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Non-standard automorphisms of branched coverings of a disk and a sphere

by

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Abstract. Let Y be a closed 2-dimensional disk or a 2-sphere. We consider a simple, d-sheeted branched covering $\pi : X \to Y$. We fix a base point A_0 in Y ($A_0 \in \partial Y$ if Y is a disk). We consider the homeomorphisms h of Y which fix ∂Y pointwise and lift to homeomorphisms ϕ of X—the automorphisms of π. We prove that if Y is a sphere then every such ϕ is isotopic by a fiber-preserving isotopy to an automorphism which fixes the fiber $\pi^{-1}(A_0)$ pointwise. If Y is a disk, we describe explicitly a small set of automorphisms of π which induce all allowable permutations of $\pi^{-1}(A_0)$. This complements our result in Fund. Math. 217 (2012), no. 2, where we found a set of generators for the group of isotopy classes of automorphisms of π which fix the fiber $\pi^{-1}(A_0)$ pointwise.

1. Introduction. Let $\pi : X \to D$ be a simple, connected, d-sheeted branched covering of a closed 2-dimensional disk D. *Simple* means that over each point of D there are either d simple points of X or $d-2$ simple points and one "double" point, a branch point. The image $A = \pi(B)$ of a branch point B is called a *branch value.* Isotopy classes of homeomorphisms of D, which are fixed on the boundary of D and permute the branch values, form the braid group $\mathbf{B}_n$, where n is the number of the branch values. Some of these homeomorphisms lift to homeomorphisms of X, which we call the *automorphisms* of the covering. We fix a base point A_0 on the boundary ∂D. An automorphism ϕ of the covering is *standard* if it leaves the fiber $\pi^{-1}(A_0) = \{B_1, \dots, B_d\}$ pointwise fixed. In [WW2] we have described a set of generators for the group $L(\pi)$ of classes of homeomorphisms of D which lift to standard automorphisms of X.

The *non-standard* automorphisms permute the points $B_1, \dots, B_d$. There are some obvious topological restrictions on such permutations. The map π restricted to a boundary component of X defines a covering of ∂D of some degree. We call it the *degree* of the component of ∂X. The automorphism ϕ

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must take each boundary component of X to a boundary component of the same degree. It must also preserve the orientation of the boundary component. Thus ϕ takes all points of $\pi^{-1}(A_0)$ belonging to one boundary component to points of $\pi^{-1}(A_0)$ belonging to the other boundary component and must preserve the cyclic order of the points along the component. It is an easy consequence of known results (Proposition 1 and Remark 2) that every permutation which satisfies the above restrictions can be realized by an automorphism of the covering (Proposition 9). On the other hand a construction of a suitable homeomorphism depends on a long inductive procedure and a homeomorphism obtained in this way is very hard to understand and to use. In Proposition 13 we describe explicitly a set of automorphisms which induce a generating set of all possible permutations.

We also consider simple coverings of degree d of a 2-dimensional sphere. We may choose a base point A_0 on the sphere, not a branch value. Every homeomorphism of the sphere is isotopic relative to the branch values to a homeomorphism which fixes A_0. Let $B_1, \ldots, B_d$ be the points in the fiber over A_0. Some homeomorphisms of the sphere lift to automorphisms of the covering. The lifting may permute the points $B_1, \ldots, B_d$. We prove in Theorem 14 that every automorphism of a simple connected covering of the sphere is isotopic by a fiber-preserving isotopy to an automorphism which fixes the fiber over A_0 pointwise. Let D be the closure of the complement of a small disk which contains A_0 in its boundary. We may restrict the covering of the sphere to a covering π of D. It follows that any set of generators for $L(\pi)$ extended by the identity to all of S^2 indeed generates the group of all isotopy classes of liftable homeomorphisms of the sphere.

Branched coverings of a disk and their equivalence classes were studied by Hurwitz in [H] and by Berstein and Edmonds in [BE]. Equivalence classes of branched coverings of surfaces of any genus were studied by Gabai and Kazez in [GK]. Lifting of homeomorphisms was considered in [BW] for 3-sheeted coverings and in [CW] for d-sheeted coverings of a disk by a disk. More recently Mulazzani and Piergallini considered d-sheeted coverings in [MP] and proved that $L(\pi)$ is always generated by powers of half-twists. Apostolakis considered 4-sheeted coverings in [A] and found generators for a certain quotient of the group $L(\pi)$. In [WW1] a small finite set of generators of $L(\pi)$ was found for every simple 4-sheeted covering of a disk and in [WW2] for simple coverings of any degree.

