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Dynamical properties of the automorphism groups of the random poset and random distributive lattice

by

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Abstract. A method is developed for proving non-amenability of certain automorphism groups of countable structures and is used to show that the automorphism groups of the random poset and random distributive lattice are not amenable. The universal minimal flow of the automorphism group of the random distributive lattice is computed as a canonical space of linear orderings but it is also shown that the class of finite distributive lattices does not admit hereditary order expansions with the Amalgamation Property.

0. Introduction. In this paper we continue the study of the dynamics of automorphism groups of countable structures and its connection with Ramsey theory in the spirit of [KPT]. We start with some basic definitions.

Let L be a countable first-order language. A class $\mathcal{K}$ of finite L-structures is called a *Fraïssé class* if it contains structures of arbitrarily large (finite) cardinality, is countable (in the sense that it contains only countably many isomorphism types) and satisfies the following:

(i) *Hereditary Property* (HP): If $\boldsymbol{B} \in \mathcal{K}$ and $\boldsymbol{A}$ can be embedded in $\boldsymbol{B}$, then $\boldsymbol{A} \in \mathcal{K}$.
(ii) *Joint Embedding Property* (JEP): If $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, there is $\boldsymbol{C} \in \mathcal{K}$ such that $\boldsymbol{A}, \boldsymbol{B}$ can be embedded in $\boldsymbol{C}$.
(iii) *Amalgamation Property* (AP): If $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and $f\colon \boldsymbol{A} \to \boldsymbol{B}$ and $g\colon \boldsymbol{A} \to \boldsymbol{C}$ are embeddings, there is $\boldsymbol{D} \in \mathcal{K}$ and embeddings $r\colon \boldsymbol{B} \to \boldsymbol{D}$ and $s\colon \boldsymbol{C} \to \boldsymbol{D}$ such that $r \circ f = s \circ g$.

(Throughout this paper embeddings and substructures will be understood in the usual model-theoretic sense (see, e.g., Hodges [Ho]); e.g., for

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graphs embeddings are induced embeddings, i.e., isomorphisms onto induced subgraphs.)

If $\mathcal{K}$ is a Fraïssé class, there is a unique, up to isomorphism, countably infinite structure $\boldsymbol{K}$ which is *locally finite* (i.e., finite generated substructures are finite), *ultrahomogeneous* (i.e., isomorphisms between finite substructures can be extended to automorphisms of the structure) and such that, up to isomorphism, its finite substructures are exactly those in $\mathcal{K}$. We call this the *Fraïssé limit* of $\mathcal{K}$, in symbols

$$\boldsymbol{K} = \mathrm{Flim}(\mathcal{K}).$$

We are interested in amenability properties of the automorphism group $G = \mathrm{Aut}(\boldsymbol{K})$, viewed as a topological group under the pointwise convergence topology. We note that the groups $\mathrm{Aut}(\boldsymbol{K})$, for $\boldsymbol{K}$ as above, are exactly the closed subgroups of the infinite symmetric group S_∞ (see [KPT]).

There are many examples of $G = \mathrm{Aut}(\boldsymbol{K})$ which are *extremely amenable*, i.e., every continuous action of such a group on a (non-empty) compact Hausdorff space, i.e., a *G-flow*, has a fixed point (see [KPT] and references therein). There are also many examples of such $G = \mathrm{Aut}(\boldsymbol{K})$ which are not extremely amenable but they are still *amenable* (i.e., every G-flow has an invariant Borel probability measure). This happens, for example, when $\mathcal{K}$ has the *Hrushovski Property* (i.e., for any $\boldsymbol{A} \in \mathcal{K}$ and for any (partial) isomorphisms $\varphi_i \colon \boldsymbol{B}_i \to \boldsymbol{C}_i, 1 \le i \le k$, where $\boldsymbol{B}_i, \boldsymbol{C}_i$ are substructures of $\boldsymbol{A}$, there is $\boldsymbol{B} \in \mathcal{K}$ containing $\boldsymbol{A}$ such that each φ_i can be extended to an automorphism ψ_i of $\boldsymbol{B}$, $1 \le i \le k$). This is because this is equivalent to the following property of $G = \mathrm{Aut}(\boldsymbol{K})$: there is an increasing sequence $C_0 \subseteq C_1 \subseteq \cdots$ of compact subgroups of G with $\bigcup_n C_n$ dense in G (see [KR]). A typical example of a class with the Hrushovski Property is the class $\mathcal{G}$ of finite graphs (see [H]). Its Fraïssé limit is the random graph $\boldsymbol{R}$, thus the automorphism group of the random graph is amenable. In fact, it moreover contains a dense locally finite subgroup (see [BM]).

There are also automorphism groups of Fraïssé structures which are not amenable. For example, the automorphism group of the countable atomless Boolean algebra (which is the Fraïssé limit of the class of finite Boolean algebras). This group is isomorphic to the group of homeomorphisms of the Cantor space $2^{\mathbb{N}}$ and the evaluation action of this homeomorphism group on $2^{\mathbb{N}}$ is a continuous action with no invariant probability Borel measure.

Let $\mathcal{P}$ be the Fraïssé class of all finite posets. Its Fraïssé limit $\mathrm{Flim}(\mathcal{P}) = \boldsymbol{P}$ is called the *random poset*. Let also $\mathcal{D}$ be the Fraïssé class of finite distributive lattices. Its Fraïssé limit $\mathrm{Flim}(\mathcal{D}) = \boldsymbol{D}$ is called the *random distributive lattice* (see Grätzer [G] for the theory of distributive lattices). In this paper we prove the following result (in Sections 3 and 4):

THEOREM 0.1. *The automorphism groups* $\mathrm{Aut}(\boldsymbol{P})$*, resp.* $\mathrm{Aut}(\boldsymbol{D})$*, of the random poset, resp. random distributive lattice, are not amenable.*

In particular, this shows that there is no amenable countable dense subgroup of $\mathrm{Aut}(\boldsymbol{P})$ or $\mathrm{Aut}(\boldsymbol{D})$ (but it is known that there are free countable dense subgroups; see [GMR], [GK]).

In Section 5 we also discuss the topological dynamics of $\mathrm{Aut}(\boldsymbol{D})$ and its connections with Ramsey properties of the class $\mathcal{D}$, in the spirit of [KPT].

Let $\boldsymbol{D} = \langle D, \wedge, \vee\rangle$ and let $X_{\mathcal{D}^*}$ be the space of linear orderings on D that have the property that for any finite Boolean sublattice $\boldsymbol{B} = \langle B, \wedge, \vee\rangle$ of $\boldsymbol{D}$ the order $<|B$ is *natural*, i.e., is the anti-lexicographical ordering induced by an ordering of the atoms of $\boldsymbol{B}$. (The notation $\mathcal{D}^*$ will be explained later.) Then $X_{\mathcal{D}^*}$ viewed as a compact subspace of 2^{D^2} endowed with the product topology and the obvious action of $\mathrm{Aut}(\boldsymbol{D})$ on it is an $\mathrm{Aut}(\boldsymbol{D})$-flow. Recall that for any topological group G, a G-flow X is *minimal* if every orbit is dense. Also a minimal G-flow X is the *universal minimal flow* if any minimal G-flow Y is a factor of X, i.e., there is a continuous surjection $\pi\colon X \to Y$ which is a G-map: $\pi(g\cdot x) = g\cdot \pi(x)$ for all $g \in G$ and $x \in X$. Such a flow exists and is unique up to isomorphism (see, e.g., [KPT]). In this paper we will demonstrate the following theorem:

THEOREM 0.2. *The universal minimal flow of* $\mathrm{Aut}(\boldsymbol{D})$ *is* $X_{\mathcal{D}^*}$.

We also consider Ramsey-theoretic properties of the class $\mathcal{D}$. Fix a countable language L and structures $\boldsymbol{A}, \boldsymbol{B}$ in L. Then $\boldsymbol{A} \subseteq \boldsymbol{B}$ means that $\boldsymbol{A}$ is a *substructure* of $\boldsymbol{B}$ and $\boldsymbol{A} \le \boldsymbol{B}$ means that $\boldsymbol{A}$ can be embedded in $\boldsymbol{B}$, i.e., $\boldsymbol{A}$ is isomorphic to a substructure of $\boldsymbol{B}$. We also let, for $\boldsymbol{A} \le \boldsymbol{B}$, $\binom{\boldsymbol{B}}{\boldsymbol{A}}$ be the set of all substructures of $\boldsymbol{B}$ isomorphic to $\boldsymbol{A}$. Given a class $\mathcal{K}$ of finite structures in L, $\boldsymbol{A} \le \boldsymbol{B} \le \boldsymbol{C}$ all in $\mathcal{K}$ and $k \ge 2$, $t \ge 1$,

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{k,t}$$

means that for any coloring $c\colon \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \{1, \dots, k\}$, there is $\boldsymbol{B}' \subseteq \boldsymbol{C}$ with $\boldsymbol{B}' \cong \boldsymbol{B}$ such that c on $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$ obtains at most t many values. We simply write $\boldsymbol{C} \to (\boldsymbol{B})^{A}_{k}$ if $t = 1$.

Let now $\mathcal{K}$ be a class of finite structures in L, and $\boldsymbol{A} \in \mathcal{K}$. The *Ramsey degree* of $\boldsymbol{A}$ in $\mathcal{K}$, in symbols $t(\boldsymbol{A}, \mathcal{K})$, is the least t, if it exists, such that for any $\boldsymbol{A} \le \boldsymbol{B}$ in $\mathcal{K}$, and any $k \ge 2$, there is $\boldsymbol{C} \ge \boldsymbol{B}$ in $\mathcal{K}$ such that

$$\boldsymbol{C} \to (\boldsymbol{B})^{A}_{k,t}.$$

Otherwise let $t(\boldsymbol{A}, \mathcal{K}) = \infty$. If $t(\boldsymbol{A}, \mathcal{K}) = 1$ we say that $\boldsymbol{A}$ is a *Ramsey object* in $\mathcal{K}$. A Fraïssé class has the *Ramsey property* (RP) if all its elements are Ramsey objects.

It is a well-known fact in the theory of distributive lattices (see [G]) that for any finite distributive lattice $\boldsymbol{L}$ there is a (unique up to isomorphism that fixes $\boldsymbol{L}$) finite Boolean lattice $\boldsymbol{B_L}$ that has the following properties, with $0^{\boldsymbol{L}}$, resp. $1^{\boldsymbol{L}}$, denoting the minimum, resp. maximum elements of a finite lattice $\boldsymbol{L}$:

(i) $\boldsymbol{L} \subseteq \boldsymbol{B_L}$, $0^{\boldsymbol{L}} = 0^{\boldsymbol{B_L}}$, $1^{\boldsymbol{L}} = 1^{\boldsymbol{B_L}}$,
(ii) $\boldsymbol{L}$ generates $\boldsymbol{B_L}$ as a Boolean algebra.

Let then $t(\boldsymbol{L})$ be defined by

$$t(\boldsymbol{L}) = \frac{|\mathrm{Aut}(\boldsymbol{B_L})|}{|\mathrm{Aut}(\boldsymbol{L})|} = \frac{n_{\boldsymbol{L}}!}{|\mathrm{Aut}(\boldsymbol{L})|},$$

where $n_{\boldsymbol{L}}$ is the number of atoms of $\boldsymbol{B_L}$. We have

Theorem 0.3 (Fouché [F]). *The Ramsey degree $t(\boldsymbol{L}, \mathcal{D})$ of a finite distributive lattice $\boldsymbol{L}$ is equal to $t(\boldsymbol{L})$.*

Corollary 0.4 (Hagedorn–Voigt [HV] (unpublished); see also Prömel –Voigt [PV, 2.2]). *The Ramsey objects in $\mathcal{D}$ are exactly the Boolean lattices.*

For example, it easily follows from 0.3 that the Ramsey degree $t(\boldsymbol{n}, \mathcal{D})$ of a linear ordering $\boldsymbol{n}$ with $n \geq 1$ elements is equal to $(n-1)!$.

It is quite common for a Fraïssé class $\mathcal{K}$ (in a language L) to admit an *order expansion* $\mathcal{K}^*$ (i.e., a class of finite structures in the language $L \cup \{<\}$ so that if $\boldsymbol{A}^* = \langle \boldsymbol{A}, < \rangle \in \mathcal{K}^*$, then $\boldsymbol{A} \in \mathcal{K}$, $<$ is a linear ordering on the universe A of $\boldsymbol{A}$, and moreover $\mathcal{K}$ consists of all reducts in the language L of the structures in $\mathcal{K}^*$) such that $\mathcal{K}^*$ is a Fraïssé class and has the Ramsey Property. This has many applications in the study of the Ramsey-theoretic properties of $\mathcal{K}$ and the dynamics of the automorphisms group of its Fraïssé limit (see [KPT]). The Fraïssé classes of posets, $\mathcal{P}$, Boolean lattices, $\mathcal{BL}$ (see Sections 1, 3, 4), and Boolean algebras, $\mathcal{BA}$ (see [KPT]), admit such order expansions. However we show in Section 5 that, rather surprisingly, $\mathcal{D}$ fails to do so. In fact we have the following result:

Theorem 0.5. *There is no order expansion of the class $\mathcal{D}$ of finite distributive lattices, which satisfies* HP *and* AP.

The following question was raised in [KPT, p. 174]: is there a Fraïssé class $\mathcal{K}$ for which $t(\boldsymbol{A}, \mathcal{K}) < \infty$ for all $\boldsymbol{A} \in \mathcal{K}$, but $\mathcal{K}$ does not admit a Fraïssé order expansion with RP? Such examples were found in [N] (see also [LNS]) and in [J]. The above theorem also provides such an example (in a somewhat stronger form). We also discuss other ones in Section 6.

