - 1)\beta.
\]
Hence
\[
|f_{jkl}(x) - f_{jkl}(y)| \geq \lambda\theta_{jkl} > d(x, y).
\]
Thus we can take for \(F\) the collection of all functions \(f_{jkl}\) for \(1 \leq j, k \leq m\) and \(1 \leq l \leq N_{jk}\).

We now state our main result.

**Theorem 2.7.** If a separable metric space \((M, d)\) has property \(\Pi(\lambda)\) for \(\lambda > 1\), then there is a Lipschitz embedding \(f : M \to c_0\) with
\[
d(x, y) < \|f(x) - f(y)\| \leq \lambda d(x, y), \quad x, y \in M, x \neq y.
\]

**Proof.** Let \((u_n)_{n=1}^{\infty}\) be a countable dense set of distinct points of \(M\). Set \(F_k = \{u_1, \ldots, u_k\}\) for \(n \geq 1\). Let \((\varepsilon_n)_{n=1}^{\infty}\) be a strictly decreasing sequence with \(\lim_{n \to \infty} \varepsilon_n = 0\).

Using Lemma 2.6 we can find an increasing sequence \((n_k)_{k=0}^{\infty}\) of integers (with \(n_0 = 0\)) and a sequence \((f_j)_{j=1}^{\infty}\) of Lipschitz functions \(f_j : M \to \mathbb{R}\) with \(\text{Lip}(f_j) \leq \lambda\) so that
\[
|f_j(x)| \leq \lambda\varepsilon_k, \quad x \in F_k, \ n_{k-1} < j \leq n_k,
\]
and if
\[
(2.1) \quad \lambda(d(x, F_{k+1}) + d(y, F_{k+1})) + \varepsilon_{k+1} \leq d(x, y) < \lambda(d(x, F_k) + d(y, F_k)) + \varepsilon_k
\]
then
\[
\max_{n_{k-1} < j \leq n_k} |f_j(x) - f_j(y)| > d(x, y).
\]
Define the map \( f : M \to \ell_\infty \) by \( f(x) = (f_j(x))_{j=1}^\infty \). Then \( \text{Lip}(f) \leq \lambda \) and since \( f \) maps each \( u_j \) into \( c_0 \), \( f(M) \subset c_0 \). Furthermore, if \( x \neq y \) the sequence 
\[
\sigma_k = \lambda(d(x, F_k) + d(y, F_k)) + \varepsilon_k
\]

is decreasing with \( \sigma_1 > d(x, y) \) and \( \lim_{k \to \infty} \sigma_k = 0 \). Hence there is exactly one choice of \( k \) so that (2.1) holds and thus \( \|f(x) - f(y)\| > d(x, y) \).

As a corollary, we obtain the following improvement of Aharoni’s theorem.

**Theorem 2.8.** For every separable metric space \((M, d)\) there is a Lipschitz embedding \( f : M \to c_0 \) so that
\[
d(x, y) < \|f(x) - f(y)\| \leq 2d(x, y), \quad x, y \in M, x \neq y.
\]

**Proof.** Combine Lemma 2.2 and Theorem 2.7. \( \blacksquare \)

**Remark.** It follows from Proposition 3 in Aharoni’s original paper \([1]\) that the above statement is optimal.

**Theorem 2.9.** For every locally compact metric space \((M, d)\) and every \( \lambda > 1 \), \((M, d)\) \( \lambda \)-embeds into \( c_0 \). This result is best possible.

**Proof.** The existence of the embedding follows immediately from the combination of Lemma 2.3 and Theorem 2.7. The optimality of the statement follows from Proposition 3.2 in \([16]\), where J. Pelant proved that \([0, 1]^\mathbb{N}\) equipped with the distance \( d((x_n), (y_n)) = \sum 2^{-n}|x_n - y_n| \) cannot be isometrically embedded into \( c_0 \).

To complete the picture we shall now give a locally finite counterexample. Let \((e_n)_{n=0}^\infty \) be the canonical basis of \( \ell_1 \) and consider the following locally finite metric subspace of \( \ell_1 \): \( M = \{0, e_0\} \cup \{ne_n, e_0 + ne_n : n \geq 1\} \). Assume that \( f = (f_k)_{k=1}^\infty \) is an isometry from \( M \) into \( c_0 \) such that \( f(0) = 0 \). Then for all \( n \neq m \) in \( \mathbb{N} \), there exists \( k = k_{n,m} \geq 1 \) such that
\[
|f_k(e_0 + ne_n) - f_k(me_m)| = 1 + n + m.
\]

Since \( f_k(0) = 0 \), we deduce that there is \( \varepsilon = \varepsilon_{n,m} \in \{-1, 1\} \) such that \( f_k(e_0 + ne_n) = \varepsilon(1 + n) \) and \( f_k(me_m) = -\varepsilon m \). Therefore \( f_k(e_0) = \varepsilon \) and \( f_k(ne_n) = \varepsilon n \).

Since \( f(e_0) \in c_0 \), there exists an integer \( K \) such that for all positive integers \( n \neq m, k_{n,m} \leq K \). Hence, if \( \alpha(k, n) \) is the sign of \( f_k(ne_n) \), we see that there exists \( k \leq K \) so that \( \alpha(k, n) \neq \alpha(k, m) \) whenever \( 1 \leq n < m \). But on the other hand, there is clearly an infinite subset \( A \) of \( \mathbb{N} \) such that for every \( k \leq K \) and every \( n, m \in A \), \( \alpha(k, n) = \alpha(k, m) \). This is a contradiction. \( \blacksquare \)

3. Embeddings of classical Banach spaces. In this section we will consider the best constants for embedding certain classical Banach spaces
into $c_0$. We start by establishing a lower bound condition, using the Borsuk–Ulam theorem.

**Proposition 3.1.** Suppose that $X$ is a Banach space and $f : X \to c_0$ is a Lipschitz embedding with constant $\lambda_0$. Then for any $u \in X$ with $\|u\| = 1$ and any infinite-dimensional subspace $Y$ of $X$ we have

$$\inf_{y \in Y, \|y\| = 1} \|u + y\| \leq \lambda_0.$$ 

**Proof.** It follows from Proposition 2.4 that $X$ has property $\Pi(\lambda)$ for any $\lambda > \lambda_0$. Let us consider $B_1 = -u + B_X$ and $B_2 = u + B_X$, where $B_X$ denotes the closed unit ball of $X$. Suppose $\mu > \lambda_0$ and select $\mu > \lambda > \lambda_0$. Then, for some $\nu > \mu$, we can find finitely many sets $(U_j, V_j)_{j=1}^N$ (which we can assume to be closed) satisfying

$$\lambda \delta(U_j, V_j) \geq 2\nu \quad \text{and} \quad \{ (x, y) \in B_1 \times B_2 : \|x - y\| > 2\mu \} \subset \bigcup_{j=1}^N (U_j \times V_j).$$

Now let $E$ be any subspace of $X$ of dimension greater than $N$ and let

$$A_j = \{ e \in E : \|e\| = 1, (-u + e, u - e) \in U_j \times V_j \}.$$ 

Thus the sets $A_j$ are all closed subsets of the unit sphere $S_E$ of $E$. Assume that for any $e \in S_E$, $\|u - e\| > \mu$. Then $A_1 \cup \cdots \cup A_N = S_E$. We now use a classical corollary of the Borsuk–Ulam theorem which is in fact due to Lyusternik and Shnirelman [13] and predates Borsuk’s work (see [14, p. 23]). This gives the existence of $e \in S_E$ and $k \leq N$ such that $e$ and $-e$ belong to $A_k$, i.e. $-u \pm e \in U_k$ and $u \pm e \in V_k$. This in turn implies $\delta(U_k, V_k) \leq 2$ and hence $\lambda \geq \nu > \mu$, which is a contradiction. Thus there exists $e \in S_E$ with $\|u - e\| \leq \mu$.

