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On the spectrum of stochastic perturbations of the shift and Julia sets

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Abstract. We extend the Killeen–Taylor study [Nonlinearity 13 (2000)] by investigating in different Banach spaces ($\ell^\alpha(\mathbb{N}), c_0(\mathbb{N}), c(\mathbb{N})$) the point, continuous and residual spectra of stochastic perturbations of the shift operator associated to the stochastic adding machine in base 2 and in the Fibonacci base. For the base 2, the spectra are connected to the Julia set of a quadratic map. In the Fibonacci case, the spectrum is related to the Julia set of an endomorphism of $\mathbb{C}^2$.

1. Introduction. In this paper, we study in detail the spectrum of some stochastic perturbations of the shift operator introduced by Killeen and Taylor in [3]. We focus on large Banach spaces for which we complete the Killeen–Taylor study. We also investigate the case of the Fibonacci base, but in this case, we have not been able to compute the residual and continuous spectra exactly.

We recall that in [3], Killeen and Taylor defined the *stochastic adding machine* as a stochastic perturbation of the shift in the following way: let N be a nonnegative integer written in base 2 as $N = \sum_{i=0}^{k(N)} \varepsilon_i(N)2^i$ where $\varepsilon_i(N) = 0$ or 1 for all i. It is known that there exists an algorithm that computes the digits of $N+1$. This algorithm can be described by introducing an auxiliary binary "carry" variable $c_i(N)$ for each digit $\varepsilon_i(N)$ in the following manner: Put $c_{-1}(N+1) = 1$ and

$$\varepsilon_i(N+1) = \varepsilon_i(N)+c_{i-1}(N+1) \bmod 2, \qquad c_i(N+1) = \left[\frac{\varepsilon_i(N)+c_{i-1}(N+1)}{2}\right]$$

for $i \geq 0$, where $[z]$ denotes the integer part of $z \in \mathbb{R}_+$.

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Let $\{e_i(n) : i \geq 0,\ n \in \mathbb{N}\}$ be an independent, identically distributed family of random variables which take the value 0 with probability $1-p$ and the value 1 with probability p. Let N be an integer. Given a sequence $(r_i(N))_{i\geq 0}$ of 0's and 1's such that $r_i(N) = 1$ for finitely many indices i, we consider the sequences $(r_i(N+1))_{i\geq 0}$ and $(c'_i(N+1))_{i\geq -1}$ defined by $c'_{-1}(N+1) = 1$ and for all $i \geq 0$,

$$r_i(N+1) = r_i(N) + e_i(N)c'_{i-1}(N+1) \bmod 2,$$

$$c'_i(N+1) = \left[\frac{r_i(N) + e_i(N)c'_{i-1}(N+1)}{2}\right].$$

With this definition a number $\sum_{i=0}^{\infty} r_i(N)2^i$ transitions to a number $\sum_{i=0}^{\infty} r_i(N+1)2^i$. In particular, an integer N having binary representation of the form $\varepsilon_n \ldots \varepsilon_{k+1}0\underbrace{11\ldots11}_{k}$ transitions to $\varepsilon_n \ldots \varepsilon_{k+1}1\underbrace{00\ldots00}_{k}$ with probability p^{k+1}, and a number having binary representation $\varepsilon_n \ldots \varepsilon_k\underbrace{11\ldots11}_{k}$ transitions to $\varepsilon_n \ldots \varepsilon_k\underbrace{00\ldots00}_{k}$ with probability $p^k(1-p)$. Equivalently, we obtain a Markov process $\psi(N)$ with state space $\mathbb{N}$ by setting $\psi(N) = \sum_{i=0}^{\infty} r_i(N)2^i$. The corresponding transition operator is denoted by S_p.

For $p = 1$ the transition operator equals the left shift, hence the stochastic adding machine can be seen as a stochastic perturbation of the left shift operator. It is also a model of Weber law in the context of counter and pacemarker errors. This law is used in biology and psychophysiology [4].

In [3], P. R. Killeen and J. Taylor studied the spectrum of the transition operator S_p (of $\psi(N)$) on ℓ^∞. They proved that the spectrum $\sigma(S_p)$ is equal to the filled Julia set of the quadratic map $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = (z-(1-p))^2/p^2$, i.e. $\sigma(S_p) = \{z \in \mathbb{C} : (f^n(z))_{n\geq 0}$ is bounded$\}$ where f^n is the nth iteration of f.

In [6], Messaoudi and Smania defined the stochastic adding machine in the Fibonacci base in the following manner. Consider the Fibonacci sequence $(F_n)_{n\geq 0}$ given by

$$F_0 = 1, \qquad F_1 = 2, \qquad F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2.$$

Using the greedy algorithm, we can write every nonnegative integer N in a unique way as $N = \sum_{i=0}^{k(N)} \varepsilon_i(N)F_i$ where $\varepsilon_i(N) = 0$ or 1 and $\varepsilon_i(N)\varepsilon_{i+1}(N) \neq 11$, for all $i \in \{0, \ldots, k(N)-1\}$ (see [11]). It is known that addition of 1 in the Fibonacci base (adding machine) is recognized by a finite state transducer. In [6], the authors defined the stochastic adding machine by introducing a "probabilistic transducer". They also computed the point spectrum of the transition operator acting in ℓ^∞ associated to the stochastic adding machine with respect to the base $(F_n)_{n\geq 0}$. In particular, they showed that the point spectrum $\sigma_{\mathrm{pt}}(S_p)$ in ℓ^∞ is related to the filled Julia set K_g of the function $g : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$g(x,y) = \left(\frac{1}{p^2}(x-1+p)(y-1+p), x\right).$$

Precisely, they proved that

$$\sigma_{\mathrm{pt}}(S_p) = \mathcal{K}_p = \{\lambda \in \mathbb{C} : (q_n(\lambda))_{n\geq 1} \text{ is bounded}\},$$

where $q_{F_0}(z) = z$, $q_{F_1}(z) = z^2$,

$$q_{F_k}(z) = \frac{1}{p} q_{F_{k-1}}(z) q_{F_{k-2}}(z) - \frac{1-p}{p} \quad \text{for all } k \geq 2$$

and for all nonnegative integers n, we have $q_n(z) = q_{F_{k_1}} \ldots q_{F_{k_m}}$ where $F_{k_1} + \cdots + F_{k_m}$ is the Fibonacci representation of n.

In particular, $\sigma_{\mathrm{pt}}(S_p)$ is contained in the set

$$\mathcal{E}_p = \{\lambda \in \mathbb{C} : (q_{F_n}(\lambda))_{n\geq 1} \text{ is bounded}\} = \{\lambda \in \mathbb{C} : (\lambda_1, \lambda) \in K_g\}$$

where $\lambda_1 = 1 - p + (1-\lambda-p)^2/p$.

Here we investigate the spectrum of the stochastic adding machines in base 2 and in the Fibonacci base in different Banach spaces. In particular, we compute exactly the point, continuous and residual spectra of the stochastic adding machine in base 2 for the Banach spaces c_0, c, ℓ^α, $\alpha \geq 1$.

For the Fibonacci base, we improve the result in [6] by proving that the spectrum of S_p acting on ℓ^∞ contains $\mathcal{E}_p$. The same result is proved for the Banach spaces c_0, c and ℓ^α, $\alpha \geq 1$.

The paper is organized as follows. In Section 2, we recall some basic facts of spectral theory. In Section 3, we state our main results (Theorems 3.1–3.3). Sections 4 and 5 contain the proofs in the case of the base 2 and the Fibonacci base, respectively.

2. Basic facts from spectral theory (see for instance [2], [8]–[10])**.** Let E be a complex Banach space and T a bounded operator on it. The spectrum of T, denoted by $\sigma(T)$, is the set of complex numbers λ for which $T - \lambda\,\mathrm{Id}_E$ is not an isomorphism (Id_E is the identity map).

If λ is in $\sigma(T)$ then one of the following assertions holds:

- (i) $T - \lambda\,\mathrm{Id}_E$ is not injective. In this case we say that λ is in the *point spectrum* denoted by $\sigma_{\mathrm{pt}}(T)$.
- (ii) $T - \lambda\,\mathrm{Id}_E$ is injective, not onto and has dense range. We say that λ is in the *continuous spectrum* denoted by $\sigma_{\mathrm{c}}(T)$.
- (iii) $T - \lambda\,\mathrm{Id}_E$ is injective and does not have dense range. We say that λ is in the *residual spectrum* denoted by $\sigma_{\mathrm{r}}(T)$.

