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On a generalization of Abelian sequential groups

by

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Abstract. Let (G,τ) be a Hausdorff Abelian topological group. It is called an s-group (resp. a bs-group) if there is a set S of sequences in G such that τ is the finest Hausdorff (resp. precompact) group topology on G in which every sequence of S converges to zero. Characterizations of Abelian s- and bs-groups are given. If (G,τ) is a maximally almost periodic (MAP) Abelian s-group, then its Pontryagin dual group $(G,\tau)^\wedge$ is a dense $\mathfrak{g}$-closed subgroup of the compact group $(G_d)^\wedge$, where G_d is the group G with the discrete topology. The converse is also true: for every dense $\mathfrak{g}$-closed subgroup H of $(G_d)^\wedge$, there is a topology τ on G such that (G,τ) is an s-group and $(G,\tau)^\wedge = H$ algebraically. It is proved that, if G is a locally compact non-compact Abelian group such that the cardinality $|G|$ of G is not Ulam measurable, then G^+ is a realcompact bs-group that is not an s-group, where G^+ is the group G endowed with the Bohr topology. We show that every reflexive Polish Abelian group is $\mathfrak{g}$-closed in its Bohr compactification. In the particular case when G is countable and τ is generated by a countable set of convergent sequences, it is shown that the dual group $(G,\tau)^\wedge$ is Polish. An Abelian group X is called characterizable if it is the dual group of a countable Abelian MAP s-group whose topology is generated by one sequence converging to zero. A characterizable Abelian group is a Schwartz group iff it is locally compact. The dual group of a characterizable Abelian group X is characterizable iff X is locally compact.

1. Introduction

I. Notations and preliminaries. A group G with the discrete topology is denoted by G_d. The subgroup generated by a subset A of G is denoted by $\langle A\rangle$. Let X be an Abelian topological group. The filter of all open neighborhoods at zero of X is denoted by $\mathcal{U}_X$. The group of all continuous characters on X is denoted by $\widehat{X}$. The group $\widehat{X}$ endowed with the compact-open topology $\sigma_{\rm co}$ is denoted by $X^\wedge$. Denote by $\mathbf{n}(X) = \bigcap_{\chi\in\widehat{X}} \ker\chi$ the von Neumann radical of X. If $\mathbf{n}(X)=\{0\}$, then X is called *maximally almost periodic* (*MAP*). Let H be a subgroup of X. The *annihilator* of H

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is denoted by $H^{\perp}$, i.e., $H^{\perp} = \{\chi \in X^{\wedge} : (\chi, h) = 1 \text{ for every } h \in H\}$. We recall that a subgroup H of X is called *dually closed* in X if for every $x \in X \setminus H$ there exists a character $\chi \in H^{\perp}$ such that $(\chi, x) \neq 1$. We call H *dually embedded* in X if every character of H can be extended to a character of X.

Let X be an Abelian topological group and $\mathbf{u} = \{u_n\}$ be a sequence of elements of $\widehat{X}$. Following Dikranjan et al. [DMT], we denote by $s_{\mathbf{u}}(X)$ the set of all $x \in X$ such that $(u_n, x) \to 1$. Let H be a subgroup of X. If $H = s_{\mathbf{u}}(X)$ we say that $\mathbf{u}$ *characterizes* H and that H is *characterized* (by $\mathbf{u}$) [DMT].

Following [DMT], the closure operator $\mathfrak{g}_X$ is defined as follows: for every subgroup H of an Abelian topological group X one puts

$$\mathfrak{g}(H) = \mathfrak{g}_X(H) := \bigcap_{\mathbf{u} \in \widehat{X}^{\mathbb{N}}} \{s_{\mathbf{u}}(X) : H \leq s_{\mathbf{u}}(X)\}.$$

We say that H is $\mathfrak{g}$-*closed* if $H = \mathfrak{g}_X(H)$, and H is $\mathfrak{g}$-*dense* if $\mathfrak{g}_X(H) = X$. For an arbitrary subset S of $\widehat{X}^{\mathbb{N}}$, one puts

$$s_S(X) := \bigcap_{\mathbf{u} \in S} s_{\mathbf{u}}(X).$$

Let $\mathbf{u} = \{u_n\}$ be a non-trivial sequence in a group G. The following important question has been studied by many authors, including Graev [Gr] and Nienhuys [N]:

PROBLEM 1.1. *Is there a Hausdorff group topology τ on G such that $u_n \to e_G$ in (G, τ)?*

Protasov and Zelenyuk [ZP, PZ] obtained a criterion that gives the complete answer to this question. Following [ZP], we say that a sequence $\mathbf{u} = \{u_n\}$ in a group G is a T-*sequence* if there is a Hausdorff group topology on G in which u_n converges to the unit. The group G equipped with the finest Hausdorff group topology $\tau_{\mathbf{u}}$ with this property is denoted by $(G, \tau_{\mathbf{u}})$. A T-sequence $\mathbf{u} = \{u_n\}$ is called *trivial* if there is n_0 such that $u_n = e_G$ for every $n \geq n_0$.

Let us denote by $\mathbb{Z}_0^{\mathbb{N}}$ the direct sum $\bigoplus_{\mathbb{N}} \mathbb{Z} \subset \mathbb{Z}^{\mathbb{N}}$. Set $\mathbf{e} = \{e_n\} \in \mathbb{Z}_0^{\mathbb{N}}$, where $e_1 = (1, 0, 0, \dots), e_2 = (0, 1, 0, \dots), \dots$. Then $\mathbf{e}$ is a T-sequence in $\mathbb{Z}_0^{\mathbb{N}}$. Let $\mathbb{T}^{\mathbb{N}}$ be the direct product of a countable set of copies of $\mathbb{T}$. Set $\mathbb{T}_0^{\mathcal{H}} := \{\omega = (z_n) \in \mathbb{T}^{\mathbb{N}} : z_n \to 1\}$. Then $\mathbb{T}_0^{\mathcal{H}}$ is a Polish group under the metric $d((z_n^1), (z_n^2)) = \sup\{|z_n^1 - z_n^2| : n = 1, 2, \dots\}$ (see [G1]). Moreover, the group $(\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})$ is reflexive and $(\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})^{\wedge} = \mathbb{T}_0^{\mathcal{H}}$ [G1].

Let X be an arbitrary infinite compact metrizable Abelian group. For any $(x, \omega) = (x, (z_n)) \in X \times \mathbb{T}_0^{\mathcal{H}}$, set $\pi_X(x, \omega) := x$ and $\pi_n(x, (z_n)) := z_n$. The following criterion for a subgroup of a compact metrizable Abelian group to be characterized by a T-sequence $\mathbf{u}$ was given in [G2]:

Theorem 1.2. *Let H be a dense subgroup of an infinite compact metrizable Abelian group X and $\mathbf{u} = \{u_n\}$ be a sequence in $\widehat{X}$. Then H is characterized by $\mathbf{u}$ if and only if there exists a dually closed subgroup $H_{\mathbf{u}}$ of $X \times \mathbb{T}_0^{\mathcal{H}}$ such that*

(a) *the restriction to $H_{\mathbf{u}}$ of the projection π_X is a bijection onto H, and*
(b) *$u_n \circ \pi_X$ and π_n coincide on $H_{\mathbf{u}}$ for every n.*

As a corollary of this criterion we obtain the following simple necessary condition for a subgroup to be characterized [G2]:

Proposition 1.3. *Each characterized subgroup $H = s_{\mathbf{u}}(X)$ of X admits a finer locally quasi-convex Polish group topology.*

We denote the group $H = s_{\mathbf{u}}(X)$ with this topology by $H_{\mathbf{u}}$. Following [G3], an Abelian Polish group G is called *characterizable* if there is a continuous monomorphism p from G into a compact metrizable group X with dense image such that $p(G) = s_{\mathbf{u}}(X)$ for some sequence $\mathbf{u}$ in $\widehat{X}$.

Let G be a countably infinite Abelian group, $X = G_d^{\wedge}$, $\mathbf{u} = \{u_n\}$ be a T-sequence in G and $H = s_{\mathbf{u}}(X)$. There is a simple dual connection between the groups $(G, \tau_{\mathbf{u}})$ and $H_{\mathbf{u}}$, and moreover we can compute the von Neumann radical $\mathbf{n}(G, \tau_{\mathbf{u}})$ of $(G, \tau_{\mathbf{u}})$ as follows:

Theorem 1.4 ([G2]). *$(G, \tau_{\mathbf{u}})^{\wedge} = H_{\mathbf{u}}$ and, algebraically, $\mathbf{n}(G, \tau_{\mathbf{u}}) = H^{\perp}$.*

The counterpart of Problem 1.1 for *precompact* group topologies on $\mathbb{Z}$ is studied by Raczkowski [R]. Following [BDMW1] and motivated by [R], we say that a sequence $\mathbf{u} = \{u_n\}$ is a *TB-sequence* in a group G if there is a *precompact* Hausdorff group topology on G in which $u_n \to e_G$. The group G equipped with the finest precompact Hausdorff group topology $\tau_{b\mathbf{u}}$ with this property is denoted by $(G, \tau_{b\mathbf{u}})$.

For an Abelian group G and an arbitrary subgroup $H \le G_d^{\wedge}$, let T_H be the weakest topology on G such that all characters of H are continuous with respect to T_H. One can easily show [CR] that T_H is a totally bounded group topology on G, and it is Hausdorff iff H is dense in $G_d^{\wedge}$.

Let (G, τ) be a MAP Abelian topological group. The Bohr compactification of (G, τ) is denoted by $bG = (\widehat{(G, \tau)}_d)^{\wedge}$. The topology $\tau^+ := T_{\widehat{(G,\tau)}}$ on G induced from bG is called the *Bohr modification* of τ. Clearly, the groups (G, τ) and $G^+ := (G, \tau^+)$ have the same continuous characters.

A subset A of a topological space Ω is called *sequentially open* if whenever a sequence $\{u_n\}$ converges to a point of A, then all but finitely many of the members u_n are contained in A. The space Ω is called *sequential* if any subset A is open if and only if A is sequentially open. Franklin [Fr] gave the following characterization of sequential spaces:

THEOREM 1.5 ([Fr]). *A topological space is sequential if and only if it is a quotient of a metric space.*

Let X and Y be topological groups. Following Siwiec [S], a continuous homomorphism $p : X \to Y$ is called *sequence-covering* if it is surjective and for every sequence $\{y_n\}$ converging to the unit e_Y there is a sequence $\{x_n\}$ converging to e_X such that $p(x_n) = y_n$. We recall that a continuous mapping $f : X \to Y$ is called *compact-covering* if for every compact subset K_Y of Y there is a compact subset K_X of X such that $f(K_X) = K_Y$. A mapping $f : X \to Y$ is called *sequentially continuous* if $x_n \to x_0$ in X implies that $f(x_n) \to f(x_0)$ in Y. Sequentially continuous mappings on products of topological spaces were considered by Mazur [M]. The following important notion was introduced by Noble [Nob1]:

DEFINITION 1.6 ([Nob1]). A Hausdorff topological group (G, τ) is called an *s-group* if each sequentially continuous homomorphism from (G, τ) to a Hausdorff topological group is continuous.

Note that this definition gives an *external* characterization of s-groups. Following [Nob1], we denote by $\mathfrak{s}$ the first cardinal such that there exists a discontinuous, sequentially continuous mapping $f : 2^{\mathfrak{s}} \to \mathbb{R}$. Mazur [M] proved that $\mathfrak{s}$ is weakly inaccessible (i.e., $\mathfrak{s}$ is an uncountable regular limit cardinal). In particular, $\mathfrak{s} > \aleph_0$. The following theorem is proved by Noble [Nob1, Theorem 5.4], who replaced Mazur's factorization method by a stronger one:

THEOREM 1.7 ([Nob1]). *Every sequentially continuous homomorphism from a product of less than $\mathfrak{s}$ s-groups into a Hausdorff group is continuous, i.e., the product is an s-group.*

A simpler proof of Noble's Theorem 1.7 was given by Hušek [H]. Other results and historical remarks about s-groups can be found in [AJP, H, Sha].

The following natural generalization of Problem 1.1 was considered in [G4]:

PROBLEM 1.8. *Let G be a group and S be a set of sequences in G. Is there a Hausdorff group topology τ on G in which every sequence of S converges to the unit? Is there a* precompact *Hausdorff group topology τ on G in which every sequence of S converges to the unit?*

In analogy with T- and TB-sequences, we define as in [G4]:

DEFINITION 1.9. Let G be a group and S be a non-empty set of sequences in G. The set S is called a *T_s-set* (resp. *T_{bs}-set*) *of sequences* if there is a Hausdorff (resp. precompact Hausdorff) group topology on G in which all sequences of S converge to the unit. The finest Hausdorff (resp. precompact Hausdorff) group topology with this property is denoted by τ_S (resp. τ_{bS}).

The family of all T_s-sets (resp. T_{bs}-sets) of sequences of a group G is denoted by $\mathcal{TS}(G)$ (resp. $\mathcal{TBS}(G)$). It is clear that, if $S \in \mathcal{TS}(G)$ (resp. $S \in \mathcal{TBS}(G)$), then $S' \in \mathcal{TS}(G)$ (resp. $S' \in \mathcal{TBS}(G)$) for every non-empty subset S' of S and every sequence $\mathbf{u} \in S$ is a T-sequence (resp. $\mathbf{u}$ is a TB-sequence). Evidently, $\tau_S \subseteq \tau_{S'}$ (resp. $\tau_{bS} \subseteq \tau_{bS'}$). Also, if S contains only trivial sequences, then $S \in \mathcal{TS}(G)$ and τ_S is discrete.