Since this is true for every finite-dimensional subspace $E$ of dimension greater than $N$ and every $\mu > \lambda_0$, the conclusion follows.

**Theorem 3.2.** Suppose $1 \leq p < \infty$. Then there is a Lipschitz embedding of $\ell_p$ into $c_0$ with constant $2^{1/p}$, and this constant is best possible.

**Proof.** The fact that $\ell_p$ does not $\lambda$-embed into $c_0$ when $\lambda < 2^{1/p}$ follows immediately from Proposition 3.1. So we only need to show that $\ell_p$ satisfies condition $\Pi(2^{1/p})$.

Let $B_1$ and $B_2$ be balls with centers $a_1, a_2$ and radii $r_1, r_2$. Suppose $\mu > 2^{1/p}$. Then $\mu < 2^{1/p}(\mu^p - 1)^{1/p}$. We pick $\nu$ such that

$$\mu < \nu < 2^{1/p}(\mu^p - 1)^{1/p}$$

and we fix $\varepsilon > 0$ so that

$$2^{1/p}(\mu^p (r_1 + r_2)^p - (r_1 + r_2 + 2\varepsilon)^p)^{1/p} - 2^{1+1/p}\varepsilon > \nu(r_1 + r_2).$$
We first select $N \in \mathbb{N}$ so that
\[
\sum_{k=N+1}^{\infty} |a_1(k)|^p, \quad \sum_{k=N+1}^{\infty} |a_2(k)|^p < \varepsilon^p.
\]
Let $E$ be the linear span of $\{e_1, \ldots, e_N\}$, where $(e_j)_{j=1}^\infty$ is the canonical basis of $\ell_p$. Let $P$ the canonical projection of $\ell_p$ onto $E$, $Q = I - P$ and $R = \max(||a_1|| + r_1, ||a_2|| + r_2)$. Then we partition $RB_E$ into finitely many sets $A_1, \ldots, A_m$ with diam $A_j < \varepsilon$.

Now, set $U_j = \{x \in B_1 : Px \in A_j\}$, $V_j = \{x \in B_2 : Px \in A_j\}$ and $S = \{(j, k) : \exists (x, y) \in U_j \times V_k, \|x - y\| > \mu(r_1 + r_2)\}$.

Thus we have
\[
\{(x, y) \in B_1 \times B_2 : \|x - y\| > \mu(r_1 + r_2)\} \subset \bigcup_{(j, k) \in S} U_j \times V_k.
\]
It remains to estimate $\delta(U_j, V_k)$ for $(j, k) \in S$. Suppose that $u \in U_j$, $v \in V_k$ and $x \in U_j$, $y \in V_k$ are such that $\|x - y\| > \mu(r_1 + r_2)$. Then
\[
\|u - v\| \geq \|Pu - Pv\| \geq \|Px - Py\| - 2\varepsilon.
\]
On the other hand,
\[
r_1 \geq \|x - a_1\| \geq \|Qx - Qa_1\| \geq \|Qx\| - \varepsilon,
\]
\[
r_2 \geq \|y - a_2\| \geq \|Qy - Qa_2\| \geq \|Qy\| - \varepsilon.
\]
Thus
\[
\|Qx - Qy\| \leq r_1 + r_2 + 2\varepsilon.
\]
Now
\[
\mu^p(r_1 + r_2)^p < \|Px - Py\|^p + \|Qx - Qy\|^p \leq \|Px - Py\|^p + (r_1 + r_2 + 2\varepsilon)^p.
\]
Hence
\[
\|Px - Py\|^p > \mu^p(r_1 + r_2)^p - (r_1 + r_2 + 2\varepsilon)^p
\]
and thus
\[
2^{1/p} \delta(U_j, V_k) \geq 2^{1/p}(\mu^p(r_1 + r_2)^p - (r_1 + r_2 + 2\varepsilon)^p)^{1/p} - 2^{1+1/p}\varepsilon > \nu(r_1 + r_2).
\]

We now give a second lower bound condition in place of Proposition 3.1. We do not know whether the conclusion can be improved replacing $\lambda_0^3$ by $\lambda_0$. If $X$ has a 1-unconditional basis, $\lambda_0^3$ can be improved to $\lambda_0^2$.

**Proposition 3.3.** If $X$ is a separable Banach space and $f : X \to c_0$ is a Lipschitz embedding with constant $\lambda_0$ then if $\|x\| = 1$ and $(x_n)_{n=1}^\infty$ is a normalized weakly null sequence in $X$, we have
\[
\limsup_{n \to \infty} \|x + x_n\| \leq \lambda_0^3.
\]

**Proof.** We assume that $\|x - y\| \leq \|f(x) - f(y)\| \leq \lambda_0\|x - y\|$ for $x, y \in X$. Let $\mathcal{U}$ be a non-principal ultrafilter on the natural numbers $\mathbb{N}$. We start by
proving that if \( x \in X \) and \((y_n)_{n=1}^{\infty}, (z_n)_{n=1}^{\infty}\) are two weakly null sequences with \(\lim_{n \in \mathcal{U}} \|y_n\| \leq \|x\| \) and \(\lim_{n \in \mathcal{U}} \|z_n\| \leq \|x\|\) then

\[
(3.3) \quad \lambda_0^{-1} \lim_{n \in \mathcal{U}} \|2x + y_n + z_n\| \leq \lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} \|2x + y_m + z_n\| \\
\leq \lambda_0 \lim_{n \in \mathcal{U}} \|2x + y_n + z_n\|.
\]

It suffices to show this under the condition \(\lim_{n \in \mathcal{U}} \|y_n\| = \alpha, \lim_{n \in \mathcal{U}} \|z_n\| = \beta\) where \(\alpha, \beta \leq 1\) and \(\|x\| = 1\). Fix any \(\varepsilon > 0\). Let \(f(x) = (f_j(x))_{j=1}^{\infty}\). Then for some \(N\) we have

\[
|f_j(x) - f_j(-x)| < \varepsilon, \quad j > N.
\]

Thus

\[
|f_j(x + y_m) - f_j(-x - z_n)| \leq \lambda_0(\|y_n\| + \|z_n\|) + \varepsilon, \quad j > N.
\]

Hence

\[
\lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} \max_{j > N} |f_j(x + y_m) - f_j(-x - z_n)| \leq \lambda_0(\alpha + \beta) + \varepsilon
\]

and

\[
\lim_{n \in \mathcal{U}} \max_{j > N} |f_j(x + y_n) - f_j(-x - z_n)| \leq \lambda_0(\alpha + \beta) + \varepsilon.
\]