Finally we conclude with an open problem. Let $\mathcal{L}$ be the class of finite lattices. It is again known that $\mathcal{L}$ is a Fraïssé class (see [G]). We do not know if the automorphism group of its Fraïssé limit, the *random lattice*, is amenable or not. Nor do we know what its universal minimal flow is and if there is any way to determine the Ramsey degrees of lattices or even the Ramsey objects in $\mathcal{L}$.

1. Distributive and Boolean lattices

(A) We review here some basic facts concerning distributive and Boolean lattices. We view lattices as structures $\boldsymbol{L} = \langle L, \wedge, \vee \rangle$ in the language with two binary function symbols $\wedge, \vee$. When a lattice $\boldsymbol{L}$ has a maximum element we will denote it by $1^{\boldsymbol{L}}$ and similarly for a minimum element, which we denote it $0^{\boldsymbol{L}}$. Clearly these exist when $\boldsymbol{L}$ is finite. We emphasize though that these are not part of our language, so embeddings have only to preserve $\wedge, \vee$ but not necessarily the maxima and minima (when they exist).

Let $\mathcal{D}$ be the class of finite distributive lattices. It is well-known that $\mathcal{D}$ is a Fraïssé class (see [G, V.4]). We denote by $\boldsymbol{D}$ the Fraïssé limit of $\mathcal{D}$ and call it the *random distributive lattice*.

A *Boolean lattice* is a distributive lattice $\boldsymbol{L}$ with maximum and minimum elements which is *relatively complemented*, i.e., for each $a \leq c$ in L if $a \leq b \leq c$ then there is a (necessarily) unique x such that $b \wedge x = a$ and $b \vee x = c$. The class of finite Boolean lattices, viewed as a class of structures in the language containing only $\wedge, \vee$, does not have the hereditary property. We will therefore enlarge the language by introducing a symbol for the operation of *relative complementation* defined as follows:

$$R(a,b,c) = \begin{cases} r(a,b,c) & \text{if } a \leq b \leq c, \\ a & \text{otherwise,} \end{cases}$$

where $r(a,b,c)$ is the relative complement of b in the interval $[a,c]$, when $a \leq b \leq c$.

We denote by $\mathcal{BL}$ the class of all finite structures of the form $\boldsymbol{B} = \langle B, \wedge, \vee, R \rangle$, where $\langle B, \wedge, \vee \rangle$ is a Boolean lattice and R is relative complementation as defined above.

We note that if $\boldsymbol{B} = \langle B, \wedge, \vee, R \rangle$ and $\boldsymbol{C} = \langle C, \wedge, \vee, R \rangle$ are in $\mathcal{BL}$, then $\pi\colon \boldsymbol{B} \to \boldsymbol{C}$ is an embedding iff π is a lattice embedding, i.e., an embedding of $\langle B, \wedge, \vee \rangle$ into $\langle C, \wedge, \vee \rangle$. If $\pi(0^{\boldsymbol{B}}) = c_0 \in C$ and $\pi(1^{\boldsymbol{B}}) = c_1 \in C$, then c_0, c_1 do not have to be equal to $0^{\boldsymbol{C}}, 1^{\boldsymbol{C}}$, resp.

We also consider Boolean algebras, which we view as structures of the form $\langle B, \wedge, \vee, -, 0, 1 \rangle$ (so beyond $\wedge, \vee$ we have a symbol for complementation and constants for 0, 1). These are essentially the same as Boolean lattices except for the choice of language. Thus the notion of embedding is different. An embedding of finite Boolean algebras preserves 0, 1 but this is not necessarily the case for an embedding of Boolean lattices. As is well known, Boolean algebras can be represented as fields of sets.

We note that if $\boldsymbol{C} \subseteq \boldsymbol{B}$ are in $\mathcal{BL}$, $\boldsymbol{B}$ has n atoms $b_1, \ldots, b_n$, and $\boldsymbol{C}$ has m atoms, then there are disjoint sets $X_i \subseteq \{1, \ldots, n\}$, $0 \leq i \leq m$, such that $X_i \neq \emptyset$ if $i \neq 0, \bigvee\{b_k \colon k \in X_0\} = 0^{\boldsymbol{C}}$, $\bigvee\{b_k \colon k \in \bigcup_{i=0}^m X_i\} = 1^{\boldsymbol{C}}$, and $c_1 = \bigvee\{b_k \colon k \in X_0 \cup X_1\}, \ldots, c_m = \bigvee\{b_k \colon k \in X_0 \cup X_m\}$ are the atoms of $\boldsymbol{C}$.

We use this to verify that $\mathcal{BL}$ is a Fraïssé class. This is well-known but we could not find an explicit reference, so we write down the proof for the convenience of the reader.

Clearly $\mathcal{BL}$ satisfies (HP). Since the smallest Boolean lattice embeds in any Boolean lattice, (JEP) follows from (AP), thus it is enough to verify the latter.

Let $f\colon \boldsymbol{A} \to \boldsymbol{B}$ and $g\colon \boldsymbol{A} \to \boldsymbol{C}$ be embeddings, where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{BL}$. Let $a_1, \dots, a_k$ be the atoms of $\boldsymbol{A}$, $b_1, \dots, b_m$ the atoms of $\boldsymbol{B}$, and $c_1, \dots, c_n$ the atoms of $\boldsymbol{C}$. Let $B_0, \dots, B_k$ be pairwise disjoint subsets of $\{1, \dots, m\}$ and $C_0, \dots, C_k$ pairwise disjoint subsets of $\{1, \dots, n\}$ such that

$$\begin{aligned} f(0^{\boldsymbol{A}}) &= \bigvee\{b_\ell\colon \ell \in B_0\}, \\ f(a_i) &= \bigvee\{b_\ell\colon \ell \in B_0 \cup B_i\}, \qquad 1 \le i \le k, \\ g(0^{\boldsymbol{A}}) &= \bigvee\{c_\ell\colon \ell \in C_0\}, \\ g(a_i) &= \bigvee\{c_\ell\colon \ell \in C_0 \cup C_i\}, \qquad 1 \le i \le k. \end{aligned}$$

Let also $B' = \{1, \dots, m\} \setminus \bigcup_{i=0}^k B_i$ and $C' = \{1, \dots, n\} \setminus \bigcup_{i=0}^k C_i$.

We will now define $\boldsymbol{D} \in \mathcal{BL}$ and embeddings $r\colon \boldsymbol{B} \to \boldsymbol{D}$, $s : \boldsymbol{C} \to \boldsymbol{D}$ so that $r \circ f = s \circ g$. The atoms of $\boldsymbol{D}$ are the points in $\bigsqcup_{i=0}^k (B_i \times C_i) \sqcup B' \sqcup C'$ (where these are understood as disjoint unions). The embeddings r, s are defined as follows:

$$r(b_i) = \begin{cases} \{i\} \times C_j & \text{if } i \in B_j, \\ i & \text{if } i \in B', \end{cases}$$
$$s(c_i) = \begin{cases} B_j \times \{i\} & \text{if } i \in C_j, \\ i & \text{if } i \in C'. \end{cases}$$

Observe that $r \circ f(a_i) = B_i \times C_i = s \circ g(a_i)$ for all i, and hence $r \circ f = s \circ g$.

We denote by $\boldsymbol{B}_\infty = \langle B_\infty, \wedge, \vee, R \rangle$ the Fraïssé limit of $\mathcal{BL}$. Note that this is *not* a Boolean lattice as it has no maximum or minimum. It is however a relatively complemented distributive lattice.

(B) We will use the following standard fact concerning distributive lattices (see [G, II.4]).

THEOREM 1.1.

(i) *Every finite distributive lattice* $\boldsymbol{L}$ *can be embedded* (*as a lattice*) *in a Boolean lattice* $\boldsymbol{B}$ *with* $0^{\boldsymbol{L}}$ *sent to* $0^{\boldsymbol{B}}$ *and* $1^{\boldsymbol{L}}$ *to* $1^{\boldsymbol{B}}$.

(ii) *Let, for* $i = 1, 2$, $\boldsymbol{L}_i$ *be a finite distributive lattice and* $\boldsymbol{B}_i$ *a finite Boolean lattice such that* $\boldsymbol{L}_i$ *is a sublattice of* $\boldsymbol{B}_i$ *and* $0^{\boldsymbol{L}_i} = 0^{\boldsymbol{B}_i}$, $1^{\boldsymbol{L}_i} = 1^{\boldsymbol{B}_i}$. *If* $\boldsymbol{L}_i$ *generates* $\boldsymbol{B}_i$ *as a Boolean algebra and* $\varphi\colon \boldsymbol{L}_1 \to \boldsymbol{L}_2$ *is an isomorphism, then there is a unique isomorphism* $\overline{\varphi}\colon \boldsymbol{B}_1 \to \boldsymbol{B}_2$ *extending* φ.

(C) Using now 1.1 we verify that if $\boldsymbol{B}_\infty = \langle B_\infty, \wedge, \vee, R\rangle$ is the Fraïssé limit of $\mathcal{BL}$, then $\langle B_\infty, \wedge, \vee\rangle \cong \boldsymbol{D}$. Thus since R is definable in $\langle B_\infty, \wedge, \vee\rangle$, we have $\mathrm{Aut}(\boldsymbol{B}_\infty) = \mathrm{Aut}(\langle B_\infty, \wedge, \vee\rangle) \cong \mathrm{Aut}(\boldsymbol{D})$.

Using 1.1(i), it is clear that, up to isomorphism, the finite substructures of $\langle B_\infty, \wedge, \vee\rangle$ are the finite distributive lattices. So to show that $\langle B_\infty, \wedge, \vee\rangle$ is isomorphic to the random distributive lattice, it is enough to show that it has the following extension property: If $\boldsymbol{L}$ is a sublattice of a finite distributive lattice $\boldsymbol{M}$, $\boldsymbol{L} \subseteq \boldsymbol{M}$, and $f\colon \boldsymbol{L} \to \langle \boldsymbol{B}_\infty, \wedge, \vee\rangle$ is an embedding, then we can extend f to an embedding $\bar{f}\colon \boldsymbol{M} \to \langle B_\infty, \wedge, \vee\rangle$. Let $\boldsymbol{L}' \subseteq \langle B_\infty, \wedge, \vee\rangle$ be the image of $\boldsymbol{L}$ under f and let $\boldsymbol{B}'_{\boldsymbol{L}}$ be the finite sublattice of $\langle B, \wedge, \vee\rangle$ with $\boldsymbol{L}' \subseteq \boldsymbol{B}'_{\boldsymbol{L}}$, $0^{\boldsymbol{L}'} = 0^{\boldsymbol{B}'_{\boldsymbol{L}}}, 1^{\boldsymbol{L}'} = 1^{\boldsymbol{B}'_{\boldsymbol{L}}}$, which is moreover a Boolean lattice and is generated as a Boolean algebra by $\boldsymbol{L}'$. Let also $\boldsymbol{B}_{\boldsymbol{M}}$ be a Boolean lattice containing $\boldsymbol{M}$ with the same 0, 1 and generated as a Boolean algebra by $\boldsymbol{M}$. Finally, let $\boldsymbol{B}_{\boldsymbol{L}}$ be the Boolean sublattice of $\boldsymbol{B}_{\boldsymbol{M}}$ with the same 0, 1 as $\boldsymbol{L}$ and generated as a Boolean algebra by $\boldsymbol{L}$. By 1.1(ii) there is an isomorphism $f'\colon \boldsymbol{B}_{\boldsymbol{L}} \to \boldsymbol{B}'_{\boldsymbol{L}}$ extending f. Then $f'\colon \boldsymbol{B}_{\boldsymbol{L}} \to \boldsymbol{B}_\infty$ is an embedding (in the language $\wedge, \vee, R$), and so there is an embedding $\bar{f}'\colon \boldsymbol{B}_{\boldsymbol{M}} \to \boldsymbol{B}_\infty$ (again in the language $\wedge, \vee, R$) extending f' and thus f. Then if $\bar{f} = \bar{f}'|M$, $\bar{f}\colon \boldsymbol{M} \to \langle B_\infty, \wedge, \vee\rangle$ is an embedding that extends f.

(D) Finally we use, in Section 4, the concept of an anti-lexicographical ordering induced by an ordering of the atoms of a Boolean lattice $\boldsymbol{B}$. Let $<$ be a linear ordering of the set of $\{b_1, \ldots, b_n\}$ of $\boldsymbol{B}$. This induces the following anti-lexicographical ordering on B. Given $x, y \in B$ we can write them uniquely as $x = \delta_1 b_1 \vee \cdots \vee \delta_n b_n$, $y = \epsilon_1 b_1 \vee \cdots \vee \epsilon_n b_n$, where $\delta_i, \epsilon_i \in \{0,1\}$ and $1b = b, 0b = 0^{\boldsymbol{B}}$ for $b \in B$. Then we put

$$x <_{\mathrm{al}} y \;\Leftrightarrow\; (\delta_n < \epsilon_n) \text{ or } (\delta_n = \epsilon_n \text{ and } \delta_{n-1} < \epsilon_{n-1}) \text{ or } \ldots .$$

2. A criterion for non-amenability. We will use ideas from [KPT] to formulate a simple sufficient criterion for non-amenability of an automorphism group as above.

Let L be a countable language. Denote by $L^* = L \cup \{<\}$ the language obtained by adding a new binary relation symbol $<$ to L. A structure $\boldsymbol{A}^*$ of L^* has the form $\boldsymbol{A}^* = \langle \boldsymbol{A}, <\rangle$, where $\boldsymbol{A}$ is a structure of L and $<$ is a binary relation on A (= the universe of $\boldsymbol{A}$). A class $\mathcal{K}^*$ on L^* is called an *order class* if ($\langle \boldsymbol{A}, <\rangle \in \mathcal{K}^* \Rightarrow <$ is a linear ordering on A). For $\boldsymbol{A}^* = \langle \boldsymbol{A}, <\rangle$ as above, we put $\boldsymbol{A}^*|L = \boldsymbol{A}$.