Note also that we can define a preorder on $\mathcal{TS}(G)$ (resp. $\mathcal{TBS}(G)$) as follows: for S' and S in $\mathcal{TS}(G)$ (resp. $\mathcal{TBS}(G)$), $S' \le S$ iff $\tau_S \subseteq \tau_{S'}$ (resp. $\tau_{bS} \subseteq \tau_{bS'}$), or, in other words, iff every sequence of S' converges to the unit in τ_S (resp. τ_{bS}).

Let S be a T_s-set (resp. a T_{bs}-set) of sequences of a group G. By definition, $\tau_{\mathbf{u}}$ is finer than τ_S (resp. $\tau_{b\mathbf{u}}$ is finer than τ_{bS}) for every $\mathbf{u} \in S$. Thus, if U is open (resp. closed) in τ_S, then it is open (resp. closed) in $\tau_{\mathbf{u}}$ for every $\mathbf{u} \in S$. Let us recall that the topology $\bigwedge_{\mathbf{u}\in S} \tau_{\mathbf{u}}$ is defined as the intersection of the topologies $\tau_{\mathbf{u}}$ (i.e., U is open in $\bigwedge_{\mathbf{u}\in S} \tau_{\mathbf{u}}$ if and only if $U \in \tau_{\mathbf{u}}$ for every $\mathbf{u} \in S$). Therefore, $\tau_S \subseteq \bigwedge_{\mathbf{u}\in S} \tau_{\mathbf{u}}$ (resp. $\tau_{bS} \subseteq \bigwedge_{\mathbf{u}\in S} \tau_{b\mathbf{u}}$). Note also that, if S is finite, then, by Proposition 5.1 of [G4], the T_s-set S can be replaced by a single T-sequence (in the sense that $\tau_S = \tau_{\mathbf{v}}$ for some T-sequence $\mathbf{v}$). Let $\{\tau_i\}_{i\in I}$ be a family of Hausdorff group topologies on a group G. Denote by $\inf_{i\in I} \tau_i$ the infimum of the topologies τ_i in the lattice of all group topologies. The following theorem is proved in Section 5:

THEOREM 1.10. *Let $S = \bigcup_{i\in I} S_i$ be the union of T_s-sets of sequences S_i in a group G. Then $S \in \mathcal{TS}(G)$ if and only if $\inf_{i\in I} \tau_{S_i}$ is Hausdorff. In that case, $\tau_S = \inf_{\mathbf{u}\in S} \tau_{\mathbf{u}} = \inf_{i\in I} \tau_{S_i}$.*

The following class of topological groups is defined in [G4]:

DEFINITION 1.11. A Hausdorff topological group (G, τ) is called an *s-group* (resp. a *bs-group*) and the topology τ is called an *s-topology* (resp. a *bs-topology*) on G if there is $S \in \mathcal{TS}(G)$ (resp. $S \in \mathcal{TBS}(G)$) such that $\tau = \tau_S$ (resp. $\tau = \tau_{bS}$).

In other words, s-groups are those Hausdorff topological groups whose topology can be described by a set of convergent sequences. In particular, for every T-sequence $\mathbf{u}$ in a group G the group $(G, \tau_{\mathbf{u}})$ is an s-group. As was shown in [G4], Definitions 1.6 and 1.11 of s-groups are equivalent; more precisely:

THEOREM 1.12. *For a topological group (G, τ) the following assertions are equivalent:*

(1) *(G,τ) is an s-group (in the sense of Definition 1.11) and $\tau = \tau_S$ for some $S \in \mathcal{TS}(G)$;*
(2) *every homomorphism p from (G,τ) to a Hausdorff topological group (X,ν) is continuous if and only if $p(u_n) \to e_X$ for every sequence $\{u_n\} \in S$;*
(3) *every sequentially continuous homomorphism from (G,τ) to a Hausdorff topological group (X,ν) is continuous.*

Let us note that Definition 1.11 of s-groups is *internal* and has many advantages over Noble's, as was pointed out in [G4]. Many results of the article demonstrate other essential advantages of this internal (sequential) approach to Abelian s-groups over Noble's external one (see below, for instance, Theorems 1.18 and 1.26). The following theorems proved in [G4] will be repeatedly used in this article:

THEOREM 1.13. *Let S be a T_s-set of sequences in a group G, let H be a closed normal subgroup of (G,τ_S) and let π be the natural projection from G onto the quotient group G/H. Then $\pi(S)$ is a T_s-set of sequences in G/H and $G/H \cong (G/H, \tau_{\pi(S)})$.*

THEOREM 1.14. *Let (G,τ) and (H,ν) be Hausdorff topological groups. The following conditions are equivalent:*

(i) *(G,τ) and (H,ν) are s-groups;*
(ii) *the direct product $(G,\tau)\times(H,\nu)$ is an s-group.*

One of the most natural ways to find T_s-sets of sequences is as follows. Let (G,τ) be a Hausdorff topological group. We denote the set of all sequences of (G,τ) converging to the unit by $S(G,\tau)$:

$$S(G,\tau) = \{\mathbf{u} = \{u_n\} \subset G : u_n \to e_G \text{ in } \tau\}.$$

It is clear that $S(G,\tau) \in \mathcal{TS}(G)$ and $\tau \subseteq \tau_{S(G,\tau)}$. The group $\mathbf{s}(G,\tau) := (G, \tau_{S(G,\tau)})$ is called the *s-refinement* of (G,τ) [G4]. In fact, Theorem 1.14 tells us that $\mathbf{s}((G,\tau)\times(H,\sigma)) = \mathbf{s}(G,\tau)\times\mathbf{s}(H,\sigma)$ for any topological groups (G,τ) and (H,σ). Thus Theorems 1.13 and 1.14 assert in fact that:

- the family **S** of all s-groups forms a full subcategory with finite products of the category **TopGr** of all Hausdorff topological groups;
- the assignment $(G,\tau) \mapsto \mathbf{s}(G,\tau)$ is a functor (coreflector) from **TopGr** to **S**.

Let us recall that sequential topological spaces form a coreflective subcategory **SeqSp** of the category **Top** of all Hausdorff topological spaces and so it is stable under quotients and coproducts in **Top**. It is worth mentioning that the class **Seq** of all sequential groups is not stable under finite products (see, for instance, the example in [Ba, Theorem 6]), but **Seq** is closed under taking closed subgroups. On the other hand, the class **S** is stable under finite products (Theorem 1.14), but it is not closed under taking closed subgroups (see [G4]).

II. Main results. By Theorems 1.13 and 1.14, the class **SA** of all Abelian s-groups is closed under taking quotients and it is finitely multiplicative. This class contains all sequential groups [G4], and hence it contains the dual groups of all separable metrizable Abelian groups [CMT, Theorem 1.7]. On the other hand, for every countable T_s-set of sequences S in an Abelian group G the space (G, τ_S) is complete and sequential (see [G4]). The main theorem of [G4] asserts that an Abelian topological group G belongs to **SA** if and only if G is a quotient group of a Graev free Abelian topological group over a sequential Tychonoff space. These facts explain our interest in the study of s-groups.

The main goal of the article is to study the theory of Abelian s-groups in the following directions: (1) characterization of Abelian s-groups (Section 2), (2) Pontryagin duality for Abelian s- and bs-groups (Section 3), (3) characterization of Abelian bs-groups and sequential properties of the Bohr modification of (in particular, locally compact) Abelian topological groups (Section 4), and (4) the countable case, i.e., the case when an s-topology is generated by a countable set of convergent sequences (Section 5). Also we give some applications of this theory (Section 6) and pose several open questions in the last section.

(a) The structure of Abelian s-groups. In Section 2 we give a characterization of Abelian s-groups. First we study countably infinite s-groups whose topology is generated by a single T-sequence. The following theorem sharpens Theorem 5.6 of [G4] in the Abelian case:

THEOREM 1.15. *Let $\mathbf{u} = \{u_n\}$ be a T-sequence in an Abelian group G such that $\langle \mathbf{u} \rangle = G$. Then $(G, \tau_{\mathbf{u}})$ is a quotient group of $(\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})$ under the sequence-covering and compact-covering homomorphism*

$$\pi((n_1, n_2, \dots, n_m, 0, \dots)) = n_1 u_1 + n_2 u_2 + \dots + n_m u_m,$$

where $m \in \mathbb{N}$ and $n_1, \dots, n_m \in \mathbb{Z}$. The dual monomorphism $\pi^\wedge : (G, \tau_{\mathbf{u}})^\wedge \to (\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})^\wedge = \mathbb{T}_0^{\mathcal{H}}$ is an embedding onto $(\ker \pi)^\perp$. In particular, $(G, \tau_{\mathbf{u}})^\wedge$ is Polish.

Let $\{G_i\}_{i \in I}$, where I is a non-empty set of indices, be a family of Abelian groups. The direct sum of G_i is denoted by

$$\bigoplus_{i \in I} G_i := \Big\{ (g_i)_{i \in I} \in \prod_{i \in I} G_i : g_i = 0 \text{ for almost all } i \Big\}.$$

We denote by j_k the natural inclusion of G_k into $\bigoplus_{i \in I} G_i$, i.e.

$$j_k(g) = (g_i) \in \bigoplus_{i \in I} G_i, \quad \text{where} \quad g_i = g \text{ if } i = k \text{ and } g_i = 0 \text{ if } i \neq k.$$

Note that $\bigoplus_{i\in I} G_i$ is the coproduct of the family $\{G_i\}_{i\in I}$ in the category of all Abelian groups.

Assume that $G_i = (G_i, \tau_i)$ is an Abelian s-group for every $i \in I$. It is easy to show that the set $\bigcup_{i\in I} j_i(S(G_i, \tau_i))$ is a T_s-set of sequences in $\bigoplus_{i\in I} G_i$ (see Section 2).

Definition 1.16. Let $\{(G_i, \tau_i)\}_{i\in I}$ be a non-empty family of Abelian s-groups. The group $\bigoplus_{i\in I} G_i$ endowed with the finest Hausdorff group topology $\mathcal{T}_s$ in which every sequence of $\bigcup_{i\in I} j_i(S(G_i, \tau_i))$ converges to zero is called the *s-sum* of G_i and it is denoted by $\bigoplus^{(\mathbf{s})}_{i\in I} G_i$.

In particular, the s-sum of s-groups is again an s-group. Note that the s-sum of s-groups can be defined also for non-Abelian s-groups.

For a family $\{(G_i, \tau_i)\}_{i\in I}$ of Abelian topological groups, denote by $\mathcal{T}_f$ the *coproduct* topology on $\bigoplus_{i\in I} G_i$. Recall that $\mathcal{T}_f$ is the *final group topology* with respect to the family of canonical homomorphisms $j_k : G_k \to \bigoplus_{i\in I} G_i$ (i.e., $\mathcal{T}_f$ is the finest group topology on $\bigoplus_{i\in I} G_i$ such that all j_k are continuous). The group $(\bigoplus_{i\in I} G_i, \mathcal{T}_f)$ is called the *coproduct* of the family $\{(G_i, \tau_i)\}_{i\in I}$ in the category **TopAbGr** of all Abelian topological groups.

Proposition 1.17. *Let $\{(G_i, \tau_i)\}_{i\in I}$ be a non-empty family of Abelian s-groups. Then the coproduct and the s-sum of the family $\{(G_i, \tau_i)\}_{i\in I}$ coincide.*

In particular, this proposition asserts that the subcategory **SA** of the category **TopAbGr** is stable under taking coproducts.

The following theorem gives a characterization of Abelian s-groups and it can be considered as a natural group analogue of Franklin's Theorem 1.5:

Theorem 1.18. *Let (X, τ) be a Hausdorff Abelian topological group. The following statements are equivalent:*

(i) *(X, τ) is an s-group;*
(ii) *(X, τ) is a quotient group of the coproduct (or the s-sum) of a non-empty family of copies of $(\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})$; moreover, the quotient map may be chosen to be sequence-covering.*

(b) Pontryagin duality for Abelian s- and bs-groups. In Section 3 we study the dual groups of Abelian s- and bs-groups and prove the following generalization of the algebraic part of Theorem 1.4:

Theorem 1.19. *Let $S \in \mathcal{TS}(G)$ for an infinite Abelian group G and $i_S : G_d \to (G, \tau_S)$, $i_S(g) = g$, be the identity map. Then algebraically*

(1) $i_S^{\wedge}((G, \tau_S)^{\wedge}) = s_S(G_d^{\wedge})$;
(2) $\mathbf{n}(G, \tau_S) = [s_S(G_d^{\wedge})]^{\perp}$.

In the following theorem we describe all bs-topologies on an infinite Abelian group G. In [DMT] (see Lemma 3.1 below), it was pointed out that $\tau_{b\mathbf{u}} = T_{s_{\mathbf{u}}(G_d^\wedge)}$ for one TB-sequence $\mathbf{u}$. The next theorem generalizes this fact:

THEOREM 1.20. *Let $S \in \mathcal{TBS}(G)$ for an infinite Abelian group G and let $j_S : G_d \to (G, \tau_{bS})$, $j_S(g) = g$, be the identity map. Then*

(1) $j_S^\wedge((G, \tau_{bS})^\wedge) = s_S(G_d^\wedge)$ *algebraically;*
(2) $\tau_{bS} = \tau_S^+ = T_{s_S(G_d^\wedge)}$.