Let \(\sigma_j = \lim_{n \in \mathcal{U}} f_j(x + y_n)\) and \(\tau_j = \lim_{n \in \mathcal{U}} f_j(-x - z_n)\). Then

\[
\lim_{n \in \mathcal{U}} |f_j(x + y_n) - f_j(-x - z_n)| = |\sigma_j - \tau_j|
\]

and

\[
\lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} |f_j(x + y_m) - f_j(-x - z_n)| = |\sigma_j - \tau_j|.
\]

Thus

\[
\lim_{n \in \mathcal{U}} \|2x + y_n + z_n\| \leq \lim_{n \in \mathcal{U}} \|f(x + y_n) - f(-x - z_n)\| \\
\leq \max \left( \max_{1 \leq j \leq N} |\sigma_j - \tau_j|, \lambda_0(\alpha + \beta) + \varepsilon \right) \\
\leq \max(\lambda_0 \lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} \|2x + y_m + z_n\|, \lambda_0(\alpha + \beta) + \varepsilon).
\]

Noting that \(\varepsilon > 0\) is arbitrary and that

\[
\alpha + \beta \\leq 2 \leq \lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} \|2x + y_m + z_n\|
\]

we deduce that

\[
\lim_{n \in \mathcal{U}} \|2x + y_n + z_n\| \leq \lambda_0 \lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} \|2x + y_m + z_n\|.
\]

The other inequality in (3.3) is similar.

Now choose \(x_n = y_n = -z_n\) in (3.3). We obtain

\[
\lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} \|2x + x_m - x_n\| \leq 2\lambda_0\|x\|
\]
provided \((x_n)_{n=1}^\infty\) is weakly null and \(\lim_{n \in U} \|x_n\| \leq \|x\|\). Hence
\[
\lim_{m \in U} \|x + \frac{1}{2} x_m\| \leq \lambda_0 \|x\|.
\]
This inequality can be iterated to show that
\[
\lim_{m \in U} \lim_{n \in U} \|x + \frac{1}{2} x_m + \frac{1}{2} x_n\| \leq \lambda_0^2 \|x\|.
\]
Now assume \(\|x\| = 1\) and \((x_n)_{n=1}^\infty\) is a normalized weakly null sequence. Then
\[
\lim_{n \in U} \|x + x_n\| = \frac{1}{2} \lim_{n \in U} \|2x + x_n + x_n\|
\leq \frac{1}{2} \lambda_0 \lim_{m \in U} \lim_{n \in U} \|2x + x_m + x_n\| \leq \lambda_0^3. \]

**Theorem 3.4.** Let \(X\) be a separable Banach space.

(i) If \(X\) isometrically embeds into \(c_0\), then \(X\) is linearly isometric to a closed subspace of \(c_0\).

(ii) If, for every \(\varepsilon > 0\), \(X\) Lipschitz embeds into \(c_0\) with constant at most \(1 + \varepsilon\), then, for every \(\varepsilon > 0\), there is a closed subspace \(Y_\varepsilon\) of \(c_0\) with Banach–Mazur distance \(d_{BM}(X,Y_\varepsilon) < 1 + \varepsilon\).

**Proof.** (i) is a direct consequence of the result of [8] that if a separable Banach space is isometric to a subset of a Banach space \(Z\) then it is also linearly isometric to a subspace of \(Z\).

(ii) Here we first observe that if \(X\) contains a subspace isomorphic to \(\ell_1\) then, for any \(\varepsilon > 0\), it contains a subspace \(Z_\varepsilon\) with Banach–Mazur distance \(d_{BM}(Z_\varepsilon,\ell_1) \leq 1 + \varepsilon\) by James’ distortion theorem [10]. Assume now that \(X\) can be \(\lambda\)-embedded into \(c_0\). Thus, for any \(\varepsilon > 0\), \(\ell_1\) can be \(\lambda(1+\varepsilon)\)-embedded into \(c_0\). Then it follows from Aharoni’s counterexample in [1] that \(\lambda \geq 2\).

Suppose now that \(X\) does not contain any isomorphic copy of \(\ell_1\). If \(\|x\| = 1\) and \((x_n)_{n=1}^\infty\) is any normalized weakly null sequence then by Proposition 3.3 we have
\[
\lim_{n \to \infty} \|x + x_n\| = 1.
\]
The conclusion then follows from [12, Theorem 3.5].

**Remark.** The modulus of asymptotic smoothness of a Banach space \(X\) has been defined in [15] as follows. If \(\tau > 0\), then
\[
\varrho_X(\tau) = \sup_{x \in S_X} \inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} \|x + \tau y\| - 1.
\]
The space \(X\) is said to be asymptotically uniformly flat if \(\varrho_X(\tau) = 0\) for some \(\tau > 0\). Clearly, this is closely related to equation (3.2). It is shown in [9] and [11] that a uniformly flat Banach space is isomorphic to a subspace of \(c_0\). This property, or rather its dual version, is used in [9] to show that a Banach space which is Lipschitz isomorphic to a subspace of \(c_0\) is linearly isomorphic to a subspace of \(c_0\).
4. Embeddings into $c_0^+$. In this section and the following we complete the already thorough study of Lipschitz embeddings into $c_0^+$ made by Pelant in [16].

**Lemma 4.1.** Let $(M,d)$ be a metric space and suppose that $A$, $B$ and $C$ are non-empty subsets of $M$. Then for $\varepsilon > 0$, there exists a Lipschitz function $f : M \to \mathbb{R}_+$ with $\text{Lip}(f) \leq 1$ such that

(i) $f(x) \leq \varepsilon$, $x \in C$,

(ii) $|f(x) - f(y)| \geq \theta = \min(\delta(A,B), \max(\delta(A,C), \delta(B,C)) + \varepsilon)$, $x \in A$, $y \in B$.

**Proof.** Assume $\delta(A,C) \geq \delta(B,C)$. Then $\theta = \min(\delta(A,B), \delta(A,C) + \varepsilon)$. Define

$$f(x) = \max(\theta - d(x,A), 0), \quad x \in M.$$  

Then $f(x) = \theta$ for $x \in A$. If $x \in B$ then $\theta - d(x,A) \leq \theta - \delta(A,B) \leq 0$, so that $f(x) = 0$. Finally, if $x \in C$ we have $\theta - d(x,A) \leq \theta - \delta(A,C) \leq \varepsilon$, so that $f(x) \leq \varepsilon$. □

We may now introduce a condition analogous to $\Pi(\lambda)$. We say that $(M,d)$ has property $\Pi_+(\lambda)$, where $\lambda > 1$, if:

(i) Whenever $\mu > \lambda$ there exists $\nu > \mu$ so that if $B_1$ and $B_2$ are two metric balls of the same radius $r$, there are a finite number of sets $(U_j)_{j=1}^N$ and $(V_j)_{j=1}^N$ so that

$$\lambda \delta(U_j, V_j) \geq \nu r$$

and

$$\{(x,y) \in B_1 \times B_2 : d(x,y) > \mu r\} \subset \bigcup_{j=1}^N (U_j \times V_j).$$

(ii) If $1 < \lambda \leq 2$, there exists $1 < \theta < \lambda$ and a function $\varphi : M \to [0, \infty)$ so that

$$|\varphi(x) - \varphi(y)| \leq d(x,y) \leq \theta \max(\varphi(x), \varphi(y)), \quad x,y \in M.$$  

Let us note here that condition (ii) is not required when $\lambda > 2$ since for any fixed $a \in X$ the function $\varphi(x) = d(x,a)$ satisfies (4.4) with $\theta = 2$.

