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Possible cardinalities of maximal abelian subgroups of quotients of permutation groups of the integers

by

Saharon Shelah (Piscataway, NJ, and Jerusalem) and
Juris Steprāns (North York and Toronto)

Abstract. If G is a group then the abelian subgroup spectrum of G is defined to be the set of all κ such that there is a maximal abelian subgroup of G of size κ. The cardinal invariant $A(G)$ is defined to be the least uncountable cardinal in the abelian subgroup spectrum of G. The value of $A(G)$ is examined for various groups G which are quotients of certain permutation groups on the integers. An important special case, to which much of the paper is devoted, is the quotient of the full symmetric group by the normal subgroup of permutations with finite support. It is shown that, if we use G to denote this group, then $A(G) \leq \mathfrak{a}$. Moreover, it is consistent that $A(G) \neq \mathfrak{a}$. Related results are obtained for other quotients using Borel ideals.

1. Introduction and definitions. The maximality of abelian subgroups plays a role in various parts of group theory. For example, Mycielski [9, 8] has extended a classical result on Lie groups and shown that a maximal abelian subgroup of a compact connected group is connected. For finite symmetric groups the question of the size of maximal abelian subgroups has been examined by Burns and Goldsmith in [4] and Winkler in [16]. It will be shown in Corollary 3.1 that there is not much interest in generalizing this study to infinite symmetric groups; the cardinality of any maximal abelian subgroup of the symmetric group of the integers is $2^{\aleph_0}$. The purpose of this paper is to examine the size of maximal abelian subgroups for a class of groups closely related to the symmetric group of the integers; these arise by taking an ideal on the integers, considering the subgroup of all permutations

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which respect the ideal and then taking the quotient by the normal subgroup of permutations which fix all integers except for a set in the ideal. It will be shown that the size of maximal abelian subgroups in such groups is sensitive to the nature of the ideal as well as to various set-theoretic hypotheses.

The reader familiar with applications of the Axiom of Choice may not be surprised by the assertion just made since one can imagine constructing ideals on the integers by transfinite induction such that the quotient group just described exhibits various desired properties. Consequently, it is of interest to restrict attention to only those ideals which do not require the Axiom of Choice for their definition. All of the ideals considered will have simple definitions—indeed, they will all be Borel subsets of $\mathcal{P}(\omega)$ with the usual topology—and, in fact, the first three sections will focus on the ideal of finite sets. It should be mentioned that there is a large body of work examining the analogous quotients of the Boolean algebra $\mathcal{P}(\omega)$ modulo an analytic ideal; the monograph [6] by Farah is a good reference for this subject. However, the analogy is far from perfect since, for example, whereas the Boolean algebra $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ can consistently have $2^{2^{\aleph_0}}$ automorphisms [10] it is shown in [1] that the quotient of the full symmetric group of the integers modulo the subgroup of finite permutations has only countably many outer automorphisms. Nevertheless, it may be possible to employ methods similar to those of [6] in order to distinguish between different quotient algebras up to isomorphism. This has been done for elementary equivalence in [14, 13] for quotients of the full symmetric group on κ by the normal subgroups fixing all but λ elements. However, since the full symmetric group of the integers has only two proper normal subgroups [11], quotients of certain, naturally arising subgroups will be considered instead. One of the goals of this study is to use the cardinal invariants associated with maximal abelian subgroups as a tool to distinguish between isomorphism types of such groups.

In order to state the main results precisely some notation is needed.

Definition 1.1. If G is a group then define the *abelian subgroup spectrum* of G to be the set of all κ such that there is a maximal abelian subgroup of G of size κ. Define $A(G)$ to be the least uncountable cardinal in the abelian subgroup spectrum of G.

The requirement that $A(G)$ be uncountable rather than just infinite is important because many groups have maximal abelian subgroups isomorphic to $\mathbb{Z}$. In particular, quotients of the symmetric group on $\mathbb{N}$ often have a fixed point free permutation of $\mathbb{N}$ consisting of a single cycle which generates a maximal abelian subgroup. The question of whether maximal abelian subgroups which are not finitely generated are uncountable will not be considered here.

NOTATION 1.1. Throughout this paper the symbol $\mathbb{S}$ will be used to denote the symmetric group on $\mathbb{N}$. For $\pi \in \mathbb{S}$ let $\operatorname{supp}(\pi)$ denote the support of π, which is defined to be $\{x \in \operatorname{domain}(\pi) : \pi(x) \neq x\}$. If $\mathcal{I}$ is an ideal (1) on $\mathbb{N}$ then $\mathbb{S}(\mathcal{I}) \subseteq \mathbb{S}$ will denote the subgroup of all permutations preserving $\mathcal{I}$; in other words, a permutation π belongs to $\mathbb{S}(\mathcal{I})$ provided that $\pi(A) \in \mathcal{I}$ if and only if $A \in \mathcal{I}$. On the other hand, $\mathbb{F}(\mathcal{I})$ will be used to denote the normal subgroup of $\mathbb{S}(\mathcal{I})$ consisting of all permutations $\pi \in \mathbb{S}(\mathcal{I})$ such that $\operatorname{supp}(\pi) \in \mathcal{I}$. The abbreviation $\mathbb{F} = \mathbb{F}([\mathbb{N}]^{<\aleph_0})$ will also be used.

The focus of this paper will be on computing $A(\mathbb{S}(\mathcal{I})/\mathbb{F}(\mathcal{I}))$ for various simply defined ideals. This cardinal will be denoted by $A(\mathcal{I})$.

NOTATION 1.2. Given a pair of permutations $\{\pi, \pi'\} \in [\mathbb{S}]^2$ define

$$\operatorname{NC}(\pi, \pi') = \{n \in \mathbb{N} : \pi(\pi'(n)) \neq \pi'(\pi(n))\}.$$

A pair of permutations $\{\pi, \pi'\} \in [\mathbb{S}]^2$ will be said to *almost commute modulo an ideal* $\mathcal{I}$ if $\operatorname{NC}(\pi, \pi') \in \mathcal{I}$ and they will be said to *almost commute* if $\operatorname{NC}(\pi, \pi')$ is finite.

