FUNDAMENTA
MATHEMATICAE
218 (2012)

Covering maps for locally path-connected spaces

by

N. Brodskiy, **J. Dydak**, **B. Labuz** and **A. Mitra** (Knoxville, TN)

Abstract. We define Peano covering maps and prove basic properties analogous to classical covers. Their domain is always locally path-connected but the range may be an arbitrary topological space. One of characterizations of Peano covering maps is via the uniqueness of homotopy lifting property for all locally path-connected spaces.

Regular Peano covering maps over path-connected spaces are shown to be identical with generalized regular covering maps introduced by Fischer and Zastrow. If X is path-connected, then every Peano covering map is equivalent to the projection $\widetilde{X}/H \to X$, where H is a subgroup of the fundamental group of X and $\widetilde{X}$ equipped with the topology introduced in Spanier's *Algebraic Topology*. The projection $\widetilde{X}/H \to X$ is a Peano covering map if and only if it has the unique path lifting property. We define a new topology on $\widetilde{X}$ called the lasso topology. Then the fundamental group $\pi_1(X)$ as a subspace of $\widetilde{X}$ with the lasso topology becomes a topological group. Also, one has a characterization of $\widetilde{X}/H \to X$ having the unique path lifting property if H is a normal subgroup of $\pi_1(X)$. Namely, H must be closed in $\pi_1(X)$ with the lasso topology. Such groups include $\pi(\mathcal{U}, x_0)$ ($\mathcal{U}$ being an open cover of X) and the kernel of the natural homomorphism $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$.

1. Introduction. As locally complicated spaces naturally appear in mathematics (examples: boundaries of groups, limits under Gromov–Hausdorff convergence) there is an effort to extend homotopy-theoretical concepts to such spaces. This paper is devoted to a theory of coverings by locally path-connected spaces. Zeeman's example [17, 6.6.14 on p. 258] demonstrates difficulty in constructing a theory of coverings by non-locally path-connected spaces (that example amounts to two non-equivalent classical coverings with the same image of the fundamental groups). For coverings in the uniform category see [1] and [3].

To simplify exposition let us introduce the following concepts:

Definition 1.1. A topological space X is an *lpc-space* if it is locally path-connected. X is a *Peano space* if it is locally path-connected and connected.

2010 *Mathematics Subject Classification*: Primary 55Q52; Secondary 55M10, 54E15.
Key words and phrases: covering maps, locally path-connected spaces.

DOI: 10.4064/fm218-1-2

Fischer and Zastrow [16] defined *generalized regular coverings* of X as functions $p\colon \bar{X} \to X$ satisfying the following conditions for some normal subgroup H of $\pi_1(X)$:

R1. $\bar{X}$ is a Peano space.
R2. The map $p\colon \bar{X} \to X$ is a continuous surjection and $\pi_1(p)\colon \pi_1(\bar{X}) \to \pi_1(X)$ is a monomorphism onto H.
R3. For every Peano space Y, for every continuous function $f\colon (Y, y) \to (X, x_0)$ with $f_*(\pi_1(Y, y)) \subset H$, and for every $\bar{x} \in \bar{X}$ with $p(\bar{x}) = x_0$, there is a unique continuous $g\colon (Y, y) \to (\bar{X}, \bar{x}))$ with $p \circ g = f$.

Our view of the above concept is that of being universal in a certain class of maps and we propose a different way of defining covering maps between Peano spaces in Section 7.

Our first observation is that each path-connected space X has its universal Peano space $P(X)$, the set X equipped with new topology, such that the identity function $P(X) \to X$ corresponds to a generalized regular covering for $H = \pi_1(X)$. That way quite a few results in the literature can be formally deduced from earlier results for Peano spaces.

The way the projection $P(X) \to X$ is characterized in Theorem 2.2 generalizes to the concept of *Peano maps* in Section 7, and our *Peano covering maps* combine Peano maps with two classical concepts: Serre fibrations and unique path lifting property. Peano covering maps possess several properties analogous to the classical covering maps [19] (example: local Peano covering maps are Peano covering maps). One of them is that they are all quotients $\widehat{X}_H$ of the universal path space $\widetilde{X}$ equipped with the topology defined in the proof of Theorem 13 on p. 82 in [24] and used successfully by Bogley–Sieradski [2] and Fischer–Zastrow [16]. It turns out the endpoint projection $\widehat{X}_H \to X$ is a Peano covering map if and only if it has the uniqueness of path lifts property (see 7.4).

In an effort to unify Peano covering maps with uniform covering maps of [1] and [3] (we will explain the connection in [4]) we were led to a new topology on $\widetilde{X}_H$ (see Section 4). We call it the lasso topology and its main advantages are that the fundamental group $\pi_1(X)$ as a subspace of $\widetilde{X}$ becomes a topological group (see Section 3) and that there is a necessary and sufficient condition for $\widetilde{X}_H \to X$ to have the unique path lifting property in case H is a normal subgroup of $\pi_1(X)$. It is H being closed in $\pi_1(X)$. That explains Theorem 6.9 of [16] as the basic groups there turn out to be closed in $\pi_1(X)$ with the lasso topology. As an application of our approach we show existence of a universal Peano covering map over a given path-connected space.

2. Constructing Peano spaces. The purpose of this section is to discuss various ways of constructing new Peano spaces.

2.1. Universal Peano space. In analogy to the universal covering spaces we introduce the following notion:

Definition 2.1. Given a topological space X its *universal lpc-space* $lpc(X)$ is an lpc-space together with a continuous map (called the *universal Peano map*) $\pi\colon lpc(X) \to X$ satisfying the following universality condition:

- For any map $f\colon Y \to X$ from an lpc-space Y there is a unique continuous lift $g\colon Y \to lpc(X)$ of f (that means $\pi \circ g = f$).

Theorem 2.2. *Every space X has a universal lpc-space. It is homeomorphic to the set X equipped with a new topology, the one generated by all path components of all open subsets of the existing topology of X.*

Proof. Let U be an open set in X containing the point x, and $c(x, U)$ be the path component of x in U. Since $z \in c(x, U) \cap c(y, V)$ implies $c(z, U \cap V) \subset c(x, U) \cap c(y, V)$, the family $\{c(x, U)\}$, where U ranges over all open subsets of X and x ranges over all elements of U, forms a basis.

Given a map $f\colon Y \to X$ and given an open set U of X containing $f(y)$ one has $f(c(y, f^{-1}(U))) \subset c(f(y), U)$. That proves $f\colon Y \to lpc(X)$ is continuous if Y is an lpc-space. It also proves $lpc(X)$ is locally path-connected as any path in X induces a path in $lpc(X)$. ∎

Remark 2.3. The topology above was mentioned in Remark 4.17 of [16]. After the first version of this paper was written we were informed by Greg Conner of his unpublished preprint [6] with David Fearnley, where that topology is discussed and its properties (compactness, metrizability) are investigated.

If X is path-connected, then $lpc(X)$ is a *universal Peano space* $P(X)$ in the following sense: given a map $f\colon Z \to X$ from a Peano space Z to X there is a unique lift $g\colon Z \to P(X)$ of f.

In the remainder of this section we give sufficient conditions for a function on an lpc-space to be continuous. Those conditions are in terms of maps from basic Peano spaces: the arc in the first-countable case and *arc-hedgehogs* (see Definition 2.8) in the arbitrary case.

Proposition 2.4. *Suppose that $f\colon Y \to X$ is a function from a first-countable lpc-space Y. Then f is continuous if $f \circ g$ is continuous for every path $g\colon I \to Y$ in Y.*

Proof. Suppose U is open in X. It suffices to show that for each $y \in f^{-1}(U)$ there is an open set V in Y containing y such that the path component of y in V is contained in $f^{-1}(U)$. Pick a basis of neighborhoods $\{V_n\}_{n\geq 1}$ of y in Y and assume for each $n \geq 1$ there is a path α_n in V_n joining y to a point $y_n \notin f^{-1}(U)$. Those paths can be spliced to one path α from y to y_1 and going through all points y_n, $n \geq 2$. The path $f \circ \alpha$ starts from $f(y)$ and goes through all points $f(y_n)$, $n \geq 1$. However, as U is open, it must contain almost all of them, a contradiction. ■

The construction of the topology on $lpc(X)$ in Theorem 2.2 can be done in the spirit of the finest topology on X that retains the same continuous maps from a class of spaces.

PROPOSITION 2.5. *Suppose X is a path-connected topological space and $\mathcal{P}$ is a class of Peano spaces. The family $\mathcal{T}$ of subsets U of X such that $f^{-1}(U)$ is open in $Z \in \mathcal{P}$ for any map $f \colon Z \to X$ in the original topology is a topology and $\mathcal{P}(X) := (X, \mathcal{T})$ is a Peano space.*

Proof. Since $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, $\mathcal{T}$ is a topology on X. Suppose $U \in \mathcal{T}$ and C is a path component of U in the new topology. Suppose $f \colon Z \to X$ is a map and $f(z_0) \in C$. As $f^{-1}(U)$ is open, there is a connected neighborhood V of z_0 in Z satisfying $f(V) \subset U$. As $f(V)$ is path-connected, $f(V) \subset C$ and $C \in \mathcal{T}$. ■

In the case of first-countable spaces X we have a very simple characterization of the universal Peano map of X:

COROLLARY 2.6. *If X is a first-countable path-connected topological space, then a map $f \colon Y \to X$ is a universal Peano map if and only if Y is a Peano space, f is a bijection, and f has the path lifting property.*

Proof. Consider $\mathcal{A}(X)$ as in 2.5, where $\mathcal{A}$ consists of the unit interval. Notice the identity function $P(X) \to \mathcal{A}(X)$ is continuous as $P(X)$ is first-countable (use 2.4). Since the topology on $\mathcal{A}(X)$ is finer than that on $P(X)$, $P(X) = \mathcal{A}(X)$. Since f induces a homeomorphism from $\mathcal{A}(Y)$ to $\mathcal{A}(X)$ (due to the uniqueness of path lifting property of f), the composition $\mathcal{A}(Y) \to \mathcal{A}(X) \to P(X)$ is a homeomorphism and $f \colon Y \to P(X)$ must be a homeomorphism (its inverse is $P(X) \to \mathcal{A}(Y) \to Y$). ■

The construction in 2.5 can be used to create counter-examples to 2.6 in case X is not first-countable.

EXAMPLE 2.7. Let X be the cone over an uncountable discrete set B. Subsets of X that miss the vertex v are declared open if and only if they are open in the CW topology on X. A subset U of X that contains v is declared open if and only if U contains all but countably many edges of the cone and $U \setminus \{v\}$ is open in the CW topology on X (that means X is an arc-hedgehog if B is of cardinality ω_1; see 2.8). Notice $\mathcal{A}(X)$ is X equipped with the CW topology, the identity function $\mathcal{A}(X) \to X$ has the path lifting property but is not a homeomorphism.

Proof. Notice every subset of $X \setminus \{v\}$ that meets each edge in at most one point is discrete. Hence a path in X has to be contained in the union of finitely many edges. That means $\mathcal{A}(X)$ is X with the CW topology. ∎

We generalize 2.7 as follows:

DEFINITION 2.8. A *directed wedge* is the wedge $(Z, z_0) = \bigvee_{s\in S}(Z_s, z_s)$ of pointed Peano spaces indexed by a directed set S and equipped with the following topology (all wedges in this paper are considered with that particular topology):

(1) $U \subset Z \setminus \{z_0\}$ is open if and only if $U \cap Z_s$ is open for each $s \in S$,
(2) U is an open neighborhood of z_0 if and only if there is $t \in S$ such that $Z_s \subset U$ for all $s > t$ and $U \cap Z_s$ is open for each $s \in S$.

An *arc-hedgehog* is a directed wedge $(Z, z_0) = \bigvee_{s\in S}(Z_s, z_s)$ such that each (Z_s, z_s) is homeomorphic to $(I, 0)$.

Observe each directed wedge is a Peano space.

LEMMA 2.9. *Let S be a basis of neighborhoods of x_0 in X ordered by inclusion (i.e., $U \le V$ means $V \subset U$). If, for each $U \in S$, $\alpha_U : I \to U$ is a path in U starting from x_0, then their wedge*

$$\bigvee_{U\in S} \alpha_U : \bigvee_{U\in S} (I_U, 0_U) \to (X, x_0)$$

is continuous, where $(I_U, 0_U) = (I, 0)$ for each $U \in S$.

Proof. Only the continuity of $g = \bigvee_{U\in S} \alpha_U$ at the base point of the hedgehog $\bigvee_{U\in S}(I_U, 0_U)$ is not totally obvious. However, if V is a neighborhood of x_0 in X, then $g^{-1}(V)$ contains all I_U if $U \subset V$ and $g^{-1}(V) \cap I_W$ is open in I_W for all $W \in S$. ∎

PROPOSITION 2.10. *Suppose $f : Y \to X$ is a function from an lpc-space Y. Then f is continuous if $f \circ g$ is continuous for every map $g : Z \to Y$ from a hedgehog Z to Y.*

Proof. Assume U is open in X and $x_0 = f(y_0) \in U$. Suppose for each path-connected neighborhood V of y_0 in Y there is a path $\alpha_V : (I, 0) \to (V, y_0)$ such that $\alpha_V(1) \notin f^{-1}(U)$. By 2.9 the wedge $g = \bigvee_{V\in S} \alpha_V$ is a map g from a hedgehog to Y (here S is the family of all path-connected neighborhoods of y_0 in Y). Hence $h = f \circ g$ is continuous and there is $V \in S$ so that $I_V \subset h^{-1}(U)$. That means $f(\alpha_V(I)) \subset U$, a contradiction. ∎

2.2. Whisker topology on $\widetilde{X}$. The philosophical meaning of this section is that many results can be reduced to those dealing with Peano spaces via the universal Peano space construction. Let us illustrate this point of view by discussing a topology on $\widetilde{X}$.

Suppose (X, x_0) is a pointed topological space. Consider the set $\widetilde{X}$ of homotopy classes of paths in X originating at x_0. It has an interesting topology (see the proof of Theorem 13 on p. 82 in [24]) that has been put to use in [2] and [16]. We call it the *whisker topology* and its basis consists of sets $B([\alpha], U)$ (U is open in X, α joins x_0 and $\alpha(1) \in U$) defined as follows: $[\beta] \in B([\alpha], U)$ if and only if there is a path γ in U from $\alpha(1)$ to $\beta(1)$ such that β is homotopic rel. endpoints to the concatenation $\alpha * \gamma$.

