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Metrization criteria for compact groups in terms of their dense subgroups

by

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Dedicated to Professor A. V. Arhangel'skiĭ
on the occasion of his 75th anniversary

Abstract. According to Comfort, Raczkowski and Trigos-Arrieta, a dense subgroup D of a compact abelian group G determines G if the restriction homomorphism $\widehat{G} \to \widehat{D}$ of the dual groups is a topological isomorphism. We introduce four conditions on D that are necessary for it to determine G and we resolve the following question: If one of these conditions holds for every dense (or G_δ-dense) subgroup D of G, must G be metrizable? In particular, we prove (in ZFC) that a compact abelian group determined by all its G_δ-dense subgroups is metrizable, thereby resolving a question of Hernández, Macario and Trigos-Arrieta. (Under the additional assumption of the Continuum Hypothesis CH, the same statement was proved recently by Bruguera, Chasco, Domínguez, Tkachenko and Trigos-Arrieta.) As a tool, we develop a machinery for building G_δ-dense subgroups without uncountable compact subsets in compact groups of weight ω_1 (in ZFC). The construction is delicate, as these subgroups must have non-trivial convergent sequences in some models of ZFC.

All spaces and topological groups are assumed to be Hausdorff. Recall that a topological space X is called:

- κ*-bounded* (for a given cardinal κ) if the closure of every subset of X of cardinality at most κ is compact,
- *countably compact* if every countable open cover of X has a finite subcover,
- *pseudocompact* if every real-valued continuous function defined on X is bounded.

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It is well known that compact $\to$ κ-bounded $\to$ ω-bounded $\to$ countably compact $\to$ pseudocompact, for every infinite cardinal κ.

The symbols $w(X)$, $nw(X)$, $\chi(X)$ and $d(X)$ denote the weight, the network weight, the character and the density of a space X, respectively. All undefined topological terms can be found in [21].

As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{P}$ denotes the set of all prime numbers, $\mathbb{Z}$ denotes the group of integers, $\mathbb{Z}(p) = \mathbb{Z}/p\mathbb{Z}$ denotes the cyclic group of order $p \in \mathbb{P}$ with the discrete topology, and $\mathbb{T}$ denotes the circle group with its usual topology. The symbol $\mathfrak{c}$ denotes the cardinality of the continuum, ω_1 denotes the first uncountable cardinal and $\omega = |\mathbb{N}|$. Clearly, $\omega < \omega_1$. By Cantor's theorem, $\omega_1 \leq \mathfrak{c}$. The Continuum Hypothesis CH says that $\omega_1 = \mathfrak{c}$. We recall that this equality is both consistent with and independent of the usual Zermelo–Fraenkel axioms ZFC of set theory [29].

Recall that a cardinal τ is *strong limit* if $2^\sigma < \tau$ for every cardinal $\sigma < \tau$. For an ordinal (in particular, for a cardinal) α, we denote by $\mathrm{cf}(\alpha)$ the cofinality of α. For a cardinal κ and a set X, the symbol $[X]^{\leq\kappa}$ denotes the family of all subsets of X having cardinality at most κ. All undefined set-theoretic terms can be found in [29].

When topological groups G and H are topologically isomorphic, we denote this fact by $G \cong H$.

1. Introduction. Let G be an abelian topological group. We denote by $\widehat{G}$ the dual group of all continuous characters endowed with the compact-open topology. Following [9, 10], we say that a dense subgroup D of G *determines* G if the restriction homomorphism $\widehat{G} \to \widehat{D}$ of the dual groups is a topological isomorphism. According to [9, 10], G is said to be *determined* if every dense subgroup of G determines G. The cornerstone in this topic is the following theorem due to Chasco and Außenhofer:

THEOREM 1.1 ([2, 7]). *Every metrizable abelian group is determined.*

A remarkable partial inverse of this theorem was proved by Hernández, Macario and Trigos-Arrieta. (Under the assumption of the Continuum Hypothesis, it was established earlier by Comfort, Raczkowski and Trigos-Arrieta in [9, 10]).

THEOREM 1.2 ([24, Corollary 5.11]). *Every compact determined abelian group is metrizable.*

While Theorem 1.1 says that *every* dense subgroup of a metrizable abelian group determines it, Theorem 1.2 asserts that every non-metrizable compact abelian group necessarily contains *some* dense subgroup that does not determine it.

A subgroup D of a topological group G is called G_δ*-dense* in G if $D \cap B \neq \emptyset$ for every non-empty G_δ-subset B of G [12]. The following classical result is due to Comfort and Ross [12]:

THEOREM 1.3. *A dense subgroup D of a compact group G is pseudocompact if and only if D is G_δ-dense in G.*

The following question was asked by Hernández, Macario and Trigos-Arrieta in [24, Question 5.12(iii)]:

QUESTION 1.4. *Does there exist* (*in ZFC*) *a non-metrizable compact abelian group G such that every G_δ-dense subgroup D of G determines G?*

This question was also repeated in [14, Question 4.12].

It is useful to state explicitly the negation of the statement in Question 1.4:

QUESTION 1.5. *Let G be a compact abelian group such that every G_δ-dense subgroup of G determines G. Must G be metrizable* (*in ZFC*)?

By Theorem 1.3, one can replace "G_δ-dense" by "dense pseudocompact" in both questions to get their equivalent versions.

Theorem 1.2 says that a compact abelian group G is metrizable provided that every dense subgroup of G determines it. Since G_δ-dense subgroups of G are dense in G, a positive answer to Question 1.5 (equivalently, a negative answer to Question 1.4) would provide a strengthening of Theorem 1.2, because one would get the same conclusion under a weaker assumption, requiring only a much smaller family of G_δ-dense subgroups of G to determine it. One of the goals of this paper is to accomplish precisely this, without recourse to any additional set-theoretic assumptions beyond Zermelo–Fraenkel axioms ZFC of set theory.

REMARK 1.6. Chasco, Domínguez and Trigos-Arrieta proved that every compact abelian group G with $w(G) \geq \mathfrak{c}$ has a G_δ-dense subgroup which does not determine G (see [8, Theorem 14]). Independently, Bruguera and Tkachenko proved that every compact abelian group G with $w(G) \geq \mathfrak{c}$ contains a proper G_δ-dense reflexive subgroup D ([6, Theorem 4.9]). As mentioned at the end of [8, Section 3], this D cannot determine G. (Indeed, $\widehat{\widehat{D}} = D \neq G = \widehat{\widehat{G}}$ implies $\widehat{D} \neq \widehat{G}$.) It is clear that, *under the assumption of the Continuum Hypothesis, these results yield a consistent positive answer to Question* 1.5, *and therefore a consistent negative answer to Question* 1.4.

An overview of the paper follows. Inspired by Questions 1.4 and 1.5, in Section 2 we introduce four properties that every dense subgroup determining a compact abelian group must have (see Diagram 1), thereby making a first attempt to clarify the "fine structure" of the notion of determination.

Section 3 collects some basic facts about the properties introduced that help the reader better understand these new notions.

In Section 4, we investigate what happens to a compact group when all its dense (or all its G_δ-dense) subgroups are assumed to have one of the four properties introduced in Section 2. Our results in Section 4 substantially clarify the "fine structure" of the notion of a determined group by addressing the following question systematically: "How much determination" of a compact group is really necessary in order to get its metrization in the spirit of Theorem 1.2 or Question 1.5? It turns out that such metrization criteria can be obtained under much weaker conditions than full determination; see Theorems 4.2 and 4.5. In turn, Theorems 4.1, 4.3 and 4.4 serve to demonstrate that the conditions equivalent to the metrization of a compact group in Theorems 4.2 and 4.5 are the best possible, thereby pinpointing the exact property among the four necessary conditions "responsible" for both the validity of Theorem 1.2 and the positive answer to Question 1.5. The answer to Question 1.5 itself comes as a particular corollary of the main result; see Corollary 4.6. An added bonus of our approach is that many results in this section hold for non-abelian compact groups as well, whereas the notion of a determined group is restricted to the abelian case. (A non-commutative version of a determined group was introduced recently in [22].)

In Section 5 we develop a machinery for constructing G_δ-dense subgroups D without uncountable compact subsets in compact groups G of weight ω_1. Furthermore, when G belongs to a fixed variety $\mathcal{V}$ of groups, the subgroup D can be chosen to be a free group in the variety $\mathcal{V}$. Our machinery works in ZFC alone. As in Section 4, results in this section do not require G to be abelian. The primary novelty here is our ability to successfully handle small weights of G (like ω_1) at the expense of "killing" only *uncountable* compact subsets of D. Known constructions in the literature usually "kill" all *infinite* compact subsets, thereby eliminating also all non-trivial convergent sequences in D, but this stronger conclusion is accomplished at the expense of having been able to handle only groups G of weight $\mathfrak{c}$. In fact, this difference is inherent in the nature of the problem and not purely coincidental. Indeed, Remark 5.6 shows that the group D we construct must have non-trivial convergent sequences under some additional set-theoretic assumptions. As a particular corollary of our results, we produce a pseudocompact group topology on the free group $F_\mathfrak{c}$ with $\mathfrak{c}$ many generators without uncountable compact subsets. A recent result by Thom [34] implies that such a topology on $F_\mathfrak{c}$ must necessarily contain a non-trivial convergent sequence; see Remark 5.7.