We can repeat the same program for property $\Pi_+(\lambda)$.

**Lemma 4.2.** Every metric space has property $\Pi_+(3)$.

**Proof.** For $\mu > 3$, let

$$U = B_1 \cap \{x : \exists y \in B_2, d(x,y) > \mu r\}, \quad V = B_2 \cap \{y : \exists x \in B_1, d(x,y) > \mu r\}.$$  

Then

$$\{(x,y) \in B_1 \times B_2 : d(x,y) > \mu r\} \subset U \times V.$$
Suppose \( x \in U, \ y \in V \). Then there exists \( x' \in U \) with \( d(x',y) > \mu r \). Hence \( d(x,y) > \mu r - d(x,x') \geq (\mu - 2)r \).

Therefore we can take \( \nu = 3\mu - 6 > \mu \).

**Lemma 4.3.** Let \( \lambda > 2 \). Then every locally compact metric space has property \( \Pi_+ (\lambda) \).

The proof is immediate. Let us mention that a locally compact metric space satisfies condition (i) for every \( \lambda > 1 \).

We also have

**Lemma 4.4.** If \( \lambda > 1 \), then any compact metric space has property \( \Pi_+ (\lambda) \).

**Proof.** Let \((K,d)\) be a compact metric space. We only have to prove condition (ii). For \( \varepsilon > 0 \), pick a finite \( \varepsilon \)-net \( F \) of \( K \) and define \( \varphi_\varepsilon(x) = \max_{z \in F} (d(x,z)) \).

For a given \( \lambda > 1 \), \( \varphi_\varepsilon \) satisfies condition (ii) of \( \Pi_+ (\lambda) \) if \( \varepsilon \) is small enough.

**Proposition 4.5.** Suppose \( \lambda_0 \geq 1 \) and \( M \) is a metric space which Lipschitz embeds into \( c_0^+ \) with constant \( \lambda_0 \). Then \( M \) has property \( \Pi_+ (\lambda) \) for all \( \lambda > \lambda_0 \).

**Proof.** We first consider (i) of the definition of \( \Pi_+ (\lambda) \). Suppose \( \mu > \lambda > \lambda_0 \).

Let \( \Delta = \{(x,y) \in B_1 \times B_2 : d(x,y) > \mu r \} \) and \( f : M \to c_0^+ \) be an embedding such that

\[
d(x,y) \leq \|f(x) - f(y)\| \leq \lambda_0 d(x,y), \quad x,y \in M.
\]

Suppose \( f(x) = (f_i(x))_{i=1}^\infty \). Then there exists \( n \) so that

\[
f_i(a_1), f_i(a_2) < (\mu - \lambda)r, \quad i \geq n + 1.
\]

Thus if \((x,y) \in \Delta \) we have

\[
|f_i(x) - f_i(y)| \leq \max_{1 \leq i \leq n} (f_i(x), f_i(y)) < (\mu - \lambda)r + \lambda_0 r < d(x,y), \quad i \geq n + 1.
\]

Hence

\[
d(x,y) \leq \max_{1 \leq i \leq n} |f_i(x) - f_i(y)|, \quad (x,y) \in \Delta.
\]

Choose \( \varepsilon > 0 \) so that \( \lambda (\mu - \varepsilon) > \lambda_0 \mu \). By a compactness argument we can find coverings \((W_k^m)_{k=1}^m \) of \( B_1 \) and \((W'_k)_{k=1}^{m'} \) of \( B_2 \) such that

\[
|f_i(x) - f_i(x')| \leq \frac{1}{2} \varepsilon r, \quad x,x' \in W_k, \ 1 \leq i \leq n, \ 1 \leq k \leq m,
\]

and

\[
|f_i(x) - f_i(x')| \leq \frac{1}{2} \varepsilon r, \quad x,x' \in W'_k, \ 1 \leq i \leq n, \ 1 \leq k \leq m'.
\]
Let
\[ S = \{(k, k') : 1 \leq k \leq m, 1 \leq k' \leq m', W_k \times W_{k'} \cap \Delta \neq \emptyset\} \]
and define \((U_j)_{j=1}^N, (V_j)_{j=1}^N\) in such a way that \((U_j \times V_j)_{j=1}^N\) is an enumeration of \((W_k \times W_{k'})_{(k,k') \in S}\). Then \(\Delta \subset \bigcup_{j=1}^N U_j \times V_j\) and the same calculations as in the proof of Proposition 2.4 give
\[
\lambda \delta(U_j, V_j) \geq \nu r \quad \text{with} \quad \nu = \lambda \lambda_0^{-1}(\mu - \varepsilon) > \mu.
\]
If \(\lambda \leq 2\) we must also consider (ii). Here we define \(\varphi(x) = \lambda_0^{-1}\|f(x)\|\) where \(f : M \to c_0^+\) is as above. Then \(\varphi\) satisfies (4.4) with \(\theta = \lambda_0\). Indeed,
\[
|\varphi(x) - \varphi(y)| \leq \lambda_0^{-1}\|f(x) - f(y)\| \leq d(x, y)
\]
and
\[
d(x, y) \leq \|f(x) - f(y)\| \leq \max(\|f(x)\|, \|f(y)\|) \leq \lambda_0 \max(\varphi(x), \varphi(y)). \quad \blacksquare
\]

Next, in place of Lemma 2.5 we have

**Lemma 4.6.** Let \(\lambda > 1\) and \((M, d)\) be a metric space with property \(\Pi_+(\lambda)\). Then for every \(\mu > \lambda\) there is a constant \(\nu > \mu\) so that if \(B_1\) and \(B_2\) are two metric balls of radius \(r\) then there are finitely many sets \((U_j)_{j=1}^N, (V_j)_{j=1}^N\) such that if \((x, y) \in B_1 \times B_2\) and \(d(x, y) > \mu r\) then there exists \(1 \leq j \leq N\) so that \(x \in U_j\), \(y \in V_j\) and
\[
\lambda \mu \delta(U_j, V_j) \geq \nu d(x, y).
\]

We omit the proof of this, which is very similar to that of Lemma 2.5 and only uses part (i) of the definition of \(\Pi_+(\lambda)\).

Then we have the following analogue of Lemma 2.6.