It follows that $\sigma(T)$ is the disjoint union

$$\sigma(T) = \sigma_{\mathrm{pt}}(T) \cup \sigma_{\mathrm{c}}(T) \cup \sigma_{\mathrm{r}}(T).$$

The spectrum of a bounded operator acting on a Banach space is a compact subset of $\mathbb{C}$ [10]. There is a connection between the spectrum of T and the spectrum of the dual operator T' acting on the dual space E' by $T' : \phi \mapsto \phi \circ T$. In particular, we have

PROPOSITION 2.1 (Phillips Theorem [9, p. 145]). *Let E be a Banach space and T a bounded operator on it. Then $\sigma(T) = \sigma(T')$.*

We also have a classical relation between the point and residual spectra of T and the point spectrum of T'.

PROPOSITION 2.2 ([1, p. 581]). *For a bounded operator T we have*

$$\sigma_{\mathrm{r}}(T) \subset \sigma_{\mathrm{pt}}(T') \subset \sigma_{\mathrm{r}}(T) \cup \sigma_{\mathrm{pt}}(T).$$

In particular, if $\sigma_{\mathrm{pt}}(T) = \emptyset$ then

$$\sigma_{\mathrm{r}}(T) = \sigma_{\mathrm{pt}}(T').$$

3. Main results. Our main results are stated in the following three theorems.

MAIN THEOREM 3.1. *The spectrum of the operator S_p acting on c_0, c and ℓ^α, $\alpha \geq 1$, is equal to the filled Julia set K_f of the quadratic map $f(z) = (z - (1-p))^2/p^2$. Precisely, in c_0 (resp. ℓ^α, $\alpha > 1$), the continuous spectrum of S_p is equal to K_f and the point and residual spectra are empty. In c, the point spectrum is $\{1\}$, the residual spectrum is empty and the continuous spectrum equals $K_f \setminus \{1\}$.*

MAIN THEOREM 3.2. *In ℓ^1, the point spectrum of S_p is empty. The residual spectrum of S_p is not empty and contains a countable dense subset of the Julia set $J_f = \partial K_f$, viz. $\bigcup_{n=0}^{\infty} f^{-n}\{1\} \subset \sigma_{\mathrm{r}}(S_p)$. The continuous spectrum of S_p is the complement of the residual spectrum in the filled Julia set K_f.*

MAIN THEOREM 3.3. *The spectra of S_p acting respectively in ℓ^∞, c_0, c and ℓ^α, $\alpha \geq 1$, associated to the stochastic Fibonacci adding machines contain the set $\mathcal{E}_p = \{\lambda \in \mathbb{C} : (\lambda_1, \lambda) \in K_g\}$ where K_g is the filled Julia set of the function g and $\lambda_1 = 1 - p + (1 - \lambda - p)^2/p$.*

CONJECTURE 3.4. We conjecture that in the case of ℓ^1, the residual spectrum of the transition operator associated to the stochastic adding machine in base 2 is $\sigma_{\mathrm{r}}(S_p) = \bigcup_{n=0}^{\infty} f^{-n}\{1\}$. For the Fibonacci stochastic adding machine, we conjecture that the spectra of S_p in the Banach spaces appearing in Theorem 3.3 are all equal to $\mathcal{E}_p$.

REMARK. The methods we use can be adapted to a large class of stochastic adding machines given by transducers. Furthermore, let us point out that from the Killeen and Taylor method one may deduce in the case of ℓ^∞ that the residual and continuous spectra are empty. By contrast, here we compute directly the residual and continuous spectra in ℓ^α, c_0 and c.

4. Proofs in base 2. We are interested in the spectrum of S_p on three Banach spaces connected by duality. The space c_0 is the space of complex sequences which converge to zero, in other words, the continuous functions on $\mathbb{N}$ vanishing at infinity. The dual space of c_0 is by the Riesz Theorem the space of bounded Borel measures on $\mathbb{N}$ with total variation norm. This space can be identified with ℓ^1, the space of summable row vectors. Finally, the dual space of ℓ^1 is ℓ^∞, the space of bounded complex sequences.

We are also interested in the spectrum of S_p as an operator on the space ℓ^α with $\alpha > 1$ and also in the space c of convergent complex sequences.

PROPOSITION 4.1. *The operator S_p (acting on the right) is well defined on each $X \in \{c_0, c, \ell^\alpha\ (\alpha \geq 1)\}$; moreover, $\|S_p\| \leq 1$.*

Since the operator S_p is doubly stochastic, the proposition is a straightforward consequence of the following more general lemma.

LEMMA 4.2. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}}$ be an infinite matrix with nonnegative entries. Assume that there exists a positive constant M such that*

$$(1)\ \sup_{i\in\mathbb{N}}\Big(\sum_{j=0}^{\infty} a_{i,j}\Big) \leq M, \qquad (2)\ \sup_{j\in\mathbb{N}}\Big(\sum_{i=0}^{\infty} a_{i,j}\Big) \leq M.$$

Then A defines a bounded operator on the spaces c_0, c, ℓ^∞ and ℓ^α with $\alpha \geq 1$. In addition the norm of A is less than M.

Proof. From (1) it is easy to see that A is well defined on ℓ^∞ and its norm is less than M.

Now, let $v = (v_n)_{n\geq 0}$, $v \neq 0$, be such that $\lim_{n\to\infty} v_n = l \in \mathbb{C}$. Then for any $\varepsilon > 0$ there exists a positive integer j_0 such that for any $j \geq j_0$, we have $|v_j - l| \leq \varepsilon/(2M)$. Let $d = \sum_{j=0}^{\infty} a_{n,j}$. Then from (1) we have, for any $n \in \mathbb{N}$,

$$|(Av)_n - dl| = \Big|\sum_{j=0}^{\infty} a_{n,j}(v_j - l)\Big| \leq \sum_{j=0}^{j_0-1} a_{n,j}|v_j - l| + \frac{\varepsilon}{2}. \tag{4.1}$$

But by (2), for any $j \in \{0, \dots, j_0 - 1\}$, we have $\sum_{n=0}^{\infty} a_{n,j} < \infty$, so there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $j \in \{0, \dots, j_0 - 1\}$, we have

$$|a_{n,j}| \leq \frac{\varepsilon}{2j_0(\delta + 1)} \quad \text{where} \quad \delta = \sup\{|v_j - l| : j \in \mathbb{N}\}. \tag{4.2}$$

Combining (4.1) with (4.2) we get

$$|(Av)_n - dl| \leq \varepsilon, \qquad \forall n \geq n_0.$$

Hence $AX \subset X$ if $X = c_0$ or c.

Now take $\alpha > 1$ and $v \in \ell^\alpha$. For any $i \in \mathbb{N}$, we have

$$|(Av)_i|^\alpha \leq \Big(\sum_{j=0}^{\infty} a_{i,j}|v_j|\Big)^\alpha.$$

Let α' be the conjugate exponent of α, i.e., $1/\alpha + 1/\alpha' = 1$. Then, by the Hölder inequality,

$$\Big(\sum_{j=0}^{\infty} a_{i,j}|v_j|\Big)^\alpha \leq \Big(\sum_{j=0}^{\infty} a_{i,j}\Big)^{\alpha/\alpha'}\Big(\sum_{j=0}^{\infty} a_{i,j}|v_j|^\alpha\Big).$$

Hence

$$\Big(\sum_{j=0}^{\infty} a_{i,j}|v_j|\Big)^\alpha \leq \Big(\sup_{l\in\mathbb{N}}\sum_{j=0}^{\infty} a_{l,j}\Big)^{\alpha/\alpha'}\Big(\sum_{j=0}^{\infty} a_{i,j}|v_j|^\alpha\Big). \tag{4.3}$$

Thus

$$\begin{aligned}\|Av\|_\alpha^\alpha &\leq M^{\alpha/\alpha'}\sum_{i=0}^{\infty}\Big(\sum_{j=0}^{\infty} a_{i,j}|v_j|^\alpha\Big) = M^{\alpha/\alpha'}\sum_{j=0}^{\infty}\Big(\sum_{i=0}^{\infty} a_{i,j}\Big)|v_j|^\alpha\\ &\leq M^{\alpha/\alpha'}\sup_{j\in\mathbb{N}}\Big(\sum_{i=0}^{\infty} a_{i,j}\Big)\|v\|_\alpha^\alpha \leq M^{1+\alpha/\alpha'}\|v\|_\alpha^\alpha.\end{aligned}$$

Therefore

$$\|Av\|_\alpha \leq M\|v\|_\alpha.$$

Hence A is a continuous operator and $\|A\| \leq M$.

The case $\alpha = 1$ is an easy exercise and it is left to the reader. ■

From Proposition 4.1, we deduce that S_p is a Markov operator and its spectrum is contained in the complex unit disc.

Consider the map $f : z \in \mathbb{C} \mapsto \big(\frac{z-(1-p)}{p}\big)^2$ and denote by K_f the associated filled Julia set defined by

$$K_f = \{z \in \mathbb{C} : |f^{(n)}(z)| \not\to \infty\}.$$

Killeen and Taylor investigated the spectrum of S_p acting on ℓ^∞. They proved that the point spectrum of S_p is equal to K_f. In addition, they showed that the spectrum is invariant under the action of f. As a consequence, one may deduce that the continuous and residual spectra in this case are empty.