Sections 6, 7 and 8 are devoted to the proofs of all theorems from Sections 4 and 5. Section 9 contains some examples showing the limits of our results, and Section 10 lists open problems related to the topic of this paper.

2. Four necessary conditions for determination of a compact abelian group. In this section we introduce four conditions and show that they are all necessary for determination of a compact abelian group.

DEFINITION 2.1. Let X be a space.

(i) We shall say that X is *w-compact* if there exists a compact subset C of X such that $w(C) = w(X)$.
(ii) We shall say that X has the *Arhangel'skiĭ property* (or is an *Arhangel'skiĭ space*) provided that $w(X) \leq |X|$.

The letter w in front of "compact" in item (i) is intended to abbreviate the word "weight", but one can also view it as an abbreviation of the word "weak", as every compact space is obviously w-compact.

The name for the class of spaces in item (ii) was chosen to pay tribute to the first paper of Professor Arhangel'skiĭ [1] where he introduced the notion of network weight and demonstrated its importance in the study of compact spaces. A celebrated result of Arhangel'skiĭ from [1] says that $w(X) = nw(X) \leq |X|$ for every compact space X. In our terminology, this means that every compact space has the Arhangel'skiĭ property. In fact, a bit more can be said. Indeed, let X be a w-compact space. Then X contains a compact subset C such that $w(C) = w(X)$. Combining this with the above result of Arhangel'skiĭ, we obtain $w(X) = w(C) \leq |C| \leq |X|$. Therefore, X has the Arhangel'skiĭ property. This argument shows that

(α) a w-compact space has the Arhangel'skiĭ property.

DEFINITION 2.2. Let G be a topological group.

(i) We shall say that G is *projectively w-compact* if every continuous homomorphic image of G is w-compact.
(ii) We shall say that G is *projectively Arhangel'skiĭ* if every continuous homomorphic image of G has the Arhangel'skiĭ property.

Since compactness is preserved by continuous images and compact spaces are w-compact, all compact groups are projectively w-compact. From (α) and Definition 2.2(ii) we get

(β) projectively w-compact groups are projectively Arhangel'skiĭ.

The following necessary condition for determination was found by the authors in [16]. Since it plays a crucial role in the present paper, we provide a shorter self-contained proof requiring no recourse to the notion of qc-density that was essential in [16].

THEOREM 2.3 ([16, Corollary 2.4]). *If a subgroup D of an infinite compact abelian group G determines G, then D contains a compact subset X such that $w(X) = w(G)$.*

Proof. Since D determines G and $\widehat{G}$ is discrete, there exists a compact subset X of D and an open neighbourhood V of 0 in $\mathbb{T}$ such that $W(X,V) = \{0\}$, where $W(X,V) = \{\chi \in \widehat{G} : \chi(X) \subseteq V\}$. Let $\pi : \widehat{G} \to C(X,\mathbb{T})$ be the restriction homomorphism defined by $\pi(\chi) = \chi\restriction_X$ for $\chi \in \widehat{G}$, where $C(X,\mathbb{T})$ denotes the group of all continuous functions from X to $\mathbb{T}$ equipped with the compact-open topology. Since $\ker\pi \subseteq W(X,V) = \{0\}$, π is a monomorphism, and so $w(G) = |\widehat{G}| = |H|$, where $H = \pi(\widehat{G})$. Furthermore, the open subset $U = \{f \in C(X,\mathbb{T}) : f(X) \subseteq V\}$ of $C(X,\mathbb{T})$ satisfies $U \cap H = \{0\}$, so H is a discrete subgroup of $C(X,\mathbb{T})$. Therefore, $|H| = w(H) \le w(C(X,\mathbb{T})) = w(X) + \omega$ by [21, Proposition 3.4.16]. This proves that $w(G) \le w(X) + \omega$. To finish the proof of the inequality $w(G) \le w(X)$, it suffices to show that X is infinite. Indeed, assume that X is finite. Then $C(X,\mathbb{T}) = \mathbb{T}^X$ is compact, and so the discrete subgroup H of $C(X,\mathbb{T})$ must be finite. This contradicts the fact that $|H| = |\widehat{G}| \ge \omega$, as G is infinite. Finally, the reverse inequality $w(X) \le w(G)$ is clear. ■

The relevance of the four notions introduced in Definitions 2.1 and 2.2 to the topic of our paper is evident from the following corollary of this theorem.

Corollary 2.4. *If a subgroup D of a compact abelian group determines it, then D is projectively w-compact.*

Proof. Let D be a dense subgroup of a compact abelian group G that determines G, and let $f : D \to N$ be a continuous homomorphism onto some topological group N. Then f can be extended to a continuous group homomorphism from G to the completion $H = \widehat{N}$ of N, and we denote this extension by the same letter f. Since D determines G, the dense subgroup $f(D)$ of the compact group $f(G) = H$ determines H [10, Corollary 3.15]. If H is finite, then $f(D) = H$ is compact, so trivially w-compact. If H is infinite, we apply Theorem 2.3 to conclude that $f(D)$ contains a compact set X with $w(X) = w(H) = w(f(D))$. That is, $f(D)$ is w-compact. This shows that D is projectively w-compact. ■

The relations between the properties introduced above in the class of precompact abelian groups can be summarized in the following diagram:

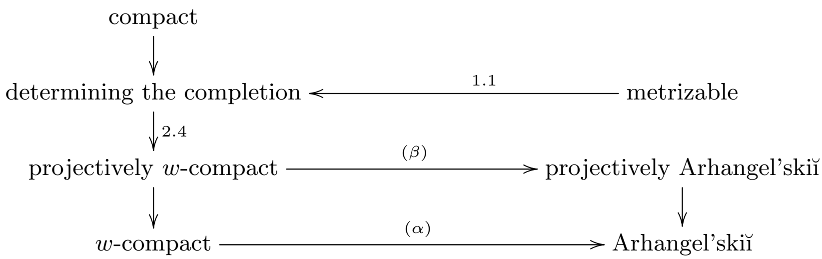

Diagram 1

This diagram shows that the four properties from Definitions 2.1 and 2.2 are necessary for the completion of a precompact abelian group to be determined. With a possible exception of the arrow 2.4, none of the other arrows in Diagram 1 is invertible.

A dense subgroup D of a compact group G that determines G need not be either compact or metrizable. To see this, it suffices to recall that the direct sum $\bigoplus_{\alpha<\omega_1} \mathbb{T}$ of ω_1 copies of $\mathbb{T}$ determines $\mathbb{T}^{\omega_1}$; see [10, Corollary 3.12].

In Example 9.1, we exhibit a pseudocompact projectively Arhangel'skiĭ group D that is not w-compact. (Furthermore, under the assumption of CH, D can even be chosen to be countably compact.) In particular, neither the arrow (α) nor the arrow (β) is reversible.

For every infinite cardinal κ, there exists a κ-bounded w-compact (thus, Arhangel'skiĭ) abelian group that is not projectively Arhangel'skiĭ (and so is not projectively w-compact); see Example 9.2.

We do not know if the arrow 2.4 in Diagram 1 is invertible. In fact, it is tempting to conjecture that Corollary 2.4 gives not only a necessary but also a sufficient condition for a compact abelian group to be determined by its dense subgroup.

Question 2.5. *Does every dense projectively w-compact subgroup of a compact abelian group determine it?*

We refer the reader to Remark 10.2(ii) for a partial positive answer to this question.

3. Properties of Arhangel'skiĭ spaces and projectively Arhangel'skiĭ groups. Our first remark shows that the Arhangel'skiĭ property is "local".

Remark 3.1. For every space X, the inequalities $w(X) \leq |X|$ and $\chi(X) \leq |X|$ are equivalent.

Proposition 3.2.

- (i) *All locally compact spaces have the Arhangel'skiĭ property.*
- (ii) *First countable (in particular, metric) spaces have the Arhangel'skiĭ property.*
- (iii) *The class of Arhangel'skiĭ spaces is closed under taking perfect preimages; that is, if $f: X \to Y$ is a perfect map from a space X onto an Arhangel'skiĭ space Y, then X has the Arhangel'skiĭ property.*
- (iv) *If $w(X)$ is a strong limit cardinal, then X has the Arhangel'skiĭ property.*

Proof. (i) Let X be a locally compact space. If X is finite, then X has the Arhangel'skiĭ property. Suppose that X is infinite. Since the one-point compactification Y of X is compact, it has the Arhangel'skiĭ property, so $w(Y) \le |Y|$. Since X is infinite and $Y \setminus X$ is a singleton, $|Y| = |X|$. Since X is a subspace of Y, we get $w(X) \le w(Y)$. This proves that $w(X) \le |X|$.

(ii) For finite spaces X, this follows from (i). If X is infinite, then the conclusion follows from Remark 3.1.