Using Theorem 1.19 we obtain the following:

THEOREM 1.21. *A dense subgroup H of an infinite compact Abelian group X is $\mathfrak{g}$-closed if and only if H algebraically is the dual group of $\widehat{X}$ endowed with some MAP s-topology.*

Note that Hart and Kunen [HK1, HK2] proved that every compact metrizable Abelian group X contains a $\mathfrak{g}$-dense proper Borel subgroup. Hence not every (even Borel) subgroup of X can be considered (algebraically) as the dual group of $(\widehat{X}, \tau)$ for some s-topology τ on $\widehat{X}$. On the other hand, non-trivial $\mathfrak{g}$-closed dense non-characterized subgroups of the torus $\mathbb{T}$ were found by Biró [Bir, §5]: every subgroup H of $\mathbb{T}$ generated by an uncountable Kronecker set K is $\mathfrak{g}$-closed (recall that K is called a *Kronecker set* if it is a compact set on which every continuous function can be uniformly approximated by characters of $\mathbb{T}$).

(c) A characterization of bs-groups and sequential properties of the Bohr modification of Abelian groups. Our main goal of this section is to obtain two characterizations of Abelian bs-groups. The first one follows from Theorem 1.20 and it is analogous to the characterization of MAP s-groups given in Theorem 1.21. Let us recall that Theorem 1.12 gives a characterization of s-groups by the requirement of continuity of all sequentially continuous *homomorphisms*. If we weaken this requirement to the continuity of only sequentially continuous *characters*, we obtain a new characterization of Abelian bs-groups that was pointed out to the author by Lukács. We summarize these results in the following theorem where the equivalence of assertions (ii) and (iii) is stated in [Luk, Theorem 4.1(a)] (here we give an independent proof of this equivalence). It is worth mentioning that by [CR] every precompact Hausdorff group topology (in particular, every bs-topology) on an Abelian group G has the form T_H for some dense subgroup H of $G_d^\wedge$.

THEOREM 1.22. *Let G be an infinite Abelian group and H be a dense subgroup of $G_d^\wedge$. The following assertions are equivalent:*

(i) (G, T_H) *is a bs-group;*
(ii) *every sequentially continuous character of* (G, T_H) *is continuous;*
(iii) H *is a* $\mathfrak{g}$*-closed subgroup of* $G_d^\wedge$.

One says (see for example [GJ]) that a cardinal κ is *not Ulam measurable* if no set X of cardinality κ admits a $\{0,1\}$-valued countably additive measure μ, defined on the family of all subsets of X, such that $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for every $x \in X$.

Let G be a *compact* Abelian group. In [G4] the question whether G is an s-group is posed. It is also natural to ask whether G is a bs-group. The next theorem gives a complete answer to the last question and it is a simple corollary of Theorem 1.22 and [CHT, HM].

THEOREM 1.23. *A compact Abelian group is a bs-group if and only if the cardinality of its dual group is not Ulam measurable.*

Let us suppose now that G is a locally compact *non-compact* Abelian group. It is also natural to ask whether G^+ is an s-group or a bs-group.

THEOREM 1.24. *Let* G *be a locally compact non-compact Abelian group whose cardinality is not Ulam measurable. Then*

(i) G^+ *is a realcompact bs-group;*
(ii) G^+ *is not an s-group;*
(iii) *the natural homomorphism* $\alpha_{G^+} : G^+ \to (G^+)^{\wedge\wedge}$ *is discontinuous.*

So this theorem gives examples of bs-groups which are not s-groups. For instance, $\mathbb{Z}^+ (= (\mathbb{Z}_d)^+)$ is a bs-group but it is not an s-group (it is well-known that $\mathbb{Z}^+$ even has no non-trivial convergent sequences, and hence $\mathbf{s}(\mathbb{Z}^+) = \mathbb{Z}_d \neq \mathbb{Z}^+$). Note also that the first example of a topological group X for which α_X is discontinuous is given in [Lep].

Let G be a precompact Abelian group. It trivially follows from the definitions that if G is an s-group then it is also a bs-group. Theorem 1.24 shows that in general the converse does not hold.

THEOREM 1.25. *Let* G *be a locally compact non-compact Abelian group. If its cardinality is not Ulam measurable* (*in particular, if* G *is separable*), *then* G *is a* $\mathfrak{g}$*-closed dense subgroup of its Bohr compactification* bG.

Theorems 1.22–1.25 are proved in Section 4.

(d) The countable case. Let G be an infinite Abelian group. In Section 5 we consider the case of countable $S \in \mathcal{TS}(G)$. In this case, the topology τ_S has a simple description (see Proposition 5.1). The main result of the section is the following:

THEOREM 1.26. *Let* G *be a countably infinite Abelian group and let* $S = \{\mathbf{u}_n\}_{n\in\omega} \in \mathcal{TS}(G)$. *Then* (G, τ_S) *is a complete sequential group with Polish* *dual group. More precisely, $(G, \tau_S)^\wedge$ is topologically isomorphic to a closed subgroup of the Polish group $\prod_{n\in\omega}(G, \tau_{\mathbf{u}_n})^\wedge$.*

(e) Applications. In Section 6 we give some applications of the results obtained.

Let (G, τ) be an Abelian non-compact MAP topological group. It is natural to ask whether (G, τ) is a $\mathfrak{g}$-closed subgroup of bG. Theorem 1.25 gives an answer for locally compact groups. For Polish groups we prove the following:

THEOREM 1.27. *Every reflexive Polish Abelian group (in particular, every separable Banach space) is $\mathfrak{g}$-closed in its Bohr compactification.*

As a consequence of Theorem 1.15 we prove the following theorem that completes Theorem 1.2:

THEOREM 1.28. *If a Polish Abelian group H is characterizable, then there is a compact subgroup K of H such that H/K embeds into $\mathbb{T}_0^{\mathcal{H}}$.*

Since a Banach space has no compact subgroups, we obtain the following necessary condition for a Banach space to be characterizable:

COROLLARY 1.29. *Every characterizable separable Banach space embeds into $\mathbb{T}_0^{\mathcal{H}}$.*

The concept of a Schwartz topological Abelian group appeared in [ACDT]. This notion generalizes the well-known notion of a Schwartz locally convex space. All nuclear groups as well as the Pontryagin dual groups of metrizable Abelian groups are Schwartz groups.

PROPOSITION 1.30. *A characterizable Abelian group is a Schwartz group if and only if it is locally compact.*

Using this proposition we prove the following:

THEOREM 1.31. *Let G be a characterizable Abelian group. Then $G^\wedge$ is characterizable if and only if G is locally compact.*

Denote by **ComplCount** the class of all complete countable Abelian groups and set $\mathbf{ComplCount}^\wedge = \{G^\wedge : G \in \mathbf{ComplCount}\}$. Problems 2.20–2.22 of [G3] concern the description of the class $\mathbf{ComplCount}^\wedge$. The following proposition shows that this class contains the countable direct products of second countable locally compact Abelian groups:

PROPOSITION 1.32. *Let $\{X_n\}_{n\in\omega}$ be a sequence of second countable locally compact Abelian groups. Then there is a complete sequential countably infinite Abelian MAP group (G, τ) such that*

$$(G, \tau)^\wedge = \prod_{n\in\omega} X_n.$$

In a particular case of Proposition 1.32 when all $X_n = \mathbb{R}$, we can find G explicitly:

Proposition 1.33. *There is a complete sequential MAP group topology τ on $\mathbb{Z}_0^{\mathbb{N}}$ such that*

$$(\mathbb{Z}_0^{\mathbb{N}}, \tau)^\wedge = \mathbb{R}^{\mathbb{N}}.$$

In the last section we pose some open questions.

2. The structure of Abelian s-groups. Let $\mathbf{u}$ be a T-sequence in an Abelian group G. To prove that the homomorphism π in Theorem 1.15 is sequence-covering we have to describe all sequences converging to zero in $(G, \tau_{\mathbf{u}})$. Let a sequence $\mathbf{v}$ converge to zero in $\tau_{\mathbf{u}}$. Then, by the definition of $\tau_{\mathbf{v}}$, we have $\tau_{\mathbf{u}} \subseteq \tau_{\mathbf{v}}$. The converse is trivially true. Thus, $\mathbf{v}$ converges to zero in $\tau_{\mathbf{u}}$ if and only if $\tau_{\mathbf{u}} \subseteq \tau_{\mathbf{v}}$. So the following proposition is a reformulation of Exercise 2.1.2 of [PZ]. Since the solution of this exercise has never been published, we give its complete proof. Following [ZP], for every $k, m \geq 0$ one puts

$$A_k^{\mathbf{u}} = A_k := \{0, \pm u_n : n \geq k\}, \qquad A^{\mathbf{u}}(m,k) = A(m,k) = \underbrace{A_k + \cdots + A_k}_{m+1}.$$

Proposition 2.1. *Let $\mathbf{u} = \{u_n\}$ be a T-sequence in an Abelian group G. A sequence $\mathbf{v} = \{v_n\}$ converges to zero in $(G, \tau_{\mathbf{u}})$ if and only if there are $m \in \omega$ and $n_0 \in \omega$ such that for every $n \geq n_0$ each member $v_n \neq 0$ can be represented in the form*

$$v_n = a_1^n u_{k_1^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n},$$

where $k_1^n < \cdots < k_{l_n}^n$, $|a_1^n| + \cdots + |a_{l_n}^n| \leq m+1$ and $k_1^n \to \infty$.

Proof. If either $\mathbf{u}$ or $\mathbf{v}$ is trivial, the proposition is evident. Assume that $\mathbf{u}$ and $\mathbf{v}$ are non-trivial. The sufficiency is clear. Let us prove the necessity. Since the subgroup $\langle \mathbf{u} \rangle$ of G is open in $\tau_{\mathbf{u}}$, there is n_0 such that $v_n \in \langle \mathbf{u} \rangle$ for every $n \geq n_0$. Thus, without loss of generality, we may assume that $\langle \mathbf{u} \rangle = G$ and hence $G = \bigcup_{m \in \omega} A^{\mathbf{u}}(m, 0)$. Since $\mathbf{v}$ converges to zero, by [PZ, Lemma 2.3.3] (see also Theorem 5.7 of [G4]) there is $m \in \omega$ such that $\mathbf{v} \subset A^{\mathbf{u}}(m, 0)$. So, if $v_n \neq 0$, then

$$v_n = a_1^n u_{k_1^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n}, \tag{2.1}$$

where $k_1^n < \cdots < k_{l_n}^n$, $a_1^n \cdots a_{l_n}^n \neq 0$ and $|a_1^n| + \cdots + |a_{l_n}^n| \leq m+1$. We can choose a representation of v_n of the form (2.1) with the minimal value of the sum $|a_1^n| + \cdots + |a_{l_n}^n|$. Clearly, for this chosen representation of v_n every sum of terms of the form $a_i^n u_{k_i^n}$ in (2.1) is non-zero (in particular, $a_i^n u_{k_i^n} \neq 0$ for $i = 1, \ldots, l_n$).

Let us show that $k_1^n \to \infty$. Assuming the converse and passing to a subsequence we may suppose that $k_1^n = k_1$, $a_1^n = a_1$ and $a_{k_1^n}^n u_{k_1^n} = a_1 u_{k_1} \neq 0$

for every n. So

$$v_n = a_1 u_{k_1} + a_2^n u_{k_2^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n} = a_1 u_{k_1} + w_n^1.$$

If $k_2^n \to \infty$, we observe that w_n^1 converges to zero. Hence $0 \neq a_1 u_{k_1} = v_n - w_n^1 \to 0$. This is impossible. Thus, there is a bounded subsequence of $\{k_2^n\}$. Passing to a subsequence we may suppose that $k_2^n = k_2$, $a_2^n = a_2$ and $a_2^n u_{k_2^n} = a_2 u_{k_2} \neq 0$ for every n. So

$$v_n = a_1 u_{k_1} + a_2 u_{k_2} + a_3^n u_{k_3^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n} = a_1 u_{k_1} + a_2 u_{k_2} + w_n^2.$$

By hypothesis, $a_1 u_{k_1} + a_2 u_{k_2} \neq 0$. Continuing this process and taking into account that

$$0 < |a_1| < |a_1| + |a_2| < \cdots \leq m + 1,$$

after at most $m+1$ steps, we see that there is a *fixed* and *non-zero* subsequence of $\mathbf{v}$. Thus $v_n \not\to 0$. This contradiction shows that $k_1^n \to \infty$. ■

A. Leiderman asked the author: when do two T-sequences define the same topology? As a corollary of Proposition 2.1 we obtain:

PROPOSITION 2.2. *Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ and $\mathbf{v} = \{v_n\}_{n\in\omega}$ be T-sequences in an Abelian group G. Then $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$ if and only if there are positive integers m and n_0 such that for every $n \geq n_0$ each $v_n \neq 0$ and each $u_n \neq 0$ can be represented in the form*

(2.2)

$$v_n = a_1^n u_{k_1^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n}, \qquad k_1^n < \cdots < k_{l_n}^n,\ k_1^n \to \infty,\ \sum_{i=1}^{l_n} |a_i^n| \leq m+1;$$

$$u_n = b_1^n v_{s_1^n} + \cdots + b_{q_n}^n v_{s_{q_n}^n}, \qquad s_1^n < \cdots < s_{q_n}^n,\ s_1^n \to \infty,\ \sum_{i=1}^{q_n} |b_i^n| \leq m+1.$$

Proof. Assume that $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$. Then $v_n \to 0$ in $\tau_{\mathbf{u}}$ and $\mathbf{v}$ has representation (2.2) for some positive integers $m(\mathbf{u})$ and $n_0(\mathbf{u})$ by Proposition 2.1. The same is true for the sequence $\mathbf{u}$. Putting $m = \max\{m(\mathbf{u}), m(\mathbf{v})\}$ and $n_0 = \max\{n_0(\mathbf{u}), n_0(\mathbf{v})\}$ we obtain (2.2).