Here we will compute exactly the residual part and the continuous part of the spectrum of S_p acting on c_0, c and ℓ^α, $\alpha \geq 1$.

THEOREM 4.3. *The spectrum of the operator S_p acting on each $X \in \{c_0, c, \ell^\alpha\ (1 \le \alpha < \infty)\}$ is equal to K_f. More precisely, in c_0 (resp. ℓ^α, $\alpha > 1$), the continuous spectrum of S_p is equal to K_f and the point and residual spectra are empty. In c, the point spectrum is the singleton $\{1\}$, the residual spectrum is empty and the continuous spectrum is $K_f \setminus \{1\}$.*

For the proof of Theorem 4.3 we shall need the following proposition.

PROPOSITION 4.4. *The spectrum of S_p in each $X \in \{c_0, c, \ell^\alpha\ (1 \le \alpha \le \infty)\}$ is contained in K_f.*

The main idea of the proof of Proposition 4.4 can be found in the Killeen–Taylor proof. The key argument is that $\widetilde{S}_p^2$ is similar to the operator $ES_p \oplus OS_p$, where

$$\widetilde{S}_p = \frac{S_p - (1-p)\,\mathrm{Id}}{p}$$

and E, O denote the even and odd operators acting on X by

$$E(h_0, h_1, \ldots) = (h_0, 0, h_1, 0, h_2, \ldots),$$
$$O(h_0, h_1, \ldots) = (0, h_0, 0, h_1, 0, h_2, \ldots),$$

for any $h = (h_0, h_1, \ldots)$ in X. Precisely, for all $v = (v_i)_{i \ge 0} \in X$, we have

$$\widetilde{S}_p^2(v) = ES_p(v_0, v_2, \ldots, v_{2n}, \ldots) + OS_p(v_1, v_3, \ldots, v_{2n+1}, \ldots).$$

As a consequence we deduce from the mapping spectral theorem [9] that the spectrum of S_p is invariant under f.

Let us start the proof of Theorem 4.3 by proving the following result.

PROPOSITION 4.5. *The point spectrum of S_p acting on each $X \in \{c_0, \ell^\alpha\ (\alpha \ge 1)\}$ is empty, and the point spectrum of S_p on c is $\{1\}$.*

For the proof, we need the following lemma from [3].

LEMMA 4.6 ([3]). *Let n be a nonnegative integer and $X_n = \{m \in \mathbb{N} : (S_p)_{n,m} \neq 0\}$. Then the following properties are valid:*

(1) *For all nonnegative integers n, we have $n \in X_n$ and $(S_p)_{n,n} = 1 - p$.*
(2) *If $n = \varepsilon_k \ldots \varepsilon_1 0$, $k \ge 2$, is an even integer then $X_n = \{n, n+1\}$ and $(S_p)_{n,n+1} = p$.*
(3) *If $n = \varepsilon_k \ldots \varepsilon_t 0 \underbrace{1 \ldots 1}_{s}$ is an odd integer with $s \ge 1$ and $k \ge t \ge s+1$, then $X_n = \{n, n+1, n - 2^m + 1\ (1 \le m \le s)\}$ and n transitions to $n + 1 = \varepsilon_k \ldots \varepsilon_t 1 \underbrace{0 \ldots 00}_{s}$ with probability $(S_p)_{n,n+1} = p^{s+1}$, and to $n - 2^m + 1 = \varepsilon_k \ldots \varepsilon_t 0 \underbrace{1 \ldots 1}_{s-m} \underbrace{0 \ldots 0}_{m}$, $1 \le m \le s$, with probability $(S_p)_{n,n-2^m+1} = p^m(1-p)$.*

Proof of Proposition 4.5. Let λ be an eigenvalue of S_p associated to the eigenvector $v = (v_n)_{n\geq 0}$ in $X \in \{c_0, c, \ell^\alpha\ (\alpha \geq 1)\}$. By Lemma 4.6, $(S_p)_{i,i+k} = 0$ for all $i, k \in \mathbb{N}$ with $k \geq 2$. Therefore, for all integers $k \geq 1$, we have

$$\sum_{i=0}^{k} (S_p)_{k-1,i} v_i = \lambda v_{k-1}. \tag{4.4}$$

Hence one can prove by induction on k that for all integers $k \geq 1$, there exists a complex number $q_k = q_k(p, \lambda)$ such that

$$v_k = q_k v_0. \tag{4.5}$$

By Lemma 4.6 and the fact that $((S_p - \lambda\,\mathrm{Id})v)_{2^n} = 0$ for all nonnegative n, we get

$$p^{n+1} v_{2^n} + (1-p-\lambda) v_{2^n-1} + \sum_{i=1}^{n} p^i (1-p) v_{2^n-2^i} = 0, \qquad \forall n \geq 0. \tag{4.6}$$

Hence

$$v_{2^n} = \frac{1}{p} A - \left(\frac{1}{p} - 1\right) v_0,$$

where

$$A = -\frac{1}{p^n}\Big((1-p-\lambda) v_{2^{n-1}+(2^{n-1}-1)} + \sum_{i=1}^{n-1} p^i (1-p) v_{2^{n-1}+(2^{n-1}-2^i)}\Big).$$

On the other hand, by the self-similarity structure of the transition matrix S_p, one can prove that if i and j are two integers such that for some positive integer n we have $2^{n-1} \leq i, j < 2^n$, then the transition probability from i to j is equal to the transition probability from $i - 2^{n-1}$ to $j - 2^{n-1}$. Using this last fact and (4.6), it follows that

$$v_{2^n} = \frac{1}{p} q_{2^{n-1}} v_{2^{n-1}} - \left(\frac{1}{p} - 1\right) v_0.$$

This gives

$$q_{2^n} = \frac{1}{p} q_{2^{n-1}}^2 - \left(\frac{1}{p} - 1\right), \tag{4.7}$$

where

$$q_{2^0} = q_1 = -\frac{1-p-\lambda}{p}.$$

Case 1: *$v \in c_0$ or ℓ^α, $\alpha \geq 1$.* We have $\lim_{n\to\infty} q_{2^n} = 0$. Thus by (4.7), we get $p = 1$, which is absurd, then the point spectrum is empty.

Case 2: *$v \in c$.* Assume that $\lim q_n = l \in \mathbb{C}$. Then by (4.7), we deduce that $l = 1$ or $l = p - 1$. On the other hand, for any $n \in \mathbb{N}$, there exist k

nonnegative integers $n_1 < \cdots < n_k$ such that $n = 2^{n_1} + \cdots + 2^{n_k}$. We can prove (see [3]) that

$$q_n = q_{2^{n_1}} \ldots q_{2^{n_k}}. \tag{4.8}$$

Then $\lim q_{2^{n-2}+2^n} = l^2 = l$, thus $l = p-1$ is excluded. Since S_p is stochastic, we conclude that $l = 1$ and $\sigma_{\mathrm{pt},c}(S_p) = \{1\}$. ∎

Remark. By the same arguments as above, Killeen and Taylor [3] proved that the point spectrum of S_p acting on ℓ^∞ is equal to K_f. In fact, it is easy to see from the arguments above that $\sigma_{\mathrm{pt},\ell^\infty}(S_p) = \{\lambda \in \mathbb{C} : q_n(\lambda) \text{ bounded}\}$. Indeed, (4.7) implies that if $(q_{2^n})_{n\geq 0}$ is bounded, then $|q_{2^n}| \leq 1$ for all $n \geq 0$. This clearly forces $\sigma_{\mathrm{pt},\ell^\infty}(S_p) = \{\lambda \in \mathbb{C} : q_{2^n}(\lambda) \text{ bounded}\}$ by (4.8). Now, since

$$q_{2^n} = h \circ f^{n-1} \circ h^{-1}(q_1) = h \circ f^{n-1}(\lambda), \quad \forall n \in \mathbb{N},$$

where $h(x) = x/p - (1-p)/p$, we conclude that $\sigma_{\mathrm{pt},\ell^\infty}(S_p) = K_f$. It follows from Proposition 4.4 that $\sigma_{\ell^\infty}(S_p) = K_f$ and the residual and continuous spectra are empty.

Proposition 4.7. *The residual spectrum of S_p acting on $X \in \{c_0, c, \ell^\alpha$ $(\alpha > 1)\}$ is empty.*

Proof. Let λ be an element of the residual spectrum of S_p acting on c_0 (resp. c). Then, by Proposition 2.2, we deduce that there exists a sequence $u = (u_k)_{k\geq 0} \in \ell^1$ such that $u(S_p - \lambda\,\mathrm{Id}) = 0$.