(iii) Since finite spaces have the Arhangel'skiĭ property by (i), we shall assume that X is infinite. Since Y has the Arhangel'skiĭ property,

$$w(Y) \le |Y| = |f(X)| \le |X|. \tag{3.1}$$

There exists a one-to-one continuous map $g : X \to Z$ onto a space Z such that $w(Z) \le nw(X) \le |X|$ [1]. Let $h : X \to Y \times Z$ be the diagonal product of f and g defined by $h(x) = (f(x), g(x))$ for all $x \in X$. Since f is a perfect map, so is h [21, Theorem 3.7.9]. Since g is one-to-one, h is an injection. It follows that X and $h(X)$ are homeomorphic, so

$$w(X) = w(h(X)) \le w(Y \times Z) = \max\{w(Y), w(Z)\} \le \max\{w(Y), |X|\}.$$

Combining this with (3.1), we conclude that $w(X) \le |X|$. Thus, X has the Arhangel'skiĭ property.

(iv) Since $d(X) \le w(X) \le 2^{d(X)}$ and $w(X)$ is a strong limit cardinal, $w(X) = d(X) \le |X|$. ∎

Proposition 3.3. *If a topological group G contains a dense subgroup H with the Arhangel'skiĭ property, then G itself has the Arhangel'skiĭ property.*

Proof. Since H is dense in G, $\chi(H) = \chi(G)$. Since H has the Arhangel'skiĭ property, $\chi(H) \le w(H) \le |H|$. Since H is a subgroup of G, $|H| \le |G|$. This shows that $\chi(G) \le |G|$. Therefore, G has the Arhangel'skiĭ property by Remark 3.1. ∎

This proposition does not hold for spaces since one may have $w(Y) < w(X)$ when Y is a dense subspace of X.

Proposition 3.4. *Every pseudocompact group G such that $w(G) \le \mathfrak{c}$ is projectively Arhangel'skiĭ.*

Proof. Indeed, let $f : G \to H$ be a continuous surjective homomorphism of G onto a topological group H. Then H is pseudocompact, as a continuous image of the pseudocompact space G. If H is finite, then H has the Arhangel'skiĭ property by Proposition 3.2(i). Assume now that H is infinite. Then $|H| \ge \mathfrak{c}$ [20, Proposition 1.3(a)]. To show that H has the Arhangel'skiĭ property, it suffices to note that $w(H) \le \mathfrak{c}$. Indeed, let $\widehat{f} : \widehat{G} \to \widehat{H}$ be the extension of f over the completion $\widehat{G}$ of G. Since $\widehat{G}$ is compact and $\widehat{f}$ is surjective, $w(H) = w(\widehat{H}) \le w(\widehat{G}) = w(G) \le \mathfrak{c}$. ∎

Item (i) of our next proposition shows that the restriction on weight in Proposition 3.4 is the best possible, while item (ii) of Proposition 3.5

shows that even groups "arbitrarily close" to compact need not have the Arhangel'skiĭ property. (Compare this with Proposition 3.2(i).)

PROPOSITION 3.5.

(i) *Every compact group G with $w(G) = \mathfrak{c}^+$ has a dense countably compact subgroup without the Arhangel'skiĭ property.*
(ii) *For every infinite cardinal κ, each compact group G of weight $\tau = 2^{2^{2^\kappa}}$ has a dense κ-bounded subgroup without the Arhangel'skiĭ property.*

Proof. (i) Since $\mathfrak{c}^+ \leq 2^\mathfrak{c}$, applying [27, Theorem 2.7] we can choose a dense subgroup H of G such that $|H| = \mathfrak{c}$. By the standard closing-off argument, we can find a countably compact subgroup D of G such that $H \subseteq D$ and $|D| \leq \mathfrak{c}$. Since H is dense in G, so is D. Since $|D| = \mathfrak{c} < \mathfrak{c}^+ = w(G) = w(D)$, D does not have the Arhangel'skiĭ property.

(ii) By [27, Theorem 2.7], G contains a dense subgroup H of size 2^{2^κ}. Let D be the κ-closure of H in G; that is, $D = \bigcup\{\overline{A} : A \in [H]^{\leq\kappa}\}$, where $\overline{A}$ denotes the closure of A in G. Clearly, D is a subgroup of G containing H, so D is dense in G. Since $|[H]^{\leq\kappa}| \leq 2^{2^\kappa}$ and $|\overline{A}| \leq 2^{2^\kappa}$ for every $A \in [H]^{\leq\kappa}$, we conclude that $|D| \leq 2^{2^\kappa} < 2^{2^{2^\kappa}} = w(D)$. Therefore, D does not have the Arhangel'skiĭ property. ■

4. Metrizability of compact groups via conditions on their dense subgroups. Our first theorem in this section demonstrates that the weakest condition in Diagram 1 is not sufficient to get the metrizability of a compact group G even when this condition is imposed on all dense subgroups of G.

THEOREM 4.1. *Every dense subgroup of a compact group G has the Arhangel'skiĭ property if and only if $w(G)$ is a strong limit cardinal.*

Our second theorem shows that the projective version of the weakest condition in Diagram 1 imposed on *all* dense subgroups of a compact group G suffices to obtain its metrizability.

THEOREM 4.2. *Every dense subgroup of a compact group G is projectively Arhangel'skiĭ if and only if G is metrizable.*

Since a dense determining subgroup of a compact abelian group is projectively Arhangel'skiĭ (see Diagram 1), the "only if" part of this result strengthens Theorem 1.2 by offering the same conclusion under a much weaker assumption.

For a cardinal σ, the minimum cardinality of a pseudocompact group of weight σ is denoted by $m(\sigma)$ [11].

The next theorem is a counterpart of Theorem 4.1 for G_δ-dense subgroups.

THEOREM 4.3. *Every G_δ-dense subgroup of a compact group G has the Arhangel'skiĭ property if and only if $m(w(G)) \geq w(G)$.*

Our next result is the counterpart of Theorem 4.2 with "dense" replaced by "G_δ-dense".

THEOREM 4.4. *For a compact group G, the following conditions are equivalent:*

(i) *every G_δ-dense (equivalently, each dense pseudocompact) subgroup of G is projectively Arhangel'skiĭ;*
(ii) *all dense countably compact subgroups of G are projectively Arhangel'skiĭ;*
(iii) $w(G) \leq \mathfrak{c}$.

This theorem shows that having all G_δ-dense subgroups of a compact group G projectively Arhangel'skiĭ is not sufficient to obtain metrizability of G. Our next theorem shows that strengthening "projectively Arhangel'skiĭ" to "projectively w-compact" yields metrizability of G in the case when G is either connected or abelian.

THEOREM 4.5. *Let G be a compact group that is either abelian or connected. If all G_δ-dense (equivalently, all dense pseudocompact) subgroups of G are projectively w-compact, then G is metrizable.*

Combining this result with Corollary 2.4, we obtain the following corollary solving Question 1.4 in the negative and Question 1.5 in the positive.

COROLLARY 4.6. *If all G_δ-dense subgroups of a compact abelian group G determine it, then G is metrizable.*

Under the assumption of the Continuum Hypothesis, one can obtain the following stronger version of Theorem 4.5 in the abelian case.

THEOREM 4.7. *Assume CH. If all dense countably compact subgroups of a compact abelian group G are projectively w-compact, then G is metrizable.*

The proofs of Theorems 4.1–4.4 are postponed until Section 6, while the proofs of Theorems 4.5 and 4.7 are given in Section 8.

Let G be any compact abelian group of weight ω_1. It follows from Theorem 4.4 that all G_δ-dense subgroups of G are projectively Arhangel'skiĭ, even though G is not metrizable. This shows that "projectively w-compact" cannot be weakened to "projectively Arhangel'skiĭ" in the assumption of Theorems 4.5 and 4.7. Furthermore, since ω_1 is not a strong limit cardinal, Theorem 4.1 implies that G has a dense subgroup without the Arhangel'skiĭ property. In Example 9.3 below, we shall exhibit compact abelian groups G

of arbitrarily large weight such that every G_δ-dense subgroup of G has the Arhangel'skiĭ property, but there exists a dense subgroup of G without the Arhangel'skiĭ property.

We finish this section with the following corollary of its main results.

COROLLARY 4.8. *For a compact abelian group G, the following conditions are equivalent:*

(i) *G is metrizable;*
(ii) *every dense subgroup of G determines G;*
(iii) *every G_δ-dense* (*equivalently, each dense pseudocompact*) *subgroup of G determines G;*
(iv) *every dense subgroup of G is projectively Arhangel'skiĭ;*
(v) *every G_δ-dense* (*equivalently, each dense pseudocompact*) *subgroup of G is projectively w-compact.*

Furthermore, under CH, the following two items can be added to the list of equivalent conditions (i)–(v):

(vi) *every dense countably compact subgroup of G determines G;*
(vii) *every dense countably compact subgroup of G is projectively w-compact.*

Proof. (i)→(ii) is Theorem 1.1, (ii)→(iv) follows from Diagram 1, (iv)→(i) follows from Theorem 4.2.

(i)→(iii) follows from Theorem 1.1, (iii)→(v) follows from Corollary 2.4, (v)→(i) is Theorem 4.5.

(i)→(vi) follows from Theorem 1.1, (vi)→(vii) follows from Corollary 2.4. Finally, (vii)→(i) is Theorem 4.7. (We note that only the last implication needs CH.) ∎

Since countable compactness is stronger than pseudocompactness and a dense pseudocompact subgroup of a compact abelian group is G_δ-dense in it (Theorem 1.3), the implication (vi)→(i) of Corollary 4.8 strengthens the consistent result typeset in italics in Remark 1.6.