The set $\widetilde{X}$ equipped with the whisker topology will be denoted by $\widehat{X}$ as in [2].

Both [2] and [16] consider quotient spaces $\widehat{X}/H$, where H is a subgroup of $\pi_1(X, x_0)$. We find it more convenient to follow [24, pp. 82–83]:

Definition 2.11. Suppose H is a subgroup of $\pi_1(X, x_0)$. Define $\widetilde{X}_H$ as the set of equivalence classes of paths in X under the relation $\alpha \sim_H \beta$ defined via $\alpha(0) = \beta(0) = x_0$, $\alpha(1) = \beta(1)$ and $[\alpha * \beta^{-1}] \in H$ (the equivalence class of α under the relation $\sim_H$ will be denoted by $[\alpha]_H$).

To introduce a topology on $\widetilde{X}_H$ we define sets $B_H([\alpha]_H, U)$ (denoted by $\langle \alpha, U \rangle$ on p. 82 in [24]), where U is open in X, α joins x_0 and $\alpha(1) \in U$, as follows: $[\beta]_H \in B_H([\alpha]_H, U)$ if and only if there is a path γ in U from $\alpha(1)$ to $\beta(1)$ such that $[\beta * (\alpha * \gamma)^{-1}] \in H$ (equivalently, $\beta \sim_H \alpha * \gamma$).

The set $\widetilde{X}_H$ equipped with the topology (which we call the *whisker topology on $\widetilde{X}_H$*) whose basis consists of $B_H([\alpha]_H, U)$, where U is open in X, α joins x_0 and $\alpha(1) \in U$, is denoted by $\widehat{X}_H$ in analogy to the notation $\widehat{X}$ in [2] that corresponds to H being trivial.

Given a path α in X and a path β in X from x_0 to $\alpha(0)$ one can define a *standard lift* $\hat{\alpha}$ of it to $\widehat{X}_H$ originating at $[\beta]_H$ by the formula $\hat{\alpha}(t) = [\beta * \alpha_t]_H$, where $\alpha_t(s) = \alpha(s \cdot t)$ for $s, t \in I$ (see [17, Proposition 6.6.3]).

Let us extract the essence of the proof of [24, Theorem 13 on pp. 82–83]:

Lemma 2.12. *Suppose X is a path-connected space and H is a subgroup of $\pi_1(X, x_0)$. An open set $U \subset X$ is evenly covered by $p_H \colon \widehat{X}_H \to X$ if and only if U is locally path-connected and the image of $h_\alpha \colon \pi_1(U, x_1) \to \pi_1(X, x_0)$ is contained in H for any path α in X from x_0 to any $x_1 \in U$.*

Proof. Recall that U is *evenly covered* by p_H (see [24, p. 62]) if $p_H^{-1}(U)$ is the disjoint union of open subsets $\{U_s\}_{s \in S}$ of $\widehat{X}_H$ each of which is mapped homeomorphically onto U by p_H. Also, recall $h_\alpha \colon \pi_1(U, x_1) \to \pi_1(X, x_0)$ is given by $h_\alpha([\gamma]) = [\alpha * \gamma * \alpha^{-1}]$.

Suppose U is evenly covered, γ is a loop in (U, x_1), and α is a path from x_0 to x_1. If $[\alpha]_H \neq [\alpha * \gamma]_H$, then they belong to two different sets U_u and U_v, $u, v \in S$. However, there is a path from $[\alpha]_H$ to $[\alpha * \gamma]_H$ in $p_H^{-1}(U)$

given by the standard lift of γ, a contradiction. Thus $[\alpha]_H = [\alpha * \gamma]_H$ and $[\alpha * \gamma * \alpha^{-1}] \in H$.

To show that U is locally path-connected, take a point $x_1 \in U$, pick a path α from x_0 to x_1 and select the unique $s \in S$ so that $[\alpha]_H \in U_s$. There is an open subset V of U satisfying $B_H([\alpha]_H, V) \subset U_s$. As $p_H|U_s$ maps U_s homeomorphically onto U, $p_H(B_H([\alpha]_H, V))$ is an open neighborhood of x_1 in U and it is path-connected.

Suppose U is locally path-connected and the image of $h_\alpha\colon \pi_1(U, x_1) \to \pi_1(X, x_0)$ is contained in H for any path α in X from x_0 to any $x_1 \in U$. Pick a path component V of U and notice sets $B_H([\beta]_H, V)$, β ranging over paths from x_0 to points of V, are either identical or disjoint. Observe $p_H|B_H([\beta]_H, V)$ maps $B_H([\beta]_H, V)$ homeomorphically onto V. Thus each V is evenly covered and that is sufficient to conclude U is evenly covered. ∎

As in [24, p. 81], given an open cover $\mathcal{U}$ of X, $\pi(\mathcal{U}, x_0)$ is the subgroup of $\pi_1(X, x_0)$ generated by elements of the form $[\alpha * \gamma * \alpha^{-1}]$, where γ is a loop in some $U \in \mathcal{U}$ and α is a path from x_0 to $\gamma(0)$.

Here is our improvement of [24, Theorem 13 on p. 82] and [16, Theorem 6.1]:

THEOREM 2.13. *If X is a path-connected space and H is a subgroup of $\pi_1(X, x_0)$, then the endpoint projection $p_H\colon \widehat{X}_H \to X$ is a classical covering map if and only if X is a Peano space and there is an open covering $\mathcal{U}$ of X so that $\pi(\mathcal{U}, x_0) \subset H$.*

Proof. Apply 2.12. ∎

PROPOSITION 2.14. *$\widehat{P(X)}_H$ is naturally homeomorphic to $\widehat{X}_H$ if X is path-connected.*

Proof. Since the continuity of $f\colon (Z, z_0) \to (P(X), x_0)$, for any Peano space Z, is equivalent to the continuity of $f\colon (Z, z_0) \to (X, x_0)$, paths in $(P(X), x_0)$ correspond to paths in (X, x_0). Also, $\pi_1(P(X), x_0) \to \pi_1(X, x_0)$ is an isomorphism so H is a subgroup of both $\pi_1(P(X), x_0)$ and $\pi_1(X, x_0)$, and the equivalence classes of relations $\sim_H$ are identical in both spaces $\widetilde{P(X)}$ and $\widetilde{X}$. Notice that basis open sets are identical in $\widehat{P(X)}_H$ and $\widehat{X}_H$. ∎

REMARK 2.15. In view of 2.14 some results in [16] dealing with maps $f\colon Y \to X$, where Y is Peano, can be derived formally from corresponding results for $f\colon Y \to P(X)$. A good example is Lemma 2.8 in [16]:

- $p\colon \widetilde{X} \to X$ has the unique path lifting property if and only if $\widetilde{X}$ is simply connected.

It follows formally from Corollary 4.7 in [2]:

- The universal endpoint projection $p\colon \widehat{Z} \to Z$ for a connected and locally path-connected space Z has the unique path lifting property if and only if $\widehat{Z}$ is simply connected.

When working in the pointed topological category the space $\widehat{X}_H$ is equipped with the base-point $\widehat{x}_0$ equal to the equivalence class of the constant path at x_0.

Let us illustrate $\widehat{X}_H$ in the case of $H = \pi_1(X, x_0)$.

PROPOSITION 2.16. *If $H = \pi_1(X, x_0)$, then:*

(a) *The endpoint projection $p_H\colon (\widehat{X}_H, \widehat{x}_0) \to (X, x_0)$ is an injection and $p_H(B([\alpha]_H, U))$ is the path component of $\alpha(1)$ in U.*
(b) *$\widehat{X}_H$ is a Peano space.*
(c) *Given a map $g\colon (Z, z_0) \to (X, x_0)$ from a pointed Peano space to (X, x_0), there is a unique lift $h\colon (Z, z_0) \to (\widehat{X}_H, \widehat{x}_0)$ of g ($p_H \circ h = g$).*

Proof. (a) Clearly, $p_H(B_H([\alpha]_H, U))$ equals the path component of $\alpha(1)$ in U. If $[\beta_1]_H$ and $[\beta_2]_H$ map to the same point x_1, then $\beta_1(1) = \beta_2(1)$ and $\gamma = \beta_1 * \beta_2^{-1}$ is a loop. Hence $[\gamma] \in H$ and $[\beta_2]_H = [\gamma * \beta_2]_H = [\beta_1]_H$ proving p_H is an injection.

(b) is well established in both [2] and [16]. Notice it follows from (a).

(c) For each $z \in Z$ pick a path α_z from z_0 to z in Z. Define $h(z)$ as $[\alpha_z]_H$ and notice h is continuous as $h^{-1}(B_H([\alpha_z]_H, U))$ equals the path component of $g^{-1}(U)$ containing z (use part (a)). As p_H is injective, there is at most one lift of g. ∎

In view of 2.16 we have a convenient definition of a universal Peano space in the pointed category:

DEFINITION 2.17. By the *universal Peano space* $P(X, x_0)$ of (X, x_0) we mean the pointed space $(\widehat{X}_H, \widehat{x}_0)$, $H = \pi_1(X, x_0)$, and the *universal Peano map of* (X, x_0) is the endpoint projection $P(X, x_0) \to (X, x_0)$. Equivalently, $P(X, x_0)$ is $(P(C), x_0)$, where C is the path component of x_0 in X.

Due to standard lifts the endpoint projection $p_H\colon (\widehat{X}_H, \widehat{x}_0) \to (X, x_0)$ always has the path lifting property. Thus the issue of interest is the uniqueness of path lifting property of p_H.

Here is a necessary and sufficient condition for p_H to have the unique path lifting property (compare it to [2, Theorem 4.5] for Peano spaces):

PROPOSITION 2.18. *If X is a path-connected space and $x_0 \in X$, then the following conditions are equivalent:*

(a) *$p_H\colon (\widehat{X}_H, \widehat{x}_0) \to (X, x_0)$ has the unique path lifting property.*
(b) *The image of $\pi_1(p_H)\colon \pi_1(\widehat{X}_H, \widehat{x}_0) \to \pi_1(X, x_0)$ is contained in H.*

Proof. (a)⇒(b). A loop α in $\widehat{X}_H$ must equal the standard lift of $\beta = p_H(\alpha)$. For the standard lift of β to be a loop in $\widehat{X}_H$ one must have $[\beta] \in H$.

(b)⇒(a). Given a lift $\bar{\alpha}$ of a path α in (X, x_0) it suffices to show $\bar{\alpha}(1) = [\alpha]_H$ as that implies $\bar{\alpha}$ is the standard lift of α (use $\alpha|[0,t]$ instead of α). Pick a path β satisfying $\hat{\alpha}(1) = [\beta]_H$ and let $\hat{\beta}$ be its standard lift. As $\bar{\alpha} * (\hat{\beta})^{-1}$ is a loop in $\widehat{X}_H$, its image $\gamma = p_H(\bar{\alpha} * (\hat{\beta})^{-1})$ generates an element $[\gamma]$ of H. Hence $\alpha \sim \gamma * \beta$ and $\bar{\alpha}(1) = [\beta]_H = [\alpha]_H$. ∎

3. Topologizing the fundamental group. In this section we discuss topological structures on $\pi_1(X, x_0)$. The two known ways of topologizing the fundamental group fail to make $\pi_1(X, x_0)$ a topological group even in the case of X being the Hawaiian earring. We introduce here the lasso topology that makes $\pi_1(X, x_0)$ a topological group.

We will consider topologizing the fundamental group in a broader context of topologizing the set of paths X^I or the set of homotopy classes of paths (rel. endpoints) $\mathcal{P}(X)$.

The set X^I carries an involution ι defined as $\iota(f) = f^{-1}$, where $f^{-1}(t) = f(1-t)$. The concatenation operation τ is defined on the subset $M = \{(a,b) \in X^I \times X^I \mid a(1) = b(0)\}$ and takes M onto X^I. The involution ι respects homotopies rel. endpoints and therefore defines an involution $\iota^\sharp$ on the set $\mathcal{P}(X)$. Similarly, the concatenation τ defines a concatenation operation $\tau^\sharp$ on the subset $\mathcal{M} = \{(a,b) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid a(1) = b(0)\}$.

We are interested in topologizing the sets X^I and $\mathcal{P}(X)$ so that the involution and the concatenation are continuous.

3.1. Compact-open topology. The first canonical topology to consider is the compact-open topology.

Lemma 3.1. *If the set X^I is equipped with the compact-open topology, then both the involution ι and the concatenation τ are continuous.*

Proof. The continuity of ι follows from the continuity of the composition $I^I \times X^I \to X^I$ defined by $(g,h) \mapsto g \circ h$ (we are only interested in the case of g being the function $g(t) = 1 - t$).

Suppose that $(g,h) \in M$ and $f * g$ is the concatenation of f and g. Let a basic open neighborhood V of $f * g$ be determined by a finite family of compacta $K_n \subset I$ and open sets $O_n \subset X$. Define the basic open neighborhood U of f (respectively W of g) by the compacta $2 * (K_n \cap [0,1/2])$ (respectively by $2 * (-1/2 + K_n \cap [1/2,1])$) and open sets $O_n \subset X$. Then $\tau(M \cap U \times W) \subset V$. ∎

The compact-open topology on X^I induces the quotient topology on the set $\mathcal{P}(X)$. The following lemma is easy to prove.

LEMMA 3.2. *If the set $\mathcal{P}(X)$ is equipped with the quotient of the compact-open topology on X^I, then the involution $\iota^\sharp$ is continuous.*

The concatenation operation $\tau^\sharp$ is not continuous [13] for the space X as simple as the Hawaiian earring (even when $\tau^\sharp$ is restricted to the set of loops).

DEFINITION 3.3. For any topological space X we denote by $\pi_1^{\mathrm{top}}(X, x_0)$ the fundamental group equipped with the subspace topology when considered as a subspace of the set $\mathcal{P}(X)$ equipped with the quotient topology of the compact-open topology on X^I.

Thus the fundamental group $\pi_1^{\mathrm{top}}(HE, x_0)$ fails to be a topological group [13].

3.2. Whisker topology. Spanier [24] introduced a new topology on $\pi_1(X, x_0)$ that was used by Bogley and Sieradski [2] and later was generalized by Fisher and Zastrow [16] to $\widetilde{X}$.

DEFINITION 3.4. For any topological space X the *whisker topology* on the set X^I is defined by the basis $B(g, U) = \{g * \beta \mid \beta \subset U\}$, where U is a neighborhood of the endpoint $g(1)$ in X, and β is a path in U originating at $g(1)$.