2. Preliminaries and notation. In this section $\pi : X \to D$ is a fixed, connected, simple d-sheeted branched covering of a disk with n branch values $A_1, \ldots, A_n$. We choose a base point A_0 on the boundary of D. Let $B_1, \ldots, B_d$ be the points of X in $\pi^{-1}(A_0)$. Let σ be a closed loop in D which starts at A_0 and misses the branch values. When we lift σ to X from any point B_i, we end up at some point B_j. This defines a permutation $\mu(\sigma)$ in the symmetric group Σ_d, which depends only on the homotopy class of σ in the complement of the branch values. We thus get the *monodromy* homomorphism μ from the fundamental group of $D-\{A_1,\dots,A_n\}$ based at A_0 to the group Σ_d. We compose loops from left to right, and similarly for permutations, but homeomorphisms are composed from right to left. The monodromy of the boundary ∂D of D, oriented clockwise, is called the *total monodromy* of the covering π. We say that coverings $\pi_1 : X_1 \to D_1$ and $\pi_2 : X_2 \to D_2$ are *equivalent* if there exist orientation preserving homeomorphisms $h : D_1 \to D_2$ and $\phi : X_1 \to X_2$ such that $hp_1 = p_2\phi$.

The following was proven in [BE] and again in [MP]:

Proposition 1. *Connected simple coverings π_1 and π_2 are equivalent if and only if they have the same degree d (number of sheets), the same number n of branch points and the total monodromy of π_1 is conjugate to the total monodromy of π_2 in the symmetric group Σ_d.*

Remark 2. Consider Proposition 1. A permutation σ which conjugates the total monodromy of π_2 to the total monodromy of π_1 plays an important role in the construction of an equivalence. Let σ be such a permutation, let $\{B_1,\dots,B_d\}$ be the fiber over the base point A_0 of π_1 and let $\{C_1,\dots,C_d\}$ be the fiber over the base point A'_0 of π_2. Then there exists a homeomorphism $h : D_1 \to D_2$ which takes A_0 to A'_0 and there exists a lifting of h which takes B_i to $C_{\sigma(i)}$.

Definition 3. A *curve* in D is a simple path which begins at A_0 and ends at some branch value and does not meet other branch values. Curves are defined up to isotopy relative to the branch values and A_0. We say that two curves are *disjoint* if they meet only at A_0.

By the *monodromy* $\mu(\alpha)$ of a curve α we mean the monodromy of a closed path $\hat{\alpha}$ which goes along α to a point very near to its end point, a branch value A_i, then goes clockwise around A_i along a small circle and then comes back along α. There is one non-trivial component of $\pi^{-1}(\alpha)$ which connects some pair of points (B_i, B_j) and then $\mu(\alpha)$ is the transposition (i,j) in Σ_d.

Definition 4. Following [MP] we say that curves $\gamma_1,\dots,\gamma_k$ form a *system of curves* if $\gamma_i \cap \gamma_j = \{A_0\}$ for any $i \neq j$ and the curves meet at A_0 in this clockwise order. If $\gamma_1,\dots,\gamma_k$ form a system of curves then the sequence of transpositions $(\mu(\alpha_1),\dots, \mu(\alpha_k))$ is called the *monodromy sequence* of the system. A maximal system of curves, consisting of n curves, is called a (*geometric*) *basis.*

Definition 5. An *arc* in D is a simple path which connects two branch values and is disjoint from the other branch values and from the boundary of D. Arcs are defined up to isotopy relative to the branch values. A closed regular neighborhood of an arc x can be identified with the closed unit disk U in the complex plane $\mathbb{C}$ with the arc x corresponding to the subarc $y = [-1/2, 1/2]$ of the real axis. A *half-twist around* x is the isotopy class of a homeomorphism of D obtained by extending with the identity the following homeomorphism T of U. The homeomorphism T rotates the disk $\{z : |z| \leq 1/2\}$ counterclockwise around 0 by 180 degrees and the rotation is damped out to the identity at the boundary of U. We denote the half-twist around x again by the letter x.