5. Pseudocompact groups of small weight without uncountable compact subsets. For a subset X of a group G we denote by $\langle X\rangle$ the subgroup of G generated by X.

By a *variety of groups* we mean, as usual, a class of groups closed under taking Cartesian products, subgroups and quotients (i.e., a *closed class* in the sense of Birkhoff [5]). Another, equivalent, way of defining a variety is by giving a fixed family of identities satisfied by all groups of the variety ([5]; see also [31, Theorem 15.51]).

Definition 5.1. Let $\mathcal{V}$ be a variety of groups.

(a) Recall that a subset X of a group G is called *$\mathcal{V}$-independent* provided that the following two conditions are satisfied:
  (i) $\langle X\rangle \in \mathcal{V}$;
  (ii) for every map $f: X \to H$ with $H \in \mathcal{V}$, there exists a homomorphism $\tilde{f}: \langle X\rangle \to H$ extending f.

(b) For every group $G \in \mathcal{V}$ the cardinal $r_{\mathcal{V}}(G) = \sup\{|X| : X$ is a $\mathcal{V}$-independent subset of $G\}$ is called the *$\mathcal{V}$-rank* of G.

(c) A group G is *$\mathcal{V}$-free* if G is generated by some $\mathcal{V}$-independent subset X of G. We call this X the *generating set* (or the set of *generators*) of G and we write $G = F_{\mathcal{V}}(X)$.

Theorem 5.2. *Let $\mathcal{V}$ be a variety of groups and L be a compact metric group that belongs to $\mathcal{V}$ such that $r_{\mathcal{V}}(L^\omega) \geq \omega$. Let I be a set such that $\omega_1 \leq |I| \leq \mathfrak{c}$. Then the group L^I contains a G_δ-dense* (*so dense pseudocompact*) *$\mathcal{V}$-free subgroup D of cardinality $\mathfrak{c}$ such that all compact subsets of D are countable. Since $w(D) = |I|$, the group D is not w-compact.*

The proof of this theorem is postponed until Section 7.

Corollary 5.3. *Let L be a compact simple Lie group. Then for every uncountable set I of size at most $\mathfrak{c}$, the group L^I contains a G_δ-dense free subgroup D of cardinality $\mathfrak{c}$ such that all compact subsets of D are countable; in particular, D is not w-compact.*

Proof. By [3, Theorem 2], $r_{\mathcal{G}}(L^\omega) \geq r_{\mathcal{G}}(L) \geq \omega$, where $\mathcal{G}$ is the variety of all groups. Now we can apply Theorem 5.2 with $\mathcal{V} = \mathcal{G}$. ∎

Corollary 5.4. *For every non-trivial compact metric abelian group L and every uncountable set I of size at most $\mathfrak{c}$, the group L^I contains a G_δ-dense subgroup D of cardinality $\mathfrak{c}$ such that all compact subsets of D are countable; in particular, D is not w-compact. Furthermore, if L is unbounded, then D can be chosen to be free.*

Proof. We consider two cases.

Case 1: *L is bounded*. Let n be the order of L, and let $\mathcal{A}_n$ be the variety of abelian groups of order n. Then $L \in \mathcal{A}_n$ and $r_{\mathcal{A}_n}(L^\omega) \geq \omega$, so the conclusion follows from Theorem 5.2 applied to $\mathcal{V} = \mathcal{A}_n$.

Case 2: *L is unbounded*. Let $\mathcal{A}$ be the variety of all abelian groups. Then $r_{\mathcal{A}}(L^\omega) \geq \omega$, so the conclusion follows from Theorem 5.2 applied to $\mathcal{V} = \mathcal{A}$. ∎

Following [15, Definition 5.2], we say that a variety $\mathcal{V}$ is *precompact* if $\mathcal{V}$ is generated by its finite groups. One can find in [15, Lemma 5.1] a host of conditions equivalent to precompactness of a variety. In particular, it is worth noting in connection with Theorem 5.2 that the existence of a compact group $L \in \mathcal{V}$ with $r_{\mathcal{V}}(L) \geq \omega$ is equivalent to precompactness of the variety $\mathcal{V}$.

Most of the well-known varieties are precompact; see [15, Lemma 5.3] and the comment following this lemma. The Burnside variety $\mathcal{B}_n$ for odd $n > 665$ is not precompact [13].

COROLLARY 5.5. *For a variety $\mathcal{V}$, the following conditions are equivalent:*

- (i) *$\mathcal{V}$ is precompact;*
- (ii) *for every cardinal σ with $\omega_1 \leq \sigma \leq \mathfrak{c}$, the $\mathcal{V}$-free group with $\mathfrak{c}$ many generators admits a pseudocompact group topology of weight σ without uncountable compact subsets; in particular, this topology is not w-compact.*

Proof. (i)→(ii). Suppose that $\mathcal{V}$ is precompact. By [15, Lemma 5.1], there exists a compact metric group $L \in \mathcal{V}$ with $r_{\mathcal{V}}(L) \geq \omega$. Since $r_{\mathcal{V}}(L^\omega) \geq r_{\mathcal{V}}(L)$, we get (ii) by applying Theorem 5.2 to L and a set I of cardinality σ.

(ii)→(i). This follows from [15, Theorem 5.5]. ∎

Our next remark shows that Theorem 5.2 and its Corollaries 5.3–5.5 are the best possible results that one can obtain in ZFC.

REMARK 5.6. Assume MA+¬CH, where MA stands for Martin's Axiom. In Theorem 5.2 and Corollaries 5.3, 5.4, take I to be a set of size ω_1, and let D be the group as in the conclusion of these results. In Corollary 5.5, let $\sigma = \omega_1$ and let D denote the $\mathcal{V}$-free group with $\mathfrak{c}$ many generators. Then D is a topological group of weight $\omega_1 < \mathfrak{c}$. Since MA holds, every countable subgroup of D is Fréchet–Urysohn [30]; in particular, D contains many non-trivial convergent sequences. Therefore, "all compact subsets of D are countable" cannot be strengthened to "all compact subsets of D are finite" in the conclusions of Theorem 5.2 and Corollaries 5.3, 5.4, and "without uncountable compact subsets" cannot be strengthened to "without infinite compact subsets" in the conclusion of Corollary 5.5.

Recall that the strongest totally bounded group topology on a group is called its *Bohr topology*.

REMARK 5.7. Thom recently proved that the free group with two generators equipped with its Bohr topology contains a non-trivial convergent sequence [34]. This easily implies that *every* precompact group topology on the free group with two generators contains a non-trivial convergent sequence. Since pseudocompact groups are precompact, it follows that *every pseudocompact free group of size $\mathfrak{c}$ contains a non-trivial convergent sequence.* Combining this with Theorem 1.3, we conclude that the group D as in the conclusion of Corollary 5.3 contains a non-trivial convergent sequence. This shows that "all compact subsets of D are countable" cannot be strengthened to "all compact subsets of D are finite" in the conclusion of Corollary 5.3, and "without uncountable compact subsets" cannot be strengthened to "without infinite compact subsets" in the conclusion of Corollary 5.5 when $\mathcal{V}$ is the variety of all groups.

6. Proofs of Theorems 4.1–4.4

Proof of Theorem 4.1. Suppose that $w(G)$ is not a strong limit cardinal. Then there exists a dense subgroup D of G such that $|D| = d(G) < w(G) = w(D)$; see [27, Theorem 2.7]. Hence, D does not have the Arhangel'skiĭ property.

Suppose now that $w(G)$ is a strong limit cardinal. Let D be a dense subgroup of G. Since $w(D) = w(G)$, the cardinal $w(D)$ is strong limit. Hence, D has the Arhangel'skiĭ property by Proposition 3.2(iv). ∎

Proof of Theorem 4.3. Let $\sigma = w(G)$. Assume that every G_δ-dense subgroup of G has the Arhangel'skiĭ property. According to [11], G has a G_δ-dense subgroup D of size $m(\sigma)$. Since D has the Arhangel'skiĭ property, this yields $m(\sigma) = |D| \geq w(D) = w(G) = \sigma$. Conversely, if $m(\sigma) \geq \sigma$ holds, then for every G_δ-dense subgroup D of G, one has $|D| \geq m(\sigma) \geq \sigma = w(D)$, so D has the Arhangel'skiĭ property. ∎

FACT 6.1 ([28, Lemma 1.5]). *Let G be an infinite compact group. For every infinite cardinal $\tau \leq w(G)$ there exists a continuous homomorphism $f : G \to H$ of G onto a compact group H with $w(H) = \tau$.*

FACT 6.2. *Suppose that $f : G \to H$ is a continuous surjective homomorphism of compact abelian groups, D is a subgroup of H and $D_1 = f^{-1}(D)$.*

(i) *If D is dense in H, then D_1 is dense in G.*
(ii) *If D is pseudocompact (countably compact, κ-bounded for some infinite cardinal κ), then D_1 has the same property.*
(iii) *If D is not (projectively) w-compact, then D_1 is not projectively w-compact either.*
(iv) *If D is not (projectively) Arhangel'skiĭ, then D_1 is not projectively Arhangel'skiĭ either.*

Proof. (i) The closure L of D_1 in G is compact, and so is its continuous image $f(L)$. Hence, $f(L)$ is closed in H. Since $D \subseteq f(L)$ and D is dense in H, we conclude that $f(L) = H$. Since $\ker f \subseteq D_1 \subseteq L$ and L is a subgroup of G, we deduce that $L = G$. Thus, D_1 is dense in G.