Neither the inverse ι nor the concatenation τ is continuous in the whisker topology, even if restricted to the set of loops based at a fixed point x_0. The reason is basically that the whisker topology on X^I is too strong: in order for a path $h \in X^I$ to belong to $B(g, U)$, the first half of the path h must coincide with g.

DEFINITION 3.5. For any topological space X the *whisker topology* on the set $\mathcal{P}(X)$ is defined by the basis $B([g], U) = \{[g*\beta] \mid \beta \subset U\}$, where U is a neighborhood of the endpoint $g(1)$ in X, and β is a path in U originating at $g(1)$. We denote by $\pi_1^{\mathrm{wh}}(X, x_0)$ the fundamental group equipped with the subspace topology when considered as a subspace of the set $\mathcal{P}(X)$ equipped with the whisker topology.

PROPOSITION 3.6. *If* HE *is the Hawaiian earring, then the topology of* $\pi_1^{\mathrm{top}}(\mathrm{HE}, x_0)$ *is different from* $\pi_1^{\mathrm{wh}}(\mathrm{HE}, x_0)$.

Proof. The main result of [12] is that $\pi_1^{\mathrm{top}}(\mathrm{HE}, x_0)$ is not metrizable. However, upon closer scrutiny the proof of that fact uses only the concept of first countability of $\pi_1^{\mathrm{top}}(\mathrm{HE}, x_0)$ at the trivial element. Since $\pi_1^{\mathrm{wh}}(\mathrm{HE}, x_0)$ has a countable base, it differs from $\pi_1^{\mathrm{top}}(\mathrm{HE}, x_0)$. ∎

PROPOSITION 3.7. *If* HE *is the Hawaiian earring, then neither the involution* $\iota^\sharp$ *nor the concatenation* $\tau^\sharp$ *is continuous on* $\pi_1^{\mathrm{wh}}(\mathrm{HE}, x_0)$.

Proof. Consider the Hawaiian earring $X = \mathrm{HE}$ as the subspace of the coordinate xy-plane being the union of all circles X_n of radius $1/n$ centered at $(1/n, 0)$. Denote by α_n the generator of $\pi_1(X_n, x_0)$ considered as an element of $\pi_1^{\mathrm{wh}}(HE, x_0)$ (we assume $x_0 = (0,0)$).

Fix a basic neighborhood $B([\alpha_1^{-1}], U)$ such that U does not contain the circle X_1. Consider the sequence $\{[\alpha_1 * \alpha_n]\} \subset \pi_1^{\mathrm{wh}}(HE, x_0)$ converging to $[\alpha_1]$ in the whisker topology. Then none of the elements of the sequence $\{\iota^\sharp([\alpha_1 * \alpha_n])\} = \{[\alpha_n^{-1} * \alpha_1^{-1}]\}$ belongs to $B([\alpha_1^{-1}], U)$. Indeed, an element $[\alpha_n^{-1} * \alpha_1^{-1}]$ cannot be equal to an element $[\alpha_1^{-1} * \gamma]$ for any γ avoiding $X_1 \setminus U$.

Fix a basic neighborhood $B([\alpha_1 * \alpha_1], U)$ such that U does not contain the circle X_1. Consider the sequence $\{[\alpha_1 * \alpha_n]\} \subset \pi_1^{\mathrm{wh}}(HE, x_0)$ converging to $[\alpha_1]$ in the whisker topology. Then none of the elements of the sequence $\{\tau^\sharp([\alpha_1 * \alpha_n], [\alpha_1 * \alpha_n])\} = \{[\alpha_1 * \alpha_n * \alpha_1 * \alpha_n]\}$ belongs to $B([\alpha_1 * \alpha_1], U)$. Indeed, an element $[\alpha_1 * \alpha_n * \alpha_1 * \alpha_n]$ cannot be equal to an element $[\alpha_1 * \alpha_1 * \gamma]$ for any γ avoiding $X_1 \setminus U$. ∎

3.3. Lasso topology. The problem with the whisker topology on the space of homotopy classes of paths at x_0 is that, when considered as a subspace of the homotopy classes of all paths in X, it is not symmetric. This calls for introducing another whisker at the beginning of a path. This whisker becomes a loop when considering paths at x_0, so the final result is adding whiskers and loops. This is a philosophical reason for introducing a new topology on the space of paths. A practical reason is continuity of the involution and concatenation.

Let us identify two problems with the whisker topology that prevent the involution $\iota^\sharp$ and the concatenation $\tau^\sharp$ from being continuous. The first problem is that there is a whisker at the end of a path, but no whisker at the beginning. So, having two whiskers would make the involution $\iota^\sharp$ continuous. The second problem is with a small loop getting stuck between the two concatenating paths. We introduce the concept of a lasso to resolve this problem and make the concatenation $\tau^\sharp$ continuous.

Definition 3.8. Let f be a path in a topological space X. Let V be a neighborhood of the point $f(0)$ in X and W be a neighborhood of the point $f(1)$ in X. A path α is called a *left V-whisker* of f if α is a path in V with the endpoint $\alpha(1) = f(0)$. A path β is called a *right W-whisker* of f if β is a path in W with the endpoint $\beta(0) = f(1)$.

Definition 3.9. Let $\mathcal{U}$ be an open cover of a topological space X and x be a point in X. A path l is called a *$\mathcal{U}$-lasso based at the point x* if l is equal to a finite concatenation of loops $\alpha_n * \gamma_n * \alpha_n^{-1}$, where γ_n is a loop in some $U \in \mathcal{U}$ and α_n is a path from x to $\gamma_n(0)$.

Definition 3.10. For any topological space X the *lasso topology* on the set $\mathcal{P}(X)$ is defined by the basis $B([g],V,\mathcal{U},W)$, where V is a neighborhood of the endpoint $g(0)$ in X, W is a neighborhood of the endpoint $g(1)$, and $\mathcal{U}$ is an open cover of X. A homotopy class $[h]\in\mathcal{P}(X)$ belongs to $B([g],V,\mathcal{U},W)$ if and only if this class has a representative of the form $\alpha*g*l*\beta$, where α is a left V-whisker of g, β is a right W-whisker of g, and l is a $\mathcal{U}$-lasso based at $g(1)$.

We denote by $\pi_1^l(X,x_0)$ the fundamental group equipped with the subspace topology when considered as a subspace of the set $\mathcal{P}(X)$ equipped with the lasso topology.

Observe $[f]\in B([g],V,\mathcal{U},W)$ implies $B([f],V,\mathcal{U},W)=B([g],V,\mathcal{U},W)$. Indeed, if $f\simeq\alpha*g*l*\beta$, then $g\simeq\alpha^{-1}*f*\beta^{-1}*l^{-1}*\beta*\beta^{-1}$ where $\beta^{-1}*l^{-1}*\beta$ is homotopic to a $\mathcal{U}$-lasso based at the point $f(1)$.

Also, $B([g],V_1\cap V_2,\mathcal{U}_1\cap\mathcal{U}_2,W_1\cap W_2)\subset B([g],V_1,\mathcal{U}_1,W_1)\cap B([g],V_2,\mathcal{U}_2,W_2)$, so the family of sets $B([g],V,\mathcal{U},W)$ forms a basis of a topology on $\mathcal{P}(X)$.

Proposition 3.11. *If the set $\mathcal{P}(X)$ is equipped with the lasso topology, then the involution $\iota^\sharp$ and the concatenation $\tau^\sharp$ are continuous.*

Proof. If $B([f^{-1}],V,\mathcal{U},W)$ is a basic neighborhood of a path f^{-1}, then the involution $\iota^\sharp$ will take the neighborhood $B([f],W,\mathcal{U},V)$ into $B([f^{-1}],V,\mathcal{U},W)$. Indeed,

$$\iota^\sharp([\alpha*f*l*\beta])=[\beta^{-1}*l^{-1}*f^{-1}*\alpha^{-1}]=[\beta^{-1}*f^{-1}*(f*l^{-1}*f^{-1})*\alpha^{-1}]$$

where $f*l^{-1}*f^{-1}$ is homotopic to a $\mathcal{U}$-lasso based at the point $f^{-1}(1)=f(0)$.

Let $B([f*g],V,\mathcal{U},W)$ be a basic neighborhood of a concatenation $[f*g]$. Fix an arbitrary element $U\in\mathcal{U}$ containing the point $f(1)=g(0)$. Consider the basic neighborhoods $B([f],V,\mathcal{U},U)$ of $[f]$ and $B([g],U,\mathcal{U},W)$ of $[g]$. Then any concatenation of classes from these neighborhoods belongs to $B([f*g],V,\mathcal{U},W)$. Indeed, if $[\alpha*f*l*\beta]\in B([f],V,\mathcal{U},U)$ and $[\gamma*g*m*\delta]\in B([g],U,\mathcal{U},W)$, then the concatenation $[\alpha*f*l*\beta*\gamma*g*m*\delta]$ can be written as $[\alpha*f*g*(g^{-1}*l*g*g^{-1}*\beta*\gamma*g*m)*\delta]$. Notice that both $g^{-1}*l*g$ and $g^{-1}*\beta*\gamma*g$ are $\mathcal{U}$-lassos based at $f*g(1)$. ∎

Corollary 3.12. *For any topological space X and any base point x_0, $\pi_1^l(X,x_0)$ is a topological group.*

4. Lasso topology on $\widetilde{X}$. We do not know how to characterize subgroups H of $\pi_1(X,x_0)$ for which $p_H\colon\widehat{X}_H\to X$ has the unique path lifting property. We define the lasso topology on $\widetilde{X}_H$ for which an analogous question has a satisfactory answer in the case of H being a normal subgroup.

Given an open cover $\mathcal{U}$ of X, a subgroup H of $\pi_1(X,x_0)$, a path α in X originating at x_0, and $V\in\mathcal{U}$ containing $x_1=\alpha(1)$ define $B_H([\alpha]_H,\mathcal{U},V)\subset$

$\widetilde{X}_H$ as follows: $[\beta]_H \in B_H([\alpha]_H, \mathcal{U}, V)$ if and only if there is a path γ_0 in V originating at $x_1 = \alpha(1)$ and a loop λ at x_1 such that $[\lambda] \in \pi(\mathcal{U}, x_1)$ and $\beta \sim_H \alpha * \lambda * \gamma_0$.

Observe $[\beta]_H \in B_H([\alpha]_H, \mathcal{U}, V)$ implies $B_H([\alpha]_H, \mathcal{U}, V) = B_H([\beta]_H, \mathcal{U}, V)$ and $B_H([\alpha]_H, \mathcal{U} \cap \mathcal{V}, V_1 \cap V_2) \subset B_H([\alpha]_H, \mathcal{U}, V_1) \cap B_H([\alpha]_H, \mathcal{V}, V_2)$, so the family of sets $\{B_H([\alpha]_H, \mathcal{U}, V)\}$ forms a basis of a new topology on $\widetilde{X}_H$ that we call the *lasso topology*. In the particular case of $H = \{1\}$, the trivial subgroup of $\pi_1(X, x_0)$, we simplify $\widetilde{X}_H$ to $\widetilde{X}$.

Notice the identity function $\widehat{X}_H \to \widetilde{X}_H$ is continuous when $\widetilde{X}_H$ is equipped with the lasso topology (recall that $\widehat{X}_H$ denotes the set $\widetilde{X}_H$ equipped with the whisker topology). Indeed, $B_H([\alpha]_H, V) \subset B_H([\alpha]_H, \mathcal{U}, V)$ for any $V \in \mathcal{U}$ containing $\alpha(1)$.

When dealing with the pointed topological category the space $\widetilde{X}_H$ is equipped with the base point $\widetilde{x}_0$ equal to the equivalence class of the constant path at x_0.

Let us prove a basic functorial property of our construction.

Proposition 4.1. *Suppose $f\colon (X, x_0) \to (Y, y_0)$ is a map of pointed topological spaces. Let H and G be subgroups of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, such that $\pi_1(f)(H) \subset G$. Then f induces a natural function $\tilde{f}\colon (\widetilde{X}_H, \widetilde{x}_0) \to (\widetilde{Y}_G, \widetilde{y}_0)$ which is continuous in the lasso topology.*

Proof. Put $\tilde{f}([\alpha]_H) = [f \circ \alpha]_G$ and notice
$$\tilde{f}(B_H([\alpha]_H, f^{-1}(\mathcal{U}), f^{-1}(V))) \subset B_G(\tilde{f}([\alpha]_H), \mathcal{U}, V)$$
for any open covering $\mathcal{U}$ of Y and any neighborhood V of $\alpha(1)$. ∎

In connection to 2.13 let us prove the following:

Proposition 4.2. *Let X be a path-connected space and H be a subgroup of $\pi_1(X, x_0)$. If the set $\widetilde{X}_H$ is equipped with the lasso topology, then the following conditions are equivalent:*

(a) *A fiber of the endpoint projection $p_H\colon \widetilde{X}_H \to X$ has an isolated point.*
(b) *The endpoint projection $p_H\colon \widetilde{X}_H \to X$ has discrete fibers.*
(c) *There is an open covering $\mathcal{U}$ of X so that $\pi(\mathcal{U}, x_0) \subset H$.*
(d) *$\widetilde{X}_H$ is a Peano space and $p_H\colon \widetilde{X}_H \to P(X)$ is a classical covering map.*

Proof. (a)⇒(c). Suppose $[\alpha]_H \in p_H^{-1}(x_1)$ is isolated. There is an open covering $\mathcal{U}$ of X and $V \in \mathcal{U}$ containing x_1 such that $B_H([\alpha]_H, \mathcal{U}, V) \cap p_H^{-1}(x_1) = \{[\alpha]_H\}$. Given γ in $\pi(\mathcal{U}, x_0)$, the homotopy class $[\alpha^{-1} * \gamma * \alpha]_H$ belongs to $\pi(\mathcal{U}, x_1)$, so $[\alpha * \alpha^{-1} * \gamma * \alpha]_H = [\gamma * \alpha]_H$ belongs to $B_H([\alpha]_H, \mathcal{U}, V) \cap p_H^{-1}(x_1)$. Hence $[\gamma * \alpha]_H = [\alpha]_H$ and $[\gamma] \in H$.

(c)⇒(d). Suppose there is an open covering $\mathcal{U}$ of X so that $\pi(\mathcal{U},x_0)\subset H$ and W is a path component of $U\in\mathcal{U}$. Notice $B_H([\alpha]_H,\mathcal{U},U)$ is mapped by p_H bijectively onto W, and that is sufficient for (d).