DEFINITION 6. A *sequence of arcs* consists of arcs $x_1, \ldots, x_{k-1}$ such that x_i meets x_{i+1} at one of its end points and there are no other intersections between x_i and x_j for $1 \leq i < j \leq k-1$. We associate a sequence of arcs $x_1, \ldots, x_{k-1}$ with any system of curves $\alpha_1, \ldots, \alpha_k$. The arc x_i connects the end point of α_i to the end point of α_{i+1} and is homotopic, relative to the branch values, to the path $\alpha_i^{-1}\alpha_{i+1}$.

2.1. Hurwitz action and Hurwitz moves. We consider n-tuples $(\tau_1, \ldots, \tau_n)$ of transpositions belonging to Σ_d. The *Hurwitz action* of the braid group $\mathbf{B}_n$ on such n-tuples is defined as follows:

$$\sigma_i(\tau_1, \ldots, \tau_n) = (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1}\tau_i\tau_{i+1}, \tau_{i+2}, \ldots, \tau_n),$$

where σ_i is the standard generator of the braid group $\mathbf{B}_n$. This action is also called *jumping with the transposition* τ_i *to the right, over the transposition* τ_{i+1}. Two n-tuples are *Hurwitz equivalent* if they belong to the same orbit of the Hurwitz action. We say that an n-tuple $(\tau_1, \ldots, \tau_n)$ is *connected* if the transpositions τ_i generate the whole group Σ_d. This happens if the graph whose vertices are the numbers $1, \ldots, d$ and edges are the transpositions τ_i is connected. The Hurwitz action takes a connected n-tuple to a connected n-tuple.

Hurwitz *moves* act on bases. If $\gamma_1, \ldots, \gamma_n$ is a basis and if $x_1, \ldots, x_{n-1}$ is the sequence of arcs associated to the basis then the Hurwitz move σ_i takes the basis to its image under the action of the half-twist x_i. We have $x_i(\gamma_i) = \gamma_{i+1}$, $x_i(\gamma_{i+1}) = \gamma'_{i+1}$ and the other curves of the basis are fixed (see Figure 1). This move is also called *jumping with* γ_i *to the right over* γ_{i+1}.

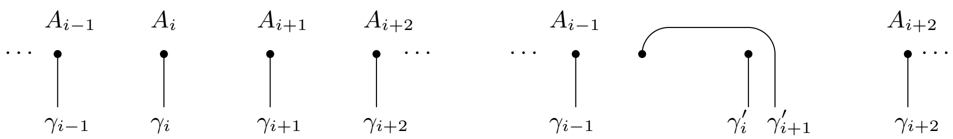

Fig. 1. Hurwitz move. Jump with γ_i to the right over γ_{i+1}.

After the jump the curve γ_{i+1} appears at position i and the new curve γ'_{i+1} appears at position $i+1$. The inverse of this move, the image of the half-twist x_i^{-1}, is called *jumping with* γ_{i+1} *to the left over* γ_i.

The monodromy of the new curve γ'_{i+1} is equal to $\mu(\gamma_{i+1})\mu(\gamma_i)\mu(\gamma_{i+1})$, therefore the Hurwitz move σ_i on the basis induces the Hurwitz action σ_i on the monodromy sequence of the basis. The covering π is connected if and only if the monodromy sequence of the basis is connected.

If an n-tuple $t = (\tau_1, \ldots, \tau_n)$ coincides with the monodromy sequence of a basis then the product $\tau_1 \ldots \tau_n$ is equal to the total monodromy of the covering. Therefore for any k-tuple $t = (\tau_1, \ldots, \tau_k)$ we shall call the product $\tau_1 \ldots \tau_k$ the *total monodromy* of t. The total monodromy of a k-tuple is preserved by the Hurwitz action.

LEMMA 7. *All connected sequences of transpositions with the given length and the given product are Hurwitz equivalent.*

This is Lemma 1.2 in [MP] used in the proof of Proposition 1 above.