**Lemma 4.7.** Let \(\lambda > 1\). Suppose \((M, d)\) has property \(\Pi_+(\lambda)\). Suppose \(0 < \alpha < \beta\). Let \(F, G\) be finite subsets of \(M\) and let \(\Delta_+(F, G, \alpha, \beta)\) be the set of \((x, y) \in M \times M\) such that
\[
\lambda \max(d(x, G), d(y, G)) + \alpha \leq d(x, y) < \lambda \max(d(x, F), d(y, F)) + \beta.
\]
Then there is a finite set \(F = F(F, G, \alpha, \beta)\) of functions \(f : M \to \mathbb{R}_+\) with \(\text{Lip}(f) \leq \lambda\) and such that
\[
f(x) \leq \lambda \beta, \quad x \in F,
\]
and
\[
d(x, y) < \max_{f \in F} |f(x) - f(y)|, \quad (x, y) \in \Delta_+(F, G, \alpha, \beta).
\]

**Proof.** We first argue that for some constant \(K\) we have
\[
d(x, y) \leq K, \quad x, y \in \Delta_+(F, G, \alpha, \beta).
\]
If \(\lambda > 2\) this follows from the fact that
\[
d(x, G) + d(y, G) \geq d(x, y) - R,
\]
where \( R \) is the diameter of \( G \). Hence
\[
d(x, y) \leq K = \lambda(\lambda - 2)^{-1} R, \quad x, y \in \Delta_+(F, G, \alpha, \beta).
\]
In the case \( 1 < \lambda \leq 2 \) let \( \varphi, \theta \) be as in the definition of \( \Pi_+(\lambda) \) and satisfy (4.4). Let \( K_0 = \max\{\varphi(z) : z \in G\} \). Thus
\[
\lambda d(x, y) \leq \lambda \theta \max(\varphi(x), \varphi(y)) \leq \lambda \theta K_0 + \lambda \theta \max(d(x, G), d(y, G)) \\
\leq \lambda \theta K_0 + \theta d(x, y), \quad x, y \in \Delta_+(F, G, \alpha, \beta),
\]
so that
\[
d(x, y) \leq K = \frac{\lambda \theta K_0}{\lambda - \theta}, \quad x, y \in \Delta_+(F, G, \alpha, \beta).
\]

We next let
\[
\mu = \lambda + \frac{\alpha \lambda}{2K}
\]
and choose \( \nu = \nu(\mu) \) according to the conclusion of Lemma 4.6. We fix \( \varepsilon > 0 \) so that \( \varepsilon < \min(\alpha/2\mu, \lambda^{-1}(\lambda - 1)\beta) \).

Let \( E = \{x : d(x, G) \leq \lambda^{-1} K\} \). Since \( E \) is metrically bounded and \( F \cup G \) is finite we can partition \( E \) into finitely many subsets \( (E_1, \ldots, E_m) \) so that for each \( z \in F \cup G \) we have
\[
|d(x, z) - d(x', z)| \leq \varepsilon, \quad x, x' \in E_j, \ 1 \leq j \leq m.
\]
For each \( j \), we define \( z_j \in G \) and \( r_j \), as in the proof of Lemma 2.6, so that
\[
\inf_{x \in E_j} d(x, z_j) = \inf_{x \in E_j} d(x, G) = r_j.
\]
Note that \( r_j \leq \lambda^{-1} K \) and \( E_j \) is contained in a ball \( B_j \) centered at \( z_j \) with radius \( r_j + \varepsilon \).

Now for each pair \( (j, k) \) we denote by \( B_{j,k} \) the ball with center \( z_j \) and radius \( \max(r_j + \varepsilon, r_k + \varepsilon) \). By Lemma 4.6, we can find finitely many pairs of sets \( (\tilde{U}_{jkl}, \tilde{V}_{jkl})_{l=1}^{N_{jk}} \) such that for every \( (x, y) \in B_{j,k} \times B_{k,j} \) with \( d(x, y) > \mu(\max(r_j, r_k) + \varepsilon) \) there exists \( 1 \leq l \leq N_{jk} \) with \( x \in \tilde{U}_{jkl}, y \in \tilde{V}_{jkl} \) and
\[
\lambda \mu \delta(\tilde{U}_{jkl}, \tilde{V}_{jkl}) \geq \nu d(x, y).
\]
Then we set \( U_{jkl} = \tilde{U}_{jkl} \cap E_j \) and \( V_{jkl} = \tilde{V}_{jkl} \cap E_k \).

We now apply Lemma 4.1 to construct Lipschitz functions \( f_{jkl} : M \to \mathbb{R}_+ \) where \( 1 \leq j, k \leq m, \ 1 \leq l \leq N_{jk} \) are such that \( \text{Lip}(f_{jkl}) \leq \lambda \),
\[
f_{jkl}(x) \leq \lambda \beta, \quad x \in F,
\]
and
\[
|f_{jkl}(x) - f_{jkl}(y)| \geq \lambda \theta_{jkl}, \quad x \in U_{jkl}, y \in V_{jkl},
\]
where
\[
\theta_{jkl} = \min(\delta(U_{jkl}, V_{jkl}), \max(\delta(U_{jkl}, F), \delta(V_{jkl}, F)) + \beta).
\]
Now let us suppose \((x, y) \in \Delta_{+}(F, G, \alpha, \beta)\). Then there exists \((j, k)\) so that \(x \in E_j, y \in E_k\). It follows from our choice of \(\mu\) and \(\varepsilon\) that
\[
d(x, y) \geq \lambda \max(d(x, G), d(y, G)) + \alpha \geq \lambda \max(r_j, r_k) + \alpha > \mu(\max(r_j, r_k) + \varepsilon).
\]

Thus there exists \(1 \leq l \leq N_{jk}\) so that \(x \in U_{jkl}, y \in V_{jkl}\) and
\[
\lambda \delta(U_{jkl}, V_{jkl}) \geq \frac{\nu}{\mu} d(x, y) > d(x, y).
\]

On the other hand, \(\varepsilon < \lambda^{-1}(\lambda - 1)\beta\), so
\[
\lambda \max(\delta(U_{jkl}, F), \delta(V_{jkl}, F)) + \beta \geq \lambda \max(d(x, F), d(y, F)) + \lambda(\beta - \varepsilon) > \lambda \max(d(x, F), d(y, F)) + \beta > d(x, y).
\]

Hence
\[
|f_{jkl}(x) - f_{jkl}(y)| \geq \lambda \theta_{jkl} > d(x, y).
\]

Thus we can take for \(F\) the collection of all functions \(f_{jkl}\) for \(1 \leq j, k \leq m, 1 \leq l \leq N_{jk}\).  

Finally, our theorem is

**Theorem 4.8.** Suppose a separable metric space \((M, d)\) has property \(\Pi_{+}(\lambda)\) with \(\lambda > 1\). Then there is a Lipschitz embedding \(f : M \to c_0^+\) with
\[
d(x, y) < \|f(x) - f(y)\| \leq \lambda d(x, y), \quad x, y \in M, x \neq y.
\]

**Proof.** We use the notation of the proof of Theorem 2.7. Then we build an increasing sequence \((n_k)_{k=0}^{\infty}\) of integers (with \(n_0 = 0\)) and a sequence \((f_j)_{j=1}^{\infty}\) of Lipschitz functions \(f_j : M \to \mathbb{R}^+\) with \(\text{Lip}(f_j) \leq \lambda\) so that
\[
f_j(x) \leq \lambda \varepsilon_k, \quad x \in F_k, n_{k-1} < j \leq n_k,
\]
and if
\[
\lambda \max(d(x, F_{k+1}), d(y, F_{k+1})) + \varepsilon_{k+1} \leq d(x, y) < \lambda \max(d(x, F_k), d(y, F_k)) + \varepsilon_k
\]
then
\[
\max_{n_{k-1} < j \leq n_k} |f_j(x) - f_j(y)| > d(x, y).
\]

If \(x \neq y\) then the sequence
\[
\tau_k = \lambda \max(d(x, F_k), d(y, F_k)) + \varepsilon_k
\]
is decreasing and tends to zero.