NOTATION 1.3. Given a permutation π and $X \subseteq \mathbb{N}$ define the *orbit* of X under π by $\operatorname{orb}_\pi(X) = \{\pi^i(x)\}_{i\in\mathbb{Z},\, x\in X}$. The abbreviation $\operatorname{orb}_\pi(n) = \operatorname{orb}_\pi(\{n\})$ will be used when no confusion is possible. If $\mathcal{S}$ is a set of permutations then define

$$\operatorname{orb}_{\mathcal{S}}(X) = \Big\{\prod_{i=1}^{n} \pi_i^{j}(x)\Big\}_{n\in\omega,\, x\in X,\, \pi_i\in\mathcal{S},\, j\in\{-1,1\}}.$$

NOTATION 1.4. Given a set of permutations $\mathcal{S} \subseteq \mathbb{S}$ define $\Omega_{\mathcal{S}}$ to be the set of all non-empty, minimal sets closed under the group generated by the permutations in $\mathcal{S}$. Define $\Omega_{\mathcal{S}}(n)$ to be the unique element of $\Omega_{\mathcal{S}}$ containing n. If A and B are in $\Omega_{\mathcal{S}}$ then define A to be *$\mathcal{S}$-isomorphic* to B if there is a bijection $\psi : A \to B$ such that $\pi(\psi(a)) = \psi(\pi(a))$ for each $\pi \in \mathcal{S}$ and $a \in A$.

NOTATION 1.5. Given two finite subsets A and B of $\mathbb{N}$ of the same cardinality, define $\Delta_{A,B} : A \to B$ to be the unique order preserving mapping between them and let $\Delta_A = \Delta_{A,|A|}$. Let $\Delta_{\{A,B\}} = \Delta_{A,B} \cup \Delta_{B,A}$.

The set-theoretic notation used throughout will adhere to the contemporary standard. In particular, $[X]^k$ will denote the family of subsets of X of cardinality k and $[X]^{<k}$ will denote the family of subsets of X of cardinality less than k. Occasionally $\equiv^*$ will be used to denote equivalence modulo a finite set. Since cardinal invariants of the continuum are closely linked to the investigation of $A(\mathcal{I})$ it is worthwhile recalling the definitions of some well known invariants.

(1) An *ideal* is a collection of subsets of the integers closed under finite unions and subsets.

DEFINITION 1.2. Given an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ let $\mathcal{P}(\omega)/\mathcal{I}$ be the quotient Boolean algebra and denote the least cardinal of a maximal, uncountable, pairwise disjoint family [2] in $\mathcal{P}(\omega)/\mathcal{I}$ by $\mathfrak{a}(\mathcal{I})$. In the special case $\mathcal{I} = [\mathbb{N}]^{<\aleph_0}$ the invariant $\mathfrak{a}(\mathcal{I})$ is denoted by $\mathfrak{a}$ and it should be noted that $\mathfrak{a}$ is also the least cardinal of an infinite, maximal almost disjoint family; namely, a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that any two of its elements have finite intersection and $\mathcal{A}$ is maximal with respect to this property. The least cardinal of an ideal $\mathcal{B} \subseteq \mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ such that there is no $C \in \mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ disjoint from all members of $\mathcal{B}$ (other than the equivalence class of the finite sets) is denoted by $\mathfrak{p}$.

In Section 2 it is shown that $\mathfrak{a}$ is an upper bound for $A([\mathbb{N}]^{<\aleph_0})$ while in Section 3 it is shown that $\mathfrak{p}$ serves as a lower bound for $A([\mathbb{N}]^{<\aleph_0})$. Sections 4 and 5 deal with consistency results. In Section 4 it is shown that $\mathfrak{a}$ is not the best possible upper bound for $A([\mathbb{N}]^{<\aleph_0})$ since in the iterated Laver model $A([\mathbb{N}]^{<\aleph_0})$ is strictly less than $\mathfrak{a}$. Sections 5 and 6 deal with quotients using ideals other than the ideal of finite sets. It is shown in Section 5 that adding $\aleph_1$ Cohen reals to a model where $2^{\aleph_0} > \aleph_1$ yields a model where $A(\mathcal{I}_{1/x}) = \aleph_1 < 2^{\aleph_0}$ and $\mathcal{I}_{1/x}$ is the ideal of sets whose reciprocals form a series with finite sum. Section 6 deals with ideals similar to the density ideal. It is shown that $A(\mathcal{I}) = 2^{\aleph_0}$ for many of these ideals $\mathcal{I}$. No extra set-theoretic axioms are used here. The final section contains some open questions.

2. An upper bound

PROPOSITION 2.1. $A([\mathbb{N}]^{<\aleph_0}) \leq \mathfrak{a}$.

Proof. Let $\mathcal{A}$ be a maximal almost disjoint family of subsets of $\mathbb{N}$ of size $\mathfrak{a}$ and let $F(\mathcal{A})$ be the free abelian group generated by $\mathcal{A}$; in other words, $F(\mathcal{A})$ consists of all $f : \mathcal{A} \to \mathbb{Z}$ such that f has finite support. For $a \in \mathcal{A}$ define $\pi_a : a \to a$ by $\pi_a(i) = \min(\{j \in a : j > i\})$ and, for $j \in \mathbb{Z}$, let π_a^j denote the j-fold composition of π_a, noting that both the domain and range of π_a^j are co-finite subsets of a. If $f \in F(\mathcal{A})$ then let $\Phi(f)$ be the set of all permutations π such that there is a finite set $F \subseteq \mathbb{N}$ such that if a and a' are distinct elements of $\mathrm{supp}(f)$ then $a \cap a' \subseteq F$ and such that

$$\pi(j) = \begin{cases} \pi_a^{f(a)}(j) & \text{if } j \in a \setminus F \text{ and } f(a) \neq 0, \\ j & \text{if } j \notin F \text{ and } (\forall a \in \mathcal{A})\ j \notin a \text{ or } f(a) = 0, \end{cases}$$

leaving $\Phi(f)$ undefined if there are no such permutations. Observe that $\Phi(0) = \mathbb{F}$ where 0 denotes the constant function with value 0. Also note that if $\pi \in \Phi(f)$ and $\sigma \in \Phi(g)$ then $\pi \circ \sigma \in \Phi(f + g)$. Since it is easy to

[2] See [15] for a more detailed discussion of this invariant.

see that if $\pi \in \mathbb{F}$ and $\sigma \in \Phi(f)$ then both $\sigma \circ \pi$ and $\pi \circ \sigma$ are in $\Phi(f)$, it follows that $\Phi(f)$ is a coset of $\mathbb{F}$ if it is defined. Since Φ is easily seen to be one-to-one, it is an isomorphism between a subgroup of $F(\mathcal{A})$ and the subgroup $\Phi(F(\mathcal{A}))$ of $\mathbb{S}/\mathbb{F}$.