3. Automorphisms of a covering of a disk. We consider a simple connected branched covering $\pi : X \to D$ of a closed disk D. The covering has degree d and has n branch values $A_1, \ldots, A_n$. We choose the base point A_0 on the boundary of D. We consider the isotopy classes, relative to the boundary ∂D and the branch values A_i, of the homeomorphisms of D which fix ∂D pointwise and permute the branch values. We say that h *lifts* if there is a homeomorphism ϕ of X such that $\pi\phi = h\pi$. Let $F_0 = \{B_1, \ldots, B_d\}$ be the fiber of π over A_0. If h lifts to ϕ then ϕ induces a permutation σ of F_0, $\phi(B_i) = B_{\sigma(i)}$.

LEMMA 8. *If $d > 2$ then a homeomorphism h of D has at most one lifting. In particular there are no deck transformations, the identity lifts only to the identity.*

Proof. If ϕ_1 and ϕ_2 are liftings of h then $\phi_2^{-1}\phi_1$ is a lifting of the identity. If we restrict the covering to the complement of the branch values we get an unbranched covering. If the deck transformation $\phi_2^{-1}\phi_1$ is non-trivial it cannot fix any point. In particular it induces a non-trivial permutation σ of F_0. As X is connected there is a curve α such that $\mu(\alpha)$ does not commute with σ, for otherwise σ belongs to the center of the group Σ_d, which is trivial. If $\mu(\alpha) = (i, j)$ then the set $\{\sigma(i), \sigma(j)\}$ is different from $\{i, j\}$. But the non-trivial component of $\pi^{-1}(\alpha)$ connects the particular pair of points $\{B_i, B_j\}$, which must be left invariant by $\phi_2^{-1}\phi_1$, and is taken to $\{B_{\sigma(i)}, B_{\sigma(j)}\}$. This is impossible. ∎

PROPOSITION 9. *Let σ be a permutation in Σ_d. There exists a liftable homeomorphism of D which induces the permutation σ of the fiber F_0 if and only if σ commutes with the total monodromy τ of the covering π. A homeomorphism h is liftable and induces σ if and only if $\mu(h(\alpha)) = \sigma^{-1}\mu(\alpha)\sigma$ for every curve α in D.*

Proof. Suppose σ commutes with τ. Consider two copies of the covering π. Since $\sigma^{-1}\tau\sigma = \tau$, by Remark 2 there is a homeomorphism h of D which is pointwise fixed on ∂D, takes branch values to branch values and lifts to a homeomorphism ϕ of X satisfying $\phi(B_i) = B_{\sigma(i)}$. Conversely if h lifts to ϕ and $\phi(B_i) = B_{\sigma(i)}$ and if α is a curve with $\mu(\alpha) = (i,j)$ then $\mu(h(\alpha)) = (\sigma(i),\sigma(j)) = \sigma^{-1}\mu(\alpha)\sigma$. Therefore the same is true for every loop α. In particular also for ∂D, which is invariant under h. Since $\mu(\partial D) = \tau$, the permutation σ commutes with τ. Finally suppose that h is a homeomorphism of D and there exists σ such that $\mu(h(\alpha)) = \sigma^{-1}\mu(\alpha)\sigma$ for every curve α. Then $\mu(h(\alpha)) = \sigma^{-1}\mu(\alpha)\sigma$ for every loop α. In particular σ commutes with τ. We now recall the standard construction of the lifting ϕ, which is very easy in this case. We let $\phi(B_1) = B_{\sigma(1)}$. For any $x \in X$ we connect B_1 to x by a path $\bar{\gamma}$. We project $\bar{\gamma}$ to D to the curve $\gamma = \pi(\bar{\gamma})$ and then we lift $h(\gamma)$ to a path in X from the point $B_{\sigma(1)}$. We let $\phi(x)$ be the end point of this lifting. Then ϕ is well defined and a bijection because $\mu(h(\alpha)) = \sigma^{-1}\mu(\alpha)\sigma$ for every loop α. It is also a local homeomorphism. ∎

We want to construct explicitly homeomorphisms which induce all possible permutations commuting with τ. In order to do this we choose a special basis and construct the homeomorphisms with respect to this basis.

Standard setup. A permutation τ has a certain number of disjoint cycles of various lengths. We order them by increasing length: all cycles of length 1 come first, if there are any, then come the cycles of length 2 and so on. The centralizer of τ is generated by cycles of τ and by permutations $(a_1,b_1)(a_2,b_2)\ldots(a_k,b_k)$ where $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ are cycles of τ.