(d)⇒(b) and (b)⇒(a) are obvious. ■

Applying 4.2 to H being trivial, one gets the following (see [11] for analogous result in case of $\pi_1^{\mathrm{top}}(X,x_0)$):

Corollary 4.3. *If X is a path-connected locally path-connected space, then $\pi_1(X,x_0)$ is discrete in the lasso topology if and only if X is semilocally simply connected.*

Proposition 4.4. *If $\pi(\mathcal{V},x_0)\subset H$ for some open cover $\mathcal{V}$ of X, then the identity function $\widehat{X}_H\to\widetilde{X}_H$ is a homeomorphism when $\widetilde{X}_H$ is equipped with the lasso topology.*

Proof. Let us show $B_H([\alpha]_H,\mathcal{U},W)=B_H([\alpha]_H,W)$ if $\mathcal{U}$ is an open cover of X refining $\mathcal{V}$ and W is an element of $\mathcal{U}$ containing $\alpha(1)$. Clearly, $B_H([\alpha]_H,W)\subset B_H([\alpha]_H,\mathcal{U},W)$, so assume $[\beta]_H\in B_H([\alpha]_H,\mathcal{U},W)$. There are $h\in H$, $[\lambda]\in\pi(\mathcal{U},\alpha(1))$, and a path γ in W such that $[\beta]=[h*\alpha*\lambda*\gamma]$. Choose $h_1\in H$ so that $[h_1*\alpha]=[\alpha*\lambda]$ ($h_1=[\alpha*\lambda*\alpha^{-1}]\in\pi(\mathcal{U},x_0)\subset H$). Now $[\beta]=[h*\alpha*\lambda*\gamma]=[h*h_1*\alpha*\gamma]$ and $[\beta]_H\in B_H([\alpha]_H,W)$.

Now we can show the identity function $\widehat{X}_H\to\widetilde{X}_H$ is open: given an open cover $\mathcal{W}$ of X and given a path α from x_0 to x_1 pick an element W of $\mathcal{U}=\mathcal{W}\cap\mathcal{V}$ containing x_1 and notice $B_H([\alpha]_H,\mathcal{U},W)\subset B_H([\alpha]_H,W)$. ■

Lemma 4.5. *If $G\subset H$ are subgroups of $\pi_1(X,x_0)$, then the projection $p\colon\widetilde{X}_G\to\widetilde{X}_H$ is open in the lasso topology.*

Proof. It suffices to show $p(B_G([\alpha]_G,\mathcal{U},V))=B_H([\alpha]_H,\mathcal{U},V)$. Clearly, $p(B_G([\alpha]_G,\mathcal{U},V))\subset B_H([\alpha]_H,\mathcal{U},V)$, so suppose $[\beta]_H\in B_H([\alpha]_H,\mathcal{U},V)$ and $[\beta]=[h*\alpha*\lambda*\gamma]$, where $[\lambda]\in\pi(\mathcal{U},\alpha(1))$ and γ is a path in V originating at $\beta(1)$. Observe $[\beta]_H=[\alpha*\lambda*\gamma]_H=p([\alpha*\lambda*\gamma]_G)$. ■

We arrived at the fundamental result for the lasso topology on $\widetilde{X}_H$:

Theorem 4.6. *Suppose $G\subset H$ are subgroups of $\pi_1(X,x_0)$. Consider the lasso topology on the sets $\widetilde{X}_G$ and $\widetilde{X}_H$. If G is normal in $\pi_1(X,x_0)$, then H/G, identified with the fiber $p^{-1}([\tilde{x}_0]_H)$ of the projection $p\colon\widetilde{X}_G\to\widetilde{X}_H$, is a topological group and acts continuously on $\widetilde{X}_G$ so that*

(a) *The natural map $(H/G)\times\widetilde{X}_G\to\widetilde{X}_G\times\widetilde{X}_G$ defined by $([\alpha]_G,[\beta]_G)\mapsto([\alpha*\beta]_G,[\beta]_G)$ is an embedding.*
(b) *The quotient map from $\widetilde{X}_G$ to the orbit space corresponds to the projection $p\colon\widetilde{X}_G\to\widetilde{X}_H$.*

Proof. The fiber F of the projection $p\colon\widetilde{X}_G\to\widetilde{X}_H$ is the set of classes $[\alpha]_G$ such that $[\alpha]\in H$, so it corresponds to H/G. Define $\mu\colon F\times\widetilde{X}_G\to\widetilde{X}_G$

as follows: given $[\alpha]_G \in F$ and given $[\beta]_G \in \widetilde{X}_G$ put $\mu([\alpha]_G, [\beta]_G) = [\alpha * \beta]_G$. To see μ is well defined assume $[\gamma_1], [\gamma_2] \in G$. Now $[\gamma_1 * \alpha * \gamma_2 * \beta]_G = [\gamma_1 * (\alpha * \gamma_2 * \alpha^{-1}) * (\alpha * \beta)]_G = [\alpha * \beta]_G$ as $[\alpha * \gamma_2 * \alpha^{-1}] \in G$ due to normality of G in H.

Suppose $\mathcal{U}$ is an open cover of X, that $V, V_1 \in \mathcal{U}$, and

(1) $[\alpha]_G \in F$, $[\beta]_G \in \widetilde{X}_G$,
(2) $[\alpha_1]_G \in B_G([\alpha]_G, \mathcal{U}, V_1) \cap F$, and $[\beta_1]_G \in B_G([\beta]_G, \mathcal{U}, V)$.

Thus $[\alpha_1] = [g_1 * \alpha * \lambda_1]$ for some $[\lambda_1] \in \pi(\mathcal{U}, x_0)$ and $[g_1] \in G$. Similarly, $[\beta_1] = [g_2 * \beta * \lambda_2 * \gamma]$, where $[g_2] \in G$, $[\lambda_2] \in \pi(\mathcal{U}, \beta(1))$, and γ is a path in V. Now,

$$\begin{aligned}
[\alpha_1^{-1} * \beta_1]_G &= [\lambda_1^{-1} * \alpha^{-1} * g_1^{-1} * g_2 * \beta * \lambda_2 * \gamma]_G \\
&= [(\lambda_1^{-1} * \alpha^{-1} * g_1^{-1} * g_2 * \alpha * \lambda_1) * \lambda_1^{-1} * \alpha^{-1} * \beta * \lambda_2 * \gamma]_G \\
&= [\lambda_1^{-1} * \alpha^{-1} * \beta * \lambda_2 * \gamma]_G \\
&= [(\alpha^{-1} * \beta) * (\beta^{-1} * \alpha * \lambda_1^{-1} * \alpha^{-1} * \beta) * \lambda_2 * \gamma]_G \\
&\quad \in B_G([\alpha^{-1} * \beta]_G, \mathcal{U}, V)
\end{aligned}$$

as $[\lambda_1^{-1} * \alpha^{-1} * g_1^{-1} * g_2 * \alpha * \lambda_1] \in G$ and $[\beta^{-1} * \alpha * \lambda_1^{-1} * \alpha^{-1} * \beta] \in \pi(\mathcal{U}, (\alpha^{-1} * \beta)(1))$.

The above calculations amount to

$$\rho((F \cap B_G(x, \mathcal{U}, V_1)) \times B_G(y, \mathcal{U}, V)) \subset B_G(\rho(x, y), \mathcal{U}, V),$$

where $\rho(x, y) := \mu(x^{-1}, y)$, which implies the following:

(1) F is a topological group,
(2) μ is continuous,
(3) $(x, y) \mapsto (\mu(x^{-1}, y), y)$ from $F \times \widetilde{X}_G$ onto its image is open.

As the map in (3) is injective, it is an embedding. Hence $(x, y) \mapsto (\mu(x, y), y)$ is an embedding.

To see (b) use 4.5 or check it directly. ■

5. Path lifting

Definition 5.1. A pointed map $f\colon (X, x_0) \to (Y, y_0)$ has the *path lifting property* if any path $\alpha\colon (I, 0) \to (Y, y_0)$ has a lift $\beta\colon (I, 0) \to (X, x_0)$.

A surjective map $f\colon X \to Y$ has the *path lifting property* if for any path $\alpha\colon I \to Y$ and any $y_0 \in f^{-1}(\alpha(0))$ there is a lift $\beta\colon I \to X$ of α such that $\beta(0) = y_0$.

Definition 5.2. A pointed map $f\colon (X, x_0) \to (Y, y_0)$ has the *uniqueness of path lifts property* if any two paths $\alpha, \beta\colon (I, 0) \to (X, x_0)$ are equal if $f \circ \alpha = f \circ \beta$.

A pointed map $f\colon (X,x_0)\to(Y,y_0)$ has the *unique path lifting property* if it has both the path lifting property and the uniqueness of path lifts property.

A map $f\colon X\to Y$ has the *uniqueness of path lifts property* if any two paths $\alpha,\beta\colon I\to X$ are equal if $f\circ\alpha=f\circ\beta$ and $\alpha(0)=\beta(0)$.

A surjective map $f\colon X\to Y$ has the *unique path lifting property* if it has both the path lifting property and the uniqueness of path lifts property.

COROLLARY 5.3. *Supppose $G\subset H$ are subgroups of $\pi_1(X,x_0)$. Consider the lasso topology on the sets $\widetilde{X}_G$ and $\widetilde{X}_H$ and identify H/G with the fiber $p^{-1}([\tilde{x}_0]_H)$ of the projection $p\colon \widetilde{X}_G\to\widetilde{X}_H$. If G is normal in $\pi_1(X,x_0)$, then the following conditions are equivalent:*

(a) *The map $p\colon \widetilde{X}_G\to\widetilde{X}_H$ has the uniqueness of path lifts property.*
(b) *$\pi_0(H/G)=H/G$, i.e. H/G has trivial path components.*

Proof. (a)$\Rightarrow$(b). If H/G has a non-trivial path component, then there is a non-trivial lift of the constant path at the base-point of $\widetilde{X}_H$.

(b)$\Rightarrow$(a). Suppose α and β are two lifts of the same path γ in $\widetilde{X}_H$ and $\alpha(0)=\beta(0)$. By 4.6 there is a path λ in H/G with the property $\lambda(t)\cdot\alpha(t)=\beta(t)$ for each $t\in I$. As $\lambda(0)=1\in H/G$ and H/G has trivial path components, $\lambda(t)=1\in H/G$ for all $t\in I$ and $\alpha=\beta$. ∎

PROPOSITION 5.4. *Supppose $G\subset H$ are subgroups of $\pi_1(X,x_0)$. Consider the lasso topology on the sets $\widetilde{X}_G$ and $\widetilde{X}_H$ and identify H/G with the fiber $p^{-1}([\tilde{x}_0]_H)$ of the projection $p\colon \widetilde{X}_G\to\widetilde{X}_H$. If G is normal in $\pi_1(X,x_0)$, then the following conditions are equivalent:*

(a) *H/G is a T_0-space.*
(b) *H/G is Hausdorff.*
(c) *Fibers of the projection $p\colon \widetilde{X}_G\to\widetilde{X}_H$ are T_0.*
(d) *Fibers of the projection $p\colon \widetilde{X}_G\to\widetilde{X}_H$ are Hausdorff.*
(e) *For each $h\in H-G$ there is a cover $\mathcal{U}$ such that $(G\cdot h)\cap\pi(\mathcal{U},x_0)=\emptyset$.*
(f) *G is closed in H if $\pi_1(X,x_0)$ is equipped with the lasso topology.*

Proof. In view of 4.6, (a)$\Leftrightarrow$(c) and (b)$\Leftrightarrow$(d).

(a)$\Rightarrow$(e). Assume H/G is T_0 and $h\in H-G$. Since $[\beta]_G\in B_G([\alpha]_G,\mathcal{U},V)$ is equivalent to $[\alpha]_G\in B_G([\beta]_G,\mathcal{U},V)$, there is an open cover $\mathcal{U}$ and $V\in\mathcal{U}$ containing x_0 such that $G\cdot h\notin B_G(G\cdot 1,\mathcal{U},V)$. That means precisely there is no $\lambda\in\pi(\mathcal{U},x_0)$ such that $G\cdot h=G\cdot\lambda$, hence $(G\cdot h)\cap\pi(\mathcal{U},x_0)=\emptyset$.

(e)$\Rightarrow$(d). Suppose α,β are two paths in (X,x_0) so that $[\alpha]_H=[\beta]_H$ but $[\alpha]_G\neq[\beta]_G$. Choose $h\in H-G$ satisfying $[h\cdot\alpha]=[\beta]$. Pick an open cover $\mathcal{U}$ of X satisfying $G\cdot h\cap\pi(\mathcal{U},x_0)=\emptyset$ and let $V\in\mathcal{U}$ contain $\alpha(1)$. Suppose $[\gamma]_G\in B_G([\alpha]_G,\mathcal{U},V)\cap B_G([\beta]_G,\mathcal{U},V)$ and $[\gamma]_H=[\alpha]_H$. Let $h_0\in H$ satisfy $[h_0\cdot\alpha]=[\gamma]$. Choose $\lambda_1,\lambda_2\in\pi(\mathcal{U},\alpha(1))$ such that $G\cdot[h_0\cdot\alpha]=G\cdot\alpha\cdot\lambda_1$ and

$G\cdot[h_0\cdot\alpha] = G\cdot[h\cdot\alpha]\cdot\lambda_2$. As G is normal in H, $G\cdot h = h\cdot G = G\cdot(\alpha\cdot\lambda_1\cdot\lambda_2^{-1}\cdot\alpha^{-1})$, a contradiction as $\alpha\cdot\lambda_1\cdot\lambda_2^{-1}\cdot\alpha^{-1}\in\pi(\mathcal{U},x_0)$.

(b)$\Rightarrow$(a) is obvious.

(e)$\Leftrightarrow$(f). G being closed in H means existence, for each $h\in H-G$, of an open cover $\mathcal{U}$ such that $G\cap B(h,\mathcal{U},V)=\emptyset$ for some $V\in\mathcal{U}$ containing x_0. That, in turn, is equivalent to non-existence of $\lambda\in\pi(\mathcal{U},x_0)$ satisfying $h\cdot\lambda\in G$, i.e. $(G\cdot h^{-1})\cap\pi(\mathcal{U},x_0)=\emptyset$. ∎

Corollary 5.5. *Suppose $G\subset H$ are subgroups of $\pi_1(X,x_0)$. If G is a normal subgroup of $\pi_1(X,x_0)$, then the following conditions are equivalent:*

(a) *H/G has trivial components.*
(b) *H/G has trivial path components.*
(c) *G is closed in H if $\pi_1(X,x_0)$ is equipped with the lasso topology.*

Proof. (b)$\Rightarrow$(c). Suppose H/G has trivial path components. In view of 5.4 it suffices to show H/G is T_0 to deduce G is closed in H. If there are two points u and v of H/G such that any open subset of H/G either contains both of them or contains neither, then any function $I\to\{u,v\}\subset H/G$ is continuous. Hence $u=v$ as H/G does not contain non-trivial paths.