If \(\lambda > 2\), we clearly have \(\tau_1 > d(x, y)\).

Assume \(1 < \lambda \leq 2\). Let \(\varphi\) be given by part (ii) of property \(\Pi_{+}(\lambda)\). We choose \(\varepsilon_1 > \lambda \varphi(u_1)\). Then
\[
d(x, y) \leq \lambda \max(\varphi(x), \varphi(y)) < \varepsilon_1 + \lambda \max(d(x, u_1), d(y, u_1)) = \tau_1.
\]
Hence, in both cases the desired embedding can be defined again by 
\[ f(x) = (f_j(x))_{j=1}^\infty. \]

As a first corollary, we obtain the following two results of Pelant’s ([16]).

**Corollary 4.9.**
(a) For every separable metric space \((M, d)\) there is a Lipschitz embedding \(f : M \to c_0^+\) so that
\[ d(x, y) < \|f(x) - f(y)\| \leq 3d(x, y), \quad x, y \in M, \ x \neq y. \]

(b) For any compact metric space \((K, d)\) and any \(\lambda > 1\), \((K, d)\) \(\lambda\)-embeds into \(c_0^+\).

It is proved in [16] that both of the above statements are optimal. This was also known to Aharoni [2] for part (a).

We also have

**Theorem 4.10.** For every locally compact metric space \((M, d)\) and every \(\lambda > 2\), \((M, d)\) \(\lambda\)-embeds into \(c_0^+\). This result is optimal.

**Proof.** The result is obtained by combining Theorem 4.8 and Lemma 4.3. We only have to show its optimality.

Let \(D\) be the set of all finite sequences with values in \(\{0, 1\}\), including the empty sequence denoted \(\emptyset\), and let \(D^* = D \setminus \{\emptyset\}\). For \(s \in D\), we denote by \(|s|\) its length. Then \((e_s)_{s \in D}\) is the canonical basis of \(\ell_1(D)\). We consider the following metric subspace of \(\ell_1(D)\):
\[ M = \{0, e_\emptyset\} \cup \{|s|e_s, e_\emptyset + |s|e_s : s \in D^*\}. \]

This is clearly a locally finite metric space. Assume now that there exists 
\[ f = (f_k)_{k=1}^\infty : M \to c_0^+ \]

such that
\[ \|x - y\|_1 \leq \|f(x) - f(y)\|_\infty \leq 2\|x - y\|_1, \quad x, y \in M. \]

There exists \(K \geq 1\) such that \(f_k(e_\emptyset) < 1\) and \(f_k(0) < 1\) for all \(k > K\). Then, using the positivity of \(f\), we obtain
\[ |f_k(e_\emptyset + ne_s) - f_k(ne_t)| \leq \max(f_k(e_\emptyset + ne_s), f_k(ne_t)) < 1 + 2n, \quad k > K, \ s \neq t, \ |s| = |t| = n. \]

On the other hand,
\[ \|f(e_\emptyset + ne_s) - f(ne_t)\|_\infty \geq 1 + 2n, \quad s \neq t, \ |s| = |t| = n. \]

Thus, for all \(s \neq t, \ |s| = |t| = n\), there exists \(k \leq K\) so that
\[ |f_k(e_\emptyset + ne_s) - f_k(ne_t)| \geq 1 + 2n. \]

Let now \(C = \max(\|f(e_\emptyset)\|_\infty, \|f(0)\|_\infty)\). Then
\[ |f_k(e_\emptyset + ne_s) - f_k(ne_t)| \leq C + 2n, \quad k \leq K, \ s \neq t, \ |s| = |t| = n. \]
Thus, for \( n \) large enough and all \( s \neq t \), \( |s| = |t| = n \), there exists \( k \leq K \) such that either
\[
f_k(\nu s) \leq C - 1 \quad \text{and} \quad f_k(\nu_0 + \nu e_t) \geq 1 + 2n
\]
or
\[
f_k(\nu s) \geq 1 + 2n \quad \text{and} \quad f_k(\nu_0 + \nu e_t) \leq C - 1.
\]
Therefore, either
\[
f_k(\nu s) \leq C - 1 \quad \text{and} \quad f_k(\nu_0 + \nu e_t) \geq 2n - 1
\]
or
\[
f_k(\nu s) \geq 1 + 2n \quad \text{and} \quad f_k(\nu_0 + \nu e_t) \leq C + 1.
\]
Let us now define \( \alpha(k, s) = \mathbb{1}_{[0, C+1]}(f_k(|s|e_s)) \). Then, for \( n \) large enough, we see that for all \( s \neq t \), \( |s| = |t| = n \), there exists \( k \leq K \) so that \( \alpha(k, s) \neq \alpha(k, t) \), which is clearly impossible if \( n > K \). This finishes our proof. ■

5. Embeddings of subsets of classical Banach spaces into \( c_0^+ \)

**Proposition 5.1.** Suppose \( X \) is a separable Banach space and that \( f : X \to c_0^+ \) is a Lipschitz embedding with constant \( \lambda_0 \). Then for any \( u \in X \) with \( \|u\| = 1 \) and any infinite-dimensional subspace \( Y \) of \( X \) we have
\[
\inf_{\|y\| = 1} \inf_{y \in Y} \|u + 2y\| \leq \lambda_0.
\]

**Proof.** The proof is almost identical to that of Proposition 3.1. It follows from Proposition 4.5 that \( X \) has property \( \Pi_+(\lambda) \) for any \( \lambda > \lambda_0 \). We consider \( B_1 = -u + 2B_X \) and \( B_2 = u + 2B_X \), where \( B_X \) denotes the closed unit ball of \( X \). Suppose \( \mu > \lambda_0 \) and select \( \mu > \lambda > \lambda_0 \). Then, for some \( \nu > \mu \), we can find finitely many closed sets \( (U_j, V_j) \) satisfying
\[
\lambda \delta(U_j, V_j) \geq 2\nu
\]
and
\[
\{(x, y) \in B_1 \times B_2 : \|x - y\| > 2\mu\} \subset \bigcup_{j=1}^N (U_j \times V_j).
\]
Now let \( E \) be any subspace of \( X \) of dimension greater than \( N \) and let
\[
A_j = \{e \in E : \|e\| = 1, (-u + 2e, u - 2e) \in U_j \times V_j\}.
\]
We then conclude the proof as in Proposition 3.1. Assume that for any \( e \in S_E, \|u + 2e\| > \mu \). Then \( A_1 \cup \cdots \cup A_N = S_E \) and so there exists \( e \in S_E \) and \( k \leq N \) such that \( e \) and \( -e \) belong to \( A_k \), i.e. \(-u \pm 2e \in U_k \) and \( u \pm 2e \in V_k \). This implies that \( \delta(U_k, V_k) \leq 2 \), which is a contradiction. So, there exists \( e \in S_E \) with \( \|u + 2e\| \leq \mu \) and we conclude as in the proof of Proposition 3.1. ■
\textbf{Theorem 5.2.} Suppose $1 \leq p < \infty$.