Claim. *$u_k = (1/q_k)u_0$ for all $k \in \mathbb{N}$.*

We have

$$\forall k \in 2\mathbb{N}, \quad (u(S_p - \lambda\,\mathrm{Id}))_{k+1} = pu_k + (1-p-\lambda)u_{k+1} = 0.$$

Hence

$$\forall k \in 2\mathbb{N}, \quad u_k = q_1 u_{k+1}. \tag{4.9}$$

If k is odd, then $k = 2^n - 1 + t$ where $t = 0$ or $t = \sum_{j=2}^{s} 2^{n_j}$ with $1 \leq n < n_2 < \cdots < n_s$. Since $(u(S_p - \lambda\,\mathrm{Id}))_{k+1} = 0$, we have

$$p^{n+1}u_k + (1-p-\lambda)u_{k+1} + \sum_{i=1}^{n} p^i(1-p)u_{k+2^i} = 0. \tag{4.10}$$

Observe that the relation (4.10) between u_k and u_{k+2^n} is similar to the relation (4.6) between v_{2^n} and v_0. We will prove, by induction on n, that

$$u_k = q_{2^n} u_{k+2^n}. \tag{4.11}$$

Indeed, if $n = 1$ then by (4.10) and (4.9), we get

$$p^2 u_k + (q_1(1-p-\lambda) + p(1-p))u_{k+2} = 0.$$

Therefore

$$u_k = \left(\frac{q_1^2}{p} - \frac{1-p}{p}\right)u_{k+2} = q_2 u_{k+2}.$$

Thus (4.11) is proved for $n = 1$.

Now, assume that (4.11) holds for $1, \dots, m-1$. Take $n = m$ and $1 \le i < m$. Then $k + 2^i = 1 + 2 + \cdots + 2^{m-1} + 2^i + t = 2^i - 1 + t'$ where $t' = 2^m + t$. Applying the induction hypothesis, we get

$$u_{k+2^i} = q_{2^i} u_{k+2^{i+1}} = q_{2^i} q_{2^{i+1}} \dots q_{2^{m-1}} u_{k+2^m}.$$

On the other hand, since $2^i + \cdots + 2^{m-1} = 2^m - 2^i$, we have

$$u_{k+2^i} = q_{2^m - 2^i} u_{k+2^m}. \tag{4.12}$$

Considering (4.10) with $n = m$ and (4.12) yields

$$u_k = -\frac{1}{p^{m+1}}\Big((1-p-\lambda)q_{2^m-1} + \sum_{i=1}^{m} p^i(1-p)q_{2^m-2^i}u_{k+2^m}\Big).$$

Combining this with (4.5) and (4.6), we obtain (4.11) for $n = m$. Thus (4.11) holds for all integers $n \ge 1$.

In particular, $u_{2^{n-1}-1} = q_{2^{n-1}} u_{2^n-1}$ for all $n \ge 1$. Thus

$$u_{2^n-1} = \frac{1}{q_{2^0} q_2 \dots q_{2^{n-1}}} u_0 = \frac{1}{q_{2^n-1}} u_0, \quad \forall n \ge 1. \tag{4.13}$$

On the other hand, for all integers $n \ge 1$, by (4.9) we have $u_{2^n} = q_{2^0} u_{2^n+2^0}$, and from (4.11) we see that

$$u_{2^n} = q_{2^0} q_{2^1} u_{2^2-1+2^n} = \cdots = q_{2^0} q_{2^1} \dots q_{2^{n-1}} u_{2^{n+1}-1}. \tag{4.14}$$

Consequently, from (4.13) and (4.14), we obtain

$$u_{2^n} = \frac{1}{q_{2^n}} u_0, \quad \forall n \ge 1. \tag{4.15}$$

Now fix an integer $k \in \mathbb{N}$ and assume that $k = \sum_{i=1}^{s} 2^{n_i}$ where $0 \le n_1 < \cdots < n_s$. We will prove by induction on s that

$$u_k = \frac{1}{q_{2^{n_1}} q_{2^{n_2}} \dots q_{2^{n_s}}} u_0 = \frac{1}{q_k} u_0. \tag{4.16}$$

Indeed, it follows from (4.15) that (4.16) is true for $s = 1$.

Now assume that (4.16) is true for all integers $1 \le i < s$.

Case 1: *k is odd.* In this case

$$k = \sum_{i=1}^{s} 2^{n_i} = 2^n - 1 + l = \sum_{j=0}^{n-1} 2^j + l$$

where $l = 0$ if $n = s + 1$ and $l = \sum_{i=n}^{s} 2^{n_i}$ if $n \le s$.

If $n \geq 2$, we use (4.11) to get $u_{k-2^{n-1}} = q_{2^{n-1}} u_k$ and by induction hypothesis, we have

$$u_k = \frac{1}{q_{2^{n-1}} q_{k-2^{n-1}}} u_0 = \frac{1}{q_k} u_0.$$

If $n = 1$, we consider (4.9) to write $u_k = (1/q_1) u_{k-1}$. Thus, we deduce, by induction hypothesis, that

$$u_k = \frac{1}{q_1 q_{k-1}} u_0 = \frac{1}{q_k} u_0.$$

Case 2: *k is even.* In this case $n_1 > 0$, and by (4.9), we deduce that

$$u_k = q_{2^0} u_{k+2^0} = q_{2^0} u_{k+2^1-1}.$$

Applying (4.11), it follows that

$$\begin{aligned} u_k = q_{2^0} q_{2^1} u_{k+2^2-1} = \cdots &= q_{2^0} q_{2^1} \ldots q_{2^{n_1}-1} u_{k+2^{n_1}-1} \\ &= q_{2^0} q_{2^1} \ldots q_{2^{n_1}-1} u_{(k-2^{n_1})+2^{n_1+1}-1}. \end{aligned}$$

Hence

$$\begin{aligned} u_k &= q_{2^0} \ldots q_{2^{n_1}-1} q_{2^{n_1+1}} u_{(k-2^{n_1})+2^{n_1+2}-1} \\ &= q_{2^0} \ldots q_{2^{n_1}-1} q_{2^{n_1+1}} \ldots q_{2^{n_2}-1} u_{(k-2^{n_1}-2^{n_2})+2^{n_2+1}-1}. \end{aligned}$$

Thus

$$u_k = \frac{\prod_{i=0}^{n_s} q_{2^i}}{\prod_{i=1}^{s} q_{2^{n_i}}} u_{2^{n_s+1}-1}. \tag{4.17}$$

By (4.17) and (4.13) we get

$$u_k = \frac{1}{\prod_{i=1}^{s} q_{2^{n_i}}} u_0 = \frac{1}{q_k} u_0.$$

Therefore we have proved that for all nonnegative integers k,

$$u_k = \frac{1}{q_k} u_0. \tag{4.18}$$

We conclude that u is in ℓ^1 if and only if $\sum_{k=1}^{\infty} |1/q_k(\lambda)| < \infty$. But this shows that the residual spectrum of S_p acting on c_0 or c satisfies

$$\sigma_{\mathrm{r}}(S_p) \subset \left\{ \lambda \in \overline{\mathbb{D}(0,1)} : \sum_{k=1}^{\infty} \left| \frac{1}{q_k(\lambda)} \right| < \infty \right\}. \tag{4.19}$$

We claim that $\sum_{k=1}^{\infty} |1/q_k(\lambda)| < \infty$ implies $|q_{2^n-1}| \geq 1$ for all $n \geq 1$. Indeed, by d'Alembert's Theorem, we have

$$\limsup \frac{|q_n|}{|q_{n+1}|} \leq 1. \tag{4.20}$$

Now assume that n is even. Then $n = 2^{k_0} + \cdots + 2^{k_m}$ where $1 \leq k_0 < k_1 < \ldots < k_m$ (representation in base 2). In this case $n+1 = 2^0 + 2^{k_0} + \cdots + 2^{k_m}$.

Using (4.8), we obtain $|q_n|/|q_{n+1}| = 1/|q_1|$, and by (4.20) we get

$$|q_1| \geq 1. \tag{4.21}$$

Since

$$q_{2^n} = \frac{1}{p} q_{2^{n-1}}^2 - \left(\frac{1}{p} - 1\right) \quad \text{for all } n \geq 0,$$

the triangle inequality yields $|q_{2^n}| \geq 1$ for all $n \geq 1$. Let i be a positive integer. Since $2^i - 1 = \sum_{j=0}^{i-1} 2^j$, we deduce by (4.8) that $q_{2^i-1} = q_{2^{i-1}} q_{2^{i-2}} \cdots q_1$. Hence

$$|q_{2^i-1}| \geq 1 \quad \text{for any } i \geq 1.$$

On the other hand, considering the first coordinate of the vector $u(S_p - \lambda \operatorname{Id}) = 0$ we have

$$(1-p-\lambda)u_0 + \sum_{i=1}^{\infty} p^i (1-p) u_{2^i-1} = 0.$$

Dividing by p, we obtain

$$q_1 = \sum_{i=1}^{\infty} p^{i-1}(1-p)/q_{2^i-1}. \tag{4.22}$$

We claim that there exists an integer $i_0 \in \mathbb{N}$ such that $|q_{2^{i_0}-1}| > 1$. Indeed, if not the series $\sum_{i\in\mathbb{N}} 1/|q_{2^i-1}|$ will diverge. Thus $|q_1| < \sum_{i\neq i_0}^{\infty} p^i(1-p) + p^{i_0-1}(1-p) < 1$, absurd. We conclude that the residual spectrum of S_p acting on c_0 (resp. c) is empty.