(c)$\Rightarrow$(a). We need the following

Claim. *If $h_1,h_2\in H$ and $G\cdot f\in B_H(G\cdot h_1,\mathcal{U},V)\cap B_H(G\cdot h_2,\mathcal{U},V)\cap(H/G)$ for some open cover $\mathcal{U}$ of X and some $V\in\mathcal{U}$ containing x_0, then $G\cdot h_1^{-1}\cdot h_2\subset G\pi(\mathcal{U},x_0)$.*

Proof. $G\cdot f = G\cdot h_1\cdot\lambda_1$ and $G\cdot f = G\cdot h_2\cdot\lambda_2$ for some $\lambda_1,\lambda_2\in\pi(\mathcal{U},x_0)$. Now $h_1\cdot G = h_2\cdot G\cdot(\lambda_2\cdot\lambda_1^{-1})$ and $(h_1^{-1}\cdot h_2)\cdot G\subset G\cdot(\lambda_1\cdot\lambda_2^{-1})\subset G\pi(\mathcal{U},x_0)$. ∎

Suppose G is closed in H and $h\in H-G$. By 5.4 there is a cover $\mathcal{U}$ such that $(G\cdot h)\cap\pi(\mathcal{U},x_0)=\emptyset$. If there is a connected subset C of H/G containing $G\cdot h_1\cdot h$ and $G\cdot h_1$ for some $h_1\in H$, we consider the open cover $\{C\cap B_G(z,\mathcal{U},V)\}_{z\in C}$ of C and define the equivalence relation on C determined by that cover ($z\sim z'$ if there is a finite chain $z=z_1,\dots,z_k=z'$ in C such that $B_G(z_i,\mathcal{U},V)\cap B_G(z_{i+1},\mathcal{U},V)\cap C\neq\emptyset$ for all $i<k$). Equivalence classes of that relation are open, hence closed and must equal C. Thus there is a finite chain $h_1,\dots,h_k=h_1\cdot h$ in H such that $B_G([h_i]_G,\mathcal{U},V)\cap B_G([h_{i+1}]_G,\mathcal{U},V)\cap(H/G)\neq\emptyset$ for all $i<k$. By the Claim there are elements $g_i\in G$ $(i<k)$ so that $g_i\cdot h_i^{-1}\cdot h_{i+1}\in\pi(\mathcal{U},x_0)$. By normality of G in H there is $g\in G$ satisfying $g\cdot\prod_{i=1}^{k-1}h_i^{-1}\cdot h_{i+1}=g\cdot h\in\pi(\mathcal{U},x_0)$, a contradiction. ∎

Theorem 5.6. *If G is a normal subgroup of $\pi_1(X,x_0)$, then the following conditions are equivalent:*

(a) *The endpoint projection $p_G\colon(\widetilde{X}_G,\widetilde{x}_0)\to(X,x_0)$ has the unique path lifting property when $\widetilde{X}_G$ is equipped with the lasso topology.*
(b) *G is closed in $\pi_1(X,x_0)$ equipped with the lasso topology.*

(c) $\pi_1(p_G)\colon \pi_1(\widetilde{X}_G, \widetilde{x}_0) \to \pi_1(X, x_0)$ *is a monomorphism and its image equals* G.

Proof. Put $H = \pi_1(X, x_0)$ and observe $\widetilde{X}_H$ is the Peanification of (X, x_0) by 2.16.

(a)⇔(b). By 5.3 the group H/G has trivial path components. Use 5.5.

(a)⇒(c). Given a loop in $(\widetilde{X}_G, \widetilde{x}_0)$ we may assume it is a canonical lift of a loop α in (X, x_0). For that lift to be a loop we must have $[\alpha] \in G$. Thus the image of $\pi_1(p_G)\colon \pi_1(\widetilde{X}_G, \widetilde{x}_0) \to \pi_1(X, x_0)$ equals G (canonical lifts of elements of G show that the image contains G). If α is null-homotopic in (X, x_0), then its canonical lift is null-homotopic as well. Thus $\pi_1(p_G)\colon \pi_1(\widetilde{X}_G, \widetilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism.

(c)⇒(a). If H/G has a non-trivial path component (we use 5.3), then there is a path from the base point to a different point $[\alpha]_G$ of H/G. Concatenating the canonical lift of α with the reverse of that path gives a loop in $(\widetilde{X}_G, \widetilde{x}_0)$ whose image in $\pi_1(X, x_0)$ is $[\alpha] \notin G$, a contradiction. ∎

Proposition 5.7. *Suppose (X, x_0) is a pointed topological space and H is a subgroup of $\pi_1(X, x_0)$. The closure of H in the lasso topology on $\pi_1(X, x_0)$ consists of all elements $g \in \pi_1(X, x_0)$ such that for each open cover $\mathcal{U}$ of X there is $h \in H$ and $\lambda \in \pi(\mathcal{U}, x_0)$ satisfying $g = h \cdot \lambda$. If H is a normal subgroup of $\pi_1(X, x_0)$, then so is its closure.*

Proof. Suppose $g \in \pi_1(X, x_0)$ and for each open cover $\mathcal{U}$ of X there is $h \in H$ and $\lambda \in \pi(\mathcal{U}, x_0)$ satisfying $g = h \cdot \lambda$. Notice $B(g, \mathcal{U})$ contains h, so g belongs to the closure of H. If H is normal, then $k \cdot g \cdot k^{-1} = (k \cdot h \cdot k^{-1}) \cdot (k \cdot \lambda \cdot k^{-1})$ also belongs to the closure of H. ∎

Corollary 5.8. *The closure of the trivial subgroup of $\pi_1(X, x_0)$ in the lasso topology equals $\bigcap_{\mathcal{U} \in \mathrm{COV}} \pi(\mathcal{U}, x_0)$, where* COV *stands for the family of all open covers of X.*

Example 5.9. The harmonic archipelago HA of Bogley and Sieradski [2] is a Peano space such that $\pi_1(X, x_0)$ equals $\bigcap_{\mathcal{U} \in \mathrm{COV}} \pi(\mathcal{U}, x_0)$. Hence $\pi_1(X, x_0)$ is the only closed subgroup of $\pi_1(X, x_0)$ in the lasso topology. HA is built by stretching disks $B(2^{-n}, 2^{-n-2})$ to form cones over their boundaries with the vertices at height 1 in the 3-space.

Corollary 5.10. *Suppose (X, x_0) is a pointed topological space. The following subgroups of $\pi_1(X, x_0)$ are closed in the lasso topology:*

(a) *subgroups H containing $\pi(\mathcal{U}, x_0)$ for some open cover $\mathcal{U}$ of X,*
(b) *$\bigcap_{\mathcal{U} \in S} \pi(\mathcal{U}, x_0)$ for any family S of open covers of X,*
(c) *the kernel of $\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ for any map $f\colon (X, x_0) \to (Y, y_0)$ to a pointed locally path-connected semilocally simply connected space,*

(d) *the kernel of the natural homomorphism* $\pi_1(X,x_0)\to\check{\pi}_1(X,x_0)$ *from the fundamental group to the Čech fundamental group.*

Proof. (a) Any subgroup containing $\pi(\mathcal{U},x_0)$ is open. Any open subgroup of a topological group is closed.

(b) easily follows from (a).

(c) follows from 4.3 and 4.1 as $\pi_1^l(f)\colon \pi_1^l(X,x_0)\to\pi_1^l(Y,y_0)$ is continuous and $\pi_1^l(Y,y_0)$ is discrete.

(d) follows from (c). Indeed $\check{\pi}_1(X,x_0)$ is defined (see [8] or [20]) as the inverse limit of an inverse system $\{\pi_1(K_s,k_s)\}_{s\in S}$, where each K_s is a simplicial complex and there are maps $f_s\colon (X,x_0)\to(K_s,k_s)$ so that for $t>s$ the map f_s is homotopic to the composition of f_t and the bonding map $(K_t,k_t)\to(K_s,k_s)$. That means the kernel of the natural homomorphism $\pi_1(X,x_0)\to\check{\pi}_1(X,x_0)$ is the intersection of kernels of all $\pi_1(f_s)$, $s\in S$. ∎

The concept of a space X being *homotopically Hausdorff* was introduced by Conner and Lamoreaux [7, Definition 1.1] to mean that for any point x_0 in X and for any non-homotopically trivial loop γ at x_0 there is a neighborhood U of x_0 in X with the property that no loop in U is homotopic to γ rel. x_0 in X. Subsequently, Fischer and Zastrow [16] defined a space X to be *homotopically Hausdorff relative to a subgroup* H *of* $\pi_1(X,x_0)$ if for any $g\notin H$ and for any path α originating at x_0 there is an open neighborhood U of $\alpha(1)$ in X such that no element of $H\cdot g$ can be expressed as $[\alpha*\gamma*\alpha^{-1}]$ for some loop γ in $(U,\alpha(1))$. We generalize this definition as follows:

Definition 5.11. Suppose $G\subset H$ are subgroups of $\pi_1(X,x_0)$. Then X is (H,G)*-homotopically Hausdorff* if for any $h\in H\setminus G$ and any path α originating at x_0 there is an open neighborhood U of $\alpha(1)$ in X such that none of the elements of $G\cdot h$ can be expressed as $[\alpha*\gamma*\alpha^{-1}]$ for any loop γ in $(U,\alpha(1))$.

Notice X being homotopically Hausdorff relative to H corresponds to X being $(\pi_1(X,x_0),H)$-homotopically Hausdorff.

Let us characterize the concept of being (H,G)-homotopically Hausdorff in terms of the whisker topology on the fundamental group. Recall that for a path α in X the homomorphism $h_\alpha\colon \pi_1(X,\alpha(1))\to\pi_1(X,\alpha(0))$ is given by $h_\alpha([\gamma])=[\alpha*\gamma*\alpha^{-1}]$.

Proposition 5.12. *If* $G\subset H$ *are subgroups of* $\pi_1(X,x_0)$*, then* X *is* (H,G)*-homotopically Hausdorff if and only if for every path* α *in* X *that terminates at* x_0 *the group* $h_\alpha(G)$ *is closed in* $h_\alpha(H)$ *in the whisker topology.*

Proof. $h_\alpha(G)$ being closed in $h_\alpha(H)$ means existence, for each $h\in H\setminus G$, of a neighborhood U of $x_1=\alpha(0)$ such that $B([\alpha*h*\alpha^{-1}],U)\cap([\alpha]\cdot G\cdot[\alpha^{-1}])=\emptyset$. Thus, for every loop γ in U at x_1, there is no $g\in G$ satisfying

$[\alpha * h * \alpha^{-1} * \gamma^{-1}] = [\alpha * g^{-1} * \alpha^{-1}]$. The last equality is equivalent to $[g * h] = [\alpha^{-1} * \gamma * \alpha]$. ∎

EXAMPLE 5.13. Proposition 5.12 allows for an easy construction of subgroups H of $\pi_1(X, x_0)$ such that X is not homotopically Hausdorff relative to H. Namely, $X = S^1 \times S^1 \times \cdots$ and $H = \bigoplus Z \subset \prod Z = \pi_1(X)$.

Let us show that G being closed in H (in the lasso topology) is a stronger condition than X being (H, G)-homotopically Hausdorff.

LEMMA 5.14. *Suppose $G \subset H$ are subgroups of $\pi_1(X, x_0)$. If G is closed in H in the lasso topology, then X is (H, G)-homotopically Hausdorff.*

Proof. Given $h \in H \setminus G$ pick an open cover $\mathcal{U}$ and $W \in \mathcal{U}$ containing x_0 so that $B(h, \mathcal{U}, W)$ does not intersect G. Given a path α in X from x_0 to x_1 choose $V \in \mathcal{U}$ containing x_1. Suppose there is a loop γ in (V, x_1) so that $[\alpha * \gamma * \alpha^{-1}] = g \cdot h$ for some $g \in G$. Now $[\alpha * \gamma^{-1} * \alpha^{-1}] \in \pi(\mathcal{U}, x_0)$ and $g^{-1} = h * [\alpha * \gamma^{-1} * \alpha^{-1}] \in G \cap B(h, \mathcal{U}, W)$, a contradiction. ∎

REMARK 5.15. The proof of 5.14 suggests that the trivial subgroup of $\pi_1(X, x_0)$ being closed is philosophically related to the concept of X being *strongly homotopically Hausdorff* (see [23]). Recall a metric space X is strongly homotopically Hausdorff if for any non-null-homotopic loop α in X there is an $\epsilon > 0$ such that α is not freely homotopic to a loop of diameter less than ϵ.

LEMMA 5.16. *Given subgroups $G \subset H$ of $\pi_1(X, x_0)$ the following conditions are equivalent:*

(a) *The fibers of the natural projection $p \colon \widehat{X}_G \to \widehat{X}_H$ are T_0.*
(b) *The fibers of the natural projection $p \colon \widehat{X}_G \to \widehat{X}_H$ are Hausdorff.*
(c) *X is (H, G)-homotopically Hausdorff.*

Proof. (a)⇒(c). Suppose $h \in H \setminus G$ and α is a path in X from x_0 to x_1. As $[h * \alpha]_G \neq [\alpha]_G$ belong to the same fiber of p, there is a neighborhood U of x_1 so that $[h * \alpha]_G \notin B_G([\alpha]_G, U)$ or $[\alpha]_G \notin B_G([h * \alpha]_G, U)$. Notice $[h * \alpha]_G \notin B_G([\alpha]_G, U)$ is equivalent to $[\alpha]_G \notin B_G([h * \alpha]_G, U)$. Suppose there is a loop γ in (U, x_1) so that $g \cdot h = [\alpha * \gamma * \alpha^{-1}]$ for some $g \in G$. Now $[h * \alpha]_G = [g \cdot h * \alpha]_G = [\alpha * \gamma]_G \in B_G([\alpha]_G, U)$, a contradiction.