We now construct a special basis. For a cycle $(a_1,\ldots,a_k)$ we choose a sequence of transpositions $(a_k,a_{k-1}),(a_{k-1},a_{k-2}),\ldots,\ (a_2,a_1)$. If $k=1$ there are no transpositions. If $(b_1,\ldots,b_s)$ is the next cycle of τ we insert two connecting transpositions (a_1,b_s) between the sequence of the first and the second cycle. Finally we add the last transposition an even number of times at the end of the sequence in order to complete the sequence to n transpositions. This sequence of transpositions is connected and has total monodromy τ. By Lemma 7 any basis in D is Hurwitz equivalent to a basis with the above monodromy sequence. We fix a basis with the above monodromy sequence and call it $\alpha_1,\ldots,\alpha_n$. We denote by $x_1,\ldots,x_{n-1}$ the sequence of arcs associated with the basis $\alpha_1,\ldots,\alpha_n$.

In order to find a homeomorphism which induces a permutation σ we construct a basis $\gamma_1,\ldots,\gamma_n$ such that $\mu(\gamma_i) = \sigma^{-1}\mu(\alpha_i)\sigma$ and we construct a homeomorphism which takes the standard basis α_i onto the new basis. The basis γ_i is constructed by Hurwitz moves. We recall the relation between Hurwitz moves and homeomorphisms.

LEMMA 10. *Let h be a homeomorphism of D and let σ_i be a Hurwitz move applied to the basis $h(\alpha_1), \ldots, h(\alpha_n)$. Let $\gamma_1, \ldots, \gamma_n$ be the new basis, after the Hurwitz move. Then $\gamma_j = hx_i(\alpha_j)$.*

Proof. The arc $y_i = h(x_i)$ is associated to the pair of curves $h(\alpha_i), h(\alpha_{i+1})$. The Hurwitz move σ_i is the result of the application of the half-twist y_i to the basis $h(\alpha_1), \ldots, h(\alpha_n)$. The half-twist y_i is equal to hx_ih^{-1}. Indeed h^{-1} takes a neighborhood of y_i to a neighborhood of x_i, then the half-twist x_i twists the neighborhood of x_i and then h transports the twisted neighborhood of x_i back to the neighborhood of y_i. Therefore σ_i takes the curve $h(\alpha_j)$ to $y_i(h(\alpha_j)) = hx_i(\alpha_j)$. ∎

We recall the notion of the Dehn twist.

DEFINITION 11. Let D_1 be a subdisk of D. Consider a collar neighborhood of the boundary ∂D_1 of D_1 which does not contain branch values. Let ∂_1 be its inner boundary. The *Dehn twist* T with respect to ∂D_1 is the isotopy class of a homeomorphism of D_1 which rotates the boundary ∂D_1 clockwise by 360 degrees and is damped to the identity at the inner boundary ∂_1 and is extended by the identity to all of D. If $y_1, \ldots, y_k$ is a sequence of arcs and if D_1 is a *regular* neighborhood of $y_1 \cup \cdots \cup y_k$ (a disk neighborhood of the union $y_1 \cup \cdots \cup y_k$ which does not contain branch values different from the end points of the arcs y_i), then $T(y_1, \ldots, y_k)$ denotes the Dehn twist with respect to ∂D_1.

Recall that if α is a curve then $\hat{\alpha}$ denotes a closed path which goes along α to a point very near to its end point, a branch value A_i, then goes clockwise around A_i along a small circle and then comes back along α.

LEMMA 12. *Let $\gamma_1, \ldots, \gamma_k$ be a system of curves in D and let $y_1, \ldots, y_{k-1}$ be the sequence of arcs associated to $\gamma_1, \ldots, \gamma_k$. Let D_1 be a regular neighborhood of $y_1 \cup \cdots \cup y_{k-1}$ and let δ_1 be equal to the product $\hat{\gamma}_1 \ldots \hat{\gamma}_k$. Let $t = T(y_1, \ldots, y_{k-1})$. Then $\mu(\delta_1) = \mu(\gamma_1) \ldots \mu(\gamma_k)$, $t(\hat{\gamma}_i)$ is isotopic to $\delta_1^{-1}\hat{\gamma}_i\delta_1$ and $\mu(t(\gamma_i)) = \mu(\delta_1)^{-1}\mu(\gamma_i)\mu(\delta_1)$ for $i = 1, \ldots, k$. Also $t = (y_1 \ldots y_{k-1})^k$.*