The same proof shows that the residual spectrum of S_p acting on ℓ^α, $\alpha > 1$, is empty. ∎

Remark. By (4.19), it follows that $\lambda \in \sigma_{\mathrm{r}}(X)$ with $X = c_0$, c or ℓ^α, $\alpha > 1$, implies $\lim |q_n(\lambda)| = +\infty$. But this contradicts Proposition 4.4, so that $\sigma_{\mathrm{r},c_0}(S_p) = \sigma_{\mathrm{r},\ell^\alpha}(S_p) = \emptyset$.

Proposition 4.8. *The following equalities are satisfied:*

$$\sigma_{\mathrm{c},c}(S_p) = K_f \setminus \{1\}, \qquad \sigma_{\mathrm{c},c_0}(S_p) = \sigma_{\mathrm{c},\ell^\alpha}(S_p) = K_f \quad \text{for all } \alpha > 1.$$

Proof. Assume that $X \in \{c_0, c\}$. Then, by the Phillips Theorem, the spectrum of S_p in X is equal to the spectrum of S_p in ℓ^∞, and from Propositions 4.5 and 4.7, we obtain the result.

Now, assume $X = \ell^\alpha$, $\alpha > 1$. According to Propositions 4.4, 4.5 and 4.7, it is enough to prove that $K_f \subset \sigma(S_p)$. Consider $\lambda \in K_f$. We will prove that λ belongs to the approximate point spectrum of S_p. For all integers $k \geq 2$, put $w^{(k)} = (1, q_1(\lambda), \ldots, q_k(\lambda), 0, 0, \ldots)^t \in \ell^\alpha$ where $(q_k(\lambda))_{k\geq 1} = (q_k)_{k\geq 1}$ is the sequence defined in (4.5), and let $u^{(k)} = w^{(k)}/\|w^{(k)}\|_\alpha$.

Claim. $\lim_{n\to\infty} \|(S_p - \lambda \operatorname{Id})u^{(2^n)}\|_\alpha = 0$.

Indeed, we have

$$\forall i \in \{0, \dots, k-1\}, \qquad ((S_p - \lambda\,\mathrm{Id})u^{(k)})_i = 0.$$

Thus

$$\sum_{i=0}^{\infty} |((S_p - \lambda\,\mathrm{Id})u^{(k)})_i|^{\alpha} = \frac{\sum_{i=k}^{\infty} |\sum_{j=0}^{k} (S_p - \lambda\,\mathrm{Id})_{i,j} w_j^{(k)}|^{\alpha}}{\|w^{(k)}\|_{\alpha}^{\alpha}}.$$

Putting $a_{i,j} = |(S_p - \lambda\,\mathrm{Id})_{i,j}|$ for all i, j and using (4.3), we get

$$\Big|\sum_{j=0}^{k} (S_p - \lambda\,\mathrm{Id})_{i,j} w_j^{(k)}\Big|^{\alpha} \le C \sum_{j=0}^{k} |(S_p - \lambda\,\mathrm{Id})_{i,j}|\, |w_j^{(k)}|^{\alpha}$$

where $C = \sup_{i\in\mathbb{N}} (\sum_{j=0}^{\infty} |(S_p - \lambda\,\mathrm{Id})_{i,j}|)^{\alpha/\alpha'}$ and α' is the conjugate of α.

Observe that C is a finite nonnegative constant because S_p is a stochastic matrix and λ belongs to K_f which is a bounded set.

In this way we have

$$\begin{aligned}\|(S_p - \lambda\,\mathrm{Id})u^{(k)}\|_{\alpha}^{\alpha} &\le C \sum_{i=k}^{\infty} \frac{(\sum_{j=0}^{k} |w_j^{(k)}|^{\alpha} |(S_p - \lambda\,\mathrm{Id})_{i,j}|)}{\|w^{(k)}\|_{\alpha}^{\alpha}} \\ &= \frac{C}{\|w^{(k)}\|_{\alpha}^{\alpha}} \sum_{j=0}^{k} |w_j^{(k)}|^{\alpha} \sum_{i=k}^{\infty} |(S_p - \lambda\,\mathrm{Id})_{i,j}|.\end{aligned}$$

Now, for $k = 2^n$, we will compute the terms

$$A_{k,j} = \sum_{i=k}^{\infty} |(S_p - \lambda\,\mathrm{Id})_{i,j}|, \qquad 0 \le j \le k.$$

Assume that $0 \le j < k = 2^n$. Then $(S_p - \lambda\,\mathrm{Id})_{i,j} = (S_p)_{i,j}$ for all $i \ge k$.

CASE 1: *j is odd.* Then by Lemma 4.6, $(S_p)_{i,j} \ne 0$ if and only $i = j - 1$ or $i = j$. Hence $(S_p)_{i,j} = 0$ for all $i \ge k$. Thus

$$A_{k,j} = 0. \tag{4.23}$$

CASE 2: $j = 0$. Then by Lemma 4.6, we have

$$A_{k,j} = \sum_{i=2^n}^{\infty} (S_p)_{i,0} = \sum_{i=n+1}^{\infty} p^i(1-p) = p^{n+1}. \tag{4.24}$$

CASE 3: *j is even and $j > 0$.* Then $j = \varepsilon_{n-1} \dots \varepsilon_s \underbrace{0 \dots 0}_{s} = \sum_{i=s}^{n-1} \varepsilon_i 2^i$ with $s \ge 1$ and $\varepsilon_s = 1$. But by Lemma 4.6, $(S_p)_{i,j} \ne 0$ if and only if $i = 2^m - 1 + j$ for some $0 \le m \le s$. Hence $i < 2^n = k$.

Therefore, in this case

$$A_{k,j} = 0. \tag{4.25}$$

Now assume $j = k = 2^n$. In this case, we have

$$A_{k,j} = |1 - p - \lambda| + \sum_{i=2^n+1}^{\infty} (S_p)_{i,2^n}.$$

On the other hand, by Lemma 4.6, $(S_p)_{i,2^n} \neq 0$ if and only if $i = 2^n + 2^m - 1$ for some $0 \leq m \leq n$, and $(S_p)_{2^n+2^m-1,2^n} = p^m(1-p)$. Therefore

$$A_{k,j} = \sum_{i=2^n}^{\infty} |(S_p - \lambda \,\mathrm{Id})_{i,2^n}| = |1 - p - \lambda| + \sum_{m=0}^{n} p^m(1-p). \tag{4.26}$$

By (4.23)–(4.26), we have for $k = 2^n$ and $0 \leq j \leq k$,

$$A_{k,j} \neq 0 \iff j = 0 \text{ or } j = k = 2^n.$$

Consequently,

$$\begin{aligned}\|(S_p - \lambda \,\mathrm{Id})u^{(2^n)}\|_\alpha^\alpha &\leq C \frac{|w_0^{(k)}|^\alpha A_{k,0} + |w_k^{(k)}|^\alpha A_{k,k}}{\|w^{(k)}\|_\alpha^\alpha} \\ &= C \frac{p^{n+1} + |q_{2^n}|^\alpha (|1 - p - \lambda| + \sum_{m=0}^{n} p^m(1-p))}{\|w^{(2^n)}\|_\alpha^\alpha}.\end{aligned}$$

We claim that $\|w^{(2^n)}\|_\alpha \to \infty$ as $n \to \infty$. Indeed, if not, since the sequence $\|w^{(2^n)}\|_\alpha$ is increasing, it must converge. Put $w = (q_i)_{i\geq 0}$ with $q_0 = 1$. It follows that the sequence $(w^{(2^n)})_{n\geq 0}$ converges to w in ℓ^α, which means that there exists a nonzero vector $w \in \ell^\alpha$ such that $(S_p - \lambda \,\mathrm{Id})w = 0$. This contradicts Proposition 4.5. Now, since λ belongs to the filled Julia set which is a bounded set and $(q_n)_{n\geq 0}$ is a bounded sequence, it follows that $\|(S_p - \lambda \,\mathrm{Id})u^{(2^n)}\|_\alpha \to 0$, and the claim is proved. We conclude that λ belongs to the approximate point spectrum of S_p and the proof of Proposition 4.8 is complete. ∎

This ends the proof of Theorem 4.3.