If $\mathcal{K}$ is a class of finite structures in L, we say that an order class $\mathcal{K}^*$ is an *order expansion* of $\mathcal{K}$ if

$$\mathcal{K} = \mathcal{K}^*|L = \{\boldsymbol{A}^*|L\colon \boldsymbol{A}^* \in \mathcal{K}^*\}.$$

In this case for any $\boldsymbol{A} \in \mathcal{K}$ and $\boldsymbol{A}^* = \langle \boldsymbol{A}, <\rangle \in \mathcal{K}^*$, we say that $<$ is a

$\mathcal{K}^*$-*admissible ordering* for $\boldsymbol{A}$. We say that the order expansion $\mathcal{K}^*$ is *reasonable* if for every $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, embedding $\pi\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{A}$, there is a $\mathcal{K}^*$-admissible ordering $<'$ on $\boldsymbol{B}$ such that π is also an embedding of $\langle \boldsymbol{A}, <\rangle$ into $\langle \boldsymbol{B}, <'\rangle$ (i.e., π also preserves $<, <'$).

Assume now that $\mathcal{K}$ is a Fraïssé class and $\mathcal{K}^*$ an order expansion of $\mathcal{K}$. Let $\boldsymbol{K} = \mathrm{Flim}(\mathcal{K})$ and denote by $X_{\mathcal{K}^*}$ the space of all linear orderings $<^*$ on K (the universe of $\boldsymbol{K}$) that have the property that for any finite substructure $\boldsymbol{A}$ of $\boldsymbol{K}$, $\boldsymbol{A}^* = \langle \boldsymbol{A}, <^*|A\rangle \in \mathcal{K}^*$. We call these $\mathcal{K}^*$-*admissible orderings* on $\boldsymbol{K}$. They clearly form a closed (thus compact) subspace of 2^{K^2} (with the product topology). If $\mathcal{K}^*$ is reasonable, then $X_{\mathcal{K}^*}$ is non-empty.

Let $G = \mathrm{Aut}(\boldsymbol{K})$ be the automorphism group of $\boldsymbol{K}$. It acts continuously on $X_{\mathcal{K}^*}$ in the obvious way, so $X_{\mathcal{K}^*}$ is a G-flow.

Recall from [KPT, 7.3] that $\mathcal{K}^*$ has the *ordering property*, OP, if for every $\boldsymbol{A} \in \mathcal{K}$ there is $\boldsymbol{B} \in \mathcal{K}$ such that for any $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{A}$ and for any $\mathcal{K}^*$-admissible ordering $<'$ on $\boldsymbol{B}$, there is an embedding $\pi\colon \langle \boldsymbol{A}, <\rangle \to \langle \boldsymbol{B}, <'\rangle$. The following was proved in [KPT, 7.4], assuming that $\mathcal{K}^*$ is a Fraïssé, reasonable order expansion of $\mathcal{K}$:

The G-flow $X_{\mathcal{K}^*}$ is minimal (*i.e., every orbit is dense*) *iff $\mathcal{K}^*$ has the ordering property.*

We now use these ideas to establish a sufficient criterion for the non-amenability of G.

Proposition 2.1. *Let $\mathcal{K}$ be a Fraïssé class in a language L, and $\mathcal{K}^*$ a Fraïssé order expansion of $\mathcal{K}$ which is reasonable and has the ordering property. Suppose that there are $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ and for each $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{A}$, an embedding $\pi_<\colon \boldsymbol{A} \to \boldsymbol{B}$ with the following properties:*

- (i) *There is a $\mathcal{K}^*$-admissible ordering $<'$ on $\boldsymbol{B}$ such that for every $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{A}$, $\pi_<$ is* not *an embedding of $\langle \boldsymbol{A}, <\rangle$ into $\langle \boldsymbol{B}, <'\rangle$.*
- (ii) *For any two distinct $\mathcal{K}^*$-admissible orderings $<_1, <_2$ on $\boldsymbol{A}$ and every $\mathcal{K}^*$-admissible ordering $<'$ on $\boldsymbol{B}$ one of $\pi_{<_1}, \pi_{<_2}$ fails to be an embedding from $\langle \boldsymbol{A}, <_1\rangle, \langle \boldsymbol{A}, <_2\rangle$, resp., into $\langle \boldsymbol{B}, <'\rangle$.*

Then if $\boldsymbol{K} = \mathrm{Flim}(\mathcal{K})$, the group $G = \mathrm{Aut}(\boldsymbol{K})$ is not amenable.

Proof. We can assume that $\boldsymbol{A}, \boldsymbol{B}$ are substructures of $\boldsymbol{K}$. Let $<_1, \dots, <_n$ enumerate all the $\mathcal{K}^*$-admissible orderings on $\boldsymbol{A}$ and let the image of $\boldsymbol{A}$ under $\pi_{<_i}$ $(1 \le i \le n)$ be denoted by $\boldsymbol{A}_i$, which is a substructure of $\boldsymbol{B}$. Denote also by $<_i'$ the image of $<_i$ under $\pi_{<_i}$, which is a $\mathcal{K}^*$-admissible ordering on $\boldsymbol{A}_i$.

For any finite substructure $\boldsymbol{C}$ of $\boldsymbol{K}$ and $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{C}$, let $N_{\langle \boldsymbol{C}, <\rangle}$ denote the nonempty basic clopen set in $X_{\mathcal{K}^*}$ consisting of all

$<^* \in X_{\mathcal{K}^*}$ with $<^*|C = <$. Condition (i) tells us that

$$\bigcup_{i=1}^{n} N_{\langle \boldsymbol{A}_i, <_i' \rangle} \neq X_{\mathcal{K}^*}.$$

Condition (ii) also says that for $1 \leq i \neq j \leq n$,

$$N_{\langle \boldsymbol{A}_i, <_i' \rangle} \cap N_{\langle \boldsymbol{A}_j, <_j' \rangle} = \emptyset.$$

Suppose now, towards a contradiction, that G is amenable, so that in particular the G-flow $X_{\mathcal{K}^*}$ admits an invariant probability Borel measure, say μ. Since $\mathcal{K}^*$ has the ordering property, this action is minimal and so μ has full support, i.e., $\mu(V) > 0$ for every open non-empty set $V \subseteq X_{\mathcal{K}^*}$.

Since for $1 \leq i \leq n$, $\pi_i \colon \boldsymbol{A} \to \boldsymbol{A}_i$ is an isomorphism, there is $\varphi_i \in G$ extending π_i. Clearly then $\varphi_i(N_{\langle \boldsymbol{A}, <_i \rangle}) = N_{\langle \boldsymbol{A}_i, <_i' \rangle}$, so $\mu(N_{\langle \boldsymbol{A}, <_i \rangle}) = \mu(N_{\langle \boldsymbol{A}_i, <_i' \rangle})$. But obviously

$$\bigcup_{i=1}^{n} N_{\langle \boldsymbol{A}, <_i \rangle} = X_{\mathcal{K}^*},$$

so $\mu(\bigcup_{i=1}^n N_{\langle \boldsymbol{A}_i, <_i' \rangle}) = \sum_{i=1}^n \mu(N_{\langle \boldsymbol{A}_i, <_i' \rangle}) = 1$. On the other hand the set $V = X_{\mathcal{K}^*} \setminus \bigcup_{i=1}^n N_{\langle \boldsymbol{A}_i, <_i' \rangle}$ is open non-empty, so $\mu(V) > 0$, a contradiction. ■

There is also another variation of this criterion which requires weaker conditions on the class $\mathcal{K}^*$ but imposes a stronger condition on $\boldsymbol{B}$.

PROPOSITION 2.2. *Let $\mathcal{K}$ be a Fraïssé class in a language L, and $\mathcal{K}^*$ an order expansion of $\mathcal{K}$ which is reasonable and has* HP. *Suppose that there are $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ and for each $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{A}$, an embedding $\pi_< \colon \boldsymbol{A} \to \boldsymbol{B}$ with the following properties:*

(i) *There is a $\mathcal{K}^*$-admissible ordering $<'$ on $\boldsymbol{B}$ such that for every $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{A}$, $\pi_<$ is* not *an embedding of $\langle \boldsymbol{A}, < \rangle$ into $\langle \boldsymbol{B}, <' \rangle$.*

(ii) *For any two distinct $\mathcal{K}^*$-admissible orderings $<_1, <_2$ on $\boldsymbol{A}$ and every $\mathcal{K}^*$-admissible ordering $<'$ on $\boldsymbol{B}$ one of $\pi_{<_1}, \pi_{<_2}$ fails to be an embedding from $\langle \boldsymbol{A}, <_1 \rangle, \langle \boldsymbol{A}, <_2 \rangle$, resp., into $\langle \boldsymbol{B}, <' \rangle$.*

Moreover assume that the automorphism group of $\boldsymbol{B}$ acts transitively on the set of $\mathcal{K}^*$-admissible orderings of $\boldsymbol{B}$.

Then if $\boldsymbol{K} = \mathrm{Flim}(\mathcal{K})$, the group $G = \mathrm{Aut}(\boldsymbol{K})$ is not amenable.

Proof. Repeat the proof of 2.1 and notice that $\mu(N_{\langle \boldsymbol{B}, <' \rangle}) = 0$. But then by the transitivity of the action of the automorphism group of $\boldsymbol{B}$ on the set of $\mathcal{K}^*$-admissible orderings of $\boldsymbol{B}$, it follows that $\mu(N_{\langle \boldsymbol{B}, < \rangle}) = 0$ for any $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{B}$, and thus $\mu(X_{\mathcal{K}^*}) = 0$, a contradiction. ■

3. The non-amenability of $\mathrm{Aut}(\boldsymbol{P})$**.** We now apply 2.1 to the class $\mathcal{P}$ of all finite posets. Here the class $\mathcal{P}^*$ consists of all $\langle \boldsymbol{A}, < \rangle$ with $\boldsymbol{A} = \langle A, \prec \rangle$ a finite poset and $<$ a linear extension of $\prec$. This is a reasonable, Fraïssé order expansion of $\mathcal{P}$. It also has the ordering property by [PTW, Theorem 16]. It only remains to verify the existence of finite posets $\boldsymbol{A}, \boldsymbol{B}$ satisfying (i), (ii) of 2.1.

Indeed, take $\boldsymbol{A} = \langle \{a_0, a_1\}, \prec \rangle$ to be the poset consisting of two elements a_0, a_1 which are not related (i.e., the partial order $\prec$ on $\boldsymbol{A}$ is empty). Let $\boldsymbol{B} = \langle \{b_0, b_1, b_2\}, \prec' \rangle$, where $(b_0, b_2), (b_1, b_2)$ are unrelated in $\prec'$ but $b_0 \prec' b_1$.

There are two $\mathcal{P}^*$-admissible orderings $<_1, <_2$ on $\boldsymbol{A}$, given by

$$a_1 <_1 a_0, \quad a_0 <_2 a_1.$$

We now define the embeddings $\pi_{<_i} \colon \boldsymbol{A} \to \boldsymbol{B}$ by $\pi_{<_1}(a_0) = b_0, \pi_{<_1}(a_1) = b_2$ and $\pi_{<_2}(a_0) = b_1, \pi_{<_2}(a_1) = b_2$. Then (in the notation of the proof of 2.1), if $<'$ is a $\mathcal{P}^*$-admissible ordering on $\boldsymbol{B}$ that extends $<'_1$, we must have $b_2 <' b_0 <' b_1$, while if it extends $<'_2$ we must have $b_0 <' b_1 <' b_2$, so condition (ii) is clear. To verify (i), note that the ordering $<'$ on $\boldsymbol{B}$ given by $b_0 <' b_2 <' b_1$ is $\mathcal{P}^*$-admissible and it extends none of $<'_1, <'_2$.

Thus the proof that the automorphism of the random poset is not amenable is complete.

Remark 3.1. Although one can easily see, as we mentioned in the introduction, that the automorphism group of the countable atomless Boolean algebra is not amenable, one can also give a proof using 2.1. Indeed, let $\mathcal{BA}$ denote the class of finite Boolean algebras and $\mathcal{BA}^*$ the class of all finite Boolean algebras with an ordering that is induced anti-lexicographically from an ordering of the atoms (see Section 1, **(D)**). These satisfy all the other conditions required in 2.1, so we only need to find $\boldsymbol{A}, \boldsymbol{B}$ satisfying (i), (ii). Below we use the notation in the proof of 2.1. Indeed, let $\boldsymbol{A}$ be the Boolean algebra with two atoms a, b and $\boldsymbol{B}$ be the Boolean algebra with three atoms x, y, z. For the ordering $<_1$ on $\boldsymbol{A}$ induced by $a <_1 b$, we let $\pi_{<_1}$ be the embedding sending a to $y \vee z$ and b to x. For the ordering $<_2$ on $\boldsymbol{A}$ induced by $b <_2 a$, we let $\pi_{<_2}$ be the embedding sending a to y and b to $x \vee z$. This easily works since any $\mathcal{K}^*$-admissible ordering on $\boldsymbol{B}$ that extends $<'_1$ must have x as maximum atom, while any such ordering that extends $<'_2$ must have y as maximum. Also any $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{B}$ in which z is the maximum atom does not extend either of $<'_1, <'_2$.

Similarly one can see that one can apply 2.2 (with the same $\boldsymbol{A}, \boldsymbol{B}, \mathcal{BA}^*$).