(i) There is a Lipschitz embedding of $\ell_p$ into $c_0^+$ with constant $(2^p + 1)^{1/p}$ and this is best possible.

(ii) There is a Lipschitz embedding of $\ell_p^+$ into $c_0^+$ with constant $3^{1/p}$ and this is best possible.

\textit{Proof.} Let us prove first that $\ell_p$ has $\Pi_+ (c_p)$ where $c_p = (1 + 2^p)^{1/p}$. The proof is very similar to that of Theorem 3.2. Let $B_1$ and $B_2$ be balls with centers $a_1, a_2$ and radius $r > 0$. Suppose $\mu > c_p$ and $\mu < \nu < c_p (\mu^p - 2^p)^{1/p}$.

Fix $\varepsilon > 0$ such that
\[ c_p (\mu^p r^p - 2^p (r + \varepsilon)^p)^{1/p} - 2 \varepsilon c_p > \nu r. \]

We select $N \in \mathbb{N}$ so that
\[ \sum_{k=N+1}^{\infty} |a_1(k)|^p, \sum_{k=N+1}^{\infty} |a_2(k)|^p < \varepsilon^p. \]

Let $E$ be the linear span of $\{e_1, \ldots, e_N\}$ where $(e_j)$ is the canonical basis of $\ell_p$. Let $P$ the canonical projection of $\ell_p$ onto $E$, $Q = I - P$ and $R = \max(\|a_1\|, \|a_2\|) + r$. Then we partition $RB_E$ into finitely many sets $A_1, \ldots, A_m$ with $\text{diam}(A_j) < \varepsilon$.

Now, set $U_j = \{x \in B_1 : Px \in A_j\}$, $V_j = \{x \in B_2 : Px \in A_j\}$ and $S = \{(j, k) : \exists (x, y) \in U_j \times V_k, \|x - y\| > \mu r\}$. Thus we have
\[ \{(x, y) \in B_1 \times B_2 : \|x - y\| > \mu r\} \subset \bigcup_{(j, k) \in S} U_j \times V_k. \]

It remains to estimate $\delta(U_j, V_k)$ for $(j, k) \in S$. Suppose $u \in U_j$, $v \in V_k$ and that $x \in U_j$, $y \in V_k$ are such that $\|x - y\| > \mu r$. Then
\[ \|u - v\| \geq \|Pu - Pv\| \geq \|Px - Py\| - 2 \varepsilon. \]

On the other hand, $r \geq \|x - a_1\| \geq \|Qx\| - \varepsilon$ and $r \geq \|y - a_2\| \geq \|Qy\| - \varepsilon$. Thus
\[ (\ref{eq:5.6}) \quad \|Qx - Qy\| \leq 2r + 2 \varepsilon. \]

Now
\[ \mu^p r^p < \|Px - Py\|^p + \|Qx - Qy\|^p \leq \|Px - Py\|^p + 2^p (r + \varepsilon)^p. \]

Hence
\[ \|Px - Py\|^p > \mu^p r^p - 2^p (r + \varepsilon)^p, \]

and so
\[ c_p \delta(U_j, V_k) \geq c_p (\mu^p r^p - 2^p (r + \varepsilon)^p)^{1/p} - 2 \varepsilon c_p > \nu r. \]

Hence $\ell_p$ has $\Pi_+(c_p)$. 

Next we show that $\ell^+_p$ has property $\Pi_+(3^{1/p})$. To do this we repeat the argument above. We take $\mu > 3^{1/p}$ and suppose that $\mu < \nu < 3^{1/p}(\mu^p - 2)^{1/p}$. Choose $\varepsilon > 0$ so that

$$3^{1/p}(\mu^p r^p - 2(r + \varepsilon)^p)^{1/p} - 2\varepsilon 3^{1/p} > \nu r.$$ 

Next repeat the construction, but working inside the positive cone $\ell^+_p$. The only difference is that (5.6) is replaced by

$$\|Qx - Qy\| \leq 2^{1/p}\max(\|Qx\|, \|Qy\|) \leq 2^{1/p}(r + \varepsilon).$$

Hence

$$\|Px - Py\| > \mu r^p - 2(r + \varepsilon)^p,$$

and so this time

$$3^{1/p}\delta(U_j, V_k) > 3^{1/p}(\mu^p r^p - 2(r + \varepsilon)^p)^{1/p} - 2\varepsilon 3^{1/p} > \nu r.$$ 

For the second half of the condition, when $3^{1/p} \leq 2$, we note that $\varphi(x) = \|x\|$ satisfies (4.4) with $\theta = 2^{1/p} < 3^{1/p}$.

These calculations combined with Theorem 4.8 show the existence of the Lipschitz embeddings in parts (i) and (ii). Proposition 5.1 shows that the constant is best possible when in (i). For (ii) suppose $f : \ell^+_p \to c^+_0$ is an embedding such that

$$\|f(x) - f(y)\| \leq \lambda \|x - y\|, \quad x, y \in \ell^+_p,$$

where $\lambda < 3^{1/p}$. Let $f(x) = (f_j(x))_{j=1}^{\infty}$. Let $\varepsilon = (3^{1/p} - \lambda)/2$. Then there exists $N$ such that

$$\max(f_j(e_1), f_j(0)) < \varepsilon, \quad j \geq N + 1.$$ 

Hence if $m, n > 1$ then

$$|f_j(e_1 + e_m) - f_j(e_n)| \leq \max(f_j(e_1 + e_m), f_j(e_n)) \leq \lambda + \varepsilon < 3^{1/p}, \quad j \geq N + 1.$$ 

Now we may pass to a subsequence so that the following limits exist:

$$\lim_{k \to \infty} f_j(e_1 + e_{n_k}) = \sigma_j, \quad \lim_{k \to \infty} f_j(e_{n_k}) = \tau_j, \quad 1 \leq j \leq N.$$ 

Clearly,

$$|\sigma_j - \tau_j| \leq \lambda, \quad 1 \leq j \leq N.$$ 

Now

$$\lim_{k \to \infty} |f_j(e_1 + e_{n_k}) - f_j(e_{n+1})| \leq \lambda, \quad 1 \leq j \leq N,$$

and we have a contradiction since $\|e_1 + e_{n_k} - e_{n+1}\| = 3^{1/p} > \lambda$. 

6. Spaces embedding isometrically into $c_0$ and $c^+_0$. In this final section we study isometric embeddings into $c_0$ and $c^+_0$. Note that a separable Banach space isometrically embeds into $c_0$ if and only if it embeds linearly and isometrically [8].
We recall that a metric space \((M,d)\) is an ultrametric space if
\[
d(x,y) \leq \max(d(x,z),d(z,y)), \quad x,y,z \in M. 
\]
Note that this implies
\[
d(x,y) = \max(d(x,z),d(z,y)), \quad d(x,z) \neq d(z,y). 
\]

**Lemma 6.1.** Let \((M,d)\) be a separable ultrametric space. Then there is
a countable subset \(\Gamma\) of \([0, \infty)\) such that \(d(x,y) \in \Gamma\) for all \(x,y \in M\).