Spectrum of S_p acting on the right on ℓ^1. Here, we will study the spectrum of S_p acting (on the right) on ℓ^1. We deduce from Proposition 4.4 that this spectrum is contained in K_f. On the other hand, using the same proof as for Proposition 4.8, we find that K_f is contained in the approximate point spectrum of S_p. This shows that the spectrum of S_p acting on ℓ^1 is equal to K_f.

THEOREM 4.9. *In ℓ^1, the residual spectrum contains a countable dense subset of the Julia set $J_f = \partial K_f$. The continuous spectrum is not empty and is equal to the complement of the residual spectrum in the filled Julia set K_f.*

Proof. The proof of Proposition 4.7 shows that the residual spectrum of S_p in ℓ^1 is equal to the point spectrum of S_p (acting on the right) in $(\ell^1)' = \ell^\infty$. By (4.18) and (4.22), we see that

$$\begin{aligned}
&\sigma_{\mathrm{r}}(S_p)\\
&= \left\{\lambda \in \mathbb{C} : (q_n(\lambda)) \text{ and } (1/q_n(\lambda)) \text{ are bounded and } q_1 = \sum_{i=1}^{\infty} \frac{p^{i-1}(1-p)}{q_{2^i-1}}\right\}\\
&\qquad = K_f \cap \left\{\lambda \in \mathbb{C} : (1/q_n(\lambda)) \text{ is bounded and } q_1 = \sum_{i=1}^{\infty} \frac{p^{i-1}(1-p)}{q_{2^i-1}}\right\}.
\end{aligned}$$

On the other hand we have

$$q_{2^n}^2 = f(q_{2^{n-1}}^2) = \cdots = f^n(q_1^2) = f^{n+1}(\lambda), \qquad \forall n \geq 0. \tag{4.27}$$

Let $n \in \mathbb{N}$ and $E_n = \{\lambda \in \mathbb{C} : q_{2^n}(\lambda) = 1\}$.

Claim 1. $\bigcup_{n=0}^{\infty} E_n = \bigcup_{n=0}^{\infty} f^{-n}\{1\}$.

Indeed, let $\lambda \in \mathbb{C}$ be such that $f^n(\lambda) = 1$ for some $n \geq 1$. Then, by (4.27), we have $q_{2^{n-1}} = 1$ or $q_{2^{n-1}} = -1$. From (4.7), we see that $q_{2^{n-1}} = -1$ implies $q_{2^n} = 1$. Hence $f^{-n}\{1\} \subset E_{n-1} \cup E_n$. Since $1 \in E_n$ for all $n \geq 0$, we conclude that $\bigcup_{n=0}^{\infty} f^{-n}\{1\} \subset \bigcup_{n=0}^{\infty} E_n$. The other inclusion follows from (4.27).

Claim 2. $\bigcup_{n=0}^{\infty} E_n \subset \sigma_{\mathrm{r}}(S_p)$.

Indeed, assume that $n \in \mathbb{N}$ and $\lambda \in E_n$. Then by (4.7), we get

$$q_{2^k} = 1, \qquad \forall k \geq n. \tag{4.28}$$

But from (4.28) and (4.8), the sequences $(q_k(\lambda))_{k\geq 0}$ and $(1/q_k(\lambda))_{k\geq 0}$ are bounded. Moreover,

$$\begin{aligned}
q_1 = \sum_{i=1}^{\infty} \frac{p^{i-1}(1-p)}{q_{2^i-1}} \ &\Leftrightarrow\ q_2 = \sum_{i=2}^{\infty} \frac{p^{i-2}(1-p)q_1}{q_{2^i-1}}\\
&\Leftrightarrow\ q_{2^k} = \sum_{i=k+1}^{\infty} p^{i-k-1}(1-p)\frac{q_{2^0}\cdots q_{2^{k-1}}}{q_{2^i-1}}, \qquad \forall k \geq 0\\
&\Leftrightarrow\ q_{2^k} = \sum_{i=k+1}^{\infty} \frac{p^{i-k-1}(1-p)}{q_{2^k}q_{2^{k+1}}\cdots q_{2^{i-1}}}, \quad \forall k \geq 0.
\end{aligned}$$

Thus

$$q_1 = \sum_{i=1}^{\infty} \frac{p^{i-1}(1-p)}{q_{2^i-1}} \ \Leftrightarrow\ 1 = \sum_{i=0}^{\infty} p^i(1-p).$$

From this, $\lambda \in \sigma_{\mathrm{r}}(S_p)$ and Claim 2 is proved.

Since 1 is a repulsing fixed point of f, it follows that $\bigcup_{n=0}^{\infty} f^{-n}\{1\}$ is a dense subset of the Julia set J_f. From this fact combined with Claims 1 and 2, we conclude that the residual spectrum contains a countable dense subset of the Julia set J_f.

On the other hand, $(p-1)^2 \in K_f$ since $f((p-1)^2) = (p-1)^2$, but $(p-1)^2 \notin \sigma_{\mathrm{r}}(S_p)$ because for any positive integer n, $q_{2^n}((p-1)^2) = p-1$, which implies that $\lim q_n = 0$ and hence $1/q_n$ is not bounded. Thus $(p-1)^2 \in \sigma_{\mathrm{c}}(S_p)$. This finishes the proof of the theorem. ∎

CONJECTURE 4.10. The residual spectrum in ℓ^1 equals $\bigcup_{n=0}^{\infty} f^{-n}\{1\}$.

Spectrum of S_p acting on the left. The Phillips Theorem combined with Proposition 2.2 and Theorems 4.3 and 4.9 leads to the following result.

THEOREM 4.11. *The spectrum of S_p (acting on the left) in the spaces c_0, c, ℓ^α with $1 \le \alpha \le \infty$ equals the filled Julia set K_f. Precisely:*

In c_0 and ℓ^α with $1 \le \alpha < \infty$, the spectrum equals the continuous spectrum.

In c, the point spectrum equals $\{1\}$ and the continuous spectrum equals $K_f \setminus \{1\}$.

In ℓ^∞, the point spectrum equals the residual spectrum in ℓ^1.

5. Fibonacci stochastic adding machine (see [6]). Let us consider the Fibonacci sequence $(F_n)_{n\ge 0}$ given by the relation

$$F_0 = 1, \qquad F_1 = 2, \qquad F_n = F_{n-1} + F_{n-2} \quad \forall n \ge 2.$$

Using the greedy algorithm, we can write (see [11]) every nonnegative integer N in a unique way as $N = \sum_{i=0}^{k(N)} \varepsilon_i(N) F_i$ where $\varepsilon_i(N) = 0$ or 1 and $\varepsilon_i(N)\varepsilon_{i+1}(N) \neq 0$, for $0 \le i \le k(N) - 1$.

It is known that addition of 1 in base $(F_n)_{n\ge 0}$ (called the *Fibonacci adding machine*) is given by a finite state transducer on $A^* \times A^*$ where $A = \{0,1\}$. This transducer is formed by two states (an initial state I and a terminal state T). The initial state is connected to itself by two arrows. One of them is labeled by $(10,00)$ and the other by $(101,000)$. There are also two arrows going from the initial state to the terminal one. One of these arrows is labeled by $(00,01)$ and the other by $(001,010)$. The terminal state is connected to itself by two arrows. One of them is labeled by $(0,0)$ and the other by $(1,1)$.

Assume that $N = \varepsilon_n \dots \varepsilon_0$. To find the digits of $N+1$, we will consider the finite path $c = (p_{k+1}, a_k/b_k, p_k) \dots (p_2, a_1/b_1, p_1)(p_1, a_0/b_0, p_0)$ where $p_i \in \{I, T\}$, $p_0 = I$, $p_{k+1} = T$, $a_i, b_i \in A^*$ where $A = \{0,1\}$ and the words

$a_k \dots a_0$ and $b_k \dots b_0$ have no two consecutive 1's. Moreover $\dots 0 \dots 0 a_k \dots a_0 = \dots 0 \dots 0 \varepsilon_n \dots \varepsilon_0$.

Hence $N+1 = \varepsilon'_n \dots \varepsilon'_0$, where

$$\dots 0 \dots 0 b_k \dots b_0 = \dots 0 \dots 0 \varepsilon'_n \dots \varepsilon'_0.$$

Example: If $N = 10 = 10010$ then N corresponds to the path

$$(T, 1/1, T)(T, 00/01, I)(I, 10/00, I).$$

Hence $N + 1 = 10100 = 11$.

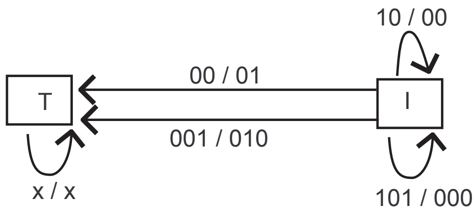

Fig. 1. Transducer of the Fibonacci adding machine

In [6], the authors define the stochastic adding machine as follows. Consider a "probabilistic" transducer $\mathcal{T}_p$ (see Fig. 2) where $0 < p < 1$, defined in the following manner.