Remark 3.2. Another example where the above method can be applied is the following: Let $O\mathcal{P}$ be the class of all finite structures of the form $\boldsymbol{A} = \langle A, \prec, < \rangle$, where $\langle A, \prec \rangle \in \mathcal{P}$ and $<$ is an *arbitrary* linear ordering on A. Let $O\mathcal{P}^*$ be the class of all structures of the form $\langle A, \prec, <, <' \rangle$, where

$\langle A, \prec, <\rangle \in \mathcal{OP}$ and $<'$ is a linear extension of $\prec$. Then one can check that $\mathcal{OP}, \mathcal{OP}^*$ are Fraïssé classes and $\mathcal{OP}^*$ is a reasonable order expansion of $\mathcal{OP}$. Moreover $\mathcal{OP}^*$ has the ordering property (see [S2]). Let $\boldsymbol{OP} = \mathrm{Flim}(\mathcal{OP})$. Then we claim that $\mathrm{Aut}(\boldsymbol{OP})$ is not amenable by verifying the criterion in Proposition 2.1. For that we take $\boldsymbol{A} = (\{a_0, a_1\}, \prec_{\boldsymbol{A}}, <_{\boldsymbol{A}})$, where a_0, a_1 are $\prec_{\boldsymbol{A}}$-unrelated and $a_0 <_{\boldsymbol{A}} a_1$, and $\boldsymbol{B} = \langle\{b_0, b_1, b_2\}, \prec_{\boldsymbol{B}}, <_{\boldsymbol{B}}\rangle$, where $b_0 \prec_{\boldsymbol{B}} b_1$ but $(b_0, b_2), (b_1, b_2)$ are unrelated in $\prec_{\boldsymbol{B}}$ and $b_0 <_{\boldsymbol{B}} b_1 <_{\boldsymbol{B}} b_2$. Then the same embedding that has been used in the argument above for $\mathcal{P}$ works for $\mathcal{OP}$ as well.

4. The non-amenability of $\mathrm{Aut}(\boldsymbol{D})$**.** We will first give a proof of this fact that is based on 2.1; this will require more of the background results in Section 1, which will also be used in the next section. At the end of this section we will give a simpler proof that uses instead criterion 2.2 and avoids most of this background.

We will verify that $\mathrm{Aut}(\boldsymbol{B}_\infty) \cong \mathrm{Aut}(\boldsymbol{D})$ is not amenable by using 2.1. We first need to define a Fraïssé class $\mathcal{BL}^*$ which is an order expansion of $\mathcal{BL}$ and is reasonable and has the ordering property. We take as $\mathcal{BL}^*$ the class of all structures of the form $\langle \boldsymbol{B}, <\rangle$, where $\boldsymbol{B} \in \mathcal{BL}$ and $<$ is a linear ordering on B induced anti-lexicographically by an ordering of the atoms of $\boldsymbol{B}$ (see Section 1, **(D)**).

We first verify that $\mathcal{BL}^*$ is a Fraïssé class.

To prove that $\mathcal{BL}^*$ satisfies HP, let $\langle \boldsymbol{A}, <'\rangle \subseteq \langle \boldsymbol{B}, <\rangle$, where $\langle \boldsymbol{B}, <\rangle \in \mathcal{BL}^*$. We need to check that the linear ordering $<|A = <'$ is induced anti-lexicographically by an ordering of the atoms of $\boldsymbol{A}$. Let $\{b_1, \ldots, b_n\}$ be the atoms of $\boldsymbol{B}$ and let A_i, $0 \le i \le m$, be pairwise disjoint subsets of $\{1, \ldots, n\}$ so that if $\bigvee\{b_j : j \in A_0 \cup A_i\} = a_i$, then $\{a_1, \ldots, a_m\}$ are the atoms of $\boldsymbol{A}$ and $a_1 < \cdots < a_m$. Let x_i be the $<$-largest element of $\{b_j : j \in A_i\}$. Then $x_1 < \cdots < x_m$. From this it easily follows that the anti-lexicographical ordering on A induced by the ordering $a_1 < \cdots < a_m$ of its atoms is exactly the same as $<'$, which completes the proof.

We next prove that $\mathcal{BL}^*$ satisfies JEP. Let $\langle \boldsymbol{A}, <\rangle, \langle \boldsymbol{B}, <'\rangle \in \mathcal{BL}^*$ and let $a_1 < \cdots < a_m$, $b_1 <' \cdots <' b_n$ be the atoms of $\boldsymbol{A}, \boldsymbol{B}$, resp. Then let $\langle \boldsymbol{C}, <''\rangle \in \mathcal{BL}^*$ have atoms $\{a_1, \ldots, a_m\} \sqcup \{b_1, \ldots, b_n\}$ ordered by $a_1 <'' \cdots <'' a_m <'' b_1 <'' \cdots <'' b_n$. Clearly $\langle \boldsymbol{A}, <\rangle, \langle \boldsymbol{B}, <'\rangle$ embed into $\langle \boldsymbol{C}, <''\rangle$.

Finally we verify AP. In Graham–Rothschild [GR] (Theorem, Section 7, p. 270, for $A = B = \{0, 1\}$, H the trivial group; see also Prömel [P1, 4.4] for $\gamma = 1$) it is shown that $\mathcal{BL}$ satisfies RP. From this it immediately follows that $\mathcal{BL}^*$ also satisfies RP. This is because $\mathcal{BL}$ is *order forgetful* (in the sense of [KPT, 5.5]), i.e., for $\langle \boldsymbol{A}, <\rangle, \langle \boldsymbol{B}, <'\rangle \in \mathcal{BL}^*$, $\boldsymbol{A} \cong \boldsymbol{B} \Leftrightarrow \langle \boldsymbol{A}, <\rangle \cong \langle \boldsymbol{B}, <'\rangle$. As noted in [KPT, 5.6], 5.6], in this situation RP for $\mathcal{BL}^*$ is equivalent to RP for $\mathcal{BL}$.

Since every structure in $\mathcal{BL}^*$ is rigid and $\mathcal{BL}^*$ has the JEP and RP, this implies that $\mathcal{BL}^*$ has the AP (see, e.g., [KPT, end of Section 3]).

Remark 4.1. We note that one can also give a direct proof of AP for $\mathcal{BL}^*$ (see Appendix 1).

To see that $\mathcal{BL}^*$ is reasonable, let $\boldsymbol{A} \subseteq \boldsymbol{B}$ be in $\mathcal{BL}$ and let $<$ be a $\mathcal{BL}^*$-admissible ordering of $\boldsymbol{A}$. Let $a_1 < \cdots < a_m$ be the atoms of $\boldsymbol{A}$ and let $\{b_1, \ldots, b_n\}$ be the atoms of $\boldsymbol{B}$. Then there are pairwise disjoint subsets A_i, $0 \le i \le m$, of $\{1, \ldots, n\}$ such that $a_i = \bigvee\{b_j : j \in A_0 \cup A_i\}$. Let also $A' = \{1, \ldots, n\} \setminus \bigcup_{i \le m} A_i$. Then let $<'$ be any ordering of $\{b_1, \ldots, b_n\}$ so that if x_j is the $<'$-maximum element of $\{b_j : j \in A_i\}$, then $x_0 <' x_1 <' \cdots <' x_m$. Denote also by $<'$ the anti-lexicographical ordering on B induced by this ordering of the atoms. Then clearly $\langle \boldsymbol{B}, <' \rangle \in \mathcal{BL}^*$ and $\langle \boldsymbol{A}, < \rangle \subseteq \langle \boldsymbol{B}, <' \rangle$.

Finally the ordering property is trivially verified for $\mathcal{BL}^*$. Given $\boldsymbol{A} \in \mathcal{BL}$ take $\boldsymbol{B} = \boldsymbol{A}$. Then it is clear that for any $\mathcal{BL}^*$-admissible orderings $<, <'$ on $\boldsymbol{A}$ there is an isomorphism between $\langle \boldsymbol{A}, < \rangle$ and $\langle \boldsymbol{A}, <' \rangle$.

To complete the proof of non-amenability using 2.1, we just use the same example as in 3.1.

(B) A proof based on criterion 2.2 goes as follows: Let $\mathcal{D}^*$ the class of all $\langle \boldsymbol{L}, < \rangle$, where $\boldsymbol{L} \in \mathcal{D}$ and $<$ is a linear ordering on L with the following property: there is a Boolean lattice $\boldsymbol{B}$ with $\boldsymbol{L} \subseteq \boldsymbol{B}$ and an ordering $<'$ induced anti-lexicographically by an ordering of the atoms of $\boldsymbol{B}$ such that $< \, = \, <'|L$ (i.e., $\langle \boldsymbol{L}, < \rangle \subseteq \langle \boldsymbol{B}, <' \rangle$). Thus if $\boldsymbol{B}$ is a Boolean lattice, then $\langle \boldsymbol{B}, < \rangle \in \mathcal{D}^* \Leftrightarrow \langle \boldsymbol{B}, < \rangle \in \mathcal{BL}^*$. Clearly $\mathcal{D}^*$ is an order expansion of $\mathcal{D}$ and satisfies HP. The fact that it is reasonable follows from 1.1 and the fact that $\mathcal{BL}^*$ is reasonable. Then use 2.2 and the same example as in 3.1.

Remark 4.2. Using the method of Section 2, the authors have also proved that the automorphism group of the ultrahomogeneous tournament $\boldsymbol{S}(2)$ is not amenable, and Andrew Zucker has proved the non-amenability of the automorphism group of the homogeneous directed graph $\boldsymbol{S}(3)$. For the definition and properties of these graphs see [N].

5. Ramsey properties of $\mathcal{D}$ and the universal minimal flow of $\mathrm{Aut}(\boldsymbol{D})$

(A) Recall the definition of the class $\mathcal{D}^*$ from Section 4, **(B)**. Denote by $X_{\mathcal{D}^*}$ the space of linear orderings $<$ on D with the property that for any finite sublattice $\boldsymbol{L} \subseteq \boldsymbol{D}$, $\langle \boldsymbol{L}, <|L \rangle \in \mathcal{D}^*$. Equivalently, $X_{\mathcal{D}^*}$ is the space of all linear orderings $<$ on D such that for any finite Boolean lattice $\boldsymbol{B} \subseteq \boldsymbol{D}$ the ordering $<|B$ is induced anti-lexicographically by an ordering of the atoms of $\boldsymbol{B}$. Then $X_{\mathcal{D}^*}$ is a closed non-empty subspace of the compact space of all orderings on D (viewed as a subspace of 2^{D^2} with the product topology), and $\mathrm{Aut}(\boldsymbol{D})$ acts continuously on $X_{\mathcal{D}^*}$ in the obvious way, so $X_{\mathcal{D}^*}$ is a $\mathrm{Aut}(\boldsymbol{D})$-flow. We now have

THEOREM 5.1. *The* $\mathrm{Aut}(\boldsymbol{D})$*-flow* $X_{\mathcal{D}^*}$ *is the universal minimal flow of* $\mathrm{Aut}(\boldsymbol{D})$.

Proof. We can identify $\boldsymbol{D}$ with the reduct $\langle B_\infty, \wedge, \vee\rangle$, where the structure $\boldsymbol{B}_\infty = \langle B_\infty, \wedge, \vee, R\rangle$ is the Fraïssé limit of $\mathcal{BL}$, and so $\mathrm{Aut}(\boldsymbol{D}) = \mathrm{Aut}(\boldsymbol{B}_\infty)$. Moreover with this identification $X_{\mathcal{D}^*} = X_{\mathcal{BL}^*}$ (see Section 2) and thus we need to verify that $X_{\mathcal{BL}^*}$ is the universal minimal flow of $\mathrm{Aut}(\boldsymbol{B}_\infty)$. By [KPT, 7.5] this will be the case provided $\mathcal{BL}^*$ is a Fraïssé class which is a reasonable order expansion of $\mathcal{BL}$ and satisfies OP and RP. We have seen in Section 4 that all of these properties are true, so the proof is complete. ∎

(B) The Ramsey degree of any distributive lattice has been computed by Fouché. Below we use the notation introduced in Section 0.

THEOREM 5.2 (Fouché [F, p. 47]). *The Ramsey degree* $t(\boldsymbol{L}, \mathcal{D})$ *of a finite distributive lattice* $\boldsymbol{L}$ *is equal to* $t(\boldsymbol{L})$.

Since a proof of 5.2 has apparently not appeared in print, we include it for the convenience of the reader in Appendix 2.

Theorem 5.2 has the following corollary.

COROLLARY 5.3 (Hagedorn–Voigt [HV] (unpublished); see also Prömel –Voigt [PV, 2.2]). *The Ramsey objects in* $\mathcal{D}$ *are exactly the Boolean lattices.*

We again include the proof in Appendix 2.

As an example of a calculation of Ramsey degrees, let $\boldsymbol{n}$ be the linear ordering with $n \geq 1$ elements viewed as a distributive lattice. Then the Boolean algebra $\boldsymbol{B_n}$ has exactly $n-1$ atoms, so $t(\boldsymbol{n}, \mathcal{D}) = (n-1)!$.

(C) Given a Fraïssé class $\mathcal{K}$ and its Fraïssé limit $\boldsymbol{K} = \mathrm{Flim}(\mathcal{K})$ a common way to compute the universal minimal flow of $\mathrm{Aut}(\boldsymbol{K})$ is to find a Fraïssé order expansion $\mathcal{K}^*$ of $\mathcal{K}$ which is reasonable and has the OP and RP. Then the space $X_{\mathcal{K}^*}$ of $\mathcal{K}^*$-admissible orderings (as defined in Section 2) is the universal minimal flow of $\mathrm{Aut}(\boldsymbol{K})$ (see [KPT]). This works for the classes $\mathcal{P}$, $\mathcal{BL}$, and $\mathcal{BA}$ with $\mathcal{P}^*$, resp. $\mathcal{BL}^*$ as defined in Section 3, resp. Section 4, and $\mathcal{BA}^*$ again consisting of Boolean algebras and orderings induced antilexicographically by an ordering of the atoms (the classes $\mathcal{BL}, \mathcal{BA}$ have essentially the same structures but different notions of embedding). It would be natural to assume that something similar can be done for the class $\mathcal{D}$ of distributive lattices, and in fact $\mathcal{D}^*$, as considered at the beginning of this section, would be the natural candidate since the corresponding space $X_{\mathcal{D}^*}$ is indeed the universal minimal flow. However it turns out that this is not the case and in fact, rather surprisingly, nothing of that sort works with $\mathcal{D}$ as opposed to $\mathcal{P}$, $\mathcal{BL}$, and $\mathcal{BA}$. More precisely, we have the following stronger result.