(ii) Since the map f is perfect, the conclusion follows from the well-known fact that the properties listed in (ii) are preserved by taking full preimages under perfect maps.

(iii) and (iv) are straightforward. ∎

Proof of Theorem 4.2. The "if" part follows from Diagram 1. Let us prove the "only if" part. Let G be a non-metrizable compact group. By Fact 6.1, there exists a continuous group homomorphism $f : G \to H$ onto a compact group H such that $w(H) = \omega_1$. Since ω_1 is not a strong limit cardinal, we can use Theorem 4.1 to find a dense subgroup D of H without the Arhangel'skiĭ property. By Fact 6.2, $D_1 = f^{-1}(D)$ is a dense subgroup of G that is not projectively Arhangel'skiĭ. ∎

Proof of Theorem 4.4. (i)→(ii). This implication is trivial, as all countably compact groups are pseudocompact.

(ii)→(iii). Assume that $w(G) \geq \mathfrak{c}^+$. By Fact 6.1, there exists a continuous surjective homomorphism $f : G \to H$ onto a compact group H such that $w(H) = \mathfrak{c}^+$. By Proposition 3.5(i), H has a dense countably compact subgroup D without the Arhangel'skiĭ property. By Fact 6.2, $D_1 = f^{-1}(D)$ is a dense countably compact subgroup of G that is not projectively Arhangel'skiĭ, in contradiction with (ii). This proves that $w(G) \leq \mathfrak{c}$.

(iii)→(i). Let D be a G_δ-dense subgroup of G. Then D is pseudocompact. Since $w(D) = w(G) \leq \mathfrak{c}$, from Proposition 3.4 we conclude that D is projectively Arhangel'skiĭ. ∎

7. Proof of Theorem 5.2

LEMMA 7.1. *Let X be a set, $\mathcal{V}$ be a variety of groups, and let e denote the identity element of $F_{\mathcal{V}}(X)$. For every $g \in F_{\mathcal{V}}(X) \setminus \{e\}$ there exists a unique non-empty finite set $F \subseteq X$ such that $g \in \langle F \rangle$ and $g \notin \langle F' \rangle$ for every proper subset F' of F.*

Proof. The existence of such an F is clear. Suppose that F_0 and F_1 are finite subsets of X such that $g \in \langle F_i \rangle$ and $g \notin \langle F_i' \rangle$ for every proper subset F_i' of F_i ($i = 0, 1$). Let $F' = F_0 \cap F_1$, so that $F' \subseteq F_i$ for $i = 0, 1$.

Fix $i = 0, 1$. Let $f : X \to F_{\mathcal{V}}(X)$ be the map that coincides with the identity on F_i and sends every element $x \in X \setminus F_i$ to $e \in F_{\mathcal{V}}(X)$. Since X is $\mathcal{V}$-independent, $F_{\mathcal{V}}(X) = \langle X \rangle \in \mathcal{V}$ by item (i) of Definition 5.1(a), so we can use item (ii) of the same definition to find a homomorphism $\tilde{f} : F_{\mathcal{V}}(X) \to F_{\mathcal{V}}(X)$ extending f. Since $g \in \langle F_i \rangle$ and f is the identity on F_i, we conclude that $\tilde{f}(g) = g$. Since $g \in \langle F_{1-i} \rangle$, we have

$$(7.1) \qquad g = \tilde{f}(g) \in \langle f(F_{1-i}) \rangle = \langle f(F_{1-i} \cap F_i) \cup f(F_{1-i} \setminus F_i) \rangle = \langle f(F') \cup \{e\} \rangle = \langle f(F') \rangle = \langle F' \rangle.$$

From $g \in \langle F_i \rangle$, (7.1), $F' \subseteq F_i$ and our assumption on F_i we conclude that $F_i = F' = F_0 \cap F_1 = F_i \cap F_{1-i}$. This proves that $F_i \subseteq F_{1-i}$.

Since the last inclusion holds for both $i = 0, 1$, it follows that $F_0 = F_1$, as required. ∎

For every $g \in F_{\mathcal{V}}(X) \setminus \{e\}$ we denote by $\mathrm{supp}_X(g)$ the unique set $F \subseteq X$ as in the conclusion of Lemma 7.1.

LEMMA 7.2. *If X is a set, $\mathcal{V}$ is a variety of groups, $g \in F_V(X) \setminus \{e\}$ and $g \in \langle Z \rangle$ for some $Z \subseteq X$, then* $\mathrm{supp}_X(g) \subseteq Z$.

Proof. Since $g \in \langle Z \rangle$, there exists a finite set $E \subseteq Z$ with $g \in \langle E \rangle$. Let F be a subset of E having the minimal size among all subsets D of E satisfying $g \in \langle D \rangle$. Then $g \in \langle F \rangle$ and $g \notin \langle F' \rangle$ for every proper subset F' of F. Lemma 7.1 implies that $\mathrm{supp}_X(g) = F \subseteq E \subseteq Z$. ∎

We shall call a space X *semi-Bernstein* provided that every compact subset of X is countable. A motivation for this definition comes from the classical notion of a Bernstein subset of the real line. One can easily see that a subset X of the real line $\mathbb{R}$ is a Bernstein set if and only if both X and its complement $\mathbb{R} \setminus X$ are semi-Bernstein spaces in our terminology.

LEMMA 7.3. *Assume that $\mathcal{V}$ is a variety of groups and X is a $\mathcal{V}$-independent subset of a separable metric group K such that $|X| = \mathfrak{c}$. Then there exists $Z \subseteq X$ such that $|Z| = \mathfrak{c}$ and $\langle Z \rangle$ is semi-Bernstein.*

Proof. Since X is $\mathcal{V}$-independent, $\langle X \rangle$ is isomorphic to $F_{\mathcal{V}}(X)$, so we can use the notation $\mathrm{supp}_X(g)$ for all $g \in \langle X \rangle \setminus \{e\}$. Since K is separable metric, the family

$$\mathcal{C} = \{C \subseteq \langle X \rangle : C \text{ is compact and } |C| = \mathfrak{c}\} \tag{7.2}$$

has size at most $\mathfrak{c}$, so we can fix an enumeration $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{c}\}$ of $\mathcal{C}$. By transfinite recursion on $\alpha < \mathfrak{c}$ we shall choose $x_\alpha, y_\alpha \in X$ satisfying conditions (i_α)–(iii_α) below:

- (i_α) $x_\alpha \notin \{x_\beta : \beta < \alpha\}$,
- (ii_α) $\{x_\beta : \beta \le \alpha\} \cap \{y_\beta : \beta \le \alpha\} = \emptyset$,
- (iii_α) $y_\alpha \in \mathrm{supp}_X(g_\alpha)$ for some $g_\alpha \in C_\alpha$.

Basis of recursion. Let $g_0 \in C_0 \setminus \{e\}$. Choose arbitrary $y_0 \in \mathrm{supp}_X(g_0)$ and $x_0 \in X \setminus \{y_0\}$. Now conditions (i_0)–(iii_0) are satisfied.

Recursive step. Suppose that $\alpha < \mathfrak{c}$ and $x_\beta, y_\beta \in X$ were already chosen for all $\beta < \alpha$ so that conditions (i_β)–(iii_β) are satisfied. We shall choose $x_\alpha, y_\alpha \in X$ satisfying conditions (i_α)–(iii_α). Let

$$H_\alpha = \langle \{x_\beta : \beta < \alpha\} \cup \{y_\beta : \beta < \alpha\} \rangle. \tag{7.3}$$

Then $|H_\alpha| \le |\alpha| \cdot \omega < \mathfrak{c}$. Since $|C_\alpha| = \mathfrak{c}$, we can choose

$$g_\alpha \in C_\alpha \setminus H_\alpha. \tag{7.4}$$

From (7.3) and (7.4) it follows that $\mathrm{supp}_X(g_\alpha) \not\subseteq \{x_\beta : \beta < \alpha\}$, so we can choose

$$y_\alpha \in \mathrm{supp}_X(g_\alpha) \setminus \{x_\beta : \beta < \alpha\}. \tag{7.5}$$

From (7.4) and (7.5) we conclude that (iii_α) holds. As $|X| = \mathfrak{c}$ and $|H_\alpha| < \mathfrak{c}$, we can choose

$$x_\alpha \in X \setminus (H_\alpha \cup \{y_\alpha\}). \tag{7.6}$$

Now (i_α) is satisfied by (7.3) and (7.6). Since (ii_β) holds for every $\beta < \alpha$, we have $\{x_\beta : \beta < \alpha\} \cap \{y_\beta : \beta < \alpha\} = \emptyset$. Combining this with (7.5) and (7.6), we get (ii_α).