Proof. The path δ_1 is isotopic to a simple loop which surrounds the loops $\hat{\gamma}_1, \ldots, \hat{\gamma}_k$ and is isotopic to the boundary of D_1 oriented clockwise, therefore t is isotopic to the Dehn twist with respect to δ_1. By the definition of Dehn twist $t(\hat{\gamma}_i)$ is isotopic to $\delta_1^{-1}\hat{\gamma}_i\delta_1$. The remaining two statements follow from the fact that μ is a homomorphism and the last statement is a standard formula in the braid group. ∎

PROPOSITION 13. *Consider the basis α_i of the standard setup.*

(1) *Let $\nu = (a_1, \ldots, a_k)$, $k > 1$, be a cycle of τ. Let $\mu(\alpha_{m+1}) = (a_k, a_{k-1})$. If ν is the first cycle of τ then $m = 0$. If ν is not the last cycle of τ, we let $t = T(x_{m-1}, x_m, \ldots, x_{m+k})$ if $m \neq 0$ and $t = T(x_1, \ldots, x_k)$ if $m = 0$. If*

ν is the last cycle of τ, we let $t = T(x_{m-1}, x_m, \ldots, x_{n-1})$ if $m \neq 0$ and $t = T(x_1, \ldots, x_{n-1})$ if $m = 0$. Then t induces the cycle ν on the fiber F_0.

(2) Let $\nu_1 = (a_1, \ldots, a_k)$ and $\nu_2 = (b_1, \ldots, b_k)$ be consecutive cycles of τ of equal length. Let $\mu(\alpha_{m+1}) = (a_k, a_{k-1})$ if $k > 1$ and $\mu(\alpha_{m+1}) = \mu(\alpha_{m+2}) = (a_1, b_1)$ if $k = 1$. If $m \neq 0$, we let $t_1 = T(x_{m-1}, \ldots, x_{m+k-1})$, and if $m = 0$, we let $t_1 = T(x_{m+1}, \ldots, x_{m+k-1})$. We further let

$$h_1 = x_{m+k} x_{m+k+1} \ldots x_{m+2k-2} x_{m+k-1} \ldots x_{m+2k-3} \ldots x_{m+1} \ldots x_{m+k-1}.$$

(The homeomorphism h_1 is the product of $(k-1)k$ half-twists.)

If ν_2 is not the last cycle of τ, we let $t_2 = T(x_{m+k+1}, \ldots, x_{m+2k+1})$. Then $h = t_1 t_2^{-1} h_1 x_{m+2k-1}^{-1} \ldots x_{m+k+1}^{-1}$ induces the permutation

$$\nu = (a_1, b_1)(a_2, b_2) \ldots (a_k, b_k) \quad \text{on } F_0.$$

If ν_2 is the last cycle of τ, we let $t_2 = T(x_{m+k+1}, \ldots, x_{n-1})$. Suppose that there are $l+1$ curves in the basis α_i with monodromy (b_1, b_2). Then $n = m + 2k + l$. Let $t_3 = T(x_{n-l-1}, x_{n-l}, \ldots, x_{n-2})$. We let

$$h = t_1 t_2^{-1} h_1 x_{n-l-1}^{-1} x_{n-l} x_{n-l+1} \ldots x_{n-1} t_3 x_{m+2k-2}^{-1} x_{m+2k-3}^{-1} \ldots x_{m+k}^{-1}.$$

Then h induces the permutation $\nu = (a_1, b_1)(a_2, b_2) \ldots (a_k, b_k)$ on F_0.

Proof. Consider (1). The Dehn twist t corresponds to a sequence of arcs as in Lemma 12. The corresponding system of curves has monodromy sequence $(c_1, a_k), (c_1, a_k), (a_k, a_{k-1}), (a_{k-1}, a_{k-2}), \ldots, (a_2, a_1), (a_1, e_s), (a_1, e_s)$ (without the first two transpositions if $m = 0$ and with the last two transpositions replaced by an even number l of transpositions (a_2, a_1) if ν is the last cycle of τ). The total monodromy of this sequence is $\mu(\delta_1) = \nu$. Then, by Lemma 12, $\mu(t(\alpha_i)) = \nu^{-1} \mu(\alpha_i) \nu$ as required. The remaining curves of the basis α_i are not changed by h and their monodromies commute with ν.