Conversely, if $v_n \neq 0$ has representation (2.2), then $v_n \to 0$ in $\tau_{\mathbf{u}}$. Thus, $\tau_{\mathbf{u}} \subseteq \tau_{\mathbf{v}}$ by the definition of $\tau_{\mathbf{v}}$. Analogously, $\tau_{\mathbf{v}} \subseteq \tau_{\mathbf{u}}$. Hence $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$. ■

Proof of Theorem 1.15. By Theorem 5.6 of [G4], π is a quotient mapping. In particular, $\pi^\wedge$ is a continuous monomorphism from $(G, \tau_{\mathbf{u}})^\wedge$ onto $(\ker \pi)^\perp$. By [G2, Lemma 2], π is compact-covering. Hence, by Lemma 5.17 of [Auß], $\pi^\wedge$ is an embedding.

Let us show that π is sequence-covering. Let $\mathbf{v} = \{v_n\} \in S(G, \tau_{\mathbf{u}})$. By Proposition 2.1, for some natural number m we can represent every $v_n \neq 0$ in the form

$$v_n = a_1^n u_{k_1^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n},$$

where $k_1^n < \cdots < k_{l_n}^n$, $|a_1^n| + \cdots + |a_{k_{l_n}^n}^n| \le m+1$ and $k_1^n \to \infty$. Set

$$z_n = a_1^n e_{k_1^n} + \cdots + a_{l_n}^n e_{k_{l_n}^n} \quad \text{if } v_n \neq 0, \quad \text{and} \quad z_n = 0 \quad \text{if } v_n = 0.$$

Then $z_n \to 0$ in $(\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})$ and $\pi(z_n) = v_n$. ∎

Note that Proposition 1.3 can be derived from this theorem and Lemma 2.2 of [BCM].

Let $\{(G_i, \tau_i)\}_{i\in I}$, where I is a non-empty set of indices, be a family of Hausdorff topological groups. For every $i \in I$ fix $U_i \in \mathcal{U}_{G_i}$ and put

$$\bigoplus_{i\in I} U_i := \Big\{ (g_i)_{i\in I} \in \bigoplus_{i\in I} G_i : g_i \in U_i \text{ for all } i \in I \Big\}.$$

Then the sets of the form $\bigoplus_{i\in I} U_i$, where $U_i \in \mathcal{U}_{G_i}$ for every $i \in I$, form a neighborhood basis at the unit of a Hausdorff group topology $\mathcal{T}_r$ on $\bigoplus_{i\in I} G_i$ that is called the *rectangular* (or *box*) topology.

Let $\mathbf{u} = \{g_n\}$ be an arbitrary sequence in $S(G_i, \tau_i)$. Evidently, the sequence $j_i(\mathbf{u})$ converges to the unit in $\mathcal{T}_r$. Thus, the set $\bigcup_{i\in I} j_i(S(G_i, \tau_i))$ is a T_s-set of sequences in $\bigoplus_{i\in I} G_i$. So, if (G_i, τ_i) is an s-group for all $i \in I$, we can define the s-sum of G_i (see Definition 1.16). Moreover, we can prove the following:

Proposition 2.3. *Let $G = \bigoplus_{i\in I} G_i$, where (G_i, τ_i) is an s-group for every $i \in I$. Set $S := \bigcup_{i\in I} j_i(S(G_i, \tau_i))$. The topology τ_S on G coincides with the finest Hausdorff group topology τ' on G for which all inclusions j_i are continuous.*

Proof. Fix $i \in I$. By construction, for every $\{u_n\} \in S(G_i, \tau_i)$, $j_i(u_n) \to e_G$ in τ_S. By Theorem 1.12, the inclusion j_i is continuous. Thus, $\tau_S \subseteq \tau'$. Conversely, if j_i is continuous with respect to τ', then $j_i(S(G_i, \tau_i)) \subset S(G, \tau')$. Hence $S \subseteq S(G, \tau')$ and $\tau' \subseteq \tau_S$ by the definition of τ_S. ∎

Proof of Proposition 1.17. The result immediately follows from Proposition 2.3. ∎

Theorem 2.4. *Let (X, τ) be an Abelian s-group. Set $I = S(X, \tau)$. For every $\mathbf{u} \in I$, let $p_{\mathbf{u}} : (\langle \mathbf{u} \rangle, \tau_{\mathbf{u}}) \to X$, $p_{\mathbf{u}}(g) = g$, be the natural inclusion of $(\langle \mathbf{u} \rangle, \tau_{\mathbf{u}})$ into X. Then the natural homomorphism*

$$p : \bigoplus_{\mathbf{u}\in S(X,\tau)}^{(\mathbf{s})} (\langle \mathbf{u} \rangle, \tau_{\mathbf{u}}) \to X, \qquad p((x_{\mathbf{u}})) = \sum_{\mathbf{u}} p_{\mathbf{u}}(x_{\mathbf{u}}) = \sum_{\mathbf{u}} x_{\mathbf{u}},$$

is a quotient map and a sequence-covering map.

Proof. Set

$$G := \bigoplus_{\mathbf{u}\in S(X,\tau)}^{(\mathbf{s})} (\langle \mathbf{u} \rangle, \tau_{\mathbf{u}}) \quad \text{and} \quad S := \bigcup_{\mathbf{u}\in S(X,\tau)} j_{\mathbf{u}}(S(\langle \mathbf{u} \rangle, \tau_{\mathbf{u}})) \in \mathcal{TS}(G).$$

Since each element of X can be regarded as the first element of some sequence $\mathbf{u} \in S(X,\tau)$, p is surjective. By construction, p is sequence-covering.

Let $\mathbf{v} = \{v_n\} \in S$. By construction, $p(v_n) = v_n \to 0$ in τ. Thus, by Theorem 1.12, p is continuous. Set $H = \ker p$. By Theorem 1.13, $G/H \cong (X, \tau_{p(S)})$. Since, by construction, $p(S) = S(X,\tau)$, we obtain $G/H \cong (X,\tau)$ by Proposition 3.3 of [G4]. ∎

To prove Theorem 1.18 we need the following proposition.

Proposition 2.5. *Let $\{(X_i,\nu_i)\}_{i\in I}$ and $\{(G_i,\tau_i)\}_{i\in I}$ be non-empty families of Abelian s-groups and let $\pi_i : G_i \to X_i$ be a quotient sequence-covering map for every $i \in I$. Set $X = \bigoplus^{(\mathbf{s})}_{i\in I} X_i$, $G = \bigoplus^{(\mathbf{s})}_{i\in I} G_i$ and $\pi : G \to X$, $\pi((g_i)) = (\pi_i(g_i))$. Then π is a quotient map.*

Proof. It is clear that π is surjective. Set

$$S_X := \bigcup_{i\in I} j_i(S(X_i,\nu_i)) \quad \text{and} \quad S_G := \bigcup_{i\in I} j_i(S(G_i,\tau_i)).$$

Since π_i is sequence-covering, we have $\pi_i(S(G_i,\tau_i)) = S(X_i,\nu_i)$. Hence $\pi(S_G) = S_X$. Thus, by Theorem 1.12, π is continuous. By Theorem 1.13, $G/\ker\pi \cong (X, \tau_{\pi(S_G)})$. Hence $G/\ker\pi \cong X$ and π is a quotient map. ∎

Proof of Theorem 1.18. Let $I = S(X,\tau)$. For every $\mathbf{u} \in I$, put $G_{\mathbf{u}} = (\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})$, $X_{\mathbf{u}} = (\langle \mathbf{u}\rangle, \tau_{\mathbf{u}})$ and let $\pi_{\mathbf{u}}$ be the unique group homomorphism from $G_{\mathbf{u}}$ onto $X_{\mathbf{u}}$ defined by $\pi_{\mathbf{u}}(e_i) = u_i$ for every $i \in \mathbb{N}$. Let $p_{\mathbf{u}} : (\langle \mathbf{u}\rangle, \tau_{\mathbf{u}}) \to X$, $p_{\mathbf{u}}(g) = g$, be the natural inclusion of $(\langle \mathbf{u}\rangle, \tau_{\mathbf{u}})$ into X. Then the result immediately follows from Theorems 1.15 and 2.4 and Proposition 2.5. ∎

The following theorem is a natural counterpart of [S, Theorem 4.1]:

Theorem 2.6. *Let (X,τ) be a non-trivial Hausdorff Abelian topological group. The following statements are equivalent:*

(i) *(X,τ) is an s-group;*
(ii) *every continuous sequence-covering homomorphism from an Abelian s-group onto (X,τ) is quotient.*

Proof. (i)⇒(ii). Let $p : G \to X$ be a sequence-covering continuous homomorphism from an s-group (G,ν) onto X. Set $H = \ker p$. We have to show that p is quotient, i.e., $X \cong G/H$. Since p is surjective, by Theorem 1.13, we have $G/H \cong (X, \tau_{p(S(G,\nu))})$. By hypothesis and Proposition 3.3 of [G4], $p(S(G,\nu)) = S(X,\tau)$ and $\tau = \tau_{S(X,\tau)}$. Thus $G/H \cong X$.

(ii)⇒(i). Let $I = S(X,\tau)$, $S := \bigcup_{\mathbf{u}\in S(X,\tau)} j_{\mathbf{u}}(S(\langle \mathbf{u}\rangle, \tau_{\mathbf{u}})) \in \mathcal{TS}(G)$, $G := \bigoplus^{(\mathbf{s})}_{\mathbf{u}\in S(X,\tau)} (\langle \mathbf{u}\rangle, \tau_{\mathbf{u}})$ and

$$p : G \to X, \qquad p((x_{\mathbf{u}})) = \sum_{\mathbf{u}} p_{\mathbf{u}}(x_{\mathbf{u}}) = \sum_{\mathbf{u}} x_{\mathbf{u}}.$$

By Theorem 1.12, p is continuous. Since p is sequence-covering, by hypothesis, p is quotient. Thus $(X,\tau) \cong G/\ker p$. By Theorem 1.13, we also have $G/\ker p \cong (X, \tau_{p(S)})$. Thus $\tau = \tau_{p(S)}$ and (X,τ) is an s-group. ∎

3. Duality. The following lemma will be used several times in what follows:

LEMMA 3.1 ([DMT, Lemma 3.1]). *Let G be an Abelian topological group and let $H \le G^\wedge$. Then, for a sequence $\mathbf{u} = \{u_n\}$ in G, one has $u_n \to 0$ in (G, T_H) if and only if $H \le s_{\mathbf{u}}(G^\wedge)$.*

We will use the following notations. For a T- or respectively TB-sequence $\mathbf{u}$ in an Abelian group G, let $i_{\mathbf{u}} : G_d \to (G,\tau_{\mathbf{u}})$ and $j_{\mathbf{u}} : G_d \to (G, \tau_{b\mathbf{u}})$, $i_{\mathbf{u}}(g) = j_{\mathbf{u}}(g) = g$, be the identity maps. The following proposition connects the notions of T- and TB-sequences and in one way or another it can be found in [DMT]:

PROPOSITION 3.2 ([DMT]). *Let $\mathbf{u} = \{u_n\}$ be a sequence in an Abelian group G. Then*

(i) *$\mathbf{u}$ is a TB-sequence if and only if it is a T-sequence and $(G,\tau_{\mathbf{u}})$ is MAP;*
(ii) *if $\mathbf{u}$ is a TB-sequence, then*

$$\tau_{b\mathbf{u}} = \tau_{\mathbf{u}}^{+} = T_{s_{\mathbf{u}}(G_d^\wedge)} \quad \text{and} \quad i_{\mathbf{u}}^\wedge((G,\tau_{\mathbf{u}})^\wedge) = j_{\mathbf{u}}^\wedge((G,\tau_{b\mathbf{u}})^\wedge) = s_{\mathbf{u}}(G_d^\wedge).$$

Proof. (i) Clearly, if a sequence $\mathbf{u} = \{u_n\}$ is a TB-sequence, then it is a T-sequence and $(G,\tau_{\mathbf{u}})$ is MAP. Conversely, if $(G,\tau_{\mathbf{u}})$ is MAP, then the sequence $\mathbf{u}$ converges to zero in $\tau_{\mathbf{u}}^{+}$ as well. Thus $\mathbf{u}$ is a TB-sequence.

(ii) Let $\mathbf{u}$ be a TB-sequence. By the definition of $\tau_{b\mathbf{u}}$ we have $\tau_{\mathbf{u}}^{+} \subseteq \tau_{b\mathbf{u}}$. On the other hand, $\tau_{b\mathbf{u}}$ is a precompact group topology coarser than $\tau_{\mathbf{u}}$ by the definition of $\tau_{b\mathbf{u}}$ and $\tau_{\mathbf{u}}$. So $\tau_{b\mathbf{u}} \subseteq \tau_{\mathbf{u}}^{+}$, since $\tau_{\mathbf{u}}^{+}$ is the finest precompact group topology on G below $\tau_{\mathbf{u}}$. Hence, $\tau_{b\mathbf{u}} = \tau_{\mathbf{u}}^{+}$ and, by Theorem 1.4, $i_{\mathbf{u}}^\wedge((G,\tau_{\mathbf{u}})^\wedge) = j_{\mathbf{u}}^\wedge((G,\tau_{b\mathbf{u}})^\wedge) = s_{\mathbf{u}}(G_d^\wedge)$. So $\tau_{b\mathbf{u}} = T_{s_{\mathbf{u}}(G_d^\wedge)}$. ∎

Proof of Theorem 1.19. (1) For every $\mathbf{u} \in S$, let $t_{\mathbf{u}} : (G,\tau_{\mathbf{u}}) \to (G,\tau_S)$, $t_{\mathbf{u}}(g) = g$, be the identity continuous map. Then $i_S = t_{\mathbf{u}} \circ i_{\mathbf{u}}$ and $i_{\mathbf{u}}^\wedge((G,\tau_{\mathbf{u}})^\wedge) = s_{\mathbf{u}}(G_d^\wedge)$ by Proposition 3.2(ii). Hence $i_S^\wedge((G,\tau_S)^\wedge) \subseteq s_{\mathbf{u}}(G_d^\wedge)$ for every $\mathbf{u} \in S$. So $i_S^\wedge((G,\tau_S)^\wedge) \subseteq s_S(G_d^\wedge)$.