THEOREM 5.4. *Let $\mathcal{K} \subseteq \mathcal{D}$ be any class which contains all the Boolean lattices and the linear ordering with three elements* (*viewed as a distributive lattice*). *Then there is no order expansion $\mathcal{K}^*$ of $\mathcal{K}$ that satisfies* HP *and* AP.

Proof. Assume that such a $\mathcal{K}^*$ exists, towards a contradiction. The argument below is inspired by the proof in [G] that $\mathcal{D}$ does not have the strong AP but additionally uses the canonization theorem below.

Denote by $\mathcal{D}^{**}$ the order expansion of $\mathcal{D}$, where

$$\langle \boldsymbol{L}, < \rangle \in \mathcal{D}^{**} \ \Leftrightarrow \ \langle \boldsymbol{L}, <^* \rangle \in \mathcal{D}^*,$$

with $<^*$ the reverse ordering of $<$.

We will use the following canonization theorem.

THEOREM 5.5 (J. Nešetřil, H. J. Prömel, V. Rödl and B. Voigt [NPRV]; see also Prömel [P2, 4.1]). *For any finite Boolean lattice $\boldsymbol{B}$, there is a finite Boolean lattice $\boldsymbol{C}$ such that for any linear ordering $<_{\boldsymbol{C}}$ on C, there is $\boldsymbol{B}' \subseteq \boldsymbol{C}$ such that $\boldsymbol{B}' \cong \boldsymbol{B}$ with $\langle \boldsymbol{B}', <_{\boldsymbol{C}}|B' \rangle \in \mathcal{D}^*$ or $\langle \boldsymbol{B}', <_{\boldsymbol{C}}|B' \rangle \in \mathcal{D}^{**}$.*

Apply this to the Boolean lattice $\boldsymbol{B}$ with two atoms a, b. It follows that there is an ordering $<$ on B such that $\langle \boldsymbol{B}, < \rangle \in \mathcal{K}^*$ and one of the following holds: $0^{\boldsymbol{B}} < a < b < 1^{\boldsymbol{B}}$, $0^{\boldsymbol{B}} < b < a < 1^{\boldsymbol{B}}$, $1^{\boldsymbol{B}} < a < b < 0^{\boldsymbol{B}}$, $1^{\boldsymbol{B}} < b < a < 0^{\boldsymbol{B}}$. Indeed, let $\boldsymbol{C}$ be as in 5.5 and let $<_{\boldsymbol{C}}$ on C be such that $\langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle \in \mathcal{K}^*$. Then there is $\boldsymbol{B}' \subseteq \boldsymbol{C}$, $\boldsymbol{B}' \cong \boldsymbol{B}$ with $\langle \boldsymbol{B}', <_{\boldsymbol{C}}|B' \rangle \in \mathcal{D}^*$ or $\langle \boldsymbol{B}', <_{\boldsymbol{C}}|B' \rangle \in \mathcal{D}^{**}$. Since $\mathcal{K}^*$ satisfies HP, $\langle \boldsymbol{B}', <_{\boldsymbol{C}}|B' \rangle \in \mathcal{K}$. Let $\langle \boldsymbol{B}, < \rangle \cong \langle \boldsymbol{B}', <_{\boldsymbol{C}}|B' \rangle$, so that $\langle \boldsymbol{B}, < \rangle \in \mathcal{D}^*$ or $\langle \boldsymbol{B}, < \rangle \in \mathcal{D}^{**}$. Then clearly one of the above four possibilities occurs.

Let now $\boldsymbol{L} = \langle \{0^{\boldsymbol{L}}, x, 1^{\boldsymbol{L}}\}, \wedge, \vee \rangle$ be the 3-element linear ordering (viewed as a distributive lattice). Then again since $\mathcal{K}^*$ has the HP, there is an ordering $<'$ on L with $\langle \boldsymbol{L}, <' \rangle \in \mathcal{K}^*$ and the maps $f\colon \langle \boldsymbol{L}, <' \rangle \to \langle \boldsymbol{B}, < \rangle$ and $g\colon \langle \boldsymbol{L}, <' \rangle \to \langle \boldsymbol{B}, < \rangle$, given by $f(0^{\boldsymbol{L}}) = 0^{\boldsymbol{B}}$, $f(1^{\boldsymbol{L}}) = 1^{\boldsymbol{B}}$, $f(x) = a$ and $g(0^{\boldsymbol{L}}) = 0^{\boldsymbol{B}}$, $g(1^{\boldsymbol{L}}) = 1^{\boldsymbol{B}}$, $g(x) = b$ are embeddings.

Suppose these could be amalgamated to $r\colon \langle \boldsymbol{B}, < \rangle \to \langle \boldsymbol{D}, <_{\boldsymbol{D}} \rangle$ and $s\colon \langle \boldsymbol{B}, < \rangle \to \langle \boldsymbol{D}, <_{\boldsymbol{D}} \rangle$ with $\langle \boldsymbol{D}, <_{\boldsymbol{D}} \rangle \in \mathcal{K}^*$ and $r \circ f = s \circ g$. Let

$$r \circ f(0^{\boldsymbol{L}}) = c_0, \qquad r \circ f(1^{\boldsymbol{L}}) = c_1,$$
$$r \circ f(x) = r(a) = d = s \circ g(x) = s(b),$$
$$r(b) = e, \qquad s(a) = e'.$$

Since $a \vee b = 1^{\boldsymbol{B}}$ and $a \wedge b = 0^{\boldsymbol{B}}$, we have $d \vee e = c_1$ and $d \wedge e = c_0$, so e is the relative complement of d in $[c_0, c_1]$. Similarly e' is the relative complement of d in $[c_0, c_1]$. Since in a distributive lattice relative complements are unique, we have $e = e'$, i.e., $r(b) = s(a) = e$. If, without loss of generality, $a < b$, then $r(a) = d <_{\boldsymbol{C}} e = r(b)$, while $s(a) = e <_{\boldsymbol{C}} d = s(b)$, a contradiction. ■

6. Some additional examples

(A) We take this opportunity to mention a few more examples of Fraïssé classes $\mathcal{K}$ for which one can calculate the universal minimal flow of the automorphism group $\mathrm{Aut}(\boldsymbol{K})$, where $\boldsymbol{K} = \mathrm{Flim}(\mathcal{K})$. It turns out to be metrizable, and we also calculate the Ramsey degree, which is finite, although the class $\mathcal{K}$ does not admit any Fraïssé order expansions.

For each $n \geq 1$, let $\mathcal{C}_n$ be the class of all finite posets $\langle P, \prec\rangle$ which consist of disjoint antichains $A_1, \ldots, A_k$ with $|A_i| \leq n$ for all i, such that

$$i < j,\ x \in A_i,\ y \in A_j \ \Rightarrow\ x \prec y.$$

Finally, let $\mathcal{E}_n$ $(n \geq 1)$ be the class of finite equivalence relations such that each equivalence class has at most n elements. (The class $\mathcal{C}_n$ has been studied in Sokić [S1].)

It is not hard to see that these are Fraïssé classes. Denote by $\boldsymbol{C}_n, \boldsymbol{E}_n$ their Fraïssé limits. Then

$$\boldsymbol{C}_n \cong \langle \mathbb{Q} \times \{1, \ldots, n\}, \prec\rangle, \quad \text{where} \quad (q, i) \prec (r, j) \ \Leftrightarrow\ q < r,$$

and

$$\boldsymbol{E}_n \cong \{\mathbb{N} \times \{1, \ldots, n\}, E\rangle, \quad \text{where} \quad (k, i)E(l, j) \ \Leftrightarrow\ k = l.$$

From this description it is straightforward to calculate the automorphism groups of these Fraïssé limits. We have

$$\mathrm{Aut}(\boldsymbol{C}_n) \cong \mathrm{Aut}(\langle \mathbb{Q}, <\rangle) \ltimes S_n^{\mathbb{Q}},$$

where $\mathrm{Aut}(\mathbb{Q})$ acts on $S_n^{\mathbb{Q}}$ by shift, and similarly

$$\mathrm{Aut}(\boldsymbol{E}_n) \cong S_\infty \ltimes S_n^{\mathbb{N}},$$

where S_∞ acts on $S_n^{\mathbb{N}}$ by shift.

We can use this to calculate the universal minimal flow of each of these groups.

In both cases, we have groups of the form $G \ltimes K$, where G is Polish and K is compact. Generalizing a result in Sokić [S1], who dealt with the case of an extremely amenable G, we compute the universal minimal flow of $G \ltimes K$ as follows.

Proposition 6.1. *Let G be a Polish group with universal minimal flow X_G and suppose that G acts continuously by automorphisms on a compact metrizable group K. Consider the semidirect product $G \ltimes K$. Then the universal minimal flow of $G \ltimes K$ is the product $X_G \times K$ with the following action of $G \ltimes K$:*

$$(g, k) \cdot (x, \ell) = (g \cdot x, k(g \cdot \ell)).$$

Proof. First notice that $G \ltimes K$ acts continuously on K by

$$(g, k) \cdot \ell = k(g \cdot \ell).$$

Thus the $(G \ltimes K)$-flow $X_G \times K$, defined as above, is the product of the action of $G \ltimes K$ on X_G given by $(g, k) \cdot x = g \cdot x$ and the action of $G \ltimes K$ on K given above. It is easy to check that this is a minimal $(G \ltimes K)$-flow.

Consider now an arbitrary $(G \ltimes K)$-flow Y. Then there is a continuous map $\rho\colon X_G \to Y$ which is G-equivariant, in the sense that

$$\rho(g \cdot x) = (g, 1) \cdot \rho(x).$$

Define then $\pi\colon X_G \times K \to Y$ by

$$\pi(x, k) = (1, k) \cdot \rho(x).$$

It is easy to check that this is $(G \ltimes K)$-equivariant and the proof is complete. ∎

It follows that the universal minimal flow of $\mathrm{Aut}(\boldsymbol{C}_n) \cong \mathrm{Aut}(\langle \mathbb{Q}, < \rangle) \ltimes S_n^{\mathbb{Q}}$ is its action on $S_n^{\mathbb{Q}}$ given by $(g, k) \cdot \ell = k(g \cdot \ell)$, since $G = \mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ is extremely amenable, so that X_G is a singleton.

Finally the universal minimal flow of $\mathrm{Aut}(\boldsymbol{E}_n) \cong S_\infty \ltimes S_n^{\mathbb{N}}$ is $X_{S_\infty} \times S_n^{\mathbb{N}}$, with the action defined as above, where X_{S_∞} is the universal minimal flow on S_∞, which was shown in Glasner–Weiss [GW] to be the space LO of all linear orderings on $\mathbb{N}$ (with the obvious action of S_∞ on LO).

Thus in all these cases the universal minimal flows are metrizable. On the other hand it is easy to see that none of these classes $\mathcal{K} = \mathcal{C}_n, \mathcal{E}_n$ for $n \geq 2$, admits an order expansion $\mathcal{K}^*$ with HP and AP. Take, for example, $\mathcal{K} = \mathcal{E}_n$ and assume such a $\mathcal{K}^*$ existed. Let $\langle \boldsymbol{B}, <_{\boldsymbol{B}} \rangle = \langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle \in \mathcal{K}^*$, where $\boldsymbol{B}$ has a single equivalence class of cardinality n. Let x_1 be the $<_{\boldsymbol{B}}$-least element of B and x_n the $<_{\boldsymbol{B}}$-largest element. Let $\boldsymbol{A} = \langle A, E \rangle \in \mathcal{K}$, where $A = \{a\}$ and let $<_{\boldsymbol{A}}$ be the empty ordering on $\boldsymbol{A}$. Then the maps $f\colon \langle \boldsymbol{A}, <_{\boldsymbol{A}} \rangle \to \langle \boldsymbol{B}, <_{\boldsymbol{B}} \rangle$ and $g\colon \langle \boldsymbol{A}, <_{\boldsymbol{A}} \rangle \to \langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle$, where $f(a) = x_1$ and $g(a) = x_n$, are clearly embeddings, so that since $\mathcal{K}^*$ satisfies HP, $\langle \boldsymbol{A}, < \rangle \in \mathcal{K}^*$. If $r\colon \langle \boldsymbol{B}, <_{\boldsymbol{B}} \rangle \to \langle \boldsymbol{D}, <_{\boldsymbol{D}} \rangle$ and $s\colon \langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle \to \langle \boldsymbol{D}, <_{\boldsymbol{D}} \rangle$ amalgamate f, g then $r \circ f(a) = s \circ g(a) = d$, so $r(x_1) = s(x_n) = d$. Let $B = \{x_1 <_{\boldsymbol{B}} \cdots <_{\boldsymbol{B}} x_n\} = \{x_1 <_{\boldsymbol{C}} \cdots <_{\boldsymbol{C}} x_n\}$. Then $d = r(x_1) <_{\boldsymbol{D}} \cdots <_{\boldsymbol{D}} r(x_n)$ and $s(x_1) <_{\boldsymbol{D}} \cdots <_{\boldsymbol{D}} s(x_n) = d$, while all $r(x_1), \ldots, r(x_n), s(x_1), \ldots, s(x_n)$ are equivalent in $\boldsymbol{D}$, thus the equivalence class of d has $2n - 1 > n$ elements, i.e., $\boldsymbol{D} \notin \mathcal{E}_n$, a contradiction.