The states of $\mathcal{T}_p$ are I and T. The labels are of the form $(0/0, 1)$, $(1/1, 1)$, $(a/b, p)$ or $(a/a, 1-p)$ where a/b is a label in $\mathcal{T}$.

The labeled edges in $\mathcal{T}_p$ are of the form $(T, (x/x, 1), T)$ where $x \in \{0, 1\}$ or of the form $(r, (a/b, p), q)$ or $(T, (a/a, 1-p), q)$ where $(r, a/b, q)$ is a labeled edge in $\mathcal{T}$, with $q = I$.

The stochastic process $\psi(N)$ is defined by $\psi(N) = \sum_{i=0}^{\infty} r_i(N) F_i$ where $(r_i(N))_{i\geq 0}$ is an infinite sequence of 0's and 1's without two consecutive 1's and with finitely many nonzero terms.

The sequence $(r_i(N))_{i\geq 0}$ is defined as follows. Put $r_i(0) = 0$ for all i, and assume that we have defined $(r_i(N-1))_{i\geq 0}$, $N \geq 1$. In the transducer $\mathcal{T}_p$, consider a path

$$\dots (T, (0/0, 1), T) \dots (T, (0/0, 1), T)(p_{n+1}, (a_n/b_n, t_n), p_n) \dots (p_1, (a_0/b_0, t_0), p_0)$$

where $p_0 = I$ and $p_{n+1} = T$, such that the words $\dots r_1(N-1) r_0(N-1)$ and $\dots 00 a_n \dots a_0$ are equal. We define the sequence $(r_i(N))_{i\geq 0}$ as the infinite sequence whose terms are 0 or 1 such that $\dots r_1(N) r_0(N) = \dots 00 b_n \dots b_0$.

We remark that $\psi(N-1)$ transitions to $\psi(N)$ with probability $p_{\psi(N-1)\psi(N)} = t_n t_{n-1} \dots t_0$.

Example: If $N = 10 = 10010$, then, in the transducer of the Fibonacci adding machine, N corresponds to the path

$$(T, 1/1, T)(T, 00/01, I)(I, 10/00, I).$$

In the stochastic Fibonacci adding machine, we have the following paths:

(1) $(T,(1/1,1),T)(T,(0/0,1),T)(T,(0/0,1),T)(T,(10/10,1-p),I)$. In this case $N = 10010$ transitions to 10010 with probability $1-p$.
(2) $(T,(1/1,1),T)(T,(00/00,1-p),I)(I,(10/00,p),I)$. In this case $N = 10$ transitions to $10000 = 8$ with probability $p(1-p)$.
(3) $(T,(1/1,1),T)(T,(00/01,p),I)(I,(10/00,p),I)$. In this case $N = 10$ transitions to $10100 = 11$ with probability p^2.

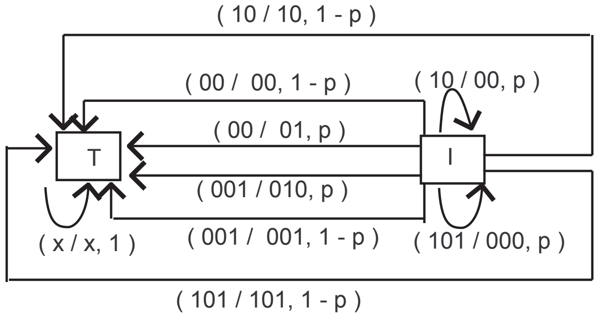

Fig. 2. Transducer of the Fibonacci fallible adding machine

By using the transducer $\mathcal{T}_p$, we can prove the following result (see [6]).

Proposition 5.1. *Let N be a nonnegative integer. Then the following statements hold:*

(1) *N transitions to N with probability $1-p$.*
(2) *$N = \varepsilon_k \dots \varepsilon_2 00$ with $k \geq 2$ transitions to $N+1 = \varepsilon_k \dots \varepsilon_2 01$ with probability p.*
(3) *$N = \varepsilon_k \dots \varepsilon_t 00 \underbrace{1010\dots1010}_{2s}$ with $s \geq 1$ and $k \geq t \geq 2s+2$ transitions to $N+1 = \varepsilon_k \dots \varepsilon_t 01 \underbrace{0\dots00}_{2s}$ with probability p^{s+1}, and to $N - \sum_{i=1}^{m} F_{2i-1} = N - F_{2m} + 1 = \varepsilon_k \dots \varepsilon_t 00 \underbrace{10\dots10}_{2s-2m} \underbrace{0\dots00}_{2m}$, $1 \leq m \leq s$, with probability $p^m(1-p)$.*
(4) *$N = \varepsilon_k \dots \varepsilon_t 0 \underbrace{0101\dots0101}_{2s}$ with $s \geq 2$ and $k \geq t \geq 2s+1$ transitions to $N+1 = \varepsilon_k \dots \varepsilon_t 0 \underbrace{1000\dots000}_{2s}$ with probability p^s, and to $N - \sum_{i=0}^{m} F_{2i} = N - F_{2m+1} + 1 = \varepsilon_k \dots \varepsilon_t 00 \underbrace{10\dots10}_{2s-2m} \underbrace{0\dots00}_{2m-1}$, $2 \leq m \leq s$, with probability $p^{m-1}(1-p)$.*
(5) *$N = \varepsilon_k \dots \varepsilon_3 001$ with $k \geq 3$ transitions to $N+1 = \varepsilon_k \dots \varepsilon_3 010$ with probability p.*

By Proposition 5.1, we construct the transition graph. We also find the associated transition operator S_p.

$$\begin{pmatrix}
1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
p(1-p) & 0 & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
p(1-p) & 0 & 0 & 0 & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0\dots \\
0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 & 0\dots \\
p^2(1-p) & 0 & 0 & 0 & 0 & p(1-p) & 0 & 1-p & p^3 & 0 & 0 & 0 & 0\dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 & 0\dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0\dots \\
0 & 0 & 0 & 0 & 0 & p(1-p) & 0 & 0 & p(1-p) & 0 & 1-p & p^2 & 0\dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}$$

Fig. 3. Transition graph of the stochastic adding machine in the Fibonacci base

Remark. In [6], the authors prove that the point spectrum of S_p in ℓ^∞ is equal to the set $\mathcal{K}_p = \{\lambda \in \mathbb{C} : (q_n(\lambda))_{n\ge 1} \text{ is bounded}\}$, where $q_{F_0}(z) = z$, $q_{F_1}(z) = z^2$, $q_{F_k}(z) = (1/p)q_{F_{k-1}}(z)q_{F_{k-2}}(z) - (1-p)/p$ for all $k \ge 2$, and for all nonnegative integers n we have $q_n = q_{F_{k_1}} \dots q_{F_{k_m}}$ where $F_{k_1} + \dots + F_{k_m}$ is the Fibonacci representation of n. In particular, $\sigma_{\mathrm{pt}}(S_p)$ is contained in

$$\mathcal{E}_p = \{\lambda \in \mathbb{C} : (q_{F_n}(\lambda))_{n\ge 1} \text{ is bounded}\} = \{\lambda \in \mathbb{C} : (\lambda_1, \lambda) \in K_g\}$$

where K_g is the filled Julia set of the function $g : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $g(x,y) = \big(\frac{1}{p^2}(x-1+p)(y-1+p), x\big)$ and $\lambda_1 = 1-p+(1-\lambda-p)^2/p$. They also investigated the topological properties of $\mathcal{E}_p$.

Proposition 5.2. *The operator S_p is well defined in the Banach spaces c_0, c and ℓ^α, $\alpha \ge 1$. The point spectra of S_p acting in c_0 and ℓ^α associated to the stochastic Fibonacci adding machines are empty. In c, the point spectrum equals $\{1\}$.*

Proof. By Proposition 5.1, we can prove that the sum of the entries in every column of S_p is bounded by a fixed constant $M > 0$.

Indeed, let $n \in \mathbb{N}$ and $s_n = \sum_{i=0}^{\infty} p_{i,n}$ be the sum of the entries in the nth column.

If $n = \varepsilon_k \dots \varepsilon_2 01$ or $n = \varepsilon_k \dots \varepsilon_3 010$ (Fibonacci representation), then by (1), (2) and (5) of Proposition 5.1, we have $s_n = 1$.