Conversely, let $x \in s_S(G_d^\wedge)$. By Proposition 3.2(ii), $x \in i_{\mathbf{u}}^\wedge((G,\tau_{\mathbf{u}})^\wedge)$ for every $\mathbf{u} = \{u_n\} \in S$. Thus, x is an algebraic homomorphism from (G,τ_S) into $\mathbb{T}$ such that, by the definition of the topology $\tau_{\mathbf{u}}$, $(u_n, x) \to 1$ for every $\mathbf{u} \in S$. By Theorem 1.12, x is a continuous character of (G,τ_S). So $x \in i_S^\wedge((G,\tau_S)^\wedge)$.

(2) By (1), algebraically we have

$$\mathbf{n}(G,\tau_S) = \bigcap_{\chi\in(G,\tau_S)^\wedge} \ker\chi = \bigcap_{x\in s_S(G_d^\wedge)} \ker x = [s_S(G_d^\wedge)]^\perp. \ ■$$

Corollary 3.3. *Let G be an infinite Abelian group and S be an arbitrary set of sequences in G. Then the following statements are equivalent:*

(1) *$S \in \mathcal{TBS}(G)$;*
(2) *$S \in \mathcal{TS}(G)$ and (G,τ_S) is MAP;*
(3) *$s_S(G_d^\wedge)$ is dense in $G_d^\wedge$.*

Proof. (1)⇒(2) is trivial.

(2)⇒(3). By Theorem 1.19, (G,τ_S) is MAP iff $s_S(G_d^\wedge)$ is dense in $G_d^\wedge$.

(3)⇒(1). For simplicity we set $H := s_S(G_d^\wedge)$. By Lemma 3.1, every $\mathbf{u}\in S$ converges to zero in T_H. Since H is dense in $G_d^\wedge$, by [CR, Theorem 1.9], T_H is Hausdorff. So $S \in \mathcal{TBS}(G)$. ■

Example 3.4. If $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$ for *TB*-sequences $\mathbf{u}$ and $\mathbf{v}$ in an Abelian group G, then, by Proposition 3.2, $\tau_{b\mathbf{u}} = \tau_{b\mathbf{v}}$. But, in general, the converse is not true, i.e., from the equality $\tau_{b\mathbf{u}} = \tau_{b\mathbf{v}}$ it does not follow that $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$. Let us consider the following example. Define two sequences $\mathbf{u} = \{u_n\}$ and $\mathbf{v} = \{v_n\}$ in $G = \mathbb{Z}$ as follows:

$$u_n = p^n, \qquad v_{2n} = p^n, \qquad v_{2n+1} = \sum_{i=0}^{n} p^{n^3-in} \qquad \text{for } n\in\omega.$$

By [Ar] (see also [BDMW2, Remark 3.8]), $s_{\mathbf{u}}(\mathbb{T}) = \mathbb{Z}(p^\infty)$. Clearly, for each $x \in \mathbb{Z}(p^\infty)$ we have $(v_{2n+1}, x) = 1$ for all sufficiently large n. Thus, also $s_{\mathbf{v}}(\mathbb{T}) = \mathbb{Z}(p^\infty)$. Since $\mathbb{Z}(p^\infty)$ is dense in $\mathbb{T}$, by Corollary 3.3, $\mathbf{u}$ and $\mathbf{v}$ are *TB*-sequences. By Proposition 3.2, $\tau_{b\mathbf{u}} = \tau_{b\mathbf{v}} = T_{\mathbb{Z}(p^\infty)}$. Let us show that $\tau_{\mathbf{v}} \subsetneq \tau_{\mathbf{u}}$. Since $\mathbf{u}$ is a subsequence of $\mathbf{v}$, we have $u_n \to 0$ in $\tau_{\mathbf{v}}$. Thus $\tau_{\mathbf{v}} \subseteq \tau_{\mathbf{u}}$ by the definition of the topology $\tau_{\mathbf{u}}$. Let us show that $\tau_{\mathbf{v}} \neq \tau_{\mathbf{u}}$. To prove this it is enough to show that $v_n \not\to 0$ in $\tau_{\mathbf{u}}$.

Suppose for a contradiction that $v_n \to 0$ in $\tau_{\mathbf{u}}$. Then $\mathbf{v}\cup\{0\}$ is compact in $\tau_{\mathbf{u}}$. Taking into account that $\mathbb{Z} = \bigcup_{m\in\omega} A^{\mathbf{u}}(m,0)$, Lemma 2.3.3 of [PZ] (see also Theorem 5.7 of [G4]) implies that $\mathbf{v} \subset A^{\mathbf{u}}(m,0)$ for some $m > 0$. Note that every element of $A^{\mathbf{u}}(m,0)$ contains at most $m+1$ non-zero summands of the form p^n. On the other hand, $v_{2(2m)+1}$ contains $2m+1$ ($> m+1$) non-zero summands of the form p^n. Hence $v_{2(2m)+1} \notin A^{\mathbf{u}}(m,0)$. This contradiction shows that $v_n \not\to 0$ in $\tau_{\mathbf{u}}$.

Proof of Theorem 1.20. Since every sequence $\mathbf{u}\in S$ converges to zero in τ_S, $\mathbf{u}$ converges to zero in τ_S^+ as well. Thus $\tau_S^+ \subseteq \tau_{bS}$ by the definition of τ_{bS}. On the other hand, τ_{bS} is a precompact group topology coarser than

τ_S by the definition of τ_{bS} and τ_S. So $\tau_{bS} \subseteq \tau_S^+$, since τ_S^+ is the finest precompact group topology on G below τ_S. Thus $\tau_{bS} = \tau_S^+$. Hence, by Theorem 1.19, $\tau_{bS} = T_{s_S(G_d^\wedge)}$ and $j_S^\wedge((G, \tau_{bS})^\wedge) = s_S(G_d^\wedge)$. ∎

As an immediate consequence of Theorems 1.19 and 1.20 we obtain:

COROLLARY 3.5. *Let $S \in \mathcal{TBS}(G)$ for an infinite Abelian group G, and $j : (G, \tau_S) \to (G, \tau_{bS})$, $j(g) = g$, be the natural continuous isomorphism. Then its conjugate homomorphism $j^\wedge : (G, \tau_{bS})^\wedge \to (G, \tau_S)^\wedge$ is a continuous isomorphism.*

The following corollary generalizes [DMT, Proposition 3.2]:

COROLLARY 3.6. *Let G be an infinite Abelian group and $S \in \mathcal{TBS}(G)$. Then:*

(1) *$w(G, \tau_{bS}) = |s_S(G_d^\wedge)|$;*
(2) *τ_{bS} is metrizable iff $s_S(G_d^\wedge)$ is countable.*

Proof. Item (1) follows from Theorem 1.20 and the property $w(G, T_Y) = |Y|$ of the topology T_Y generated by any subgroup $Y \le G_d^\wedge$ (see [CR, Theorem 1.2] and [HR, 24.15]).

Item (2) immediately follows from item (1) since a precompact group is metrizable precisely when it is second countable. ∎

REMARK 3.7. For τ_S, in general, the equality $w(G, \tau_S) = |s_S(G_d^\wedge)|$ does not hold. The following examples demonstrate that the weight $w(G, \tau_S)$ may be both less than $|s_S(G_d^\wedge)|$ and greater than $|s_S(G_d^\wedge)|$ even for countable G. Note that $|s_S(G_d^\wedge)| = |(G, \tau_S)^\wedge|$ by Theorem 1.19.

(a) Let G be a countably infinite Abelian group and H be a countably infinite dense subgroup of $G_d^\wedge$. By [DK], there is a TB-sequence $\mathbf{u}$ such that $s_{\mathbf{u}}(G_d^\wedge) = H$. Set $S = \{\mathbf{u}\}$. Then, by Theorem 1.19, $|(G, \tau_S)^\wedge| = |s_S(G_d^\wedge)| = \aleph_0$. On the other hand, by corollary from Theorem 6 of [ZP], (G, τ_S) is not metrizable and hence $w(G, \tau_S) > \aleph_0$. Thus, $w(G, \tau_S) > |(G, \tau_S)^\wedge| = |s_S(G_d^\wedge)|$.

(b) Let $G = G_d$ be a discrete countably infinite Abelian group. Then $w(G) = \aleph_0$ and $S = S(G_d)$ consists of only trivial sequences. So $s_S(G_d^\wedge) = G_d^\wedge$ has size $\mathfrak{c}$. Thus $w(G) < |s_S(G_d^\wedge)|$. The same conclusion holds true for every dense countable subgroup of a locally compact non-compact Abelian metrizable group.

Note that Theorem 1.20 describes the dual group of (G, τ_{bS}) only algebraically. In a simple partial case the next corollary gives also a topological description of $(G, \tau_{bS})^\wedge$.

COROLLARY 3.8. *Let G be a countably infinite Abelian group and let $S \in \mathcal{TBS}(G)$. If $s_S(G_d^\wedge)$ is countable, then $(G, \tau_{bS})^\wedge$ is discrete. In particular, (G, τ_{bS}) is not reflexive.*

Proof. By Corollary 3.6, (G, τ_{bS}) is a metrizable precompact countably infinite group. Hence the dual group $(G, \tau_{bS})^\wedge$ is countably infinite and discrete by [Auß, 4.10] (see also [Cha]). So the bidual group $(G, \tau_{bS})^{\wedge\wedge}$ is compact and has size continuum by [HR, 24.47]. Therefore, the countable group (G, τ_{bS}) is not reflexive. ∎

COROLLARY 3.9. *Let G be a countably infinite Abelian group, $\mathbf{u}$ be a TB-sequence and $j : (G, \tau_{\mathbf{u}}) \to (G, \tau_{b\mathbf{u}})$ be the identity continuous isomorphism. If $s_{\mathbf{u}}(G_d^\wedge)$ is countable, then $j^\wedge$ is a topological isomorphism.*

Proof. By Corollary 3.5, $j^\wedge$ is a continuous isomorphism. Since $s_{\mathbf{u}}(G_d^\wedge)$ is countable, by Theorem 1.4, $(G, \tau_{\mathbf{u}})^\wedge$ is a countable Polish group. So $(G, \tau_{\mathbf{u}})^\wedge$ is discrete. Thus $(G, \tau_{b\mathbf{u}})^\wedge$ is also discrete and $j^\wedge$ is a topological isomorphism. ∎

Theorem 1.21 immediately follows from the following:

THEOREM 3.10. *Let G be an infinite Abelian group. Set $X = G_d^\wedge$.*

(i) *If $S \in \mathcal{TS}(G)$ and $i_S : G_d \to (G, \tau_S)$ is the natural continuous isomorphism, then $i_S^\wedge((G, \tau_S)^\wedge)$ is a $\mathfrak{g}$-closed subgroup of X.*
(ii) *If H is a $\mathfrak{g}$-closed subgroup of X, then there is $S \in \mathcal{TBS}(\widehat{\mathrm{cl}\, H})$ such that $H = (\widehat{\mathrm{cl}\, H}, \tau_S)^\wedge$ algebraically.*

Proof. (i) follows from Theorem 1.19(1) and the definition of $\mathfrak{g}$-closed subgroup.

(ii) It is clear that H is a $\mathfrak{g}$-closed dense subgroup of $\mathrm{cl}\, H$. Put

$$S := \{\mathbf{u} \in (\widehat{\mathrm{cl}\, H})^{\mathbb{N}} : H \le s_{\mathbf{u}}(\mathrm{cl}\, H)\}.$$

By the definition of $\mathfrak{g}$-closed subgroups, $H = s_S(\mathrm{cl}\, H)$. Since H is dense in $\mathrm{cl}\, H$, $S \in \mathcal{TBS}(\widehat{\mathrm{cl}\, H})$ by Corollary 3.3. Now the assertion follows from Theorem 1.19(1). ∎

Since every sequential group is an s-group [G4], we obtain:

COROLLARY 3.11. *The dual group of a sequential* (*in particular, metrizable*) *group (G, τ) is a $\mathfrak{g}$-closed subgroup of the compact group $G_d^\wedge$.*

Let S be a T_s-set of sequences of an Abelian group G. As was noted in the introduction, if S is finite it can be replaced by a single T-sequence. If S is countable, then (G, τ_S) is a complete sequential group [PZ, G4], and $(G, \tau_S)^\wedge$ is Polish by Theorem 1.26. On the other hand, if (G, τ_S) is a metrizable group, then the cardinality $|S|$ of S must be uncountable by Corollary 1.18 of [G4]. Hence topological properties of an s-group depend on the cardinality of the T_s-set of sequences that generates the original topology (and conversely). These facts justify the following definition:

Definition 3.12. Let G be an Abelian group.