(c)⇒(b). Any two different elements of the same fiber of p can be represented as $[h * \alpha]_G \neq [\alpha]_G$ for some path α in X from x_0 to x_1 and some $h \in H \setminus G$. Choose a neighborhood U of x_1 with the property that none of the elements of $G \cdot h$ can be expressed as $[\alpha * \gamma * \alpha^{-1}]$ for any loop γ in (U, x_1). Suppose $[\beta]_G \in (H/G) \cap B_G([\alpha]_G, U) \cap B_G([h * \alpha]_G, U)$. That means existence of loops γ_1, γ_2 in (U, x_1) so that $[\beta]_G = [h * \alpha * \gamma_1]_G = [\alpha * \gamma_2]_G$. Hence $[h]_G = [\alpha * (\gamma_2 * \gamma_1^{-1}) * \alpha^{-1}]_G$, a contradiction. ∎

LEMMA 5.17. *Supppose $G \subset H$ are subgroups of $\pi_1(X, x_0)$, G is normal in $\pi_1(X, x_0)$, and X is (H, G)-homotopically Hausdorff. If $\alpha, \beta\colon (I, 0) \to (\widehat{X}_G, \widehat{x}_0)$ are two continuous lifts of the same path $\gamma\colon (I, 0) \to (\widehat{X}_H, \widehat{x}_0)$, then for every $h \in H$ the set*

$$S = \{t \in I \mid \alpha(t) = h \cdot \beta(t)\}$$

is closed.

Proof. Choose paths u_t, v_t in (X, x_0) so that $\alpha(t) = [u_t]_G$ and $\beta(t) = [v_t]_G$ for all $t \in I$. Assume $[u_t]_G \neq [h \cdot v_t]_G$ for some $t \in I$. Pick a neighborhood U of $x_1 = u_t(1)$ so that $[v_t * u_t^{-1}] \cdot h \cdot G \neq [v_t * \gamma * v_t^{-1}] \cdot G$ for any loop γ in (U, x_1). There is a neighborhood V of t in I so that $[u_s]_G \in B_G([u_t]_G, U)$ and $[v_s]_G \in B_G([v_t]_G, U)$ for all $s \in V$. That means $[u_s] = [g_1 * u_t * \gamma_1]$ and $[v_s] = [g_2 * v_t * \gamma_2]$ for some $g_1, g_2 \in G$ and some paths γ_1, γ_2 in U joining x_1 and $u_1(1) = v_s(1)$. Put $\gamma = \gamma_1 * \gamma_2^{-1}$ and notice $[u_s * v_s^{-1}] = [g_1 * u_t * v_t^{-1} * (v_t * \gamma * v_t^{-1}) * g_2^{-1}]$. As G is normal in $\pi_1(X, x_0)$, there is $g_3 \in G$ satisfying $[g_1 * u_t * v_t^{-1} * (v_t * \gamma * v_t^{-1}) * g_2^{-1}] = [g_3 * u_t * v_t^{-1} * (v_t * \gamma * v_t^{-1})]$ and that element cannot belong to $G \cdot h$ by the choice of U. ∎

COROLLARY 5.18. *Supppose $G \subset H$ are subgroups of $\pi_1(X, x_0)$. If H/G is countable, G is normal in $\pi_1(X, x_0)$, and X is (H, G)-homotopically Hausdorff, then the natural map $\widehat{X}_G \to \widehat{X}_H$ has the uniqueness of path lifts property.*

Proof. Pick representatives $h_i \in H$, $i \geq 1$, of all right cosets of H/G so that $h_1 = 1$. If α and β are two continuous lifts in $\widehat{X}_G$ of the same path in $\widehat{X}_H$, then each set $S_i = \{t \in I | \alpha(t) = h_i \cdot \beta(t)\}$ is closed, they are disjoint, and their union is the whole interval I. Hence only one of them is non-empty and it must be S_1. Thus $\alpha = \beta$. ∎

6. Peano maps. This section is about one of the main ingredients of our theory of covering maps for lpc-spaces. It amounts to the following generalization of Peano spaces:

DEFINITION 6.1. A map $f\colon X \to Y$ is a *Peano map* if the family of path components of $f^{-1}(U)$, U open in Y, forms a basis of neighborhoods of X.

Notice X is an lpc-space if $f\colon X \to Y$ is a Peano map. One may reword the above definition as follows: X is an lpc-space and lifts of short paths in Y are short in X. Indeed, given a neighborhood U of $x_0 \in X$ there is a neighborhood V of $f(x_0)$ in Y such that any path α in $(f^{-1}(V), x_0)$ (i.e. $f \circ \alpha$ is contained in V, hence short) must be contained in U.

PROPOSITION 6.2. *Any product of Peano maps is a Peano map.*

Proof. Suppose $f_s\colon X_s \to Y_S$, $s \in S$, are Peano maps. Observe $X = \prod_{s \in S} X_s$ is an lpc-space. Given a neighborhood U of $x = \{x_s\}_{s \in S} \in X$, we find a finite subset T of S and neighborhoods U_s of x_s in X_s such that $\prod_{s\in S} U_s \subset U$ and $U_s = X_s$ for $s \notin T$. Choose neighborhoods V_s of $f_s(x_s)$ in Y_s, $s \in T$, so that the path component of x_s in $f_s^{-1}(V_s)$ is contained in U_s. Put $V_s = X_s$ for $s \notin T$ and observe that the path component of x in $f^{-1}(V)$, $f = \prod_{s\in S} f_s$ and $V = \prod_{s\in S} V_s$, is contained in U. ∎

Here is our basic class of Peano maps:

Proposition 6.3. *If H is a subgroup of $\pi_1(X, x_0)$, then the endpoint projection $p_H\colon \widehat{X}_H \to X$ is a Peano map.*

Proof. It suffices to show that for any U open in X the path component of any $[\alpha]_H$ in $p_H^{-1}(U)$ is precisely $B_H([\alpha]_H, U)$. It is straightforward that $B_H([\alpha]_H, U)$ is path-connected so suppose β is a path in $p_H^{-1}(U)$ starting at $[\alpha]_H$. We wish to show that $\beta([0,1]) \subset B_H([\alpha]_H, U)$. Let $T = \{t : \beta(t) \in B_H([\alpha]_H, U)\}$. Now T is non-empty since $\beta(0) = [\alpha]_H$ and open as the inverse image of an open set. It suffices to prove $[0,t) \subset T$ implies $[0,t] \subset T$. Set $\beta(t) = [b]_H$. Now $p_H\beta([0,1]) \subset U$ so in particular $p_H([b]_H) \in U$. Consider $B_H([b]_H, U)$. There is an $\varepsilon > 0$ such that $\beta(t-\varepsilon, t] \subset B_H([b]_H, U)$. Pick $s \in (t-\varepsilon, t)$. Then $\beta(s) = [c_1]_H$ and $[b]_H = [b_1]_H$ so that $c_1 \simeq b_1 * \gamma_1$ for some γ_1 with $\gamma_1[0,1] \subset U$. But $\beta(s) \in B_H([\alpha]_H, U)$ so $\beta(s) = [c_2]_H$ and $[\alpha]_H = [a_1]_H$ so that $c_2 \simeq a_1 * \gamma_2$ for some γ_2 with $\gamma_2([0,1]) \subset U$. Then $b \simeq_H b_1 \simeq c_1 * \gamma_1^{-1} \simeq_H c_2 * \gamma_1^{-1} \simeq a_1 * \gamma_2 * \gamma_1^{-1} \simeq_H a * \gamma_2 * \gamma_1^{-1}$ and $(\gamma_2 * \gamma_1^{-1})([0,1]) \subset U$ so $[b]_H \in B_H([\alpha]_H, U)$ and $t \in T$. Therefore $T = [0,1]$. ∎

In analogy to path lifting and unique path lifting properties (see 5.1 and 5.2) one can introduce the corresponding concepts for hedgehogs:

Definition 6.4. A surjective map $f\colon X \to Y$ has the *hedgehog lifting property* if for any map $\alpha\colon \bigvee_{s\in S} I_s \to Y$ from a hedgehog and any $y_0 \in f^{-1}(\alpha(0))$ there is a continuous lift $\beta\colon \bigvee_{s\in S} I_s \to X$ of α such that $\beta(0) = y_0$.

Definition 6.5. $f\colon X \to Y$ has the *unique hedgehog lifting property* if it has both the hedgehog lifting property and the uniqueness of path lifts property.

Theorem 6.6. *If $f\colon X \to Y$ has the unique hedgehog lifting property, then $f\colon lpc(X) \to Y$ is a Peano map.*

Proof. Assume U is open in X and $x_0 \in U$. Suppose for each neighborhood V of $f(x_0)$ in X there is a path $\alpha_V\colon (I,0) \to (f^{-1}(V), x_0)$ such that $\alpha_V(1) \notin U$. By 2.9 the wedge $\bigvee_{V\in S} f \circ \alpha_V$ is a map g from a hedgehog to Y (here S is the family of all neighborhoods of $f(x_0)$ in Y). Its lift must be the wedge $h = \bigvee_{V\in S} \alpha_V$. However $h^{-1}(U)$ is not open in $lpc(X)$, a contradiction. ∎

Definition 6.7. Given a map $f\colon X \to Y$ of topological spaces its *Peano map* $P(f)\colon P_f(X) \to Y$ is f on X equipped with the topology generated by path components of sets $f^{-1}(U)$, U open in Y.

Notice that in the case of $f = \mathrm{id}_X$ the range $P_{\mathrm{id}_X}(X)$ of $P(\mathrm{id}_X)$, where $\mathrm{id}_X\colon X \to X$ is the identity map, is identical to $lpc(X)$ as defined in Theorem 2.2.

Recall $f\colon X \to Y$ is a *Hurewicz fibration* if every commutative diagram

$$\begin{array}{ccc} K \times \{0\} & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow f \\ K \times I & \xrightarrow{H} & Y \end{array}$$

has a filler $G\colon K \times I \to X$ (that means $f \circ G = H$ and G extends α). If the above condition is satisfied for K being any n-cell I^n, $n \geq 0$ (equivalently, for any finite polyhedron K), then f is called a *Serre fibration.* Notice for K being a point this is the classical *path lifting property.*

If the above condition is satisfied for K being any hedgehog, then f is called a *hedgehog fibration.* If the above condition is satisfied for K being any Peano space, then f is called a *Peano fibration.*

We will modify those concepts for maps between pointed spaces as follows:

Definition 6.8. A map $f\colon (X, x_0) \to (Y, y_0)$ is a *Serre* 1*-fibration* if any commutative diagram

$$\begin{array}{ccc} (I \times \{0\}, (\frac{1}{2}, 0)) & \xrightarrow{\alpha} & (X, x_0) \\ \downarrow & & \downarrow f \\ (I \times I, (\frac{1}{2}, 0)) & \xrightarrow{H} & (Y, y_0) \end{array}$$

has a filler $G\colon (I \times I, (\frac{1}{2}, 0)) \to (X, x_0)$ (that means $f \circ G = H$ and G extends α).

Observe Serre 1-fibrations have the path lifting property in the sense that any path in Y starting at y_0 lifts to a path in X originating at x_0.

Theorem 6.9. *Suppose*

$$\begin{array}{ccc} (T, z_0) & \xrightarrow{g_1} & (X, x_0) \\ i\downarrow & & \downarrow f \\ (Z, z_0) & \xrightarrow{g} & (Y, y_0) \end{array}$$

is a commutative diagram in the topological category such that (Z, z_0) *is a Peano space and* i *is the inclusion from a path-connected subspace* T *of* Z*. If* f *is a Serre* 1*-fibration, then there is a continuous lift* $h\colon (Z, z_0) \to$

$(P_f(X), x_0)$ of g extending g_1 if the image of $\pi_1(g)\colon \pi_1(Z, z_0) \to \pi_1(Y, y_0)$ is contained in the image of $\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Proof. For each point $z \in Z$ pick a path α_z in Z from z_0 to z and let β_z be a lift of $g\colon \alpha_z \mapsto Y$. In the case of $z = z_0$ we pick the constant paths α_z and β_z. In case $z \in T$ the path α_z is contained in T and $\beta_z = g_1 \circ \alpha_z$. Define $h\colon (Z, z_0) \to (P_f(X), x_0)$ by $h(z) = \beta_z(1)$. Given a neighborhood U of $g(z)$ in Y, let V be the path component of $h(z)$ in $f^{-1}(U)$ and let W be the path component of $g^{-1}(U)$ containing z. Our goal is to show $h(W) \subset V$, as that is sufficient for $h\colon (Z, z_0) \to (P_f(X), x_0)$ to be continuous. For any $t \in W$ choose a path μ_t in W from z to t. Let γ be a loop in X at x_0 so that $f(\gamma)$ is homotopic to $g(\alpha_z * \mu_t * \alpha_t^{-1})$. Notice $f(\beta_z)$ is homotopic to $f(\gamma * \beta_t)$ via a homotopy H so that $H(\{1\} \times I) \subset U$. By lifting that homotopy to X we get a path in $f^{-1}(U)$ from $h(z)$ to $h(t)$, i.e., $h(t) \in V$. ∎

Corollary 6.10. *A Peano map $f\colon X \to Y$ is a Peano fibration if and only if it is a Serre 1-fibration.*

Proof. Assume $f\colon X \to Y$ is a Peano map and a Serre 1-fibration (in the other direction 6.10 is left as an exercise), $g\colon Z \times \{0\} \to X$ is a map from a Peano space, and $H\colon Z \times I \to Y$ is a homotopy starting from $f \circ g$. Pick $z_0 \in Z$ and put $x_0 = g(z_0, 0)$, $y_0 = f(x_0)$. Notice the image of $\pi_1(g)\colon \pi_1(Z \times \{0\}, (z_0, 0)) \to \pi_1(Y, y_0)$ is contained in the image of $\pi_1(f)$. Use 6.9 to produce an extension $G\colon Z \times I \to X$ of g that is a lift of H. ∎

7. Peano covering maps. 6.9 suggests the following concept:

Definition 7.1. A map $f\colon X \to Y$ is called a *Peano covering map* if the following conditions are satisfied:

(1) f is a Peano map.
(2) f is a Serre fibration.
(3) The fibers of f have trivial path components.

Notice (3) above can be replaced by f having the unique path lifting property (see 9.3). Also notice that, in case fibers of a Peano map $f\colon X \to Y$ are T_0 spaces, path components of fibers are trivial. Indeed, two points in a path component of a fiber are always in any open set that contains one of them.