Consider (2). The twists t_1, t_2 correspond to disjoint disks and to two consecutive systems of consecutive curves of the basis α_i (see Lemma 12). The systems of curves have monodromy sequences

$$(c_1, a_k), (c_1, a_k), (a_k, a_{k-1}), \ldots, (a_2, a_1), (a_1, b_k)$$

(possibly without the first two transpositions) and

$$(a_1, b_k), (b_k, b_{k-1}), \ldots, (b_2, b_1), (b_1, e_s), (b_1, e_s)$$

(possibly with the last two transpositions replaced by an even number l of transpositions (b_2, b_1)). The other curves of the basis have monodromy which commutes with ν and the curves are not changed by h. As in Lemma 12 let δ_1 be the loop surrounding the curves corresponding to t_1 and let δ_2 be the loop surrounding the curves corresponding to t_2. Then $\mu(\delta_1) = (a_1, \ldots, a_k, b_k)$ and $\mu(\delta_2) = (a_1, b_1, \ldots, b_k)$. After application of $t_1 t_2^{-1}$ the first part of the monodromy sequence gets conjugated by $\mu(\delta_1)$ and the second part gets conjugated by $\mu(\delta_2)^{-1}$ and we get a system of curves with the monodromy sequence

$$(c_1, b_k), (c_1, b_k), (b_k, a_k), (a_k, a_{k-1}), \dots, (a_2, a_1),\\ (b_k, b_{k-1}), \dots, (b_2, b_1), (b_1, a_1), (a_1, e_s), (a_1, e_s).$$

We now perform Hurwitz moves on curves (and Hurwitz action on sequences). The transposition (a_2, a_1) has number $m+k$. We jump with it to the right over the next $k-1$ transpositions performing $\sigma_{m+k}, \sigma_{m+k+1}, \dots, \sigma_{m+2k-2}$. We next jump with (a_3, a_2) to the right over the same transpositions and so on, ending with the transposition (b_k, a_k) which also jumps to the right. We get the sequence

$$(c_1, b_k), (c_1, b_k), (b_k, b_{k-1}), \dots, (b_2, b_1), (b_1, a_k),\\ (a_k, a_{k-1}), \dots, (a_2, a_1), (b_1, a_1), (a_1, e_s), (a_1, e_s)$$

(with the suitable changes if ν_1 is the first cycle or ν_2 is the last cycle of τ).

If ν_2 is not the last cycle of τ then we next jump with (b_1, a_1) to the left. The transposition (b_1, a_1) has number $m+2k$ and we jump $k-1$ times to the left. The transposition becomes (b_1, a_k) and we get the required sequence

$$(c_1, b_k), (c_1, b_k), (b_k, b_{k-1}), \dots, (b_2, b_1), (b_1, a_k), (b_1, a_k),\\ (a_k, a_{k-1}), \dots, (a_2, a_1), (a_1, e_s), (a_1, e_s).$$

By Lemma 10 the jumps correspond to the product of half-twists

$$x_{m+k} x_{m+k+1} \dots x_{m+2k-2} x_{m+k-1} \dots x_{m+2k-3}\\ \dots x_{m+1} \dots x_{m+k-1} x^{-1}_{m+2k-1} \dots x^{-1}_{m+k+1}.$$

Together with the initial twists $t_1 t_2^{-1}$ we get the homeomorphism h. Therefore h induces the permutation $\nu = (a_1, b_1)(a_2, b_2) \dots (a_k, b_k)$.

If ν_2 is the last cycle of τ then after the initial Dehn twists and jumps we obtain the sequence of transpositions

$$(c_1, b_k), (c_1, b_k), (b_k, b_{k-1}), \dots, (b_2, b_1), (b_1, a_k), (a_k, a_{k-1}),\\ \dots, (a_2, a_1), (b_1, a_1), \dots, (b_1, a_1).$$