(1) If $S \in \mathcal{TS}(G)$, one puts

$$r_s(S) = \min\{|B| : B \in \mathcal{TS}(G) \text{ and } \tau_B = \tau_S\},$$
$$r_s^{\wedge}(S) = \min\{|B| : B \in \mathcal{TS}(G) \text{ and } s_B(G_d^{\wedge}) = s_S(G_d^{\wedge})\}.$$

(2) If $S \in \mathcal{TBS}(G)$, one puts

$$r_b(S) = \min\{|B| : B \in \mathcal{TBS}(G) \text{ and } \tau_{bB} = \tau_{bS}\},$$
$$r_b^{\wedge}(S) = \min\{|B| : B \in \mathcal{TBS}(G) \text{ and } s_B(G_d^{\wedge}) = s_S(G_d^{\wedge})\}.$$

Remark 3.13. Let (G, τ) be an s-group and $\tau = \tau_S$ for some $S \in \mathcal{TS}(G)$. Then the number $r_s(S)$ coincides with the number $r_s(G, \tau)$ that is defined in [G4].

Proposition 3.14. *Let G be an infinite Abelian group.*

(1) *If $S \in \mathcal{TS}(G)$, then $r_s^{\wedge}(S) \le r_s(S) \le |S|$.*
(2) *If $S \in \mathcal{TBS}(G)$, then $r_s^{\wedge}(S) = r_b^{\wedge}(S) = r_b(S)$.*
(3) *If $S \in \mathcal{TS}(G)$ is finite, then $r_s(S) = r_s^{\wedge}(S) = 1$.*

Proof. (1) Let $B \in \mathcal{TS}(G)$ be such that $\tau_B = \tau_S$. By Theorem 1.19(1), algebraically,

$$s_B(G_d^{\wedge}) = \widehat{(G, \tau_B)} = \widehat{(G, \tau_S)} = s_S(G_d^{\wedge}).$$

So $|B| \ge r_s^{\wedge}(S)$. Thus $r_s^{\wedge}(S) \le r_s(S)$.

(2) Let $S \in \mathcal{TBS}(G)$. By Corollary 3.3, $s_S(G_d^{\wedge})$ is dense in $G_d^{\wedge}$. Hence, if $s_B(G_d^{\wedge}) = s_S(G_d^{\wedge})$ for $B \in \mathcal{TS}(G)$, then, by Corollary 3.3, $B \in \mathcal{TBS}(G)$. Thus, $r_s^{\wedge}(S) = r_b^{\wedge}(S)$.

Let $B \in \mathcal{TBS}(G)$. By Theorem 1.20 and [CR, Theorem 1.3], $s_B(G_d^{\wedge}) = s_S(G_d^{\wedge})$ if and only if $\tau_{bB} = \tau_{bS}$. So $r_b(S) = r_b^{\wedge}(S)$.

(3) By Proposition 5.1 of [G4], $r_s(S) = 1$ and the assertion follows from item (1). ∎

Example 3.15. Let (G, τ) be a dense countably infinite subgroup of a compact infinite metrizable Abelian group X with the induced topology. Then (G, τ) is an s-group. Set $S = S(G, \tau)$. It is known [ZP] that for every T-sequence $\mathbf{v}$ in G the group $(G, \tau_{\mathbf{v}})$ is either discrete or non-metrizable. So, by Proposition 3.14(3), we have $r_s(S) \ge \aleph_0$. On the other hand, by Theorem 1.19, algebraically, $s_S(G_d^{\wedge}) = \widehat{(G, \tau)}$ is a countable subgroup of $G_d^{\wedge}$. Since G is dense in X, we know that $(G, \tau)^{\wedge} = X^{\wedge}$ is dense in $G_d^{\wedge}$ by [HR, 24.21]. So, by [DK], there exists a TB-sequence $\mathbf{u}$ in G such that $s_{\mathbf{u}}(G_d^{\wedge}) = s_S(G_d^{\wedge})$. Thus $r_s^{\wedge}(S) = 1$ and hence $r_s(S) > r_s^{\wedge}(S)$. We do not know any characterization of those Abelian s-groups (G, τ_S) for which $r_s(S) = r_s^{\wedge}(S)$.

4. A characterization of bs-groups and sequential properties of the Bohr modification of Abelian groups

Proof of Theorem 1.22. (i)⇒(ii). Set $S = S(G, T_H)$. Since (G, T_H) is a bs-group, $\tau_{bS} = T_H$. By [CR] and Theorems 1.19 and 1.20, we have $H = (G, \tau_{bS})^\wedge = (G, \tau_S)^\wedge$ algebraically. Let χ be a sequentially continuous character of (G, T_H), i.e., $(\chi, u_n) \to 1$ for every $\{u_n\} \in S$. By Theorem 1.12, $\chi \in (G, \tau_S)^\wedge$. Thus $\chi \in H$ is a continuous character of (G, T_H).

(ii)⇒(iii). Set $S = S(G, T_H)$. By Theorem 1.12, every sequentially continuous character of (G, T_H) is continuous in τ_S. Hence, by hypothesis, $H = (G, T_H)^\wedge = (G, \tau_S)^\wedge$ algebraically. By Theorem 1.19, $H = s_S(G_d^\wedge)$, and hence H is $\mathfrak{g}$-closed in $G_d^\wedge$.

(iii)⇒(i). Let S be a set of sequences of G such that $H = s_S(G_d^\wedge)$. Since H is dense in $G_d^\wedge$, $S \in \mathcal{TBS}(G)$ by Corollary 3.3. By Theorem 1.20 we have $\tau_{bS} = T_{s_S(G_d^\wedge)} = T_H$. Thus (G, T_H) is a bs-group. ∎

Denote by **BSA** the class of all Abelian bs-groups.

Corollary 4.1. *The class* **BSA** *is closed under taking finite products and quotients.*

Proof. Let $G, H \in$ **BSA**. By Theorem 1.22, to prove that $G \times H$ is a bs-group it is enough to show that every sequentially continuous character χ of $G \times H$ is continuous. Set $(\eta, g) := (\chi, (g, 0))$ and $(\psi, h) := (\chi, (0, h))$, where $g \in G$ and $h \in H$. Then η and ψ are sequentially continuous characters of G and H respectively. By hypothesis, η and ψ are continuous. Thus $(\chi, (g, h)) = (\eta, g) \cdot (\psi, h)$ is continuous as well.

Let $G \in$ **BSA** and Q be a closed subgroup of G. Denote by $q : G \to G/Q$ the quotient homomorphism. Then for every sequentially continuous character χ of G/Q the composition $\chi \circ q$ is a sequentially continuous character of G. By hypothesis, $\chi \circ q$ is continuous. Since q is quotient, χ is continuous. Now Theorem 1.22 implies that G/Q is a bs-group. ∎

Some sequential properties of the Bohr modification of s-groups are considered in the next proposition:

Proposition 4.2. *Let (G, τ) be a MAP s-group. Then*

(a) *(G, τ^+) is a bs-group.*
(b) *For the s-modification $\mathbf{s}(G, \tau^+)$ of (G, τ^+) the following hold:*
  (b1) *the topology of $\mathbf{s}(G, \tau^+)$ is included in τ;*
  (b2) *$\mathbf{s}(G, \tau^+) = (G, \tau)$ if and only if $S(G, \tau^+) = S(G, \tau)$, i.e., when (G, τ^+) and (G, τ) have the same set of convergent sequences.*
(c) *If $S(G, \tau^+) = S(G, \tau)$, then (G, τ^+) is an s-group if and only if (G, τ) is precompact. In that case $\tau = \tau^+$.*

Proof. (a) By Theorem 1.22 it is enough to show that every sequentially continuous character χ of (G,τ^+) is continuous in τ^+. Let $(\chi,u_n)\to 1$ for every $\{u_n\}\in S(G,\tau^+)$. In particular, $(\chi,u_n)\to 1$ for every $\{u_n\}\in S(G,\tau)$. Thus χ is continuous in τ by Theorem 1.12. So χ is continuous in τ^+ as well by Theorem 1.20.

(b) Set $S = S(G,\tau)$ and $Q = S(G,\tau^+)$. Denote by ν the topology of $\mathbf{s}(G,\tau^+)$. Set $Q' = S(G,\tau_Q)$. By Proposition 3.3 and [G4, Lemma 4.2] we have

$$\tau=\tau_S \quad \text{and} \quad S=S(G,\tau_S); \quad \nu=\tau_Q=\tau_{Q'} \quad \text{and} \quad Q'=S(G,\tau_{Q'}). \tag{4.1}$$

Since $S\subseteq Q$, we have $\nu=\tau_Q\subseteq\tau_S=\tau$ and (b1) follows.

Let us prove (b2). Clearly, $\mathbf{s}(G,\tau^+)=(G,\tau)$ if and only if $S(\mathbf{s}(G,\tau^+)) = S(G,\nu) = S(G,\tau_Q) = Q'$ coincides with $S(G,\tau)=S$. So to prove (b2), by (4.1) it is enough to show that $Q=Q'$. Clearly, $Q\subseteq Q'$. Let us prove the reverse inclusion. Take a sequence $\mathbf{u}\in Q'$. This means that $\mathbf{u}\to 0$ in $\tau_{Q'}=\tau_Q$. Since $\tau^+\subseteq\tau_Q$, $\mathbf{u}\to 0$ in τ^+ as well. Thus $\mathbf{u}\in S(G,\tau^+)=Q$.

(c) By hypothesis, the identity map $\mathrm{id}:(G,\tau^+)\to(G,\tau)$ is sequentially continuous. So, if (G,τ^+) is an s-group, id is continuous and $\tau\subseteq\tau^+$. Since $\tau^+\subseteq\tau$, we obtain $\tau^+=\tau$ and (G,τ) is precompact.

Conversely, if (G,τ) is precompact, then $\tau=\tau^+$. Hence (G,τ^+) is an s-group. ∎

REMARK 4.3. Let us denote by **MAPAb** the subcategory of **TopGrAb** consisting of all MAP abelian groups. Let $\mathfrak{B}:\mathbf{MAPAb}\to\mathbf{MAPAb}$ be the Bohr functor, i.e., $\mathfrak{B}(G,\tau)=(G,\tau^+)$ for every $(G,\tau)\in\mathbf{MAPAb}$. (Note that $\mathfrak{B}$ is finitely multiplicative.) Proposition 4.2(b1) says that $\mathbf{s}\circ\mathfrak{B}(\tau)\subseteq\tau$ for every MAP s-group (G,τ). Example 3.4 shows that in general this inclusion is strict.

Proposition 4.2(b2) shows that the functors $\mathbf{s}\circ\mathfrak{B}\circ\mathbf{s}$ and $\mathbf{s}$ on **MAPAb** coincide only on those groups $X\in\mathbf{MAPAb}$ for which $\mathfrak{B}\circ\mathbf{s}(X)$ and $\mathbf{s}(X)$ have the same set of convergent sequences.

Theorem 1.23 is a part of the following theorem in which, for the convenience of the reader, we summarize also some known results from [CHT], [HM] and [GJ].

THEOREM 4.4. *Let G be a compact Abelian group and $X=G^\wedge$. Then the following are equivalent:*

(i) *G is a bs-group;*
(ii) *X is realcompact;*
(iii) *X^+ is realcompact;*
(iv) *$|X|$ is not Ulam measurable.*
(v) *$|G|$ is not Ulam measurable.*

Proof of Theorem 4.4. (i)$\Leftrightarrow$(iii) follows from Theorem 1.22 and [HM, Theorem 3.1].

(iii)$\Leftrightarrow$(iv) is Theorem 3.3 of [CHT].

(ii)$\Leftrightarrow$(iv) is Theorem 12.2 of [GJ].

(iv)$\Leftrightarrow$(v) follows from [HR, 24.15 and 24.47] and [GJ, 12.5]. ∎

Even if the following proposition is proved in [F, Re, V2], we give here its simple proof using Glicksberg's theorem [Gl] in order to make the paper more self-contained.

PROPOSITION 4.5. *Let G be a locally compact Abelian group. Then G^+ and G have the same convergent to zero sequences. In particular, the identity map* $\mathrm{id} : G^+ \to G$ *is sequentially continuous.*

Proof. Let a sequence $\mathbf{u} = \{u_n\}$ converge to zero in G^+. Then the set $K = \mathbf{u} \cup \{0\}$ is compact in G^+. By Glicksberg's theorem [Gl], K is compact in G. Since only zero is a non-isolated point in K, $\mathbf{u}$ converges to zero also in G. ∎

PROPOSITION 4.6. *Let G be a locally compact non-compact Abelian group. Then G^+ is not an s-group.*

Proof. Suppose for a contradiction that G^+ is an s-group. Then, by Proposition 4.5 and Theorem 1.12, the identity map $\mathrm{id} : G^+ \to G$ is continuous. Hence G is bounded, a contradiction. ∎

Proof of Theorem 1.24. (i) Taking into consideration [GJ, 12.5], the group G^+ is realcompact by [CHT, Theorem 3.8]. Let us show that G^+ is a bs-group. By Theorem 1.22, it is enough to show that every sequentially continuous character of G^+ is continuous. Let χ be a sequentially continuous character of G^+. Clearly, χ is also a sequentially continuous character of G. It immediately follows from Varopoulos [V1] that χ is a continuous character of G, i.e., $\chi \in G^\wedge$. Thus χ is a continuous character of $(G, T_{G^\wedge}) = G^+$ as well.

(ii) follows from Proposition 4.6.

(iii) follows from Example 6.1 of [CDM]. ∎

Note that the local compactness in Theorem 1.24 is important. Indeed, for $(\mathbb{Z}, \tau_{\mathbf{u}})$ from Example 3.4 we have the following: (1) $(\mathbb{Z}, \tau_{\mathbf{u}})$ is an s-group that is not locally precompact by [PZ, 2.3.12]; (2) $(\mathbb{Z}, \tau_{\mathbf{u}}^+) = (\mathbb{Z}, T_{\mathbb{Z}(p^\infty)})$ is a metrizable bs-group by Corollary 3.6 and hence it is also an s-group.