Another example, discussed in Sokić [S1], is the class $\mathcal{B}_n$ of finite posets which consist of a disjoint union of at most n chains $C_1, \ldots, C_k$ $(k \leq n)$, so that if $x \in C_i$ and $y \in C_j$ with $i \neq j$, then x, y are incomparable. The Fraïssé limit $\boldsymbol{B}_n$ of this class is

$$\boldsymbol{B}_n \cong \langle \mathbb{Q} \times \{1, \ldots, n\}, \prec \rangle, \quad \text{where} \quad (q, i) \prec (r, j) \ \Leftrightarrow\ i = j \wedge q < r.$$

Then

$$\operatorname{Aut}(\boldsymbol{B}_n) \cong S_n \ltimes \operatorname{Aut}(\langle \mathbb{Q}, < \rangle)^n,$$

where S_n acts on $\operatorname{Aut}(\langle \mathbb{Q}, < \rangle)^n$ by shift. In this case $\operatorname{Aut}(\boldsymbol{B}_n)$ is of the form $K \ltimes G$, where K is compact and G is Polish extremely amenable. In Appendix 3, we will compute in general the universal minimal flow of $K \ltimes G$ from the universal minimal flow of G and show that it is metrizable if the universal minimal flow of G is metrizable. In the particular case when G is extremely amenable, as in the present example, the universal minimal flow of $K \ltimes G$, will be the action of this group on K given by $(k, g) \cdot \ell = k\ell$.

In [S1] it is shown that the class $\mathcal{K}_e^*$ consisting of all $\langle \boldsymbol{A}, < \rangle$ with $\boldsymbol{A} \in \mathcal{B}_n$ and $<$ a linear ordering on A that extends the partial ordering of $\boldsymbol{A}$ is a Fraïssé class, which is a reasonable order expansion of $\mathcal{K}$. However one can see that there is no order expansion of $\mathcal{B}_n$ that satisfies RP, where $n \geq 2$. Indeed suppose such a $\mathcal{K}^*$ existed and let $\langle \boldsymbol{A}, <_{\boldsymbol{A}} \rangle \in \mathcal{K}^*$ be such that A is a singleton. Let $\langle \boldsymbol{B}, <_{\boldsymbol{B}} \rangle \in \mathcal{K}^*$ be such that $\boldsymbol{B}$ is an antichain of cardinality n. Suppose, towards a contradiction, that $\langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle \in \mathcal{K}^*$, $\langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle \geq \langle \boldsymbol{B}, <_{\boldsymbol{B}} \rangle$ and

$$\langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle \to \langle \boldsymbol{B}, <_{\boldsymbol{B}} \rangle_n^{\langle \boldsymbol{A}, <_{\boldsymbol{A}} \rangle}.$$

Then $\boldsymbol{C}$ contains n incomparable chains $C_1, \ldots, C_n$, so we can define

$$c\colon \binom{\langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle}{\langle \boldsymbol{A}, <_{\boldsymbol{A}} \rangle} \to \{1, \ldots, n\}$$

by

$$c(\langle \boldsymbol{A}', <_{\boldsymbol{A}'} \rangle) = i$$

iff the point on A' is in C_i. Clearly there is no homogeneous copy of $\boldsymbol{B}$, since $n \geq 2$.

We will now calculate the Ramsey degrees of the classes $\mathcal{B}_n, \mathcal{C}_n, \mathcal{E}_n$.

(B) We start with $\mathcal{C}_n$, $n \geq 2$. Let $\boldsymbol{A} = \langle A, \prec_{\boldsymbol{A}} \rangle \in \mathcal{C}_n$. Then we have a decomposition

$$A = A_1 \sqcup \cdots \sqcup A_k$$

into maximal non-empty antichains, where

$$A_1 \prec_{\boldsymbol{A}} \cdots \prec_{\boldsymbol{A}} A_k$$

(i.e., $i < j$, $x \in A_i$, $y \in A_j \Rightarrow x \prec_{\boldsymbol{A}} y$). The number of antichains is called the *length* of $\boldsymbol{A}$, in symbols $\operatorname{length}(\boldsymbol{A})$. The structure $\boldsymbol{A}$ also gives a sequence called the *code* of A, defined by

$$\operatorname{code}(\boldsymbol{A}) = (|A_1|, \ldots, |A_k|).$$

We finally define the *character* of $\boldsymbol{A}$ by

$$\operatorname{char}(\boldsymbol{A}) = \binom{n}{|A_1|} \cdots \binom{n}{|A_k|}.$$

Proposition 6.2. *For $n \geq 2$ and $\boldsymbol{A} \in \mathcal{C}_n$,*

$$t(\boldsymbol{A}, \mathcal{C}_n) = \mathrm{char}(\boldsymbol{A}).$$

Proof. We will first show that

$$t(\boldsymbol{A}, \mathcal{C}_n) \leq \mathrm{char}(\boldsymbol{A}).$$

Fix a natural number r giving the number of colors. Let $\boldsymbol{B} = \langle B, <_{\boldsymbol{B}} \rangle \in \mathcal{C}_n$ with $\boldsymbol{A} \leq \boldsymbol{B}$. Since every $\boldsymbol{E} \in \mathcal{C}_n$ can be embedded into some $\boldsymbol{F} \in \mathcal{C}_n$ with $\mathrm{length}(\boldsymbol{E}) = \mathrm{length}(\boldsymbol{F})$ and $\mathrm{code}(\boldsymbol{F}) = (n, \ldots, n)$ we can assume that $\mathrm{code}(\boldsymbol{B}) = (n, \ldots, n)$. Note also that $\mathrm{length}(\boldsymbol{A}) \leq \mathrm{length}(\boldsymbol{B})$.

We will define $\boldsymbol{C} = \langle C, \prec_{\boldsymbol{C}} \rangle \in \mathcal{C}_n$ such that $\mathrm{length}(\boldsymbol{C}) = m$ (for some m) and $\mathrm{code}(\boldsymbol{C}) = (n, \ldots, n)$. To define m let $(m_i)_{i=0}^{\mathrm{char}(\boldsymbol{A})}$ be given by

$$m_0 = \mathrm{length}(\boldsymbol{B}), \quad m_{i+1} \to (m_i)_r^{\mathrm{length}(\boldsymbol{A})}, \quad 0 \leq i < \mathrm{char}(\boldsymbol{A}),$$

using the classical Ramsey theorem. Finally take

$$\mathrm{length}(\boldsymbol{C}) = m = m_{\mathrm{char}(\boldsymbol{A})}.$$

Now consider $\boldsymbol{G} \cong \boldsymbol{A}, \boldsymbol{G} = \langle G, \prec_{\boldsymbol{G}} \rangle \subseteq \boldsymbol{C}$. Then $\boldsymbol{G}$ is described by a $\mathrm{length}(\boldsymbol{A})$ subset

$$\{g_1 < \cdots < g_{\mathrm{length}(\boldsymbol{A})}\}$$

of $\{1, \ldots, m\}$ and a sequence of sets $(G_1, \ldots, G_{\mathrm{length}(\boldsymbol{A})})$ where $G_i \subseteq C_{g_i}$. On fixing an ordering of each maximal antichain of $\boldsymbol{C}$, this determines uniquely a sequence $(\overline{G}_1, \ldots, \overline{G}_{\mathrm{length}(\boldsymbol{A})})$, where $G_i \subseteq \{1, \ldots, n\}$.

Note that two substructures of $\boldsymbol{C}$ isomorphic to $\boldsymbol{A}$ which are described by the same $\mathrm{length}(\boldsymbol{A})$ subsets of $\{1, \ldots, m\}$ are different iff they have different sequences of subsets of $\{1, \ldots, n\}$.

Let T be the set of all sequences $(s_1, \ldots, s_{\mathrm{length}(\boldsymbol{A})})$ of subsets of $\{1, \ldots, n\}$ which are given by some substructure of $\boldsymbol{C}$ isomorphic to $\boldsymbol{A}$. Then we have a bijection

$$\varphi \colon \{1, \ldots, \mathrm{char}(\boldsymbol{A})\} \to T.$$

Now let

$$p \colon \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \{1, \ldots, r\}$$

be any coloring. There is an induced sequence $(p_i)_{i=0}^{\mathrm{char}(\boldsymbol{A})}$ of colorings given by

$$p_i \colon \binom{m}{\mathrm{length}(\boldsymbol{A})} \to \{1, \ldots, r\}, \quad p_i(K) = p(\boldsymbol{G}),$$

where $\boldsymbol{G}$ is the substructure of $\boldsymbol{C}$ isomorphic to $\boldsymbol{A}$ given by K and $\varphi(i)$.

By the definition of m, there is a decreasing sequence

$$S_{\mathrm{char}(\boldsymbol{A})} \supseteq \cdots \supseteq S_0$$

of subsets of $\{1,\dots,m\}$ with

$$|S_{i-1}| = m_{i-1}, 0 < i \le \mathrm{char}(\boldsymbol{A}),$$
$$p_i|\binom{S_{i-1}}{\mathrm{length}(\boldsymbol{A})} = \text{constant}.$$

In particular the colorings $p_1,\dots,p_{\mathrm{char}(\boldsymbol{A})}$ are constant on $\binom{S_0}{\mathrm{length}(\boldsymbol{A})}$. Let $\boldsymbol{D} \in \mathcal{C}_n$ be the substructure of $\boldsymbol{C}$ given by S_0 and with all the maximal antichains of size n. Then the p-color of a substructure of $\mathcal{D}$ isomorphic to $\boldsymbol{A}$ depends only on the sequence of subsets of $\{1,\dots,n\}$ by which it is given. Since $\mathrm{length}(\boldsymbol{B}) = m_0 = |S_0|$, we have $\boldsymbol{B} \le \boldsymbol{D}$, which shows that $t(\boldsymbol{A},\mathcal{C}_n) \le \mathrm{char}(\boldsymbol{A})$.

In order to show the opposite inequality we take the number of colors to be $r = \mathrm{char}(\boldsymbol{A})$. We consider $\boldsymbol{B} \in \mathcal{C}_n$ such that $\mathrm{length}(\boldsymbol{B}) = \mathrm{length}(\boldsymbol{A})$ and $\mathrm{code}(\boldsymbol{B}) = (n,\dots,n)$. Let $\boldsymbol{C} \in \mathcal{C}_n$ be such that $\boldsymbol{B} \le \boldsymbol{C}$. Define the coloring

$$p\colon \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \{1,\dots,r\}, \qquad p(\boldsymbol{H}) = \varphi^{-1}((H_1,\dots,H_{\mathrm{length}(\boldsymbol{A})})),$$

where $\boldsymbol{H} \subseteq \boldsymbol{C}$ is given by a $\mathrm{length}(\boldsymbol{A})$ subset K and the sequence of subsets $(H_1,\dots,H_{\mathrm{length}(\boldsymbol{A})}) \in T$. Clearly any copy of $\boldsymbol{B}$ inside $\boldsymbol{C}$ will realize all different colors, so $t(\boldsymbol{A},\mathcal{C}_n) \ge \mathrm{char}(\boldsymbol{A})$. ∎

Corollary 6.3. *The Ramsey objects in $\mathcal{C}_n$, $n \ge 2$, are exactly the $\boldsymbol{A} \in \mathcal{C}_n$ that decompose into maximal antichains of size n.*

(C) Next we discuss $\mathcal{B}_n, n \ge 2$. Let $\boldsymbol{A} = (A, \prec_{\boldsymbol{A}}) \in \mathcal{B}_n$. Then we have a decomposition

$$A = A_1 \sqcup \dots \sqcup A_k, \qquad \text{for some } 1 \le k \le n,$$

into maximal chains with respect to $\prec_{\boldsymbol{A}}$. The number k of chains is called the *length* of $\boldsymbol{A}$, in symbols $\mathrm{length}(\boldsymbol{A})$. To the structure $\boldsymbol{A}$ we assign the set

$$\{|A_1|,\dots,|A_k|\}$$

which we write as an increasing sequence, called its *dimension*, and denoted by

$$\dim(\boldsymbol{A}) = (a_1,\dots,a_s).$$

In addition we have the *multiplicity sequence*,

$$\mathrm{mult}(\boldsymbol{A}) = (m_1,\dots,m_s), \qquad m_i = |\{j\colon |A_j| = a_i\}|, \quad 1 \le i \le s.$$

The *character* of the structure $\boldsymbol{A}$ is the number

$$\mathrm{char}(\boldsymbol{A}) = \binom{n}{k} \cdot \frac{k!}{m_1!\dots m_s!}.$$

By using similar arguments to the proof of 5.2 (employing this time the product Ramsey theorem) we obtain the following:

Proposition 6.4. *For $n \geq 2$ and $\boldsymbol{A} \in \mathcal{B}_n$, we have $t(\boldsymbol{A}, \mathcal{B}_n) = \mathrm{char}(\boldsymbol{A})$.*

Corollary 6.5. *The Ramsey objects in $\mathcal{B}_n$, $n \geq 2$, are exactly the $\boldsymbol{A} \in \mathcal{B}_n$ that decompose into n maximal chains of the same size.*

(D) Finally, we consider $\mathcal{E}_n$, $n \geq 2$. Let $\boldsymbol{A} = \langle A, E_{\boldsymbol{A}} \rangle \in \mathcal{E}_n$, $n \geq 2$. Then we have a decomposition of the set A into $E_{\boldsymbol{A}}$-equivalence classes, $A = A_1 \sqcup \cdots \sqcup A_k$, for some k with $|A_i| \leq n$. The number of classes is called the *length* of $\boldsymbol{A}$, in symbols $\mathrm{length}(\boldsymbol{A})$. In addition we have the set

$$\{|A_i| \colon i \leq k\}$$

which we present as an increasing sequence $(d_1, \ldots, d_s) = \dim(\boldsymbol{A})$, called the *dimension* of $\boldsymbol{A}$. Also we have the sequence $(m_1, \ldots, m_s) \in \mathrm{mult}(\boldsymbol{A})$, the *multiplicity* of $\boldsymbol{A}$, given by

$$m_i = |\{A_j \colon |A_j| = d_i\}|.$$

The *character* of the structure $\boldsymbol{A}$ is

$$\mathrm{char}(\boldsymbol{A}) = \frac{k!}{m_1! \ldots m_s!} \binom{n}{d_1}^{m_1} \cdots \binom{n}{d_s}^{m_s}.$$

Again by similar arguments to the proof of 5.2 we deduce:

Proposition 6.6. *For $n \geq 2$ and $\boldsymbol{A} \in \mathcal{E}_n$, we have $t(\boldsymbol{A}, \mathcal{E}_n) = \mathrm{char}(\boldsymbol{A})$.*

Corollary 6.7. *The Ramsey objects in $\mathcal{E}_n$, $n \geq 2$, are exactly the equivalence relations which have all equivalence classes of size n.*

Remark 6.8. Consider also the class $\mathcal{E}_n^*$ consisting of all finite equivalence relations with at most n equivalence classes. Then, by similar arguments, one can obtain for $\mathcal{E}_n^*$ completely analogous results to those we obtained for $\mathcal{B}_n$.