We jump with the first transposition (b_1, a_1) (it has number $n-l = m+2k$) to the left over (a_2, a_1) and it becomes (a_2, b_1). This corresponds to the move σ^{-1}_{n-l-1}. Next we jump with (a_2, a_1) to the right to the end over an even number of equal transpositions. The transposition is still (a_2, a_1). The jumps correspond to the sequence $\sigma_{n-l}, \sigma_{n-l+1}, \dots, \sigma_{n-1}$. Next we perform the Dehn twist $T(y_{n-l-1}, \dots, y_{n-2})$ where y_i are associated to curves number $n-l-1, n-l, \dots, n-1$ in the last basis. The curves have monodromy sequence $(a_2, b_1), (b_1, a_1), \dots, (b_1, a_1)$ and total monodromy (a_2, b_1). The monodromy of each of these curves gets conjugated by (a_2, b_1) and we obtain the sequence of transpositions

$$(c_1, b_k), (c_1, b_k), (b_k, b_{k-1}), \ldots, (b_2, b_1), (b_1, a_k),$$
$$(a_k, a_{k-1}), \ldots, (a_3, a_2), (a_2, b_1), (a_2, a_1), \ldots, (a_2, a_1).$$

By Lemma 10 the last action corresponds to multiplication by t_3 on the right.

In the final step, we jump with the transposition (a_2, b_1), which has number $n - l - 1 = m + 2k - 1$, to the left $k - 1$ times. We get the required monodromy sequence

$$(c_1, b_k), (c_1, b_k), (b_k, b_{k-1}), \ldots, (b_2, b_1), (b_1, a_k), (b_1, a_k),$$
$$(a_k, a_{k-1}), \ldots, (a_2, a_1), \ldots, (a_2, a_1).$$

and the action corresponds to h from the Proposition. ■

4. Automorphisms of a covering of a sphere. We now consider a d-sheeted, connected, simple branched covering of a sphere $\pi : X \to S^2$ with n branch values $A_1, \ldots, A_n$. We consider the isotopy classes of homeomorphisms of S^2 which permute the branch values. We are interested in the subgroup of those classes of homeomorphisms of S^2 which lift to homeomorphisms of X. We may choose a base point A_0. Every homeomorphism is isotopic to a homeomorphism h which fixes A_0 so we may consider homeomorphisms fixing A_0, but we shall not require the isotopies to fix A_0. If a homeomorpism h is liftable then it induces (its lifting induces) a permutation of the fiber $F_0 = \pi^{-1}(A_0)$. It follows from Proposition 9 that every permutation of the fiber F_0 is induced by some liftable homeomorphism of S^2.

THEOREM 14. *Every liftable homeomorphism of S^2 is isotopic, relative to the branch values, to a homeomorphism which fixes the base point A_0 and induces the trivial permutation of the fiber F_0.*

Proof. The easy proof is based on the notion of a spin-map introduced in [B]. Let α be a simple closed path in S^2 issuing from A_0. We consider a regular neighborhood N of α, an annulus. We think of α as the circular core of the round annulus. We rotate α by 360 degrees and damp the rotation down to the identity at the boundary of the annulus. We extend the map by the identity to all of S^2. We get the *spin-map* $s(\alpha)$. It is equivalent to a product of a Dehn twist on one side of α and the inverse of the Dehn twist on the other side of α. This map is isotopic to the identity on S^2 and even within the annulus, therefore it is liftable. If α has a non-trivial monodromy then $s(\alpha)$ induces a non-trivial permutation of the fiber—the monodromy of α. Indeed if the lifting of α from a point B_i ends at B_j then the lifting of $s(\alpha)$ takes B_i to B_j. It suffices to consider the loops $\hat{\gamma}_i$ for any basis γ_i in D. The corresponding transpositions $\mu(\gamma_i)$ generate the whole group Σ_d.

Multiplying a liftable homeomorphism h by a suitable product of spin-maps, which are isotopic to the identity, we get a homeomorphism which induces the trivial permutation of F_0. ∎

COROLLARY 15. *Let $\pi : X \to S^2$ be a simple, connected branched covering of a sphere. Let D be a disk in S^2 which contains all branch values of π inside and let A_0 be a base point on the boundary of D. Let $h_1, \dots, h_k$ generate the group $L(D)$ of all classes of homeomorphisms of D, pointwise fixed on ∂D, which lift to $\pi^{-1}(D)$ and induce the trivial permutation of the fiber F_0. Then $h_1, \dots, h_k$, extended to S^2 by the identity, generate all classes of liftable homeomorphisms of S^2.*

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