Proposition 7.2. *Any product of Peano covering maps is a Peano covering map.*

Proof. Suppose $f_s\colon X_s \to Y_S$, $s \in S$, are Peano covering maps. Put $f = \prod_{s\in S} f_s$, $X = \prod_{s\in S} X_s$, and $Y = \prod_{s\in S} Y_s$. By 6.2, f is a Peano map. It is obvious f is a Serre fibration and has the uniqueness of path lifting property. ∎

COROLLARY 7.3. *Suppose $f\colon (X,x_0)\to(Y,y_0)$ is a Peano covering map. If (Z,z_0) is a Peano space, then any map $g\colon (Z,z_0)\to(Y,y_0)$ has a unique continuous lift $h\colon (Z,z_0)\to(X,x_0)$ if the image of $\pi_1(g)$ is contained in the image of $\pi_1(f)$.*

Proof. By 6.9 a lift h exists and is unique by the uniqueness of path lifting property. ∎

Our basic example of Peano covering maps is related to the whisker topology:

THEOREM 7.4. *If X is a path-connected space and $x_0\in X$, then the following conditions are equivalent:*

(a) *$p_H\colon (\widehat{X}_H,\widehat{x}_0)\to(X,x_0)$ has the unique path lifting property.*
(b) *$p_H\colon \widehat{X}_H\to X$ is a Peano covering map.*

Proof. (a)⇒(b). In view of 6.3 and 9.4 it suffices to show $p_H\colon (\widehat{X}_H,\widehat{x}_0)\to(X,x_0)$ is a Serre fibration. Suppose $f\colon (Z,z_0)\to(X,x_0)$ is a map from a simply connected Peano space Z (the case of $Z=I^n$ is of interest here). There is a standard lift $g\colon (Z,z_0)\to\widehat{X}_H$ of f defined as $g(z)=[\alpha_z]_H$, where α_z is a path in Z from z_0 to z. If T is a path-connected subspace of Z containing z_0 and $h\colon (T,z_0)\to(\widehat{X}_H,\widehat{x}_0)$ is any continuous lift of $f|T$, then $h=g|T$ due to the uniqueness of path lifting property of p_H. That proves p_H is a Serre fibration in view of 9.4 again.

(b)⇒(a) is obvious. ∎

THEOREM 7.5. *If $f\colon X\to Y$ is a map and X is an lpc-space, then the following conditions are equivalent:*

(a) *f is a Peano covering map.*
(b) *f is a Peano fibration and has the uniqueness of path lifting property.*
(c) *f is a hedgehog fibration and has the uniqueness of path lifting property.*
(d) *For any $x_0\in X$ and any map $g\colon (Z,z_0)\to(Y,f(x_0))$ from a simply-connected Peano space there is a lift $h\colon (Z,z_0)\to(X,x_0)$ of g and that lift is unique.*

Proof. (a)⇒(b). Suppose $H\colon Z\times I\to Y$ is a homotopy, Z is a Peano space, and $G\colon Z\times\{0\}\to X$ is a lift of $H|Z\times\{0\}$. Pick $z_0\in Z$, put $x_0=G(z_0,0)$ and $y_0=f(x_0)$, and notice $\operatorname{im}(\pi_1(Z\times I,(z_0,0)))\subset\operatorname{im}(\pi_1(f))$. Using 6.9 we get a lift of H and that lift is unique, hence it agrees with G on $Z\times\{0\}$.

(b)⇒(c) is obvious.

(d)⇒(c) is obvious.

(a)⇒(d) follows from 6.9.

(c)$\Rightarrow$(a). Notice f has the unique hedgehog lifting property and is a Serre 1-fibration. By 6.6, f is a Peano map. ∎

Corollary 7.6. *Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are maps of path-connected spaces and Y is a Peano space. If any two of f, g, $h = g \circ f$ are Peano covering maps, then so is the third provided its domain is an lpc-space.*

Proof. In view of 7.5 this amounts to verifying that the map has uniqueness of lifts of simply connected Peano spaces, an easy exercise. ∎

Proposition 7.7. *Suppose $f\colon X \to Y$ is a map.*

(a) *If $f\colon X \to Y$ is a Peano covering map, then $f\colon f^{-1}(U) \to U$ is a Peano covering map for every open subset U of Y.*
(b) *If every point $y \in Y$ has a neighborhood U such that $f\colon f^{-1}(U) \to U$ is a Peano covering map, then f is a Peano covering map.*

Proof. (a) $f\colon f^{-1}(U) \to U$ is clearly a Peano map, is a fibration, and has the unique path lifting property.

(b) f is a Serre 1-fibration and path components of fibers are trivial. If V is an open subset of Y containing y we pick an open subset U of X containing $f(y)$ such that $f\colon f^{-1}(U) \to U$ is a Peano covering map. There is an open neighborhood W of $f(y)$ in U so that the path component of y in $f^{-1}(W)$ is open and is contained in $V \cap f^{-1}(U)$. That proves $f\colon Y \to X$ is a Peano map. ∎

In analogy to regular classical covering maps let us introduce regular Peano covering maps:

Definition 7.8. A Peano covering map $f\colon X \to Y$ is *regular* if lifts of loops in Y are either always loops or always non-loops.

Corollary 7.9. *Given a map $f\colon X \to Y$ the following conditions are equivalent if X is path-connected:*

(a) *f is a regular Peano covering map.*
(b) *f is a Peano covering map and the image of $\pi_1(f)$ is a normal subgroup of $\pi_1(Y, f(x_0))$ for all $x_0 \in X$.*
(c) *$f\colon X \to Y$ is a generalized covering map in the sense of Fischer–Zastrow.*

Proof. (a)$\Rightarrow$(b). If the image of $\pi_1(f)$ is not a normal subgroup of $\pi_1(Y, f(x_0))$ for some $x_0 \in X$, then there is a loop α in Y at $y_0 = f(x_0)$ that lifts to a loop in X at x_0 and there is a loop β in Y at y_0 such that $\beta * \alpha * \beta^{-1}$ does not lift to a loop in X at x_0. Let γ be a lift of α originating at x_0. Let $x_1 = \beta(1)$. Notice the lift of α originating at x_1 cannot be a loop, a contradiction.

(b)⇒(c). As $\mathrm{im}(\pi_1(f))$ is a normal subgroup H of $\pi_1(Y, y_0)$, it does not depend on the choice of the base point of X in $f^{-1}(y_0)$. Using 6.9 one deduces f is a generalized covering map.

(c)⇒(a). Since each hedgehog is contractible, f has the unique hedgehog lifting property and is a Peano map by 6.6. It is also a Serre fibration, hence a Peano covering map. Also, as $H = \mathrm{im}(\pi_1(f))$ is a normal subgroup of $\pi_1(Y, y_0)$, it does not depend on the choice of the base point of X in $f^{-1}(y_0)$. Hence a loop in Y lifts to a loop in X if and only if it represents an element of H. Thus f is a regular Peano covering map. ■

In the remainder of this section we will discuss the relation of Peano covering maps to classical covering maps.

Proposition 7.10. *If $f\colon Y \to X$ is a Peano covering map and U is an open subset of X such that every loop in U is null-homotopic in X, then $f^{-1}(V) \to P(V)$ is a trivial discrete bundle for every path component V of U.*

Proof. Consider a path component W of $f^{-1}(U)$ intersecting $f^{-1}(V)$. Then f maps W bijectively onto V and it is easy to see $f|W\colon W \to V$ is equivalent to $P(V) \to V$. ■

Corollary 7.11. *If X is a semilocally simply connected Peano space, then $f\colon Y \to X$ is a Peano covering map if and only if it is a classical covering map and Y is connected.*

Proof. If f is a classical covering map and Y is connected, then Y is locally path-connected, f has the unique path lifting property and is a Serre 1-fibration. Thus it is a Peano covering map.

Suppose f is a Peano covering map and $x \in X$. Choose a path-connected neighborhood U of x in X such that any loop in U is null-homotopic in X. By 7.10, U is evenly covered by f. ■

Corollary 7.12. *If $f\colon Y \to P(X)$ is a classical covering map, then $f\colon Y \to X$ is a Peano covering map.*

Proof. By 7.7, $f\colon Y \to P(X)$ is a Peano covering map. As the identity function induces a Peano covering map $P(X) \to X$, $f\colon Y \to X$ is a Peano covering map by 7.6. ■

Proposition 7.13. *If $f\colon Y \to X$ is a Peano covering map and X is path-connected, then all fibers of f have the same cardinality.*

Proof. Given two points $x_1, x_2 \in X$ fix a path α from x_1 to x_2 and notice lifts of α establish bijectivity of fibers $f^{-1}(x_1)$ and $f^{-1}(x_2)$. ■

The following result has its origins in Lemma 2.3 of [7] and Proposition 6.6 of [16].

Proposition 7.14. *Suppose $f\colon Y \to X$ is a regular Peano covering map. If $f^{-1}(x_0)$ is countable and x_0 has a countable basis of neighborhoods in X, then there is a neighborhood U of x_0 in X such that $f^{-1}(V) \to P(V)$ is a classical covering map, where V is the path component of x_0 in U.*

Proof. Switch to X being Peano by considering $f\colon Y \to P(X)$. Notice x_0 has a countable basis of neighborhoods and f is open. Suppose there is no open subset U of X containing x_0 such that U is evenly covered. That means path components of $f^{-1}(U)$ are not mapped bijectively onto their images.

First, we plan to show there is a neighborhood U of x_0 in X such that the image of $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is contained in the image of $\pi_1(f)\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$. In particular, there is a lift $P(U, x_0) \to (Y, y_0)$ of the inclusion induced map $P(U, x_0) \to (X, x_0)$.

Suppose no such U exists. By induction we will find a basis of neighborhoods $\{U_i\}$ of x_0 in X and elements $[\alpha_i] \in \pi_1(U_i, x_0)$ that are not contained in the image of $\pi_1(U_{i+1}, x_0) \to \pi_1(X, x_0)$ and whose lifts are not loops and end at points y_i such that $y_i \neq y_j$ if $i \neq j$. Given a neighborhood U_i pick a loop α_i in (U_i, x_0) whose lift (as a path) in (Y, y_0) is not a loop and ends at $y_i \neq y_0$. There is a neighborhood U_{i+1} of x_0 in U_i such that no path component of $f^{-1}(U_{i+1})$ contains both y_0 and some y_j, $j \leq i$. Pick a loop α_{i+1} in (U_{i+1}, x_0) whose lift is not a loop.

As in [22] one can create infinite concatenations $\alpha_{i(1)} * \cdots * \alpha_{i(k)} * \cdots$ for any increasing sequence $\{i(k)\}_{k\geq 1}$. By looking at lifts of those infinite concatenations, there are two different infinite concatenations $\alpha_{i(1)} * \cdots * \alpha_{i(k)} * \cdots$ and $\alpha_{j(1)} * \cdots * \alpha_{j(k)} * \cdots$ whose lifts end at the same point $y \in f^{-1}(x_0)$. Pick the smallest k_0 so that $i(k_0) \neq j(k_0)$. We may assume $i(k_0) < j(k_0)$ and conclude there are loops β in (U_{k_0+1}, x_0) and γ in (Y, y_0) so that $\alpha_{i(k_0)} \sim f(\gamma) * \beta$, in which case the lift of $\alpha_{i(k_0)}$ in (Y, y_0) ends in the path component of $f^{-1}(U_{i(k_0)+1})$ containing y_0, a contradiction.

As f is a regular Peano covering map, we can find lifts $(U, x_0) \to (Y, y)$ of the inclusion map $(U, x_0) \to (X, x_0)$ for any $y \in f^{-1}(x_0)$. ∎

8. Peano subgroups

Definition 8.1. Suppose (X, x_0) is a pointed path-connected space. A subgroup H of $\pi_1(X, x_0)$ is a *Peano subgroup* of $\pi_1(X, x_0)$ if there is a Peano covering map $f\colon Y \to X$ such that H is the image of $\pi_1(f)\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ for some $y_0 \in f^{-1}(x_0)$.

Proposition 8.2. *If H is a Peano subgroup of $\pi_1(X, x_0)$, then X is homotopically Hausdorff relative to H. In particular, H is closed in $\pi_1(X, x_0)$ equipped with the whisker topology.*

Proof. Choose a Peano covering map $f\colon Y \to X$ so that $\operatorname{im}(\pi_1(f)) = H$ for some $y_0 \in f^{-1}(x_0)$. If $g \in \pi_1(X, x_0) \setminus H$ and α is a path in X from x_0 to x_1, then lifts of α and $g \cdot \alpha$ end in two different points y_1 and y_2 of the fiber $f^{-1}(x_1)$ and there is a neighborhood U of x_1 in X such that no path component of $f^{-1}(U)$ contains both y_1 and y_2. Suppose there is a loop γ in (U, x_1) with the property $[\alpha * \gamma * \alpha^{-1}] \in H \cdot g$. In that case the lifts of both $\alpha * \gamma$ and $g \cdot \alpha$ end at y_2. Since the lift of α ends in the same path component of $f^{-1}(U)$ as the lift of $\alpha * \gamma$, both y_1 and y_2 belong to the same component of $f^{-1}(U)$, a contradiction.

Use 5.12 to conclude H is closed in $\pi_1(X, x_0)$ equipped with the whisker topology. ∎

Proposition 8.3. *If H is a Peano subgroup of $\pi_1(X, x_0)$, then any conjugate of H is a Peano subgroup of $\pi_1(X, x_0)$.*

Proof. Choose a Peano covering map $f\colon Y \to X$ so that $\operatorname{im}(\pi_1(f)) = H$ for some $y_0 \in f^{-1}(x_0)$. Suppose $G = g \cdot H \cdot g^{-1}$ and choose a loop α in (X, x_0) representing g^{-1}. Let β be a path in (Y, y_0) that is the lift of α. Put $y_1 = \beta(1)$ and notice the image of $\pi_1(f)\colon \pi_1(Y, y_1) \to \pi_1(X, x_0)$ is G. ∎

Proposition 8.4. *Suppose (X, x_0) is a pointed path-connected topological space. If $f\colon (Y, y_0) \to (X, x_0)$ is a Peano covering map with image of $\pi_1(f)$ equal to H, then f is equivalent to the projection $p_H\colon \widehat{X}_H \to X$.*

Proof. Define $h\colon (\widehat{X}_H, \widehat{x}_0) \to (Y, y_0)$ by choosing a lift $\widehat{\alpha}$ of every path α in X starting at x_0 and declaring $h([\alpha]_H) = \widehat{\alpha}(1)$. Note h is a bijection. Given $y_1 = \widehat{\alpha}(1)$ and given a neighborhood U of y_1 in Y choose a neighborhood V of $f(y_1) = \alpha(1)$ in X so that the path component of $f^{-1}(V)$ containing y_1 is a subset of U. Observe $B_H([\alpha]_H, V) \subset h^{-1}(U)$, which proves h is continuous.