In fact, $\Phi(f)$ is defined precisely when $\sum_{a\in\mathcal{A}} f(a) = 0$. To see this, let $f \in F(\mathcal{A})$ and suppose that the support of f is B and $\sum_{b\in B} f(b) = 0$. Let $F \subseteq \mathbb{N}$ be a finite set such that $b \cap b' \subseteq F$ for any two distinct b and b' in B and such that $F \cap b$ is an initial segment of b for each $b \in B$. Let $B^+ = \{b \in B : f(b) > 0\}$ and $B^- = \{b \in B : f(b) < 0\}$. If $b \in B^+$ let b^* be the first $f(b)$ elements of $b \setminus F$ and if $b \in B^-$ let b^* be the first $-f(b)$ elements of $b \setminus F$. Let $\theta : \bigcup_{b\in B^-} b^* \to \bigcup_{b\in B^+} b^*$ be any bijection and define π as follows:

$$\pi(j) = \begin{cases} j & \text{if } j \notin \bigcup_{b\in B} b \setminus F, \\ \pi_b^{f(b)} & \text{if } j \in b \in B^+, \\ \pi_b^{f(b)} & \text{if } j \in b \setminus b^* \text{ and } b \in B^-, \\ \theta(j) & \text{if } j \in \bigcup_{b\in B^-} b^*, \end{cases}$$

and observe that π is a bijection. Moreover, $F \cup \bigcup_{b\in B^-} b^*$ witnesses that $\phi \in \Phi(f)$. It is an easy exercise to use the maximality of $\mathcal{A}$ to show that if $\sum f(a) \neq 0$ then $\Phi(f) = \emptyset$.

To see that $\Phi(F(\mathcal{A}))$ is maximal abelian let $\pi/\mathbb{F} \in \mathbb{S}/\mathbb{F} \setminus \Phi(F(\mathcal{A}))$. Before continuing, some notation will be introduced. Given two distinct elements a and a' of $\mathcal{A}$ let $f_{a,a'} \in F(\mathcal{A})$ be such that $\operatorname{supp}(f_{a,a'}) = \{a, a'\}$ and $f_{a,a'}(a) = 1 = -f_{a,a'}(a')$. Choose $\pi_{a,a'} \in \Phi(f_{a,a'})$.

Claim 1. *If $a \in \mathcal{A}$ is such that* $\operatorname{supp}(\pi) \cap a$ *is infinite then* $\operatorname{supp}(\pi) \cap a$ *is a co-finite subset of a.*

Proof. Let $a' \in \mathcal{A} \setminus \{a\}$. If the claim fails then there are infinitely many $n \in a$ such that $n \notin \operatorname{supp}(\pi)$ but $\pi_{a,a'}(n) \in \operatorname{supp}(\pi)$. For any such n it follows that $\pi \circ \pi_{a,a'}(n) \neq \pi_{a,a'}(n)$ while $\pi_{a,a'} \circ \pi(n) = \pi_{a,a'}(n)$. Hence $\pi/\mathbb{F}$ and $\pi_{a,a'}/\mathbb{F}$ do not commute. ∎

Claim 2. *If $a \in \mathcal{A}$ is such that* $\operatorname{supp}(\pi) \cap a$ *is infinite then $\pi(a) \subseteq^* a$.*

Proof. If not, there are infinitely many $n \in a$ such that $\pi(n) \notin a$. Let X be the set of all such n and choose $a' \in \mathcal{A} \setminus \{a\}$ such that $\pi(X) \setminus a'$ is infinite. Then $\pi_{a,a'} \circ \pi(n) = \pi(n)$ and $\pi_{a,a'}(n) \neq n$ for all but finitely many $n \in \pi^{-1}(\pi(X) \setminus a')$. From the last inequality it follows that $\pi(\pi_{a,a'}(n)) \neq \pi(n)$ and hence $\pi_{a,a'}/\mathbb{F}$ does not commute with $\pi/\mathbb{F}$. ∎

Claim 3. *If $a \in \mathcal{A}$ is such that* $\operatorname{supp}(\pi) \cap a$ *is infinite then there is some $i \in \mathbb{Z}$ such that $\pi{\restriction}a \equiv^* \pi_a^i{\restriction}a$.*

Proof. From Claims 1 and 2 it follows that for almost all $n \in a$ there is some $k(n)$ such that $\pi(n) = \pi_a^{k(n)}(n)$. Let $a' \in \mathcal{A} \setminus \{a\}$. If the claim is false then there are infinitely many $n \in a$ such that $k(n) \neq k(\pi_a(n)) = k(\pi_{a,a'}(n))$. For any such n it follows that

$$\begin{aligned}\pi_{a,a'} \circ \pi(n) &= \pi_{a,a'}^{k(n)+1}(n)\\ &\neq \pi_{a,a'}^{k(\pi_{a,a'}(n))+1}(n)\\ &= \pi_{a,a'}^{k(\pi_{a,a'}(n))}(\pi_{a,a'}(n)) = \pi \circ \pi_{a,a'}(n)\end{aligned}$$

and hence $\pi_{a,a'} \circ \pi$ and $\pi \circ \pi_{a,a'}$ disagree on infinitely many integers. ∎

There are now two cases to consider.

Case 1: There is a finite subset $\{a_1, \ldots, a_n\} \subseteq \mathcal{A}$ such that $\mathrm{supp}(\pi) \subseteq^* \bigcup_{i=1}^n a_i$. In this case, use Claim 3 to choose integers $k_i \in \mathbb{Z}$ such that

$$\pi \restriction a_i \equiv^* \pi_{a_i}^{k_i}$$

for each $i \leq n$. This contradicts the assumption that $\pi/\mathbb{F} \notin \Phi(F(\mathcal{A}))$.