Recall that a topological group G is called a *k-group* if every homomorphism from G to a topological group H which is continuous on every compact subset of G is continuous [Nob2]. The next proposition is of independent interest:

Proposition 4.7.

(i) *Every s-group is a k-group.*
(ii) *If G is an Abelian MAP s-group, then the natural homomorphism $\alpha_G : G \to G^{\wedge\wedge}$ is a continuous monomorphism.*

Proof. (i) Let G be an s-group. Since every convergent sequence with its limit point is compact in G, the group G is a k-group by the definition of k-groups and Theorem 1.12.

(ii) immediately follows from Theorem 2.3 of [Nob2] and item (i). ∎

Proof of Theorem 1.25. Set $X = G^{\wedge}$. Assuming that $|X|$ is not Ulam measurable, by Theorem 1.24 we deduce that $X^+ = (X, T_G)$ is a bs-group. Hence G is $\mathfrak{g}$-closed in $(X_d)^{\wedge} = bG$ by Theorem 1.22. Thus it only remains to show that $|X|$ is not Ulam measurable.

Denote by $\mathfrak{m}$ the least Ulam measurable cardinal. It is known that $\mathfrak{m}$ is strongly inaccessible (i.e., regular and a strong limit cardinal, which means that $2^{\lambda} < \mathfrak{m}$ whenever $\lambda < \mathfrak{m}$). By hypothesis, $|G|$ is not Ulam measurable, hence $|G| < \mathfrak{m}$ and $2^{|G|} < \mathfrak{m}$. To prove that $|X|$ is not Ulam measurable it suffices to note that $w(X) = w(G) \le |G|$ (see [HR, 24.14]), so that $|X| \le 2^{w(X)} = 2^{w(G)} < \mathfrak{m}$. ∎

In general, from the $\mathfrak{g}$-closedness of a locally compact non-compact Abelian group G in bG it does not follow that $|G|$ is not Ulam measurable. That is, the condition of Ulam non-measurability of $|G|$ in Theorem 1.25 is essential (we do not discuss the existence of Ulam measurable cardinals). To show this we need the next lemma.

Lemma 4.8. *Let $\{G_\alpha\}_{\alpha\in I}$ be a family of Abelian topological groups and, for every $\alpha \in I$, H_α be a $\mathfrak{g}$-closed subgroup of G_α. Then $H := \prod_{\alpha\in I} H_\alpha$ is $\mathfrak{g}$-closed in $G := \prod_{\alpha\in I} G_\alpha$.*

Proof. Let $H_\alpha = \bigcap_{\mathbf{u}\in J_\alpha} s_{\mathbf{u}}(G_\alpha)$, where J_α is a set of sequences in the dual of G_α. Naturally considering every sequence $\mathbf{u} \in J_\alpha$ as a sequence in $G^{\wedge} = \bigoplus_{\alpha\in I} G_\alpha^{\wedge}$ we set $J = \bigcup_{\alpha\in I} J_\alpha$. Then

$$\bigcap_{\mathbf{u}\in J} s_{\mathbf{u}}(G) = \bigcap_{\alpha\in I}\Big(\bigcap_{\mathbf{u}\in J_\alpha} s_{\mathbf{u}}(G)\Big) = \bigcap_{\alpha\in I}\Big(H_\alpha \times \prod_{\beta\in I\setminus\{\alpha\}} G_\beta\Big) = H. \ \blacksquare$$

Example 4.9. Let us show that there exists a locally compact non-compact Abelian group G that is $\mathfrak{g}$-closed in bG but $|G|$ is Ulam measurable. Indeed, choose a compact Abelian group G_1 such that $|G_1|$ is Ulam measurable. And let G_2 be a discrete countably infinite Abelian group. By Lemma 4.8 and Theorem 1.25, the locally compact non-compact group $G := G_1 \times G_2$ is $\mathfrak{g}$-closed in $bG = bG_1 \times bG_2 = G_1 \times bG_2$, but $|G|$ is Ulam measurable.

5. The countable case

Proof of Theorem 1.10. Let S be a T_s-set of sequences in G. Then, by definition, $\tau_S = \inf_{\mathbf{u}\in S}\tau_{\mathbf{u}}$ is Hausdorff and $\tau_S \subseteq \tau_{S_i}$ for every $i \in I$. Thus, by the definition of inf, $\tau_S \subseteq \inf_{i\in I}\tau_{S_i}$. So $\inf_{i\in I}\tau_{S_i}$ is Hausdorff.

Conversely, let $\inf_{i\in I}\tau_{S_i}$ be Hausdorff. Then, by definition, every $\mathbf{u} \in S_i \subseteq S$ converges to the unit in τ_{S_i} and hence in $\inf_{i\in I}\tau_{S_i}$ as well. Thus, $\inf_{i\in I}\tau_{S_i} \subseteq \tau_S$ by the definition of τ_S. Hence τ_S is Hausdorff. Therefore, if τ_S and $\inf_{i\in I}\tau_{S_i}$ are Hausdorff, we have $\tau_S = \inf_{\mathbf{u}\in S}\tau_{\mathbf{u}} = \inf_{i\in I}\tau_{S_i}$. ■

As an immediate corollary of Proposition 7 of [CD] and Theorem 1.10 we obtain:

PROPOSITION 5.1. *Let $S = \{S_n\}_{n\in\omega} \in \mathcal{TS}(G)$ for an Abelian group G. Then the family $\mathcal{U}$ of all sets of the form*

$$\sum_n W_n = \bigcup_{n\in\omega}(W_0 + W_1 + \cdots + W_n), \qquad \text{where } 0 \in W_n \in \tau_{S_n},$$

forms an open basis at 0 of τ_S.

Let us recall that $\mathcal{T}_r$, $\mathcal{T}_a$ and $\mathcal{T}_f$ denote the rectangular, the asterisk and the coproduct group topologies respectively on the direct sum of Abelian topological groups. We will use the following important facts:

PROPOSITION 5.2.

- (i) ([CD, Proposition 11]) *Let $\{(G_n,\tau_n)\}_{n\in\omega}$ be a countable family of Abelian topological groups. Then $\mathcal{T}_r = \mathcal{T}_a = \mathcal{T}_f$ on $\bigoplus_{n\in\omega} G_n$.*
- (ii) ([CD, Corollary 22]) *Let $\{(G_i,\tau_i)\}_{i\in I}$ be a family of locally quasi-convex Abelian groups. Then the asterisk topology on $\bigoplus_{i\in I} G_i$ is the finest locally quasi-convex topology on $\bigoplus_{i\in I} G_i$ which is coarser than the coproduct topology $\mathcal{T}_f$.*
- (iii) ([Nic, Theorem 4.3]) *Let $\{(G_i,\tau_i)\}_{i\in I}$ be a family of reflexive Abelian topological groups. Then*

$$\Big(\bigoplus_{i\in I} G_i, \mathcal{T}_f\Big)^\wedge = \Big(\bigoplus_{i\in I} G_i, \mathcal{T}_a\Big)^\wedge = \prod_{i\in I}(G_i,\tau_i)^\wedge.$$

The next corollary immediately follows from Propositions 5.2 and 1.17.

COROLLARY 5.3. *Let $\{(G_n,\tau_{S_n})\}_{n\in\omega}$ be a countable family of Abelian s-groups. Set $G = \bigoplus_{n\in\omega} G_n$ and $S = \bigcup_{n\in\omega} j_n(S_n)$. Then $\tau_S = \mathcal{T}_a$ on $\bigoplus_{n\in\omega} G_n$.*

Let us note that Higgins [Hig] was the first who proved that in general the s-topology on an *uncountable* coproduct of s-groups may be strictly finer than the asterisk topology $\mathcal{T}_a$ (see also [CD, Nic]). For example, the s-topology of each uncountable coproducts of the reals $\mathbb{R}$ is strictly finer than the asterisk topology (see a simple right proof of this fact in [Pr]).

Taking into account that the group $(\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})$ is reflexive and $(\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})^\wedge = \mathbb{T}_0^{\mathcal{H}}$ (see [G1]), as a corollary of Propositions 5.2 and 1.17 and Theorem 1.18 we obtain:

Corollary 5.4. *The dual group of an Abelian s-group imbeds into some direct product of the group* $\mathbb{T}_0^{\mathcal{H}}$.

To prove Theorem 1.26, we need the following proposition:

Proposition 5.5. *Let* $\{G_n\}_{n\in\omega}$ *be a sequence of Abelian groups and let* $\mathbf{u}_n$ *be a* T*-sequence in* G_n *for every* $n \in \omega$. *Set* $G = \bigoplus_{n\in\omega} G_n$ *and* $S = \{j_n(\mathbf{u}_n)\}_{n\in\omega}$. *Then* (G, τ_S) *is a complete sequential group,* $\tau_S = \mathcal{T}_a$ *and*

$$(G, \tau_S)^\wedge = \prod_{n\in\omega} (G_n, \tau_{\mathbf{u}_n})^\wedge.$$

Moreover, if all G_n *are countably infinite, then* $(G, \tau_S)^\wedge$ *is a Polish group.*

Proof. (G, τ_S) is a complete sequential group by [G4, Theorem 5.2]. By Corollary 5.3, $\tau_S = \mathcal{T}_a$. Hence $(G, \tau_S)^\wedge = \prod_{n\in\omega} (G_n, \tau_{\mathbf{u}_n})^\wedge$ by [K]. If all G_n are countably infinite, then, by Theorem 1.4, all $(G_n, \tau_{\mathbf{u}_n})^\wedge$ are Polish. Hence $(G, \tau_S)^\wedge$ is a Polish group as well. ∎

Proof of Theorem 1.26. Set $G' = \bigoplus_{n\in\omega} G_n$, where $G_n = G$ for every $n \in \omega$, and $S' = \{j_n(\mathbf{u}_n)\}_{n\in\omega}$. Then, by Proposition 5.5,

$$(G', \tau_{S'})^\wedge = \prod_{n\in\omega} (G, \tau_{\mathbf{u}_n})^\wedge$$

is a Polish group.

Set $p : (G', \tau_{S'}) \to (G, \tau_S)$, $p((g_n)) = \sum_n g_n$. Since $p(j_n(\mathbf{u}_n)) = \mathbf{u}_n$ converges to zero in (G, τ_S), p is continuous by Theorem 1.12. Set $H = \ker p$. Since $p(S') = S$, by Theorem 1.13, $(G, \tau_S) \cong (G', \tau_{S'})/H$. Then the conjugate homomorphism $p^\wedge$ is a continuous isomorphism from $(G, \tau_S)^\wedge$ onto the annihilator $H^\perp$ of H in $(G', \tau_{S'})^\wedge$. Since the subgroup $\langle \mathbf{u}_0, \mathbf{u}_1, \dots \rangle$ is open in τ_G, by [G4, Theorem 5.2], every compact subset of (G, τ_S) is contained in a compact subset K_n of the form

$$K_n := \Big[\bigcup_{i=0}^{n} (\mathbf{u}_i \cup (-\mathbf{u}_i))\Big] + \dots + \Big[\bigcup_{i=0}^{n} (\mathbf{u}_i \cup (-\mathbf{u}_i))\Big] + \{g_i\}_{i=1}^t$$

with $n+1$ summands in square brackets and a finite $\{g_i\}_{i=1}^t \subset G$. It is clear that a subset K'_n of G' of the form

$$K'_n := \Big[\bigcup_{i=0}^{n} (j_i(\mathbf{u}_i) \cup (-j_i(\mathbf{u}_i)))\Big] + \dots + \Big[\bigcup_{i=0}^{n} (j_i(\mathbf{u}_i) \cup (-j_i(\mathbf{u}_i)))\Big] + \{j_0(g_i)\}_{i=1}^t$$

with $n+2$ summands is compact. Since $p(K'_n) = K_n$ and p is onto and continuous, p is compact-covering. Thus, by [Auß, Lemma 5.17], $p^\wedge$ is an embedding of $(G,\tau_S)^\wedge$ into the Polish group $(G',\tau_{S'})^\wedge$. So $(G,\tau_S)^\wedge \cong H^\perp$ is a Polish group.

REMARK 5.6. Let K be an uncountable Kronecker subset of the torus $\mathbb{T}$ and H a subgroup of $\mathbb{T}$ generated by K. Then H is $\mathfrak{g}$-closed by [Bir, §5]. So, by Theorem 1.21, H is (algebraically) the dual group of $(\mathbb{Z},\tau_S)$ for some MAP s-topology τ_S on $\mathbb{Z}$ generated by a T_s-set S of sequences. On the other hand, since H does not admit any Polish group topology [G2, Theorem 2], by Theorem 1.26, the set S must be uncountable, i.e., $r_s(\mathbb{Z},\tau_S) \ge \omega_1$. It is interesting to know the explicit value of this number. Also we do not know whether the group $(\mathbb{Z},\tau_S)$ is sequential.