The recursive construction has been completed. Since (i_α) holds for every $\alpha < \mathfrak{c}$, the set $Z = \{x_\alpha : \alpha < \mathfrak{c}\} \subseteq X$ has cardinality $\mathfrak{c}$. It only remains to show that $\langle Z\rangle$ contains no uncountable compact subsets.

Indeed, suppose that C is an uncountable compact subset of $\langle Z\rangle$. By [21, Exercise 1.7.11], every separable metric space is a union of a perfect set and a countable set. Since a perfect set has size $\mathfrak{c}$, it follows that $|C| = \mathfrak{c}$. Since $C \subseteq \langle Z\rangle \subseteq \langle X\rangle$, (7.2) implies $C \in \mathcal{C}$, and so $C = C_\alpha$ for some $\alpha < \mathfrak{c}$. By (iii_α), there exists $g_\alpha \in C_\alpha$ such that $y_\alpha \in \mathrm{supp}_X(g_\alpha)$. Since $g_\alpha \in C_\alpha = C \subseteq \langle Z\rangle$, Lemma 7.2 implies that $\mathrm{supp}_X(g_\alpha) \subseteq Z$. In particular, $y_\alpha \in Z$. On the other hand, since (ii_β) holds for every $\beta < \mathfrak{c}$, we conclude that $y_\alpha \notin \{x_\beta : \beta < \mathfrak{c}\} = Z$. This contradiction shows that $\langle Z\rangle$ is semi-Bernstein. ∎

LEMMA 7.4. *Let $\mathcal{V}$ be a variety of groups and let I be a set with $\omega_1 \le |I| \le \mathfrak{c}$. Assume that K is a compact metric group, $X \subseteq K^I$ and $\varphi : X \to K$ is an injection such that:*

(i) *$\varphi(X)$ is $\mathcal{V}$-independent,*
(ii) *$\langle\varphi(X)\rangle$ is semi-Bernstein,*
(iii) *$\langle X\rangle \in \mathcal{V}$,*
(iv) *for every $x \in X$ there exists $J_x \in [I]^{\le\omega}$ such that $\pi_i(x) = \varphi(x)$ for each $i \in I \setminus J_x$, where $\pi_i : K^I \to K$ is the projection on the ith coordinate.*

Then X is $\mathcal{V}$-independent and $\langle X\rangle$ is semi-Bernstein.

Proof. From (iv) one immediately gets the following claim:

CLAIM 1. *For every $Y \in [X]^{\le\omega}$ we have:*

(a) *the set $I_Y = I \setminus \bigcup_{x\in Y} J_x$ is uncountable;*
(b) *$\pi_i\restriction_Y = \varphi\restriction_Y$ for all $i \in I_Y$.*

Let Y be a finite subset of X. Since $\langle Y\rangle \subseteq \langle X\rangle \in \mathcal{V}$ by (iii), it follows that $\langle Y\rangle \in \mathcal{V}$. By Claim 1(a), we can choose $i \in I_Y$. By Claim 1(b), $\pi_i\restriction_Y = \varphi\restriction_Y$. Since φ is an injection, $\pi_i\restriction_Y$ is an injection as well. Since $\pi_i(Y) = \varphi(Y) \subseteq \varphi(X)$ and $\varphi(X)$ is $\mathcal{V}$-independent by (i), we conclude that Y is $\mathcal{V}$-independent [15, Lemma 2.4]. Since this holds for every finite subset Y of X, the set X is $\mathcal{V}$-independent as well [15, Lemma 2.3].

Since X and $\varphi(X)$ are both $\mathcal{V}$-independent, there exists a unique isomorphism $\Phi : \langle X\rangle \to \langle \varphi(X)\rangle$ extending φ. The next claim is an immediate corollary of Claim 1(b) and our definition of Φ.

CLAIM 2. *For every $Y \in [X]^{\le\omega}$ one has $\pi_i\restriction_{\langle Y\rangle} = \Phi\restriction_{\langle Y\rangle}$ for all $i \in I_Y$.*

For every subset J of I let $p_J : K^I \to K^J$ denote the projection.

Assume that C is an uncountable compact subset of $\langle X\rangle$. Then $\Phi(C)$ is an uncountable subset of $\langle \varphi(X)\rangle$, so the closure F of $\Phi(C)$ in K is an uncountable compact subset of K. By (ii), $F \setminus \langle \varphi(X)\rangle \ne \emptyset$, so we can select

$$b \in F \setminus \langle \varphi(X)\rangle \subseteq F \setminus \Phi(C). \tag{7.7}$$

Since K is a metric space, $b \in F\setminus\Phi(C)$ and $\Phi(C)$ is dense in F, we can choose a faithfully indexed sequence $\{c_n : n \in \mathbb{N}\} \subseteq C$ such that the sequence $\{\Phi(c_n) : n \in \mathbb{N}\}$ converges to b in K. Fix $Y \in [X]^{\le\omega}$ such that $\{c_n : n \in \mathbb{N}\} \subseteq \langle Y\rangle$. From Claim 2 we conclude that

$$\{\pi_i(c_n) : n \in \mathbb{N}\} = \{\Phi(c_n) : n \in \mathbb{N}\} \quad \text{for all } i \in I_Y. \tag{7.8}$$

Use Claim 1(a) to fix $j \in I_Y$. Since the sequence $\{c_n : n \in \mathbb{N}\}$ is faithfully indexed and Φ is an injection, it follows from (7.8) that the sequence $\{\pi_j(c_n) : n \in \mathbb{N}\}$ is faithfully indexed. Therefore, the sequence $\{p_S(c_n) : n \in \mathbb{N}\}$ is faithfully indexed as well, where $S = \{j\} \cup \bigcup_{x\in Y} J_x$. Since K^S is compact, the sequence $\{p_S(c_n) : n \in \mathbb{N}\}$ has an accumulation point $y \in K^S$. Define $g \in K^I$ by

$$g(i) = \begin{cases} y(i) & \text{if } i \in S, \\ b & \text{if } i \in I \setminus S \end{cases} \quad \text{for all } i \in I. \tag{7.9}$$

CLAIM 3. *g belongs to the closure of the set $\{c_n : n \in \mathbb{N}\}$ in K^I.*

Proof. Let W be an open neighbourhood of g in K^I. Then there exist an open set $U \subseteq K^S$ and an open set $V \subseteq K^{I\setminus S}$ such that $g \in U\times V \subseteq W$. Since $I \setminus S \subseteq I_Y$ and the sequence $\{\Phi(c_n) : n \in \mathbb{N}\}$ converges to b in K, applying (7.8) and (7.9) we can find $n_0 \in \mathbb{N}$ such that $p_{I\setminus S}(c_n) \in V$ for all $n \in \mathbb{N}$ with $n \ge n_0$. Since y is an accumulation point of $\{p_S(c_n) : n \in \mathbb{N}\}$, there exists an integer $m \ge n_0$ such that $p_S(c_m) \in U$. Now $c_m \in U \times V \subseteq W$. ∎

Since C is compact, it is closed in K^I. From $\{c_n : n \in \mathbb{N}\} \subseteq C$ and Claim 3 we get $g \in C$. Since $C \subseteq \langle X\rangle$, it follows that $g \in \langle X\rangle$. Let E be a finite subset of X with $g \in \langle E\rangle$. Since I_E is uncountable by Claim 1(a) and S is countable, we can choose $i \in I_E\setminus S$. Then $b = \pi_i(g) = \Phi(g)$ by (7.9) and Claim 2. Thus, $b = \Phi(g) \in \Phi(\langle X\rangle) = \langle \varphi(X)\rangle$, in contradiction with (7.7).

This proves that all compact subsets of $\langle X\rangle$ are countable. ∎

LEMMA 7.5. *Let $\mathcal{V}$ be a variety of groups and let I be a set with $\omega_1 \le |I| \le \mathfrak{c}$. Assume that $K \in \mathcal{V}$ is a compact metric group containing a $\mathcal{V}$-*

independent subset Z of K such that $|Z| = \mathfrak{c}$ and $\langle Z\rangle$ is semi-Bernstein. Then there exists a subset X of $H = K^I$ with the following properties:

(a) *X is a $\mathcal{V}$-independent subset of H of size $\mathfrak{c}$;*
(b) *$\langle X\rangle$ is semi-Bernstein;*
(c) *X is G_δ-dense in H.*

Proof. For every $J \in [I]^{\le\omega}$ let $K^J = \{y_{\alpha,J} : \alpha < \mathfrak{c}\}$ be an enumeration of K^J.

From $|I| \le \mathfrak{c}$ it follows that $\left|[I]^{\le\omega}\right| \le \mathfrak{c}$, so we can fix a faithful enumeration $Z = \{z_{\alpha,J} : \alpha < \mathfrak{c},\ J \in [I]^{\le\omega}\}$ of Z.