If $n = \varepsilon_k \dots \varepsilon_t 01\underbrace{0\dots 00}_{s}$, $s \ge 2$, then for all integers $i \in \mathbb{N}$, $p_{i,n} > 0$ implies that $i = n$ or $i = n-1$ or $i = \varepsilon_k \dots \varepsilon_t 01\underbrace{0\dots 0}_{s-2m}\underbrace{01\dots 01}_{2m}$, $s \ge 2m$, or

$i = \varepsilon_k \ldots \varepsilon_t 01 \underbrace{0\ldots0}_{s-2m}\underbrace{10\ldots10}_{2m}$, $s \geq 2m$. Hence

$$s_n \leq 1 - p + p^{\lceil s/2 \rceil} + 2\sum_{m=1}^{\infty} p^m(1-p) \leq 1 + 2p.$$

If $n = 0$, then $s_n \leq 1 + p$.

On the other hand, since S_p is a stochastic matrix, Proposition 4.1 shows that S_p is well defined in c_0, c (resp. in ℓ^α, $\alpha \geq 1$).

Now, let λ be an eigenvalue of S_p in X where $X \in \{c_0, c, \ell^\alpha\ (\alpha \geq 1)\}$ associated to the eigenvector $v = (v_i)_{i\geq 0} \in X$. Since the transition probability from any nonnegative integer i to any integer $i+k$, $k \geq 2$, is $p_{i,i+k} = 0$ (see Proposition 5.1), the operator S_p satisfies $(S_p)_{i,i+k} = 0$ for all $i, k \in \mathbb{N}$ with $k \geq 2$. Thus for every integer $k \geq 1$, we have

$$\sum_{i=0}^{k} p_{k-1,i} v_i = \lambda v_{k-1}. \tag{5.1}$$

Hence we can prove by induction on k that for any integer $k \geq 1$, there exists a complex number $c_k = c_k(p, \lambda)$ such that

$$v_k = c_k v_0. \tag{5.2}$$

Using the fact that the matrix S_p is self-similar, we can prove that $c_k = q_k$ for all $k \in \mathbb{N}$ (see Theorem 1 in [6]). Since

$$q_{F_n}(z) = \frac{1}{p} q_{F_{n-1}}(z) q_{F_{n-2}}(z) - \frac{1-p}{p}, \qquad \forall n \in \mathbb{N},$$

and (q_{F_n}) converges to 0 as $n \to \infty$, we deduce that the point spectrum of S_p acting in c_0 (resp. in ℓ^α, $\alpha \geq 1$) is empty. Using the same idea as in Proposition 4.5, we see that $\sigma_{\mathrm{pt},c} = \{1\}$. ∎

Remark. By the Phillips Theorem and duality, it follows that the spectra of S_p acting in $X \in \{c_0, c, \ell^1, \ell^\infty\}$ associated to the stochastic Fibonacci adding machine are all equal.

Theorem 5.3. *The spectrum of S_p acting in $X \in \{\ell^\infty, c_0, c, \ell^\alpha\ (\alpha \geq 1)\}$ contains the set $\mathcal{E}_p = \{\lambda \in \mathbb{C} : (q_{F_n}(\lambda))_{n\geq 0}$ is bounded$\}$.*

Proof. The proof is similar to the proof of Proposition 4.8 and will be done in the case ℓ^α, $\alpha > 1$. Let $\lambda \in \mathcal{E}_p$ and let us prove that λ belongs to the approximate point spectrum of S_p in ℓ^α, $\alpha > 1$.

For every integer $k \geq 2$, consider $w^{(k)} = (1, q_1(\lambda), \ldots, q_k(\lambda), 0, 0, \ldots)^t \in \ell^\alpha$ where $(q_k(\lambda))_{k\geq 1} = (q_k)_{k\geq 1}$ is defined in the proof of Proposition 5.2. Let $u^{(k)} = w^{(k)}/\|w^{(k)}\|_\alpha$.

Claim. $\lim_{n\to\infty} \|(S_p - \lambda\,\mathrm{Id})u^{(F_n)}\|_\alpha = 0$.

By using the same proof as for Proposition 4.8, we have

$$\|(S_p - \lambda\,\mathrm{Id})u^{(F_n)}\|_\alpha^\alpha \le \frac{D}{\|w^{(F_n)}\|_\alpha^\alpha} \sum_{j=0}^{F_n} |w_j^{(F_n)}|^\alpha B_{F_n,j}$$

where D is a positive constant and $B_{F_n,j} = \sum_{i=F_n}^{\infty} |(S_p - \lambda\,\mathrm{Id})_{i,j}|$.

We can prove as in Proposition 4.8 that for $0 \le j \le F_n$,

$$B_{F_n,j} \ne 0 \iff j = 0 \text{ or } j = F_n.$$

Indeed, if $j \in \{1, \ldots, F_n - 1\}$, then since $i \ge F_n$, we have $(S_p - \lambda\,\mathrm{Id})_{i,j} = p_{i,j}$. If the Fibonacci representation of j is $j = \varepsilon_k \ldots \varepsilon_2 01$ or $j = \varepsilon_k \ldots \varepsilon_t 10 \ldots 0$, it is easy to see by Proposition 5.1 that $p_{i,j} \ne 0$ implies $i < F_n$.

On the other hand, if $j = 0$ then $B_{F_n,j} = \sum_{l=F_n}^{\infty} p_{l,0}$. Since $p_{l,0} \ne 0$ if and only $l = F_i - 1$, and since $p_{F_i-1,0} = p^{\lceil i/2 \rceil}(1-p)$, we have $B_{F_n,j} \le 2\sum_{i=m}^{\infty} p^i(1-p) = 2p^m$ where $m = \lceil (n+1)/2 \rceil$.

Now assume $j = F_n$. In this case, $B_{F_n,j} = |1-p-\lambda| + \sum_{i=F_n+1}^{\infty} p_{i,F_n}$. On the other hand, by Proposition 5.1, $p_{i,F_n} \ne 0$ if and only if $i = F_n + F_m - 1$ for some $0 \le m \le n$ and $p_{F_n+F_m-1,F_n} = p^{[m/2]}(1-p)$. Therefore

$$B_{F_n,F_n} = |1-p-\lambda| + \sum_{m=0}^{n} p^{\lceil m/2 \rceil}(1-p) \le |1-p-\lambda| + 2. \tag{5.3}$$

Hence

$$\|(S_p - \lambda\,\mathrm{Id})u^{(F_n)}\|_\alpha^\alpha \le D\frac{2p^m + |q_{F_n}|^\alpha(|1-p-\lambda|+2)}{\|w^{(F_n)}\|_\alpha^\alpha}.$$

Since $\|w^{(F_n)}\|_\alpha \to \infty$ as $n \to \infty$ goes to infinity and $(q_{F_n})_{n\ge 0}$ is bounded, it follows that $\|(S_p - \lambda\,\mathrm{Id})u^{(F_n)}\|_\alpha \to 0$. Therefore λ belongs to the approximate point spectrum of S_p. Thus the spectrum of S_p acting on ℓ^α, $\alpha > 1$, contains $\mathcal{E}_p$.

The case of ℓ^1 can be handled in the same way. The details are left to the reader. ■

Open questions. We are not yet able to compute the residual and continuous spectrum of S_p acting in ℓ^∞, c_0, c or in ℓ^α, $\alpha \ge 1$. We conjecture that $\sigma(S_p) = \mathcal{E}_p$. Moreover, in the case of ℓ^∞ we conjecture that the residual spectrum is empty and the continuous spectrum is $\mathcal{E}_p \setminus \mathcal{K}_p$. The difficulty here is that the matrix S_p is not doubly stochastic. One may also look for a characterization of all real numbers $0 < p < 1$ for which $\mathcal{E}_p \ne \mathcal{K}_p$.

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References

[1] N. Dunford and J. T. Schwartz, *Linear Operators. Part I. General Theory*, Wiley, New York, 1988.

[2] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Grad. Texts in Math. 19, Encyclopaedia Math. Appl. 17, Springer, New York, 1982.

[3] P. R. Killeen and T. J. Taylor, *A stochastic adding machine and complex dynamics*, Nonlinearity 13 (2000), 1889–1903.

[4] P. R. Killeen and T. J. Taylor, *How the propagation of error through stochastic counters affects time discrimination and other psychophysical judgments*, Psychological Rev. 107 (2000), 430–459.

[5] S. G. Krantz, *Complex Analysis: the Geometric Viewpoint*, 2nd ed., Carus Math. Monogr. 23, Math. Assoc. Amer., Washington, DC, 2004.

[6] A. Messaoudi and D. Smania, *Eigenvalues of Fibonacci stochastic adding machine*, Stoch. Dynam. 10 (2010), 291–313.

[7] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. 11 (1960), 401–416.

[8] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.

[9] M. Schechter, *Principles of Functional Analysis*, 2nd ed., Grad. Stud. in Math. 36, Amer. Math. Soc., 2002.

[10] K. Yosida, *Functional Analysis*, Springer, 1980.

[11] E. Zeckendorff, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liège 42 (1972), 179–182.

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