Appendix 1. A direct proof of AP for $\mathcal{BL}^*$. Let $\langle \boldsymbol{A}, <_{\boldsymbol{A}} \rangle$, $\langle \boldsymbol{B}, <_{\boldsymbol{B}} \rangle$, $\langle \boldsymbol{C}, <_{\boldsymbol{B}} \rangle \in \mathcal{BL}^*$ and let $f \colon \langle \boldsymbol{A}, <_{\boldsymbol{A}} \rangle \to \langle \boldsymbol{B}, <_{\boldsymbol{B}} \rangle$ and $g \colon \langle \boldsymbol{A}, <_{\boldsymbol{A}} \rangle \to \langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle$ be embeddings. We will find $\langle \boldsymbol{D}, <_{\boldsymbol{D}} \rangle \in \mathcal{BL}^*$ and embeddings $r \colon \langle \boldsymbol{B}, <_{\boldsymbol{B}} \rangle \to \langle \boldsymbol{D}, <_{\boldsymbol{D}} \rangle$ and $s \colon \langle \boldsymbol{C}, <_{\boldsymbol{C}} \rangle \to \langle \boldsymbol{D}, <_{\boldsymbol{D}} \rangle$ with $r \circ f = s \circ g$.

Let $a_1 <_{\boldsymbol{A}} \cdots <_{\boldsymbol{A}} a_k$ be the atoms of $\boldsymbol{A}$ and $b_1 <_{\boldsymbol{B}} \cdots <_{\boldsymbol{B}} b_m$, $c_1 <_{\boldsymbol{C}} \cdots <_{\boldsymbol{C}} c_n$ the atoms of $\boldsymbol{B}$, $\boldsymbol{C}$, resp. Let also $B_0, B_1, \ldots, B_k$ be pairwise disjoint subsets of $\{1, \ldots, m\}$ and $C_0, C_1, \ldots, C_k$ be pairwise disjoint subsets of $\{1, \ldots, n\}$ such that $f(0^{\boldsymbol{A}}) = \bigvee\{b_j : j \in B_0\}$, $f(a_i) = \bigvee\{b_j : j \in B_0 \cup B_i\}$, $g(0^{\boldsymbol{A}}) = \bigvee\{c_j : j \in C_0\}$, $g(a_i) = \bigvee\{c_j : j \in C_0 \cup C_i\}$. Also let $\overline{b}_i$ be the $<_{\boldsymbol{B}}$-maximum element of $\{b_j \colon j \in \boldsymbol{B}_i\}$, $1 \leq i \leq k$, and similarly for $\overline{c}_i$. Then $\overline{b}_i <_{\boldsymbol{B}} \cdots <_{\boldsymbol{B}} \overline{b}_k$ and $\overline{c}_i <_{\boldsymbol{C}} \cdots <_{\boldsymbol{C}} \overline{c}_k$. Finally, let $B' = \{1, \ldots, m\} \setminus \bigcup_{i \leq k} B_i$ and similarly define C'.

The set $A_{\boldsymbol{D}}$ of atoms of $\boldsymbol{D}$ is the disjoint union

$$A_{\boldsymbol{D}} = \{b_j : j \in B_0\} \sqcup \{c_j : j \in C_0\} \sqcup \bigsqcup_{1\le i\le k} (\{b_j : j \in B_i\} \times \{c_j : j \in C_i\}) \\ \sqcup \{b_j : j \in B'\} \sqcup \{c_j : j \in C'\}.$$

We now define r, s as follows:

$$\begin{aligned} r(b_j) &= \bigvee\{c_i : i \in C_0\} \vee b_j \quad \text{if } j \in B_0 \cup B', \\ s(c_j) &= \bigvee\{b_i : i \in B_0\} \vee c_j \quad \text{if } j \in C_0 \cup C', \\ r(b_j) &= \bigvee\{c_i : i \in C_0\} \vee \bigvee\{(b_j, c_k) : k \in C_i\} \quad \text{if } j \in B_i, \\ s(c_j) &= \bigvee\{b_i : i \in B_0\} \vee \bigvee\{(b_k, c_j) : k \in B_i\} \quad \text{if } j \in C_i. \end{aligned}$$

In particular, $r(0^{\boldsymbol{B}}) = \bigvee\{b_j : j \in C_0\}$ and $s(0^{\boldsymbol{C}}) = \bigvee\{c_j : j \in B_0\}$. Thus $r \circ f(0^{\boldsymbol{A}}) = \bigvee\{b_j : j \in B_0\} \vee \bigvee\{c_j : j \in C_0\} = s \circ g(0^{\boldsymbol{A}})$ and

$$\begin{aligned} r \circ f(a_i) &= \bigvee\{b_j : j \in B_0\} \vee \bigvee\{c_j : j \in C_0\} \vee \bigvee\{(b_j, c_k) : j \in B_i,\, k \in C_i\} \\ &= s \circ g(a_i), \end{aligned}$$

so $r \circ f = s \circ g$.

It remains to define $<_{\boldsymbol{D}}$ and show that r, s preserve the orderings.

The map $F : \{b_1, \ldots, b_m\} \to A_{\boldsymbol{D}}$ given by $F(b_j) = b_j$ if $j \in B_0 \cup B'$, and $F(b_j) = (b_j, \bar{c}_i)$ if $j \in B_i, 1 \le i \le k$, is an injection with image $A'_{\boldsymbol{B}} \subseteq A_{\boldsymbol{D}}$ and F carries the ordering $<_{\boldsymbol{B}}$ on $\{b_1, \ldots, b_m\}$ to an ordering $<'_{\boldsymbol{B}}$ on $A'_{\boldsymbol{B}}$. Similarly define $G : \{c_1, \ldots, c_n\} \to A_{\boldsymbol{D}}$, $A'_{\boldsymbol{C}} \subseteq A_{\boldsymbol{D}}$ and $<'_{\boldsymbol{C}}$ on $A'_{\boldsymbol{C}}$. Clearly $A'_{\boldsymbol{B}} \cap A'_{\boldsymbol{C}} = \{(\bar{b}_i, \bar{c}_i) : 1 \le i \le k\}$ and the orderings $<'_{\boldsymbol{B}}, <'_{\boldsymbol{C}}$ agree on $A'_{\boldsymbol{B}} \cap A'_{\boldsymbol{C}}$, so there is an ordering $<'$ on $A'_{\boldsymbol{B}} \cup A'_{\boldsymbol{C}}$ extending $<'_{\boldsymbol{B}} \cup <'_{\boldsymbol{C}}$. We further extend $<'$ to an ordering $<'_i$ on $A'_{\boldsymbol{B}} \cup A'_{\boldsymbol{C}} \cup (\{b_j : j \in B_i\} \times \{c_j : j \in C_i\}$, so that $y <_i z$ if $y \notin A'_{\boldsymbol{B}} \cup A'_{\boldsymbol{C}}$ and $z \in A'_{\boldsymbol{B}} \cup A'_{\boldsymbol{C}}$. Again $<'_1, \ldots, <'_k$ agree on their common domain $A'_{\boldsymbol{B}} \cup A'_{\boldsymbol{C}}$, so there is an ordering $<_{\boldsymbol{D}}$ of the atoms of $\boldsymbol{D}$ which extends all $<'_1, \ldots, <'_k$. We also denote by $<_{\boldsymbol{D}}$ the anti-lexicographical ordering it induces on $\boldsymbol{D}$. It is easy to check now that r, s preserve the corresponding orderings.

Appendix 2. Calculation of the Ramsey degree of distributive lattices. We give here the proof of Fouché's Theorem 5.2. Let t be the number of isomorphic copies $\boldsymbol{L}'$ of $\boldsymbol{L}$ which are contained in $\boldsymbol{B}_{\boldsymbol{L}}$ and are such that $\boldsymbol{L}', \boldsymbol{B}_{\boldsymbol{L}}$ have the same 0, 1 and $\boldsymbol{L}'$ generates $\boldsymbol{B}_{\boldsymbol{L}}$ as a Boolean algebra. We claim that $t = t(\boldsymbol{L})$. To see this recall from 1.1 that any $\varphi \in \mathrm{Aut}(\boldsymbol{L})$ has a unique extension $\overline{\varphi} \in \mathrm{Aut}(\boldsymbol{B}_{\boldsymbol{L}})$ and the map $\varphi \mapsto \overline{\varphi}$ is a group embedding of $\mathrm{Aut}(\boldsymbol{L})$ into $\mathrm{Aut}(\boldsymbol{B}_{\boldsymbol{L}})$. Denote by $\overline{\mathrm{Aut}}(\boldsymbol{L})$ its image. Let $X = \{\boldsymbol{L}'_1, \ldots, \boldsymbol{L}'_t\}$ be the set of copies of $\boldsymbol{L}$ in $\boldsymbol{B}_{\boldsymbol{L}}$ satisfying the above condition, where we put $\boldsymbol{L}'_1 = \boldsymbol{L}$. Clearly $\mathrm{Aut}(\boldsymbol{B}_{\boldsymbol{L}})$ acts transitively on X (by 1.1 again) and the stablizer of $\boldsymbol{L}$ is exactly $\overline{\mathrm{Aut}}(\boldsymbol{L})$, thus

$$t = |X| = \frac{|\mathrm{Aut}(\boldsymbol{B_L})|}{|\overline{\mathrm{Aut}}(\boldsymbol{L})|} = \frac{|\mathrm{Aut}(\boldsymbol{B_L})|}{|\mathrm{Aut}(\boldsymbol{L})|} = t(\boldsymbol{L}).$$

Let $\boldsymbol{K} \in \boldsymbol{D}$ with $\boldsymbol{L} \leq \boldsymbol{K}$. Let also $k \geq 2$. Using the RP for $\mathcal{BL}$ define a sequence $(\boldsymbol{C}_i)_{i=1}^t$ of Boolean lattices as follows:

$$\boldsymbol{C}_0 = \boldsymbol{B_K}, \quad \boldsymbol{C}_{i+1} \to (\boldsymbol{C}_i)_k^{\boldsymbol{B_L}}, \quad 0 \leq i < t.$$

Fix a linear ordering $<$ on C_t such that $\langle \boldsymbol{C}_t, < \rangle \in \mathcal{BL}^*$, i.e., $<$ is induced anti-lexicographically by an ordering of the atoms of $\boldsymbol{C}_t$.

We will prove that

$$\boldsymbol{C}_t \to (\boldsymbol{K})_{k,t}^{\boldsymbol{L}},$$

which shows that $t(\boldsymbol{L}, \mathcal{D}) \leq t$.