Conversely, given a neighborhood W of $\alpha(1)$ in X the image $h(B_H([\alpha]_H, W))$ of $B_H([\alpha]_H, W)$ equals the path component of $\widehat{\alpha}(1)$ in $f^{-1}(W)$ and is open in Y. ∎

Theorem 8.5. *If X is a path-connected space, $x_0 \in X$, and H is a subgroup of $\pi_1(X, x_0)$, then the following conditions are equivalent:*

(a) *H is a Peano subgroup of $\pi_1(X, x_0)$.*
(b) *The endpoint projection $p_H\colon (\widehat{X}_H, \widehat{x}_0) \to (X, x_0)$ is a Peano covering map.*
(c) *The image of $\pi_1(p_H)\colon \pi_1(\widehat{X}_H, \widehat{x}_0) \to \pi_1(X, x_0)$ is contained in H.*
(d) *$p_H\colon (\widehat{X}_H, \widehat{x}_0) \to (X, x_0)$ has the unique path lifting property.*

Proof. (c)⇔(d) is done in 2.18. (b)⇔(d) is contained in 7.4.
(a)⇒(b) follows from 8.4.
(b)⇒(a) holds as (c) implies the image of $\pi_1(p_H)$ is H. ∎

Let us state a straightforward consequence of 8.5:

Corollary 8.6. *If X is a path-connected space and $x_0 \in X$, then the following conditions are equivalent:*

(a) *The endpoint projection $p\colon (\widehat{X}, \widehat{x}_0) \to (X, x_0)$ is a Peano covering map.*
(b) *$\pi_1(p)\colon \pi_1(\widehat{X}, \widehat{x}_0) \to \pi_1(X, x_0)$ is trivial.*
(c) *$\widehat{X}$ is simply connected.*
(d) *$p\colon (\widehat{X}, \widehat{x}_0) \to (X, x_0)$ has the unique path lifting property.*

Corollary 8.7. *Closed and normal subgroups of $\pi_1(X, x_0)$ with the lasso topology are Peano subgroups of $\pi_1(X, x_0)$.*

Proof. By 5.6 the endpoint projection $p_H\colon (\widetilde{X}_H, \widetilde{x}_0) \to (X, x_0)$ has unique path lifting property. Since $p_H\colon (\widehat{X}_H, \widehat{x}_0) \to (X, x_0)$ has path lifting property, this implies $p_H\colon (\widehat{X}_H, \widehat{x}_0) \to (X, x_0)$ has the unique path lifting property. ∎

Corollary 8.8. *If $H(s)$ is a Peano subgroup of $\pi_1(X, x_0)$ for each $s \in S$, then $G = \bigcap_{s\in S} H(s)$ is a Peano subgroup of $\pi_1(X, x_0)$.*

Proof. The projection $p_G\colon (\widehat{X}_G, \widehat{x}_0) \to (X, x_0)$ factors through $p_{H(s)}$: $(\widehat{X}_{H(s)}, \widehat{x}_0) \to (X, x_0)$ for each $s \in S$. Therefore $\mathrm{im}(\pi_1(p_G)) \subset \bigcap_{s\in S} H(s) = G$ and 7.4 (in conjunction with 2.18) says G is a Peano subgroup of $\pi_1(X, x_0)$. ∎

Corollary 8.9. *For each path-connected space X there is a universal Peano covering map $p\colon Y \to X$. Thus, for each Peano covering map $q\colon Z \to X$ and any points $z_0 \in Z$ and $y_0 \in Y$ satisfying $q(z_0) = p(y_0)$, there is a Peano covering map $r\colon Y \to Z$ so that $r(y_0) = z_0$. Moreover, the image of $\pi_1(Y)$ is normal in $\pi_1(X)$.*

Proof. Let H be the intersection of all Peano subgroups of $\pi_1(X, x_0)$; by 8.8 and 8.3 it is a normal Peano subgroup of $\pi_1(X, x_0)$. Put $Y = \widehat{X}_H$ and use 7.3. ∎

It would be of interest to characterize path-connected spaces X admitting a universal Peano covering that is simply connected (that amounts to $\widehat{X}$ being simply connected). Here is an equivalent problem:

Problem 8.10. *Characterize path-connected spaces X so that the trivial group is a Peano subgroup of $\pi_1(X, x_0)$.*

So far the following classes of spaces belong to that category:

(1) Any product of spaces admitting a simply connected Peano cover (see 7.2).
(2) Subsets of closed surfaces: it is proved in [15] that if X is any subset of a closed surface, then $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is injective.

(3) 1-dimensional, compact and Hausdorff, or 1-dimensional, separable and metrizable spaces: $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is injective by [10, Corollary 1.2 and Final Remark]. It is shown in [9] (see proof of Theorem 1.4) that the projection $\widehat{X} \to X$ has the uniqueness of path lifting property if X is 1-dimensional and metrizable. See [5] for results on the fundamental group of 1-dimensional spaces.

(4) Trees of manifolds: If X is the limit of an inverse system of closed PL-manifolds of some fixed dimension, whose consecutive terms are obtained by connect summing with closed PL-manifolds, which in turn are trivialized by the bonding maps, then X is called a *tree of manifolds*. Every tree of manifolds is path-connected and locally path-connected, but it need not be semilocally simply connected at any one of its points. Trees of manifolds arise as boundaries of certain Coxeter groups and as boundaries of certain negatively curved geodesic spaces [14]. It is shown in [14] that if X is a tree of manifolds (with a certain denseness of the attachments in the case of surfaces), then $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is injective.

Notice Example 2.7 in [16] gives X so that $p\colon \widehat{X} \to X$ does not have the unique path lifting property (one can construct a simpler example with X being the harmonic archipelago). However, X is not homotopically Hausdorff.

Problem 8.11. *Let X be a path-connected space and $x_0 \in X$. Are the following conditions equivalent?*

(1) $\bigcap_{\mathcal{U} \in \mathrm{COV}} \pi(\mathcal{U}, x_0) = \{e\}$, *where* COV *stands for the family of all open covers of X.*
(2) $p\colon (\widehat{X}, \widehat{x}_0) \to (X, x_0)$ *has the unique path lifting property.*
(3) *X is homotopically Hausdorff.*

The implication (1)⇒(2) follows from 5.8, 8.7, and 8.6. The implication (2)⇒(3) follows from 8.2 and 8.6.

Corollary 8.12. *Suppose H is a normal subgroup of $\pi_1(X, x_0)$. If there is a Peano subgroup G of $\pi_1(X, x_0)$ containing H such that G/H is countable, then H is a Peano subgroup of $\pi_1(X, x_0)$ if and only if X is homotopically Hausdorff relative to H.*

Proof. By 8.2, X is homotopically Hausdorff relative to H if H is a Peano subgroup of $\pi_1(X, x_0)$.

Suppose X is homotopically Hausdorff relative to H. Given two lifts in $\widehat{X}_H$ of the same path in X, their compositions with $\widehat{X}_H \to \widehat{X}_G$ are the same by 8.5. By 5.18 the two lifts are identical and 8.5 says H is a Peano subgroup of $\pi_1(X, x_0)$. ∎

Corollary 8.13. *Suppose H is a normal subgroup of $\pi_1(X,x_0)$. If $\pi_1(X,x_0)/H$ is countable, then H is a Peano subgroup of $\pi_1(X,x_0)$ if and only if X is homotopically Hausdorff relative to H.*

9. Appendix: Pointed versus unpointed. In this section we discuss relations between pointed and unpointed lifting properties.

Proposition 9.1. *If $f\colon (X,x_0)\to(Y,y_0)$ has the uniqueness of path lifts property and X is path-connected, then $f\colon X\to Y$ has the uniqueness of path lifts property.*

Proof. Given two paths α and β in X originating at the same point and satisfying $f\circ\alpha = f\circ\beta$, choose a path γ in X from x_0 to $\alpha(0)$. Now $f\circ(\gamma*\alpha) = f\circ(\gamma*\beta)$, so $\gamma*\alpha=\gamma*\beta$ and $\alpha=\beta$. ∎

Proposition 9.2. *If $f\colon (X,x_0)\to(Y,y_0)$ has the unique path lifting property and X is path-connected, then $f\colon X\to Y$ has the unique path lifting property.*

Proof. In view of 9.1 it suffices to show $f\colon X\to Y$ is surjective and has the path lifting property. If $y_1\in Y$, we pick a path α from y_0 to y_1 and lift it to (X,x_0). The endpoint of the lift maps to y_1, hence f is surjective. Suppose α is a path in Y and $f(x_1)=\alpha(0)$. Choose a path β in X from x_0 to x_1 and lift $(f\circ\beta)*\alpha$ to a path γ in (X,x_0). Due to the uniqueness of path lifts property of $f\colon (X,x_0)\to(Y,y_0)$ one has $\gamma(t)=\beta(2t)$ for $t\le 1/2$. Hence $\gamma(1/2)=x_1$ and λ defined as $\lambda(t)=\gamma(1/2+t/2)$ for $t\in I$ is a lift of α originating from x_1. ∎

Proposition 9.3 (Lemma 15.1 in [18]). *If $f\colon X\to Y$ is a Serre 1-fibration, then f has the unique path lifting property if and only if the path components of fibers of f are trivial.*

Proof. Suppose the fibers of f have trivial path components and α,β are two lifts of the same path in Y that originate at $x_1\in X$. Let $H\colon I\times I\to Y$ be the standard homotopy from $f\circ(\alpha^{-1}*\beta)$ to the constant path at $f(x_1)$. There is a lift $G\colon I\times I\to X$ of H starting from $\alpha^{-1}*\beta$. As path components of f are trivial, $\alpha=\beta$ due to the way the standard homotopy H is defined. ∎

Proposition 9.4. *Suppose $n\ge 1$. If $f\colon (X,x_0)\to(Y,y_0)$ is a Serre n-fibration, both X and Y are path-connected, and f has the uniqueness of path lifts property, then $f\colon X\to Y$ is a Serre n-fibration.*

Proof. Suppose $H\colon I^n\times I\to Y$ is a homotopy and $G\colon I^n\times\{0\}\to X$ is its partial lift. Choose a path α in X from x_0 to $G(b,0)$, where b is the center of I^n. We can extend G to a homotopy $G\colon I^n\times[-1,0]\to X$ starting from the constant map to x_0. By splicing $f\circ G$ with the original H, we can extend H to $H\colon I^n\times[-1,1]\to Y$. The new H can be lifted to X and the lift must agree with G on $I^n \times [-1, 0]$ due to the uniqueness of path lifts property of f. ■

Acknowledgments. We thank Sasha Dranishnikov for bringing the work of Fischer–Zastrow [16] to our attention. We thank Greg Conner, Katsuya Eda, Aleš Vavpetić, and Ziga Virk for helpful comments.

References

[1] V. Berestovskii and C. Plaut, *Uniform universal covers of uniform spaces*, Topology Appl. 154 (2007), 1748–1777.

[2] W. A. Bogley and A. J. Sieradski, *Universal path spaces*, http://oregonstate.edu/~bogleyw/#research.

[3] N. Brodskiy, J. Dydak, B. Labuz and A. Mitra, *Rips complexes and covers in the uniform category*, Houston J. Math., to appear.

[4] N. Brodskiy, J. Dydak, B. Labuz and A. Mitra, *Topological and uniform structures on the fundamental group*, in preparation.

[5] J. W. Cannon and G. R. Conner, *On the fundamental groups of one-dimensional spaces*, Topology Appl. 153 (2006), 2648–2672.

[6] G. R. Conner and D. Fearnley, *Fundamental groups of spaces which are not locally path connected*, preprint, 1998.

[7] G. R. Conner and J. Lamoreaux, *On the existence of universal covering spaces for metric spaces and subsets of the Euclidean plane*, Fund. Math. 187 (2005), 95–110.

[8] J. Dydak and J. Segal, *Shape Theory: An Introduction*, Lecture Notes in Math. 688, Springer, Berlin, 1978.

[9] K. Eda, *The fundamental groups of one-dimensional spaces and spatial homomorphisms*, Topology Appl. 123 (2002), 479–505.

[10] K. Eda and K. Kawamura, *The fundamental group of one-dimensional spaces*, Topology Appl. 87 (1998), 163–172.

[11] P. Fabel, *Metric spaces with discrete topological fundamental group*, Topology Appl. 154 (2007), 635–638.

[12] P. Fabel, *The Hawaiian earring group and metrizability*, arXiv:math.GT/0603252.

[13] P. Fabel, *Multiplication is discontinuous in the Hawaiian earring group (with the quotient topology)*, Bull. Polish Acad. Sci. Math. 59 (2011), 77–83.

[14] H. Fischer and C. R. Guilbault, *On the fundamental groups of trees of manifolds*, Pacific J. Math. 221 (2005), 49–79.

[15] H. Fischer and A. Zastrow, *The fundamental groups of subsets of closed surfaces inject into their first shape groups*, Algebr. Geom. Topol. 5 (2005), 1655–1676.

[16] H. Fischer and A. Zastrow, *Generalized universal covering spaces and the shape group*, Fund. Math. 197 (2007), 167–196.

[17] P. J. Hilton and S. Wylie, *Homology Theory: An Introduction to Algebraic Topology*, Cambridge Univ. Press, New York, 1960.

[18] S. T. Hu, *Homotopy Theory*, Academic Press, New York, 1959.

[19] E. L. Lima, *Fundamental Groups and Covering Spaces*, A K Peters, Natick, MA, 2003.

[20] S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982.

[21] J. R. Munkres, *Topology*, Prentice-Hall, Upper Saddle River, NJ, 2000.

[22] J. Pawlikowski, *The fundamental group of a compact metric space*, Proc. Amer. Math. Soc. 126 (1998), 3083–3087.

[23] D. Repovš and A. Zastrow, *Shape injectivity is not implied by being strongly homotopically Hausdorff*, preprint.

[24] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

N. Brodskiy, J. Dydak, B. Labuz, A. Mitra
Department of Mathematics
University of Tennessee
Knoxville, TN 37996, U.S.A.
E-mail: brodskiy@math.utk.edu
dydak@math.utk.edu
labuz@math.utk.edu
ajmitra@math.utk.edu

Received 21 February 2011;
in revised form 5 June 2012