Case 2: There is no finite subset $\{a_1, \ldots, a_n\} \subseteq \mathcal{A}$ such that $\mathrm{supp}(\pi) \subseteq^* \bigcup_{i=1}^n a_i$. In this case there are uncountably many $a \in \mathcal{A}$ such that $\mathrm{supp}(\pi) \cap a$ is infinite. Use Claim 3 to conclude that there is some $i \in \mathbb{Z}$ such that, without loss of generality, $\pi \restriction a \equiv^* \pi_a^i$ for uncountably many $a \in \mathcal{A}$. Hence there is some $k \in \mathbb{N}$ such that $\pi \restriction \{n \in a : n \geq k\} = \pi_a^i \restriction \{n \in a : n \geq k\}$ for uncountably many $a \in \mathcal{A}$. Therefore, there are distinct a and b in $\mathcal{A}$ such that $\{n \in a : n \geq k\} \cap \{n \in b : n \geq k\}$ is not empty. If j is the maximal element of this intersection then $\pi(j) = \pi_a^i(j) \neq \pi_b^i(j) = \pi(j)$. ∎

3. A lower bound. The next series of preliminary lemmas will be used in the proof of Theorem 3.2 which establishes a lower bound for $A([\mathbb{N}]^{<\aleph_0})$. Corollary 3.1 has as a trivial consequence the fact that any maximal abelian subgroup of the full symmetric group of the integers has cardinality $2^{\aleph_0}$; however, this can also be shown by using the topology of pointwise convergence on this group and noting that any maximal abelian subgroup must be closed and uncountable, and hence have cardinality $2^{\aleph_0}$.

Lemma 3.1. *Let $\mathcal{S}$ be a finite subset of $\mathbb{S}$ whose elements almost commute with each other.*

(1) *If all the orbits of each $\pi \in \mathcal{S}$ are finite then each element of $\Omega_{\mathcal{S}}$ is finite.*
(2) *If, in addition, for each $\pi \in \mathcal{S}$ all the orbits of π have size less than or equal to $m(\pi)$ then the cardinality of all but finitely many elements of $\Omega_{\mathcal{S}}$ is no greater than $\prod_{\pi \in \mathcal{S}} m(\pi)$.*

Proof. Proceed by induction on $n = |\mathcal{S}|$, the case $n = 1$ being trivial. If the lemma is true for n let $\mathcal{S} = \{\pi_i\}_{i=1}^{n+1}$ and $\mathcal{S}' = \{\pi_i\}_{i=1}^{n}$. Define

$$B = \bigcup_{i=1}^{n} \mathrm{NC}(\pi_i, \pi_{n+1}),$$

and if the orbit of each $\pi \in \mathcal{S}$ is bounded by $m(\pi)$ then let B' be the union of those finitely many $A \in \Omega_{\mathcal{S}'}$ whose cardinality is not bounded by $\prod_{i=1}^{n} m(\pi_i)$. Define

$$B^* = \bigcup_{m \in \mathrm{orb}_{\pi_{n+1}}(B \cup B')} \Omega_{\mathcal{S}'}(m).$$

Observe that B^* is finite by the induction hypothesis and the fact that the orbits of π_{n+1} are finite. Hence, it suffices to show that if $C \in \Omega_{\mathcal{S}'}$ and $C \cap B^* = \emptyset$ then $C' = \mathrm{orb}_{\pi_{n+1}}(C)$ belongs to $\Omega_{\mathcal{S}}$. The fact that it is finite is immediate since all orbits of π_{n+1} are finite; similarly, if $|C| \leq \prod_{i=1}^{n} m(\pi_i)$ then it follows that $|C'| \leq \prod_{i=1}^{n+1} m(\pi_i)$.

To see that $C' \in \Omega_{\mathcal{S}}$ it suffices to show that if $i \leq n$ and $c \in C'$ then $\mathrm{orb}_{\pi_i}(c) \subseteq C'$. There is some $d \in C$ such that $c \in \mathrm{orb}_{\pi_{n+1}}(d)$. Since $\mathrm{orb}_{\pi_i}(d) \subseteq C \subseteq C'$ it follows that if $\mathrm{orb}_{\pi_i}(c) \not\subseteq C'$ then there must be some $e \in \mathrm{orb}_{\pi_{n+1}}(d)$ such that $\pi_i(e) \in C'$ and $\pi_i(\pi_{n+1}(e)) \notin C'$. But $\pi_{n+1}(\pi_i(e)) \in C'$ by definition. Hence $\pi_{n+1} \circ \pi_i(e) \neq \pi_i \circ \pi_{n+1}(e)$, contradicting the fact that $e \notin B$. ∎

LEMMA 3.2. *Let $\mathcal{S} \subseteq \mathbb{S}$ be finite and suppose that $\pi \in \mathbb{S}$ and $\theta \in \mathbb{S}$ almost commute with each member of $\mathcal{S}$. Then there is a finite set Y such that for any set X, if $\pi \restriction X \cup Y = \theta \restriction X \cup Y$ then $\pi \restriction \mathrm{orb}_{\mathcal{S}}(X) = \theta \restriction \mathrm{orb}_{\mathcal{S}}(X)$. Moreover, if π and θ actually commute with each member of $\mathcal{S}$ then Y can be taken to be the empty set.*

Proof. Let $Y' = \bigcup_{\sigma \in \mathcal{S}} \mathrm{NC}(\sigma, \pi) \cup \mathrm{NC}(\sigma, \theta)$ and $Y = \bigcup_{\sigma \in \mathcal{S}} \sigma(Y') \cup \sigma^{-1}(Y')$. Note that $\mathrm{orb}_{\mathcal{S}}(X \cup Y) = \bigcup_{i=0}^{\infty} X^{(i)}$ where $X^{(0)} = X \cup Y$ and $X^{(n+1)} = \bigcup_{\sigma \in \mathcal{S}} \mathrm{orb}_{\sigma}(X^{(n)})$, and hence it suffices to show by induction that $\pi \restriction X^{(n)} = \theta \restriction X^{(n)}$ for each n assuming that $\pi \restriction X^{(0)} = \theta \restriction X^{(0)}$. To this end, suppose that $\pi \restriction X^{(n)} = \theta \restriction X^{(n)}$ and $x \in X^{(n+1)} \setminus X^{(n)}$. Then there are $\overline{x} \in X^{(n)}$ and $\sigma \in \mathcal{S}$ such that $x \in \mathrm{orb}_{\sigma}(\overline{x})$. But $\pi(\overline{x}) = \theta(\overline{x})$ and hence $\sigma^k(\pi(\overline{x})) = \sigma^k(\theta(\overline{x}))$ for any k. If $n > 1$ then $\overline{x} \notin Y$ and it follows that $\pi(\sigma^k(\overline{x})) = \theta(\sigma^k(\overline{x}))$ for all k. Since $x = \sigma^k(\overline{x})$ for some k the result follows.