For $\alpha < \mathfrak{c}$ and $J \in [I]^{\le\omega}$ define $x_{\alpha,J} \in H$ by

$$x_{\alpha,J}(i) = \begin{cases} y_{\alpha,J}(i) & \text{if } i \in J \\ z_{\alpha,J} & \text{if } i \in I \setminus J \end{cases} \quad \text{for all } i \in I. \tag{7.10}$$

We claim that $X = \{x_{\alpha,J} : \alpha < \mathfrak{c},\ J \in [I]^{\le\omega}\}$ has the desired properties. Define the bijection $\varphi : X \to Z$ by $\varphi(x_{\alpha,J}) = z_{\alpha,J}$ for $(\alpha, J) \in \mathfrak{c}\times[I]^{\le\omega}$. Then items (i), (ii) and (iv) of Lemma 7.4 are satisfied. Since $\langle X\rangle$ is a subgroup of $H = K^I$ and $K \in \mathcal{V}$, it follows that $\langle X\rangle \in \mathcal{V}$, so item (iii) of Lemma 7.4 is satisfied as well. Applying this lemma, we conclude that X is $\mathcal{V}$-independent and (b) holds. Since $\varphi : X \to Z$ is a bijection, $|X| = |Z| = \mathfrak{c}$. Thus, (a) also holds.

It remains to check (c). For this, it suffices to show that $p_J(X) = K^J$ for every $J \in [I]^{\le\omega}$, where $p_J : K^I \to K^J$ is the projection. Fix such a J. Let $y \in K^J$. There exists $\alpha < \mathfrak{c}$ such that $y = y_{\alpha,J}$. Now $p_J(x_{\alpha,J}) = y_{\alpha,J} = y$ by (7.10). Since $x_{\alpha,J} \in X$, we conclude that $y \in p_J(X)$. Since $y \in K^J$ was chosen arbitrarily, this proves that $p_J(X) = K^J$. ∎

Proof of Theorem 5.2. Let $K = L^\omega$. Note that $K^I \cong L^I$, as I is uncountable. Therefore, we can work in $H = K^I$ instead of L^I.

Since $L \in \mathcal{V}$, we also have $K \in \mathcal{V}$. Since $r_V(K) \ge \omega$ and $K^\omega \cong K$, it follows from [15, Lemma 4.1] that K contains a $\mathcal{V}$-independent set of size $\mathfrak{c}$. This set (taken as X) and K satisfy the assumptions of Lemma 7.3. Let Z be as in the conclusion of this lemma. Applying Lemma 7.5 to Z and K, we obtain the set X as in the conclusion of Lemma 7.5. We claim that the subgroup $D = \langle X\rangle$ of H has the desired properties. Indeed, D is G_δ-dense in H by item (c) of the lemma. All compact subsets of D are countable by item (b). Since $D = \langle X\rangle$, from item (a) we deduce that D is a $\mathcal{V}$-free group with $\mathfrak{c}$ many generators. ∎

8. Proofs of Theorems 4.5 and 4.7. The proof of the following well-known fact can be found, for example, in [10, Theorem 4.15 and Discussion 4.14].

FACT 8.1. *Let G be a compact abelian group.*

(i) *If G is connected, then there exists a continuous surjective homomorphism of G onto $\mathbb{T}^{w(G)}$.*
(ii) *If τ is a cardinal such that $\omega < \mathrm{cf}(\tau) \le \tau \le w(G)$, then there exists a continuous surjective homomorphism $f : G \to H = K^{\tau}$, where $K = \mathbb{T}$ or $K = \mathbb{Z}(p)$ for some prime number p.*

The proof of the following fact can be found in [25, Proposition 3.2].

FACT 8.2. *If N is a totally disconnected closed normal subgroup of a compact connected group K, then $w(K/N) = w(K)$.*

We denote by G' the commutator subgroup of a group G. Recall that a group G is *perfect* if $G = G'$. A *semisimple* group is a perfect compact connected group [26, Definition 9.5]. For a topological group G, we use $c(G)$ to denote the connected component of G and we use $Z(G)$ to denote the center of G. We need the following well-known fact.

FACT 8.3. *Let G be a non-trivial compact connected group and let $A = c(Z(G))$.*

(i) *$G = A \cdot G'$ and $\Delta = A \cap G'$ is totally disconnected.*
(ii) *$G \cong (A \times G')/\Delta$ and $G/\Delta \cong A/\Delta \times G'/\Delta$.*
(iii) *$w(G) = \max\{w(A), w(G')\}$.*
(iv) *$w(A) = w(A/\Delta) = w(G/G')$.*
(v) *If $G = G'$ is semisimple, then $A = \Delta = \{e\}$, $G/Z(G)$ is a product of compact simple Lie groups and $w(G/Z(G)) = w(G)$.*
(vi) *The group G/Δ admits a continuous surjective homomorphism onto $\mathbb{T}^{w(A)} \times \prod_{i\in I} L_i$, where each L_i is a compact simple Lie group and $w(G') = \omega \cdot |I|$.*
(vii) *If $\mathrm{cf}(w(G)) > \omega$, then G admits a continuous surjective homomorphism onto $\mathbb{T}^{w(G)}$, or onto $L^{w(G)}$ for some compact simple Lie group L.*

Proof. One can find (i) in [26, Theorem 9.24] and (ii) in [26, Corollary 9.25].

(iii) From (i) it follows that G is a continuous image of $A\times G'$, so $w(G) \le w(A \times G') = \max\{w(A), w(G')\}$. Since both A and G' are subgroups of G, $\max\{w(A), w(G')\} \le w(G)$.

(iv) Since A is connected, the first equality follows from (i) and Fact 8.2. From (i) one easily gets the isomorphism $G/G' \cong A/\Delta$, which gives the second equality.

(v) This is a particular case of a theorem of Varopoulos [36]. The equality $w(G/Z(G)) = w(G)$ follows from Fact 8.2 since $Z(G)$ is totally disconnected [26, Theorem 9.19].

(vi) By (iv) and Fact 8.1(i), the connected compact abelian group A/Δ admits a continuous surjective homomorphism onto $\mathbb{T}^{w(A)}$.

Since $\Delta \subseteq Z(G) \subseteq Z(G')$, the group G'/Δ has $G'/Z(G')$ as its quotient. Since G' is semisimple [26, Corollary 9.6], from this and item (v) it follows that G'/Δ admits a continuous surjective homomorphism onto a product $\prod_{i\in I} L_i$, where each L_i is a compact simple Lie group and $w(G') = w(G'/Z(G')) = \omega \cdot |I|$.

Since $G/\Delta \cong A/\Delta \times G'/\Delta$ by (ii), we get the conclusion of item (vi).

(vii) This follows from (iii), (vi) and the fact that there are only countably many pairwise non-isomorphic (as topological groups) compact simple Lie groups. ∎

Proof of Theorem 4.5. Suppose that G is not metrizable.

Claim 4. *There exists a continuous surjective homomorphism $f : G \to H = L^{\omega_1}$, where either $L = \mathbb{T}$ or $L = \mathbb{Z}(p)$ for some prime number p, or L is a compact simple Lie group.*

Proof. If G is abelian, then the conclusion follows from Fact 8.1(ii). If G is connected, we first use Fact 6.1 to find a continuous homomorphism of G onto a (compact connected) group K of weight ω_1, and then apply Fact 8.3(vii) to K. ∎

When L is abelian, we apply Corollary 5.4 with $I = \omega_1$ to get a subgroup D of H as in the conclusion of this corollary. When L is a compact simple Lie group, we apply Corollary 5.3 with $I = \omega_1$ to get a subgroup D of H as in the conclusion of this corollary. In both cases, we use Fact 6.2 to conclude that $D_1 = f^{-1}(D)$ is a G_δ-dense subgroup of G that is not projectively w-compact. This contradicts the assumption of our theorem. Therefore, G must be metrizable. ∎

Lemma 8.4. *Assume CH. If $K = \mathbb{T}$ or $K = \mathbb{Z}(p)$ for some prime number p, then $H = K^{\omega_1}$ has a dense countably compact subgroup D without infinite compact subsets.*

Proof. We consider two cases.

Case 1: $K = \mathbb{T}$. Tkachenko [35] constructed a dense countably compact subgroup D of K^{ω_1} such that $|D| = \mathfrak{c} = \omega_1$ and D has no non-trivial convergent sequences.

Case 2: $K = \mathbb{Z}(p)$ *for some prime number* p. In this case we can argue as follows. Since CH implies Martin's Axiom MA, and the group $L = \mathbb{Z}(p)^\omega$ is compact (in the Tychonoff product topology), by the implication (a)→(c) of [19, Theorem 3.9], the group L admits a countably compact group topology without non-trivial convergent sequences. An analysis of that proof shows that this topology comes from a monomorphism $j : L \to \mathbb{Z}(p)^\mathfrak{c}$ such that $D =$

$j(L)$ is a dense subgroup of $\mathbb{Z}(p)^{\mathfrak{c}}$. Under CH, we conclude that $H = K^{\omega_1}$ has a dense countably compact subgroup D without non-trivial convergent sequences ([1]).