6. Applications. Let G be a MAP Abelian topological group, $X = \widehat{G}$ and α be the natural homomorphism from G into $G^{\wedge\wedge}$. Since G is MAP, α is injective. The weak and weak* group topologies on X are denoted by σ_w and σ_{w^*} respectively, i.e., $\sigma_w = \sigma(X, G^{\wedge\wedge})$ and $\sigma_{w^*} = \sigma(X,G)$. Then $\sigma_{w^*} \subseteq \sigma_w \subseteq \sigma_{\mathrm{co}}$. Let $t: X_d \to (X,\sigma_{\mathrm{co}})$ $(= G^\wedge)$, $t(x) = x$, be the identity map and $\mathfrak{b} := t^\wedge$ be its conjugate continuous monomorphism. Set $bG := X_d^\wedge$. It is well-known that the compact group bG with the continuous monomorphism $\mathfrak{b}\circ\alpha$ is the Bohr compactification of G (although α need not be continuous, $\mathfrak{b}\circ\alpha$ is always continuous since $(\mathfrak{b}\circ\alpha(g),x) = (\alpha(g),t(x)) = (x,g)$ for every $g\in G$ and $x \in X_d$). We shall algebraically identify G and $G^{\wedge\wedge}$ with their images $\mathfrak{b}\circ\alpha(G)$ and $\mathfrak{b}(G^{\wedge\wedge})$ respectively saying that they are subgroups of bG. It is clear that

$$\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) \subseteq \mathfrak{g}_{bG}(\mathfrak{b}(G^{\wedge\wedge})). \tag{6.1}$$

If α is surjective, the reverse inclusion is trivially satisfied. In the general case we prove the following:

PROPOSITION 6.1. *Let G be a MAP Abelian topological group and $X = \widehat{G}$. The following statements are equivalent:*

(i) $\mathbf{s}(X,\sigma_w) = \mathbf{s}(X,\sigma_{w^*})$;
(ii) $\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) = \mathfrak{g}_{bG}(\mathfrak{b}(G^{\wedge\wedge}))$.

Proof. (i)⇒(ii). By (6.1), we have to show $\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) \supseteq \mathfrak{g}_{bG}(\mathfrak{b}(G^{\wedge\wedge}))$. Let $\mathbf{u} = \{u_n\}_{n\in\omega} \subset X$ be such that $\mathfrak{b}\circ\alpha(G) \subseteq s_{\mathbf{u}}(bG)$. This means that $(\mathfrak{b}\circ\alpha(g),u_n) = (u_n,g) \to 1$ for every $g\in G$, i.e., $\mathbf{u}\in S(X,\sigma_{w^*})$. By hypothesis, $\mathbf{u}\in S(X,\sigma_w)$ too. Hence

$$(\mathfrak{b}(\chi),u_n) = (\chi,u_n) \to 1 \quad \text{for every } \chi\in G^{\wedge\wedge},$$

i.e., $\mathfrak{b}(G^{\wedge\wedge}) \subseteq s_{\mathbf{u}}(bG)$. So $\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) \supseteq \mathfrak{g}_{bG}(\mathfrak{b}(G^{\wedge\wedge}))$.

(ii)⇒(i). Since $\sigma_{w^*} \subseteq \sigma_w$, we have only to show that if $\mathbf{u} = \{u_n\}_{n\in\omega} \in S(X,\sigma_{w^*})$, then also $\mathbf{u}\in S(X,\sigma_w)$. Assuming the converse we can find

$\chi \in G^{\wedge\wedge}$ such that

$$(\chi, u_n) \not\to 1 \quad \text{as } n \to \infty.$$

Then $\mathfrak{b}(\chi) \notin s_{\mathbf{u}}(bG)$. Thus $\mathfrak{b}(\chi) \notin \mathfrak{g}_{bG}(\mathfrak{b} \circ \alpha(G))$. This is a contradiction. ∎

Since $G^{\wedge}$ is MAP, Theorem 1.21 and Corollary 3.3 imply:

Corollary 6.2. *Let an Abelian topological group (G, τ) be such that $G^{\wedge}$ is an s-group. Then $G^{\wedge\wedge}$ is a dense $\mathfrak{g}$-closed subgroup of bG.*

Proof of Theorem 1.27. By [CMT, Theorem 2.4], the dual group $G^{\wedge}$ is sequential. By [G4, Theorem 1.14], $G^{\wedge}$ is an s-group. Hence, by Corollary 6.2, $\mathfrak{b} \circ \alpha(G) = \mathfrak{b}(G^{\wedge\wedge})$ is a $\mathfrak{g}$-closed subgroup of bG. ∎

Proof of Theorem 1.28. Let H be characterizable and p be a continuous monomorphism from H into a compact metrizable group X with dense image such that $p(H) = s_{\mathbf{u}}(X)$ for some sequence $\mathbf{u}$ in $\widehat{X}$. By Corollary 3.3, $\mathbf{u}$ is a TB-sequence. Since $\langle \mathbf{u} \rangle$ is open in $(\widehat{X}, \tau_{\mathbf{u}})$, the subgroup $K := \langle \mathbf{u} \rangle^{\perp}$ is compact in $(\widehat{X}, \tau_{\mathbf{u}})^{\wedge}$ and $\langle \mathbf{u} \rangle^{\wedge} \cong (\widehat{X}, \tau_{\mathbf{u}})^{\wedge}/K$ by [BCM, Lemma 2.2]. By Proposition 1.3 and the uniqueness of the Polish group topology, $H \cong (\widehat{X}, \tau_{\mathbf{u}})^{\wedge}$. Thus, by Theorem 1.15, H/K embeds into $\mathbb{T}_0^{\mathcal{H}}$. ∎

Remark 6.3. We do not know whether the image of H/K in $\mathbb{T}_0^{\mathcal{H}}$ is dually closed (as in Theorem 1.2), but it may not be dually embedded. Indeed, by [G3], $\mathbb{R}$ is characterizable and hence it embeds into $\mathbb{T}_0^{\mathcal{H}}$. Since $\mathbb{R}^{\wedge} = \mathbb{R}$ is uncountable and $(\mathbb{T}_0^{\mathcal{H}})^{\wedge} = \mathbb{Z}_0^{\mathbb{N}}$ is countable, $\mathbb{R}$ is not dually embedded in $\mathbb{T}_0^{\mathcal{H}}$.

Proof of Proposition 1.30. Let H be a Schwartz group. Choose a compact subgroup K of H such that the quotient group H/K embeds into $\mathbb{T}_0^{\mathcal{H}}$ (Theorem 1.28). In particular, H/K is locally quasi-convex. By [ACDT, Proposition 3.6], H/K is a Schwartz Polish locally quasi-convex group. By [CDT, Theorem 15], the embedding $H/K \to \mathbb{T}_0^{\mathcal{H}}$ is compact. Thus H/K is locally compact. So H is a locally compact Polish group.

The converse assertion is trivial. ∎

Proof of Theorem 1.31. Assume that $G^{\wedge}$ is characterizable. Since $G^{\wedge}$ is a Schwartz group [ACDT, Corollary 5.6], by Proposition 1.30, $G^{\wedge}$ is locally compact. So $G^{\wedge}$ and $G^{\wedge\wedge}$ are reflexive. As was shown before the section "The proofs" in [G2], G is also reflexive. So $G = G^{\wedge\wedge}$ is locally compact.

Conversely, if G is locally compact, then the locally compact group $G^{\wedge}$ is characterizable by [G3, Theorem 2.7]. ∎

Proof of Proposition 1.32. For every X_n there is a countably infinite Abelian group G_n and a TB-sequence $\mathbf{u}_n$ in G_n such that $(G_n, \tau_{\mathbf{u}_n})^{\wedge} \cong X_n$ (see [G3]). Set $G = \bigoplus_{n \in \omega} G_n$ and $S = \{j_n(\mathbf{u}_n)\}_{n \in \omega}$. Then the proposition follows from Proposition 5.5. ∎

Proof of Proposition 1.33. By [G3, Proposition 2.9], there is a *TB*-sequence $\mathbf{u}$ in $\mathbb{Z}^2$ such that $(\mathbb{Z}^2, \tau_{\mathbf{u}})^\wedge \cong \mathbb{R}$. Since $\bigoplus_{n\in\omega} \mathbb{Z}^2 \cong \mathbb{Z}_0^{\mathbb{N}}$, the assertion follows from Proposition 5.5. ∎

7. Open questions. We start with a question which is related to Theorem 3.10:

PROBLEM 7.1. *Let X be a compact Abelian group and H be a $\mathfrak{g}$-closed non-dense subgroup of X. Is there $S \in \mathcal{TS}(\widehat{X})$ such that $i_S^\wedge((\widehat{X}, \tau_S)^\wedge) = H$?*

As already noted, if G is a *separable* metrizable Abelian group, then the dual group $G^\wedge$ is sequential by [CMT, Theorem 1.7]. Hence $G^\wedge$ is an s-group. The following questions are open:

PROBLEM 7.2. *Let an Abelian topological group G have one of the properties: non-separable metrizable, Fréchet–Urysohn, sequential or an s-group. When is $G^\wedge$ an s-group?*

PROBLEM 7.3. *Let an Abelian topological group G be such that G and $G^\wedge$ have one of the properties: metrizable, Fréchet–Urysohn, sequential or an s-group. What more can we say about G and $G^\wedge$?*

For example, if G is metrizable and $G^\wedge$ is Fréchet–Urysohn, then $G^\wedge$ is a locally compact metrizable group by [CMT, Theorem 2.2].

Let G be an Abelian group and $S \in \mathcal{TBS}(G)$. Theorem 1.20 gives a complete description of the topology τ_{bS} on G. On the other hand, we do not know any description of the topology on the dual group.

PROBLEM 7.4. *Describe the topology of $(G, \tau_{bS})^\wedge$.*

By Corollary 3.5, $(G, \tau_{bS})^\wedge = (G, \tau_S)^\wedge$ algebraically. It is natural to ask:

PROBLEM 7.5. *When are the groups $(G, \tau_{bS})^\wedge$ and $(G, \tau_S)^\wedge$ topologically isomorphic? In particular, when $(G, \tau_{\mathbf{u}})^\wedge \cong (G, \tau_{b\mathbf{u}})^\wedge$?*

Let G be a countably infinite Abelian group and $S \in \mathcal{TBS}(G)$. By Corollary 3.8, if $(G, \tau_{bS})^\wedge$ is countable, then (G, τ_{bS}) is not reflexive.

PROBLEM 7.6. *Is there an $S \in \mathcal{TBS}(G)$ for a* countable *Abelian group G of infinite exponent such that (G, τ_{bS}) is reflexive? Does there exist a TB-sequence $\mathbf{u}$ in G such that $(G, \tau_{b\mathbf{u}})$ is reflexive?*

Note that the positive answer to this question will give the positive answer to the following general problem posed to the author by Professor M. G. Tkachenko at the 2009 conference in Eilat:

PROBLEM 7.7 ([BT, Problem 5.3]). *Is there a reflexive precompact group topology on a* countable *Abelian group of infinite exponent (for example, on $\mathbb{Z}$)?*

In the next remark we comment on Problems 7.6 and 7.7.

Remark 7.8. (i) Let G be a countable Abelian group of *finite* exponent. Then a reflexive group topology on G must be discrete [AG]. Thus, in Problems 7.6 and 7.7 the assumption that G has infinite exponent is essential.

(ii) If there exists a reflexive precompact group topology τ on a countable Abelian group G of infinite exponent, then (G,τ) contains a non-trivial convergent sequence. (Indeed, otherwise, (G,τ), being countable, does not contain infinite compact subsets. Hence $(G,\tau)^\wedge$ is precompact with countable dual group. So $(G,\tau)^\wedge$ is metrizable and precompact. Thus $(G,\tau)^{\wedge\wedge} = (G,\tau)$ is discrete by [Auß, Cha]. But since G has infinite exponent, it is countably infinite. Hence the precompact group topology τ cannot be discrete. This contradiction shows that (G,τ) contains a non-trivial convergent sequence.) Therefore the case of a single *TB*-sequence in Problem 7.6 is of independent interest.

(iii) If there exists a *TB*-sequence $\mathbf{u}$ in a countable Abelian group G of infinite exponent such that $(G,\tau_{b\mathbf{u}})$ is reflexive, then $\widehat{(G,\tau_{b\mathbf{u}})} = s_{\mathbf{u}}(G_d^\wedge)$ has size continuum. (It is well-known that a Polish space is either countable or has size continuum. Now the assertion follows from Proposition 1.3 and Corollary 3.8.)

By Proposition 4.2(a), for every MAP Abelian s-group (G,τ) the group (G,τ^+) is a bs-group. If, in addition, (G,τ) is locally compact non-compact, then (G,τ^+) is not an s-group by Theorem 1.24.

Problem 7.9. *Characterize those MAP Abelian s-groups (G,τ) for which (G,τ^+) is an s-group as well.*

Taking into account Theorems 1.25 and 1.27, one can ask:

Problem 7.10. *Which MAP Abelian groups are $\mathfrak{g}$-closed in their Bohr compactification?*

Taking into consideration Theorem 1.28, one can ask whether the converse is true:

Problem 7.11. *Let a Polish group H and its compact subgroup K be such that H/K embeds into $\mathbb{T}_0^{\mathcal{H}}$. When is H characterizable? Is there a closed subgroup of $\mathbb{T}_0^{\mathcal{H}}$ that is not characterizable?*

It would be of interest to know an answer to the following question:

Problem 7.12. *Which infinite-dimensional Banach spaces embed into $\mathbb{T}_0^{\mathcal{H}}$? In particular, does c_0 or ℓ_p, $p \geq 1$, embed into $\mathbb{T}_0^{\mathcal{H}}$?*

Under the assumption that one of the spaces in Problem 7.12 does not embed into $\mathbb{T}_0^{\mathcal{H}}$, by Corollary 1.29, we find that this space is not characterizable and hence the answer to Problem 2.10 of [G3] is negative.

Since $\mathbb{T}_0^{\mathcal{H}} \cong c_0/\mathbb{Z}_0^{\mathbb{N}}$ (see [G1]), the next problem is:

Problem 7.13. *Describe all closed subgroups of* $\mathbb{T}_0^{\mathcal{H}}$ *and* c_0.

It is of interest to study hereditary properties of bs-groups.

Problem 7.14. *Let* (G,τ) *be a bs-group. Which subgroups of* (G,τ) *are bs-group as well?*

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