If $n = 1$ it will be shown by induction on $|k|$ that if $x \in X^{(1)}$ and $\overline{x} \in X^{(0)}$ and $\sigma \in \mathcal{S}$ are such that $x = \sigma^k(\overline{x})$ then $\theta(x) = \pi(x)$. If $|k| = 0$ this is immediate. First assume that $k > 0$ and $\theta(\sigma^{k-1}(\overline{x})) = \pi(\sigma^{k-1}(\overline{x}))$. If $\sigma^{k-1}(\overline{x}) \notin Y'$ then $\sigma^{k-1}(\overline{x}) \notin \mathrm{NC}(\pi, \sigma) \cup \mathrm{NC}(\theta, \sigma)$ and so

$$\theta(\sigma^k(\overline{x})) = \sigma(\theta(\sigma^{k-1}(\overline{x}))) = \sigma(\pi(\sigma^{k-1}(\overline{x}))) = \pi(\sigma^k(\overline{x}))$$

as required. On the other hand, if $\sigma^{k-1}(\overline{x}) \in Y'$ then $\sigma(\sigma^{k-1}(\overline{x})) \in Y$ and so $\theta(\sigma^k(\overline{x})) = \pi(\sigma^k(\overline{x}))$ in this case also. The case that $k < 0$ is handled similarly. ∎

Definition 3.1. If $H \subseteq \mathbb{S}$ is a subgroup then define H to be *strongly almost abelian* if for each $h \in H$ there is a finite set $F(h) \subseteq \mathbb{N}$ such that if h_1 and h_2 belong to H then $\mathrm{NC}(h_1, h_2) \subseteq F(h_1) \cup F(h_2)$.

Lemma 3.3. *If $H \subseteq \mathbb{S}$ is an uncountable subgroup and $F : H \to [\mathbb{N}]^{<\aleph_0}$ attests to the fact that H is strongly almost abelian then there is a perfect set $P \subseteq \mathbb{S}$ and a finite $W \subseteq \mathbb{N}$ such that*:

- *There is some $g^* \in H$ such that for all $n \in \mathbb{N} \setminus W$ and $\pi \in P$ either $\pi(n) = n$ or $\pi(n) = g^*(n)$.*
- *$\mathrm{NC}(\pi, h) \subseteq W \cup F(h) \cup h^{-1}(W)$ for $\pi \in P$ and $h \in H$.*

Moreover, if H is actually abelian and not just strongly almost abelian then W can be assumed to be empty and it can be concluded that each $\pi \in P$ commutes with each $h \in H$.

Proof. Given $X \subseteq \mathbb{N}$ and a finite $W \subseteq \mathbb{N}$ define $\mathrm{cl}^0_W(X) = X$,

$$\mathrm{cl}^1_W(X) = \{z \in \mathbb{N} \setminus W : (\exists h \in H)(\exists x \in X \setminus F(h))\ z = h(x) \text{ and } z \notin F(h^{-1})\} \cup X$$

and let $\mathrm{cl}^{n+1}_W(X) = \mathrm{cl}^1_W(\mathrm{cl}^n_W(X))$; finally, let $\mathrm{cl}_W(X) = \bigcup_{i=1}^{\infty} \mathrm{cl}^i_W(X)$. Observe first that it follows from an argument similar to that in Lemma 3.2 that if $F(g_1) \subseteq W$ and $F(g_2) \subseteq W$ and $g_1 \restriction X = g_2 \restriction X$ then $g_1 \restriction \mathrm{cl}^1_W(X) = g_2 \restriction \mathrm{cl}^1_W(X)$, and hence $g_1 \restriction \mathrm{cl}_W(X) = g_2 \restriction \mathrm{cl}_W(X)$. If for every $W \in [\mathbb{N}]^{<\aleph_0}$ there is some $A_W \in [\mathbb{N}]^{<\aleph_0}$ such that $A_W \cup \mathrm{cl}_W(A_W) = \mathbb{N}$ then it follows that each $g \in H$ is determined by its values on $F(g) \cup A_{F(g)}$. This contradicts the assumption that H is uncountable.

Therefore there must be some $W \in [\mathbb{N}]^{<\aleph_0}$ such that $\mathrm{cl}_W(A) \neq \mathbb{N}$ for every $A \in [\mathbb{N}]^{<\aleph_0}$. Furthermore, it is possible to choose some finite $W' \supseteq W$ such that the set of all $g \in H$ such that $F(g) \subseteq W'$ is uncountable. Observe that $\mathrm{cl}_W(A) \supseteq \mathrm{cl}_{W'}(A)$ for any A, so it is possible to select $\{a_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ such that $\{\mathrm{cl}_{W'}(\{a_i\})\}_{i=1}^{\infty}$ is an infinite family covering $\mathbb{N} \setminus W'$. Observe that if $g \in H$ is such that $F(g) \subseteq W'$ and $g \restriction \mathrm{cl}_{W'}(\{a_i\})$ is the identity for all but finitely many i then g is determined by its values on $W \cup \{a_i : (\exists n \in \mathrm{cl}_{W'}(\{a_i\}))\ g(n) \neq n\}$. Hence, there must be some $\overline{g} \in H$ such that $F(\overline{g}) \subseteq W'$ and $\overline{g} \restriction \mathrm{cl}_{W'}(\{a_i\})$ is not the identity for infinitely many i. Let

$$Z = \left\{ i \in \mathbb{N} : (\exists n \in \mathrm{cl}_{W'}(\{a_i\}))\ \overline{g}(n) \neq n \text{ and } \overline{g}^{-1}(W') \cap \mathrm{cl}_{W'}(\{a_i\}) = \emptyset \right\}.$$

First notice that it follows from the definition of $\mathrm{cl}_{W'}$ and the inclusion $F(\overline{g}) \subseteq W'$ that $\overline{g}(\mathrm{cl}_{W'}(\{a_i\})) \subseteq \mathrm{cl}_{W'}(\{a_i\}) \cup W'$. Hence $\overline{g} \restriction \mathrm{cl}_{W'}(\{a_i\})$ is a permutation of $\mathrm{cl}_{W'}(\{a_i\})$ for each $i \in Z$. Therefore, if for each $t : Z \to 2$

the function g_t is defined by

$$g_t(n) = \begin{cases} \overline{g}(n) & \text{if } n \in \mathrm{cl}_{W'}(\{a_i\}) \text{ and } t(i) = 0, \\ n & \text{otherwise,} \end{cases}$$

then g_t is a permutation of $\mathbb{N}$. It is routine to check that each $g_t(h(n)) = h(g_t(n))$ provided that $n \notin W' \cup F(h) \cup h^{-1}(W')$. ∎

COROLLARY 3.1. *If $H \subseteq \mathbb{S}$ is an uncountable, maximal strongly almost abelian subgroup then $|H| = 2^{\aleph_0}$.*

Proof. The maximality of H implies that it must contain the perfect set of the conclusion of Lemma 3.3. ∎

LEMMA 3.4. *If H is a maximal abelian subgroup of $\mathbb{S}/\mathbb{F}$ and there are*

$$\{\pi_1/\mathbb{F}, \ldots, \pi_n/\mathbb{F}\} \subseteq H$$

such that, letting $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$, the set $\{|a|\}_{a \in \Omega_{\mathcal{S}}}$ is infinite, then $|H| = 2^{\aleph_0}$.