The rest of the proof is common for both cases. Suppose that X is an infinite compact subset of D. Since D has no non-trivial convergent sequences, X does not have any point of countable character. Then $|X| \geq 2^{\omega_1} > \omega_1 = \mathfrak{c}$ by the Čech–Pospišil theorem. This contradicts the inequality $|X| \leq |D| = \mathfrak{c}$. Therefore every compact subset X of D is finite. ∎

Proof of Theorem 4.7. Suppose that G is not metrizable. Use Fact 8.1(ii) to find a continuous surjective homomorphism $f : G \to H = K^{\omega_1}$, where K is either $\mathbb{T}$ or $\mathbb{Z}(p)$ for some prime number p. Let D be a dense countably compact subgroup of H without infinite compact subsets constructed in Lemma 8.4. Since D is dense in H, $w(D) = w(H) = \omega_1$. This shows that D is not w-compact. By Fact 6.2, $D_1 = f^{-1}(D)$ is a dense countably compact subgroup of G that is not projectively w-compact. This contradicts the assumption of our theorem. Therefore, G must be metrizable. ∎

9. Examples

Example 9.1. *For every cardinal τ such that $\omega_1 \leq \tau \leq \mathfrak{c}$, there exists a pseudocompact projectively Arhangel'skiĭ group D of weight τ that is not w-compact. Furthermore, under CH, D can even be chosen to be countably compact.* Indeed, let $K = \mathbb{T}$ or $K = \mathbb{Z}(p)$ for some prime number p. Apply Corollary 5.4 to $L = K$ and $I = \tau$ to find a G_δ-dense subgroup D of K^τ such that all compact subsets of D are countable; in particular, D is not w-compact. By Theorem 1.3, D is pseudocompact. Under CH, we can use Lemma 8.4 to choose D to be even countably compact. Since $w(D) = w(K^\tau) = \tau \leq \mathfrak{c}$, from Proposition 3.4 we conclude that D is projectively Arhangel'skiĭ.

Recall that a subgroup D of a topological abelian group G is called *essential* in G if $D \cap N = \{0\}$ implies $N = \{0\}$ for every closed subgroup N of G [4, 32, 33]. A topological group G is called *minimal* if there exists no Hausdorff group topology on G strictly coarser than the topology of G. A dense subgroup D of a compact abelian group G is minimal if and only if D is essential in G [4, 32, 33].

Example 9.2. *Let p be a prime number and κ be an infinite cardinal. Define $\tau = 2^{2^{2^\kappa}}$. Then there exists a dense essential (= minimal) κ-bounded w-compact subgroup of $\mathbb{Z}(p^2)^\tau$ that is not projectively Arhangel'skiĭ.* Indeed,

([1]) In case $p = 2$, one can also make a recourse to an old result of Hajnal and Juhász [23] asserting the existence of a subgroup D of K^{ω_1} that is an HFD set. Such D is a dense countably compact subgroup of K^{ω_1} without infinite compact subsets.

let $G = \mathbb{Z}(p^2)^\tau$ and let $f : G \to G$ be the (continuous) homomorphism defined by $f(g) = pg$ for $g \in G$. Let $H = f(G)$. Then $H \cong \mathbb{Z}(p)^\tau$. From Proposition 3.5(ii), we get a dense κ-bounded subgroup D of $H \cong \mathbb{Z}(p)^\tau$ without the Arhangel'skiĭ property. Applying Fact 6.2, we conclude that $D_1 = f^{-1}(D)$ is a dense κ-bounded subgroup of G that is not projectively Arhangel'skiĭ. Since $pG = \ker f$ is easily seen to be an essential subgroup of G, from $\ker f \subseteq D_1$ it follows that D_1 is an essential subgroup of G. Finally, note that $\ker f \cong Z(p)^\tau$ is a compact subset of D_1 such that $w(\ker f) = w(Z(p)^\tau) = \tau = w(G) = w(D_1)$, which shows that D_1 is w-compact.

For an infinite cardinal σ, define $\log \sigma = \min\{\tau \geq \omega : \sigma \leq 2^\tau\}$. Let $\beth_0 = \omega$, and let $\beth_{\alpha+1} = 2^{\beth_\alpha}$ for every ordinal α and $\beth_\beta = \sup\{\beth_\alpha : \alpha < \beta\}$ for every limit ordinal $\beta > 0$.

Example 9.3. *Let G be a compact group of weight $\sigma > \omega$.*

(i) *If* $\mathrm{cf}(\log \sigma) = \omega$ *and* $\sigma = (\log \sigma)^+$, *then every G_δ-dense subgroup of G has the Arhangel'skiĭ property.* Indeed, by Theorem 4.3, it suffices to show that $m(\sigma) \geq \sigma$. It is known that $\log \sigma \leq m(\sigma)$ and $\mathrm{cf}(m(\sigma)) > \omega$ [11, Theorem 2.7]. Therefore, $m(\sigma) > \log \sigma$ and $m(\sigma) \geq (\log \sigma)^+ = \sigma$ by our hypothesis.

(ii) *If α is an ordinal of countable cofinality and $\sigma = \beth_\alpha^+$, then all G_δ-dense subgroups of G have the Arhangel'skiĭ property.* Indeed, it suffices to check that $\sigma = \beth_\alpha^+$ satisfies the hypothesis of item (i). Obviously, $\log \sigma = \beth_\alpha$, so $\mathrm{cf}(\log \sigma) = \mathrm{cf}(\beth_\alpha) = \mathrm{cf}(\alpha) = \omega$ and $\sigma = \beth_\alpha^+ = (\log \sigma)^+$.

(iii) *If σ is a cardinal as in* (ii), *then G has a dense subgroup without the Arhangel'skiĭ property.* Indeed, σ is not a strong limit cardinal, so the conclusion follows from Theorem 4.1.

Here is an alternative proof of item (ii) of this example that makes no recourse to its item (i) and the cardinal function $m(-)$. Assume that D is a G_δ-dense subgroup of G without the Arhangel'skiĭ property. Then $|D| < w(D) = w(G) = \beth_\alpha^+$, so $|D| \leq \beth_\alpha$. Since $\beth_\alpha$ is a strong limit cardinal and $\beth_\alpha^+ = w(D) \leq 2^{|D|}$, we deduce that $|D| = \beth_\alpha$. Therefore, D is a pseudocompact group such that $|D|$ a strong limit cardinal of countable cofinality. This contradicts a well-known theorem of van Douwen [20].

10. Final remarks and open questions

Remark 10.1. While "projectively w-compact" and "projectively Arhangel'skiĭ" can differ for a single topological group, from Diagram 1 and the equivalence of items (ii), (iv) and (v) of Corollary 4.8 one concludes that these two properties and the property "determining the completion" coincide when imposed uniformly on *all* dense subgroups of a given compact abelian group. Similarly, while it is unclear whether "determining the completion" and "projectively w-compact" may differ for a single topological group, the equivalence of items (iii) and (v) of Corollary 4.8 shows that these two properties coincide when imposed uniformly on *all* G_δ-dense subgroups of a given compact abelian group.

Recall that a topological group G is called *totally minimal* if all (Hausdorff) quotient groups of G are minimal.

Remark 10.2. (i) In a forthcoming paper [17] we prove that *every dense totally minimal subgroup of a compact abelian group G determines G*. This shows that, in contrast with the results in Section 4, a weaker form of "determination" asking all dense totally minimal subgroups of G to determine G imposes no restrictions whatsoever on a compact abelian group G.

(ii) In a forthcoming paper [18] we prove that *totally minimal abelian groups are projectively w-compact*. Therefore, the italicized statement in item (i) shows that the answer to Question 2.5 is positive for this (proper) subclass of the class of projectively w-compact groups.

Question 10.3. *What can one say about a compact* (*abelian*) *group G such that all dense subgroups of G are w-compact?*

From Theorem 4.1 and Diagram 1 it follows that $w(G)$ must be a strong limit cardinal, but we do not know if G must be metrizable.

Question 10.4. *What is the minimal weight σ of an ω-bounded abelian group that is not projectively Arhangel'skiĭ? Is $\sigma = \mathfrak{c}^+$?*

We only know that $\mathfrak{c}^+ \leq \sigma \leq 2^{2^{\mathfrak{c}}}$. The first inequality follows from Proposition 3.4, and the second from Example 9.2 (with $\kappa = \omega$).

Question 10.5. *Does Theorem* 4.5 *hold for all compact groups?*

Question 10.6. *Does Theorem* 4.7 *hold in ZFC? Does the implication* (vi)$\to$(i) *of Corollary* 4.8 *hold in ZFC?*

As an intermediate step towards solving this question, one may also wonder if CH can be weakened to Martin's Axiom MA in Theorem 4.7 and in the implication (vi)$\to$(i) of Corollary 4.8.

We conjecture that the following question has a negative answer (although we have no counterexample at hand):

Question 10.7. *If every dense ω-bounded subgroup of a compact abelian group G determines it, must G be metrizable?*

Here comes the counterpart of Question 10.3 for G_δ-dense subgroups:

Question 10.8. *Describe the compact* (*abelian*) *groups G such that every G_δ-dense subgroup of G is w-compact.*

QUESTION 10.9. *Let $K = \mathbb{T}$ or $K = \mathbb{Z}(p)$ for some prime number p. In ZFC, does there exist a dense countably compact subgroup D of K^{ω_1} without uncountable compact subsets?*

As one can see from the proof of Theorem 4.5, a positive answer to this question for $K = \mathbb{T}$ and $K = \mathbb{Z}(p)$ for all $p \in \mathbb{P}$ would yield a positive answer to Question 10.6.

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