**Proof.** For each fixed \(x \in M\) let \(\Gamma_x = \{d(x,y) : y \in M\}\). Suppose \(\Gamma_x\) is
uncountable; then for some \(\delta > 0\) the set \(\Gamma_x \cap (\delta, \infty)\) is uncountable. Pick an
uncountable set \((y_i)_{i \in I}\) in \(M\) so that \(d(x,y_i) > \delta\) and the values of \(d(x,y_i)\)
are distinct for \(i \in I\). Then for \(i \neq j\) we have \(d(y_i,y_j) > \delta\) by (6.9). This
contradicts the separability of \(M\).

Thus each \(\Gamma_x\) is countable. Let \(D\) be a countable dense subset of \(M\) and
let \(\Gamma = \bigcup_{x \in D} \Gamma_x\). If \(y, z \in M\) with \(y \neq z\), pick \(x \in D\) with \(d(x,y) < d(x,z)\).
Then \(d(y,z) = d(x,z) \in \Gamma\) by (6.9). \(\blacksquare\)

**Theorem 6.2.** Every separable ultrametric space embeds isometrically
into \(c_0^+\).

**Proof.** Pick \(\Gamma\) as in Lemma 6.1. Let \((a_j)_{j=1}^\infty\) be a countable dense
subset of an ultrametric space \(M\). Let \(D\) be the collection of finite sequences
\((r_1, \ldots, r_n)\) with \(r_j \in \Gamma\) for \(1 \leq j \leq n\). For each \((r_1, \ldots, r_n) \in D\) we define
a function \(f_{r_1, \ldots, r_n}\) by
\[
f_{r_1, \ldots, r_n}(x) = \begin{cases} 
\min(r_1, \ldots, r_n) & \text{if } d(x,a_j) = r_j, 1 \leq j \leq n, \\
0 & \text{otherwise.} 
\end{cases}
\]

If \(x \in M\) let \(d(x,a_j) = s_j\). Then \(\lim_{n \to \infty} \min(s_1, \ldots, s_n) = 0\) and it
follows that \(f(x) = (f_{r_1, \ldots, r_n}(x))_{(r_1, \ldots, r_n) \in D}\) is a map from \(M\) into \(c_0^+ (D)\).

If \(x, y \in M\) and \(f_{r_1, \ldots, r_n}(x) \neq f_{r_1, \ldots, r_n}(y)\) we can assume without loss
of generality that \(d(x,a_j) = r_j\) for \(1 \leq j \leq n\) but \(d(y,a_k) \neq r_k\) for some
\(1 \leq k \leq n\). Then from (6.9) we get
\[
|f_{r_1, \ldots, r_n}(x) - f_{r_1, \ldots, r_n}(y)| = \min(r_1, \ldots, r_n) 
\leq r_k \leq \max(d(x,a_k),d(y,a_k)) = d(x,y). 
\]
Thus \(\|f(x) - f(y)\| \leq d(x,y)\) for \(x,y \in M\).

On the other hand, if \(x \neq y\) there is a least \(k\) so that \(d(x,a_k) \neq d(y,a_k)\).
Assume that \(d(x,a_k) > d(y,a_k)\) and \(r_j = d(x,a_j)\) for \(1 \leq j \leq k\). Then
\(d(x,y) = r_k\). On the other hand, \(d(x,y) \leq r_j\) for \(1 \leq j \leq k\). Hence
\[
d(x,y) = r_k = |f_{r_1, \ldots, r_k}(x) - f_{r_1, \ldots, r_k}(y)|. 
\]
Thus \(f\) is an isometry. \(\blacksquare\)

As a final example we consider an infinite branching tree \(T\) defined as the
set of all ordered subsets (nodes) \(a = (m_1, \ldots, m_k)\) (where \(m_1 < \cdots < m_k\))
Hence \( f \) so that \( |a| = k \) be the length of \( a \) so that \( |\emptyset| = 0 \). If \( a = (m_1, \ldots, m_k) \), \( b = (n_1, \ldots, n_l) \) are two nodes then we define \( a \wedge b \) to be the node \((m_1, \ldots, m_r)\) where \( r \leq \min(k, l) \) is the greatest integer such that \( m_j = n_j \) for \( 1 \leq j \leq r \). We write \( a \prec b \) if \( b \wedge a = a \). Furthermore, \( T \) is a graph if we define two nodes \( a, b \) to be adjacent if \( |a| - |b| \) is an isometry.

Theorem 6.3. The infinite branching tree embeds isometrically into \( c_0 \).

Proof. For each \((a, n) \in T \times \mathbb{N}\) we define

\[
   f_{a,n}(b) = \begin{cases} 
   |b| - |a|, & a \prec b, b \neq a, b_{|a|+1} = n, \\
   |a| - |b|, & a \prec b, b \neq a, b_{|a|+1} > n, \\
   0, & \text{otherwise}.
   \end{cases}
\]

For fixed \( b \) we have \( f_{a,n}(b) \neq 0 \) only when \( a \prec b, b \neq a \) and \( n \leq b_{|a|+1} \), and this is a finite set. Hence \( f(b) = (f_{a,n}(b))_{(a, n) \in T \times \mathbb{N}} \) defines a map of \( T \) into \( c_0(T \times \mathbb{N}) \).

Suppose that \( d(b, b') = 1 \) and \( |b'| = |b| + 1 \). Then by examining cases it is clear that \( |f_{a,n}(b) - f_{a,n}(b')| \leq 1 \) for all \((a, n) \in T \times \mathbb{N}\). It follows that \( \|f(b) - f(b')\| \leq d(b, b') \) for arbitrary \( b, b' \in T \).

If \( b \neq b' \) pick \( a = b \wedge b' \) and assume as we may that either \( b' = a \wedge b = a \) or \( b_{|a|+1} < b'_{|a|+1} \). Put \( n = b_{|a|+1} \). Then

\[
   f_{a,n}(b) = |b| - |a|, \quad f_{a,n}(b') = |a| - |b'|,
\]

so that

\[
   |f_{a,n}(b) - f_{a,n}(b')| = d(b, b').
\]

Hence \( f \) is an isometry. \( \blacksquare \)

Remarks. Since \( c_0 \) 2-embeds into \( c_0^+ \), so does \( T \). It follows from the fact that \( T \) contains a copy of \( \mathbb{Z} \) that it is again optimal. However, the set of nodes of the same level of a tree equipped with the geodesic distance, like \( T \), is a fundamental example of ultrametric space, and therefore, by Theorem 6.2, embeds isometrically into \( c_0^+ \).

Let us also mention that we do not know whether the metric spaces \( c_0 \) and \( c_0^+ \) are Lipschitz isomorphic.

Extending a work of J. Bourgain \([7]\), F. Baudier recently proved in \([4]\) that the infinite dyadic tree equipped with the geodesic distance metrically embeds into a Banach space \( X \) if and only if \( X \) is not super-reflexive. Together with the second named author, F. Baudier also showed in \([5]\) that any locally finite metric space metrically embeds into any Banach space without cotype.
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