Proof. Let $A_j = \bigcup(\Omega_{\mathcal{S}} \cap [\mathbb{N}]^j)$ and note that $\mathcal{A} = \{A_j\}_{j=2}^{\infty}$ is an infinite family of pairwise disjoint sets. For each non-empty $A_j \in \mathcal{A}$ choose some $j^* \leq n$ such that $\pi_{j^*} \upharpoonright A_j$ is different from the identity. For each $F : \mathcal{A} \to 2$ define

$$\theta_F(k) = \begin{cases} \pi_{j^*}(k) & \text{if } k \in A_j \text{ and } F(A_j) = 1, \\ k & \text{otherwise,} \end{cases}$$

and observe that θ_F is a bijection and that if F and G differ on an infinite set then so do θ_F and θ_G. It suffices to show that $\mathrm{NC}(\theta_F, \pi)$ is finite for each $\pi \in H$ and $F : \mathcal{A} \to 2$.

To see this, let $\pi \in H$ and let j be so large that

$$\Big(\bigcup_{i=1}^{n} \mathrm{NC}(\pi_i, \pi)\Big) \cap \Big(\bigcup_{i=1}^{\infty} A_i\Big) \subseteq \bigcup_{i=1}^{j} A_i.$$

Hence, if $k \geq j$ then $\pi \upharpoonright A_k$ commutes with $\pi_i \upharpoonright A_k$ for $i \leq n$. It suffices to show that $\pi \upharpoonright A_k$ is a permutation of A_k for $k \geq j$, because it will then follow that for $m \in A_k$,

$$\pi \circ \theta_F(m) = \pi \circ \pi_{k^*}(m) = \pi_{k^*} \circ \pi(m) = \theta_F \circ \pi(m)$$

if $F(A_k) = 1$, and

$$\pi \circ \theta_F(m) = \pi(m) = \theta_F \circ \pi(m)$$

if $F(A_k) = 0$. To see that $\pi \upharpoonright A_k$ is a permutation of A_k let $a \in A_k$. Then π is an $\mathcal{S}$-isomorphism from $\Omega_{\mathcal{S}}(a)$ onto $\Omega_{\mathcal{S}}(\pi(a))$. If $\pi(a) \in A_l$ for some $l \neq k$ then $|\Omega_{\mathcal{S}}(a)| = k \neq |\Omega_{\mathcal{S}}(\pi(a))|$, and this contradicts the fact that π is a bijection. ∎

Definition 3.2. If $g \in \mathbb{S}$ then define $I(g) = \bigcup\{a \in \Omega_{\{g\}} : |a| = \aleph_0\}$. For a finite set $\mathcal{S} \subseteq \mathbb{S}$ define

$$I^*(\mathcal{S}) = \bigcup_{\sigma\in\mathcal{S}} \bigcup_{m\in I(\sigma)} \Omega_{\mathcal{S}}(m).$$

Lemma 3.5. *If $H \subseteq \mathbb{S}$ is an uncountable, maximal, almost commuting subgroup of size less than $2^{\aleph_0}$ then $[\mathbb{N}]^{<\aleph_0} \cup \{I^*(\mathcal{S})\}_{\mathcal{S}\in[H]^{<\aleph_0}}$ generates a proper ideal.*

Proof. If not, let $B \subseteq H$ and $C \subseteq \mathbb{N}$ be finite sets such that $I^*(B) \cup C = \mathbb{N}$. Without loss of generality it may be assumed that $\mathrm{NC}(b, b') \subseteq C$ for each b and b' in B. Let $S = \{A \in \Omega_B : A \cap C = \emptyset\}$. Observe that each set in S is infinite. In order to see this, let $A \in S$ and note that

$$A \subseteq I^*(B) = \bigcup_{b\in B} \bigcup_{m\in I(b)} \Omega_B(m)$$

since $A \cap C = \emptyset$. Hence there are b and $m \in I(b)$ such that $A \cap \Omega_B(m) \neq \emptyset$. But since $A \in \Omega_B$ it must also be the case that $m \in A$. Since $b \in B$ it follows that $\mathrm{orb}_b(m) \subseteq A$ and so A is infinite by the definition of $I(b)$.

Moreover, S itself is an infinite set, since otherwise Lemma 3.2 would imply that H is countable. To see this, let Y be a finite set given by Lemma 3.2 such that if θ and π almost commute with each $b \in B$ and $\theta\restriction(X \cup Y) = \pi\restriction(X \cup Y)$ then $\theta\restriction\mathrm{orb}_B(X) = \pi\restriction\mathrm{orb}_B(X)$. Assuming S is finite, choose a finite set X such that $X \cap A \neq \emptyset$ for all $A \in S$. Then $\bigcup_{x\in X} \mathrm{orb}_B(x) \supseteq \bigcup S$ and so any $h \in H$ is determined by its values on $X \cup Y \cup C$.

With these observations in hand, let $\mathbb{S}_S$ be the symmetric group on S and define $\Phi : H \to \mathbb{S}_S$ by $\Phi(h)(s) = t$ if and only if $|h(s) \cap t| = \aleph_0$. First observe that Φ is well defined. To see this, suppose that $s \in S$ and $h \in H$ and there are distinct t and t' in S such that $|h(s)\cap t| = |h(s)\cap t'| = \aleph_0$. Then there exist i and j in $s \setminus \bigcup_{b\in B} \mathrm{NC}(h, b)$ such that $h(i) \in t$ and $h(j) \in t'$. But then, since $\{i, j\} \subseteq s \in \Omega_B$, there is some g in the subgroup generated by B such that $g(i) = j$. Since $i \notin \bigcup_{b\in B} \mathrm{NC}(h, b)$ it follows that $h(j) = h(g(i)) = g(h(i))$. Furthermore, $g(h(i)) \in t$ because $h(i) \in t$ and g belongs to the subgroup generated by B. However, $h(j) \in t'$ and t and t' are disjoint. Therefore $h(g(i)) \neq g(h(i))$, contradicting the choice of i. A similar argument shows that Φ is a homomorphism.