Indeed, let

$$c : \binom{\boldsymbol{C}_t}{\boldsymbol{L}} \to \{1, \dots, k\}$$

be a coloring. Fix an ordering $<_{\boldsymbol{L}}$ on $\boldsymbol{B_L}$ given lexicographically by an ordering of the atoms of $\boldsymbol{B_L}$. Also let $\boldsymbol{L}'_1, \dots, \boldsymbol{L}'_t$ be the copies of $\boldsymbol{L}$ in $\boldsymbol{B_L}$ with the same 0, 1 as $\boldsymbol{B_L}$ that generate $\boldsymbol{B_L}$ as a Boolean algebra. For each $1 \leq i \leq t$, define the coloring

$$c_i : \binom{\boldsymbol{C}_t}{\boldsymbol{B_L}} \to \{1, \dots, k\}$$

as follows: Let $\boldsymbol{B}' \in \boldsymbol{C}_t$ with $\boldsymbol{B}' \cong \boldsymbol{B_L}$. Then there is a unique isomorphism $\pi : \langle \boldsymbol{B}', <|B' \rangle \to \langle \boldsymbol{B_L}, <_{\boldsymbol{L}} \rangle$ (notice here that $<|B'$ is also given anti-lexicographically by an ordering of the atoms of $\boldsymbol{B}'$). Let $\boldsymbol{L}'$ be the preimage of $\boldsymbol{L}'_i$ under π. Then put

$$c_i(\boldsymbol{B}') = c(\boldsymbol{L}').$$

There is now $\overline{\boldsymbol{C}}_{t-1} \cong \boldsymbol{C}_{t-1}$ with $\overline{\boldsymbol{C}}_{t-1} \subseteq \boldsymbol{C}_t$ such that c_t is constant on $\binom{\overline{\boldsymbol{C}}_{t-1}}{\boldsymbol{B_L}}$. Similarly there is $\overline{\boldsymbol{C}}_{t-2} \cong \boldsymbol{C}_{t-2}$ with $\overline{\boldsymbol{C}}_{t-2} \subseteq \overline{\boldsymbol{C}}_{t-1}$ such that c_{t-1} is constant on $\binom{\overline{\boldsymbol{C}}_{t-2}}{\boldsymbol{B_L}}$, etc. So we obtain inductively $\overline{\boldsymbol{C}}_0 \subseteq \overline{\boldsymbol{C}}_1 \subseteq \dots \subseteq \boldsymbol{C}_t$ with $\overline{\boldsymbol{C}}_{t-1} \cong \boldsymbol{C}_{t-1}, \dots, \overline{\boldsymbol{C}}_0 \cong \boldsymbol{C}_0 = \boldsymbol{B_K}$ such that c_i is constant on $\binom{\overline{\boldsymbol{C}}_0}{\boldsymbol{B_L}}$, say with value $\bar{c}_i$, for every $1 \leq i \leq t$. Let $\overline{\boldsymbol{K}} \subseteq \overline{\boldsymbol{C}}_0 \subseteq \boldsymbol{C}$ be a copy of $\boldsymbol{K}$ with the same 0, 1 as $\overline{\boldsymbol{C}}_0$ and which generates $\overline{\boldsymbol{C}}_0$ as a Boolean algebra. We claim that c on $\binom{\overline{\boldsymbol{K}}}{\boldsymbol{L}}$ takes at most the t values $\bar{c}_1, \dots, \bar{c}_t$. Let $\boldsymbol{L}' \cong \boldsymbol{L}$ with $\boldsymbol{L}' \subseteq \overline{\boldsymbol{K}}$. Let $\boldsymbol{B}_{\boldsymbol{L}'} \subseteq \boldsymbol{C}_0$ be the Boolean lattice with the same 0, 1 as $\boldsymbol{L}'$ and which is generated as a Boolean algebra by $\boldsymbol{L}'$. Thus $\boldsymbol{B}'_{\boldsymbol{L}'} \subseteq \boldsymbol{C}_0$ and $\boldsymbol{B}_{\boldsymbol{L}'} \cong \boldsymbol{B_L}$. Consider the unique isomorphism $\pi : \langle \boldsymbol{B}'_{\boldsymbol{L}'}, <|B'_{\boldsymbol{L}'} \rangle \to \langle \boldsymbol{B_L}, <_{\boldsymbol{L}} \rangle$. Then $\pi(\boldsymbol{L}') = \boldsymbol{L}'_i$ for some $1 \leq i \leq t$, and so $c_i(\boldsymbol{B}'_{\boldsymbol{L}'}) = c(\boldsymbol{L}') = \bar{c}_i$.

We will next show that $t(\boldsymbol{L}, \mathcal{D}) \geq t$. For that we will prove that for any $\boldsymbol{K} \in \mathcal{D}$ with $\boldsymbol{K} \geq \boldsymbol{B_L}$, there is a coloring $c : \binom{\boldsymbol{K}}{\boldsymbol{L}} \to \{1, \dots, t\}$ so that for any copy $\boldsymbol{B}$ of $\boldsymbol{B_L}$ in $\boldsymbol{K}$ the coloring c on $\binom{\boldsymbol{B}}{\boldsymbol{L}}$ takes all t colors. Let $<$ be an ordering on $\boldsymbol{B_K}$ given anti-lexicographically by an ordering of its atoms. Then define $\overline{c}\colon \binom{\boldsymbol{B_K}}{\boldsymbol{L}} \to \{1,\dots,t\}$ as follows: Let $\boldsymbol{L}' \cong \boldsymbol{L}$ with $\boldsymbol{L}' \subseteq \boldsymbol{B_K}$. Then let $\boldsymbol{B}'_{\boldsymbol{L}'} \subseteq \boldsymbol{B}_K$ be defined as before and let $\pi\colon \langle \boldsymbol{B}_{\boldsymbol{L}'}, <|B'_{\boldsymbol{L}'}\rangle \to \langle \boldsymbol{B_L}, <_{\boldsymbol{L}}\rangle$ be the unique isomorphism. If $\pi(\boldsymbol{L}') = \boldsymbol{L}'_i$, then we put

$$\overline{c}(\boldsymbol{L}') = i.$$

Finally, let c be the restriction of $\overline{c}$ to $\binom{\boldsymbol{K}}{\boldsymbol{L}}$. If $\boldsymbol{B} \cong \boldsymbol{B_L}$ and $\boldsymbol{B} \subseteq \boldsymbol{K}$, then it is clear that c on $\binom{\boldsymbol{B}}{\boldsymbol{L}}$ takes all t colors.

This finishes the proof of 5.2. Next we derive from this Corollary 5.3.

If $\boldsymbol{L}$ is a Boolean lattice, then clearly $t(\boldsymbol{L},\mathcal{D}) = t(\boldsymbol{L}) = 1$. Conversely, assume that $\boldsymbol{L}$ is not a Boolean lattice and let $\boldsymbol{B_L}$ be as before. Then $\boldsymbol{L} \neq \boldsymbol{B}_L$. To show that $t(\boldsymbol{L}) > 1$ we will show that there is $\varphi \in \mathrm{Aut}(\boldsymbol{B_L})$ such that $\varphi(L) \neq L$. Assume this fails, towards a contradiction, i.e., for all $\varphi \in \mathrm{Aut}(\boldsymbol{B_L})$, $\varphi(L) = L$ (i.e., every automorphism of $\boldsymbol{B}_L$ fixes L setwise). We will view $\boldsymbol{B}_L$ as the Boolean lattice of all subsets of a finite set $X = \{1,\dots,n\}$ and thus $L \subseteq \{Y\colon Y \subseteq X\}$. Since $\boldsymbol{L}$ is not a Boolean lattice there is $Y \notin L$ (thus $Y \neq \emptyset$) such that $Y^c = X \setminus Y \in L$. Let $A \in L \setminus \{\emptyset\}$ have the smallest cardinality among all non-empty elements of L. Let $a_0 \in A$, $y_0 \in Y$ and let $\varphi \in \mathrm{Aut}(\boldsymbol{B_L})$ exchange a_0, y_0. If $\overline{A} = \varphi(A)$, then $|\overline{A}| = |A|$, $\overline{A} \in L$ and $\overline{A} \cap Y \neq \emptyset$, so by replacing A with $\overline{A}$ if necessary, we can assume that $A \cap Y \neq \emptyset$. We now claim that actually $A \subseteq Y$. Otherwise $A \setminus Y = A \cap Y^c \in L$ and $A \setminus Y \neq \emptyset$ but $|A \setminus Y| < |A|$, a contradiction.

Decompose $Y = A \sqcup A_0 \sqcup \cdots \sqcup A_{k-1} \sqcup B$, where $|A_i| = |A|$ and $|B| < |A|$. Then one of $A_0,\dots,A_{k-1},B$ is not in L. If $A_i \notin L$, then there is $\varphi \in \mathrm{Aut}(\boldsymbol{B_L})$ with $\varphi(A) = A_i$. However, $A \in L$ but $A_i \notin L$, which is a contradiction. So $B \notin L$ (thus $B \neq \emptyset$). Then fix $B_0 \subseteq A$ with $|B_0| = |B|$. Let $C = A \setminus B_0 \neq \emptyset$, so that $|C| < |A|$, therefore $C \notin L$. Also $C \cup B_0 = A \in L$, so if $C \cup B \in L$, then $C = (C \cup B_0) \cap (C \cup B) \in L$. Therefore $C \cup B \notin L$. However $|C \cup B| = |C \cup B_0|$, so as before there is $\varphi \in \mathrm{Aut}(\boldsymbol{B_L})$ with $\varphi(C \cup B_0) = C \cup B$, which contradicts the fact that $C \cup B_0 \in L$ but $C \cup B \notin L$. ∎

Appendix 3. The universal minimal flow of $K \ltimes G$, K compact. Let K be a compact metrizable group and G a Polish group on which K acts continuously by automorphisms. If X_G is the universal minimal flow of G, we will calculate the universal minimal flow of $K \ltimes G$. In fact we will do this in a more general situation.

Consider a Polish group H and a normal closed subgroup $G \lhd H$. Assume moreover that there is a compact transversal $K \subseteq H$ for the left cosets of G in H. By translating we can always assume that $1 \in K$. We consider the selector map $s\colon H \to K$, where $s(h)$ is the unique element of $K \cap hG$. We next verify that s is continuous. Indeed, every $h \in H$ is uniquely written as

$h = kg$, where $g \in G$ and $s(h) = k \in K$. To prove the continuity of s, assume that $h_n \to h$ and let $h_n = k_n g_n$, $h = kg$. If $k_n \not\to k$, towards a contradiction, then by the compactness of k we can assume, by going to a subsequence, that $k_n \to \ell \in K$ for some $\ell \neq k$. Then $g_n = k_n^{-1} h_n \to \ell^{-1} h = g' \in G$, since G is closed. Thus $h = \ell g' = kg$, so $\ell = k$, a contradiction.

In the special case $H = K \ltimes G$, we can identify G with the closed normal subgroup $\{(1, g) \colon g \in G\}$ and K with the transversal (which is actually a closed subgroup) $\{(k, 1) \colon k \in K\}$. Then $s(h) = s(k, g) = (k, 1)$.

Returning to the general case, observe that $s(h_1 s(h_2 k)) = s(h_1 h_2 k)$. Therefore

$$h \cdot k = s(hk)$$

defines an action of H on K. Let also $\rho \colon H \times K \to G$ be defined by

$$\rho(h, k) = (h \cdot k)^{-1} hk.$$

Then ρ is a *cocycle* for this action, i.e.,

$$\rho(h_1 h_2, k) = \rho(h_1, h_2 \cdot k) \rho(h_2, k).$$

Clearly $h \cdot k = s(hk)$ and $\rho(h, k) = s(hk)^{-1}(hk)$ are continuous.

Suppose now X_G is a G-flow. Then we can define an H-flow X_H, called the *induced flow* as follows:

$$X_H = X_G \times K$$

and the action of H on X_H is defined by

$$h \cdot (x, k) = (\rho(h, k) \cdot x, h \cdot k).$$

CLAIM 1. *If X_G is minimal, so is X_H.*

Proof. Fix $(x_0, k_0) \in X_G \times K$. Let $(x, k) \in X_G \times K$ and let $V_1 \subseteq X_G$ and $V_2 \subseteq K$ be open with $(x, k) \in V_1 \times V_2$. We will find $h \in H$ with $h \cdot (x_0, k_0) \in V_1 \times V_2$.

Since the action of H on K is transitive, let $h_0 \in H$ be such that $h_0 \cdot k_0 = k$. Let $g \in G$. By the normality of G in H, $h_0 k_0 G = G h_0 k_0 = kG$ and $g h_0 k_0 \in kG$. Thus $(g h_0) \cdot k_0 = k = h_0 \cdot k_0$.

We also have

$$(h_0 \cdot k_0) \rho(h_0, k_0) = h_0 k_0,$$

so

$$g(h_0 \cdot k_0) \rho(h_0, k_0) = (g h_0) k_0 = (g h_0 \cdot k_0) \rho(g h_0, k_0) = (h_0 \cdot k_0) \rho(g h_0, k_0).$$

Now find $g' \in G$ such that $g' \cdot x_0 \in V_1$. Then let $g'' \in G$ be defined by

$$(h_0 \cdot k_0)^{-1} g'' (h_0 \cdot k_0) \rho(h_0, k_0) = g',$$

so that $\rho(g'' h_0, k_0) = g'$. (This follows from the above formulas by putting

$g = g''$.) Then if $h = g''h_0$, we have

$$h \cdot (x_0, k_0) = g''h_0 \cdot (x_0, k_0) = (\rho(g''h_0, k_0) \cdot x_0, g''h_0 \cdot k_0)$$
$$= (g' \cdot x_0, k) \in V_1 \times V_2. \blacksquare$$

CLAIM 2. *If X_G is the universal minimal flow of G, then X_H is the universal minimal flow of H.*

Proof. We have seen that X_H is minimal, so it is enough to show that if X is an arbitrary H-flow, then there is a continuous H-map $\varphi\colon X_H \to X$.

The restriction of the H-action on X to G gives a G-flow on X and therefore there is a continuous G-map $\pi\colon X_G \to X$. Then define $\varphi\colon X_H \to X$ by

$$\varphi(x, k) = k \cdot \pi(x).$$

Clearly φ is continuous, so we only need to verify that $\varphi(h \cdot (x, k)) = h \cdot \varphi(x, k)$.

Now

$$\begin{aligned}\varphi(h \cdot (x, k)) &= \varphi(\rho(h, k) \cdot x, h \cdot k) = (h \cdot k) \cdot \pi(\rho(h, k) \cdot x)\\ &= (h \cdot k) \cdot \rho(h, k) \cdot \pi(x) = (h \cdot k)\rho(h, k) \cdot \pi(x)\\ &= hk \cdot \pi(x) = h \cdot k \cdot \pi(x) = h \cdot \varphi(x, k). \blacksquare\end{aligned}$$

Putting these together we have thus shown the following:

THEOREM. *Let H be a Polish group, $G \lhd H$ a closed normal subgroup, and assume that there is a compact transversal for the left cosets of G in H. If X_G is the universal minimal flow of G, then the induced action of H on $X_H = X_G \times K$ is the universal minimal flow of H.*

COROLLARY. *Let K be a compact metrizable group acting continuously by automorphisms on a Polish group G. If G is extremely amenable, then the universal minimal flow of $K \ltimes G$ is the action of this group on K given by $(k, g) \cdot l = kl$.*

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