Moreover, its image $\Phi(H)$ is an abelian subgroup of $\mathbb{S}_S$. To see this, let $s \in S$. If $\Phi(g)(\Phi(h)(s)) \neq \Phi(h)(\Phi(g)(s))$ then $g(h(s)) \not\equiv^* h(g(s))$, and hence there are infinitely many $i \in s$ such that $g(h(i)) \neq h(g(i))$, contradicting the fact that h almost commutes with g.

To begin, it will be shown that there cannot be a perfect set $P \subseteq \mathbb{S}_S$ such that:

(1) There is some $g^* \in H$ such that for all $s \in S$ and $\pi \in P$ either $\pi(s) = s$ or $\pi(s) = \Phi(g^*)(s)$.
(2) Every element of P commutes with every element of $\Phi(H)$.

To see this, suppose that P and g^* contradict the assertion. For $\pi \in P$ define $\pi^* \in \mathbb{S}$ by

$$\pi^*(i) = \begin{cases} i & \text{if } i \in s \in S \text{ and } \pi(s) = s, \\ g^*(i) & \text{if } i \in s \in S \text{ and } \pi(s) \neq s. \end{cases}$$

It suffices to show that π^* almost commutes with each $h \in H$. To see that let $i \in \mathbb{N} \setminus \mathrm{NC}(g^*, h)$ and $s \in S$ be such that $i \in s$. If $\pi(s) = s$ then $h(\pi^*(i)) = h(i) = \pi(h(i))$, the last equality holding because $h(i) \in \Phi(h)(s)$ and $\pi(\Phi(h)(s)) = \Phi(h)(\pi(s)) = \Phi(h)(s)$. On the other hand, if $\pi(s) \neq s$ then $h(\pi^*(i)) = h(g^*(i)) = g^*(h(i)) = \pi^*(h(i))$, the last equality holding because $h(i) \in \Phi(h)(s)$ and $\pi(\Phi(h)(s)) = \Phi(h)(\pi(s)) \neq \Phi(h)(s)$.

It will now be shown that $\Phi(H)$ is uncountable. Once this is done, since $\Phi(H)$ has already been shown to be abelian, it will follow from Lemma 3.3 that there exist P and g^* satisfying conditions (1) and (2). So suppose that $\Phi(H)$ is countable. To begin, notice that there must be some $A \in \Omega_{\Phi(H)}$ such that $\{h\restriction\bigcup A\}_{h\in H}$ is uncountable—keep in mind that $A \subseteq \Omega_B$. This is so because if not, then it is easy to find P and g^* satisfying conditions (1) and (2). Simply let $g^* \in H$ be any permutation such that $\Phi(g^*)\restriction A$ is different from the identity on infinitely many sets in $\Omega_{\Phi(H)}$. Then let P be the set of all $g \in \mathbb{S}_S$ such that for all $A \in \Omega_{\Phi(H)}$ either $g\restriction A = \Phi(g^*)\restriction A$ or else $g\restriction A$ is the identity. If no such g^* exists then it follows that H is countable because $\{h\restriction\bigcup A\}_{h\in H}$ is countable for each $A \in \Omega_{\Phi(H)}$ and each h is the identity on all but finitely many $A \in \Omega_{\Phi(H)}$.

Hence, there must be some $A \in \Omega_{\Phi(H)}$ such that $\{h\restriction\bigcup A\}_{h\in H}$ is uncountable and so there is some $h^* \in H$ such that $\{h\restriction\bigcup A : \Phi(h)\restriction A = \Phi(h^*)\restriction A\}$ is uncountable. Observe that if $\Phi(h)\restriction A = \Phi(h')\restriction A$ and there is some $i \in \bigcup A$ such that $h(i) = h'(i)$ then $h\restriction\bigcup A \equiv^* h'\restriction\bigcup A$. To see this, note first that if $\{i, j\} \subseteq s \in A$ then there is some b in the group generated by B such that $b(i) = j$. Since $s \notin C$ it follows that $h(j) = h(b(i)) = b(h(i)) = b(h'(i)) = h'(j)$. If $j \in \bigcup A$ then there are s_i and s_j in A such that $i \in s_i$ and $j \in s_j$ and there is $\overline{h} \in H$ such that $\Phi(\overline{h})(s_i) = s_j$. Since s_i is infinite, there is some $i^* \in s_i \setminus (\mathrm{NC}(\overline{h}, h) \cup \mathrm{NC}(\overline{h}, h'))$ such that $\overline{h}(i^*) \in s_j$. Hence $h(\overline{h}(i^*)) = \overline{h}(h(i^*)) = \overline{h}(h'(i^*)) = h'(\overline{h}(i^*))$ and since $\{\overline{h}(i^*), j\} \subseteq s_j$ it follows that $h(j) = h'(j)$. But since $\{h\restriction\bigcup A : \Phi(h)\restriction A = \Phi(h^*)\restriction A\}$ is uncountable it is possible to find h and h' such that there are i and j in $\bigcup A$ such that $h(i) = h'(i)$, $h(j) \neq h'(j)$ and $\Phi(h)\restriction A = \Phi(h^*)\restriction A$. This is a contradiction. ∎

The following alternative characterization, due to M. Bell, of the cardinal invariant $\mathfrak{p}$ of Definition 1.2 will be used in the proof of Theorem 3.2.

THEOREM 3.1 ([3]). *The cardinal $\mathfrak{p}$ is the least cardinal such that there is a σ-centerd partially ordered set* ([3]) $\mathbb{P}$ *and a collection D of $\mathfrak{p}$ dense subsets of $\mathbb{P}$ for which there is no centered subset $G \subseteq \mathbb{P}$ intersecting each member of D.*

THEOREM 3.2. *If $H \subseteq \mathbb{S}/\mathbb{F}$ is an uncountable, maximal abelian subgroup then $|H| \geq \mathfrak{p}$—in other words, $A([\mathbb{N}]^{<\aleph_0}) \geq \mathfrak{p}$.*

Proof. If $H \subseteq \mathbb{S}/\mathbb{F}$ is an uncountable, maximal abelian subgroup and $|H| < \mathfrak{p}$ then it follows from Lemma 3.5 that $\{I^*(\mathcal{S}) : \{\sigma/\mathbb{F} : \sigma \in \mathcal{S}\} \in [H]^{<\aleph_0}\}$ generates a proper ideal.

Let $\mathbb{P}$ be the partial order consisting of all $p = (h^p, \mathcal{S}^p)$ such that:

(1) h^p is a finite involution ([4]),
(2) $\mathcal{S}^p$ is a finite subset of $\mathbb{S}$ such that $\{\sigma/\mathbb{F} : \sigma \in \mathcal{S}^p\} \subseteq H$,

and define $p \leq q$ if and only if

(1) $h^p \supseteq h^q$,
(2) $\mathcal{S}^p \supseteq \mathcal{S}^q$,
(3) the domain of $h^p \setminus h^q$ is disjoint from $I^*(\mathcal{S}^q)$,
(4) if j is in the domain of $h^p \setminus h^q$ and $\sigma \in \mathcal{S}^q$ then $\sigma(j)$ is in the domain of $h^p \setminus h^q$ and $\sigma(h^p(j)) = h^p(\sigma(j))$.

It is clear that $\mathbb{P}$ is σ-centered because if $h^p = h^q$ then the conditions p and q have the common extension $(h^p, \mathcal{S}^p \cup \mathcal{S}^q)$. Moreover, the sets $D_\pi = \{p \in \mathbb{P} : \pi \in \mathcal{S}^p\}$ are dense for all $\pi \in \mathbb{S}$. Furthermore, so are the sets

$$E_n = \{p \in \mathbb{P} : n \in \text{domain}(h^p) \cup I^*(\mathcal{S}^p)\}.$$

To see this, let $p \in \mathbb{P}$ and suppose that $n \notin I^*(\mathcal{S}^p)$. This, together with Lemma 3.1, implies that $\Omega_{\mathcal{S}^p}(n)$ is finite since all of the infinite orbits under $\mathcal{S}^p$ are contained in $I^*(\mathcal{S}^p)$. Now let h^q be the union of h^p and the identity on $\Omega_{\mathcal{S}^p}(n)$ and let $q = (h^q, \mathcal{S}^p)$. Then $q \in E_n$ and $q \leq p$.

Hence, if $|H| < \mathfrak{p}$ then there is a filter $G \subseteq \mathbb{P}$ meeting each D_π for $\pi \in H$ and E_n for $n \in \mathbb{N}$. Define $\pi_G : \mathbb{N} \to \mathbb{N}$ by

$$\pi_G(j) = \begin{cases} h^p(j) & \text{if } (\exists p \in G)\ j \in \text{domain}(h^p), \\ j & \text{if } (\forall p \in G)\ j \notin \text{domain}(h^p). \end{cases}$$

It is easily verified that $\pi_G \in \mathbb{S}$. To see that $\pi_G/\mathbb{F}$ commutes with each member of H let $\pi/\mathbb{F} \in H$. Let $p \in G$ be such that $\pi \in \mathcal{S}^p$. Then if $j \in \mathbb{N} \setminus \text{domain}(h^p)$ there are two possibilities. If there is some $q \in G$ such that j belongs to the domain of h^q it is clear that $\pi(\pi_G(j)) = \pi(h^q(j)) =$

([3]) A partially ordered set is said to be *σ-centerd* if it is the union of countably many centered subsets—in other words, it is the union of countably many subsets any finite subset of which has a lower bound.

([4]) In other words, h^p is it own inverse.

$h^q(\pi(j)) = \pi_G(\pi(j))$. However, if there is no $q \in G$ such that $j \in \mathrm{domain}(h^q)$ then, by the fact that $E_j \cap G \neq \emptyset$, there is some $q \in G$ such that $j \in I^*(\mathcal{S}^q)$. Since $\pi \in \mathcal{S}^q$ it follows that $\pi(j) \in I^*(\mathcal{S}^q)$. Hence $\pi_G(j) = j$ and $\pi_G(\pi(j)) = \pi(j)$ and so $\pi(\pi_G(j)) = \pi(j) = \pi_G(\pi(j))$.

All that remains to be shown is that the following sets are dense:

$$D_{\pi,k} = \{p \in \mathbb{P} : (\exists j \geq k)\ h^p(j) \neq \pi(k)\}$$

for $\pi \in H$ and $k \in \mathbb{N}$ because then a filter $G \subseteq \mathbb{P}$ can be chosen meeting each relevant $D_{\pi,k}$. To establish this, let $p \in \mathbb{P}$ be given. By Lemma 3.1 each element of $\Omega_{\mathcal{S}^p}$ which is disjoint from $I^*(\mathcal{S}^p)$ is finite. Moreover, by Lemma 3.4 there must be infinitely many of the same cardinality, and hence there must be distinct A and B in $\Omega_{\mathcal{S}^p}$ such that:

(1) both A and B are disjoint from $I^*(\mathcal{S}^p)$,
(2) both A and B are disjoint from the domain of h^p,
(3) $k < \min(A)$,
(4) $k < \min(B)$,
(5) $\Phi : A \to B$ is an $\mathcal{S}^p$-isomorphism.

There are two possibilities. If $\Phi = \pi \restriction A$ then let h^q be the union of h^p and the identity on A and let $q = (h^q, \mathcal{S}^p)$. Otherwise, let $h^q = h^p \cup \Phi \cup \Phi^{-1}$ and let $q = (h^q, \mathcal{S}^p)$. In either case $q \leq p$ and $q \in D_{\pi,k}$. ∎

4. $A([\mathbb{N}]^{<\aleph_0})$ can be smaller than $\mathfrak{a}$. This section will establish that $A([\mathbb{N}]^{<\aleph_0})$ is smaller than $\mathfrak{a}$ in the Laver model. The argument will require some preliminary definitions and observations.

Definition 4.1. If $\mathcal{F} \subseteq [\mathbb{N}]^{<\aleph_0}$ then $\mathcal{F}$ will be said to be *small* provided that:

- If $b \subseteq a \in \mathcal{F}$ then $b \in \mathcal{F}$.
- If $\mathcal{G}$ is an infinite subset of $\mathcal{F}$ then $\mathcal{G}$ contains an infinite Δ-system.

A collection $\mathcal{X}$ of subsets of $\mathbb{N}$ will be said to be *$\mathcal{F}$-splitting* if for every sequence $\{\mathcal{G}_n\}_{n=0}^{\infty}$ each element of which is an infinite set of pairwise disjoint subsets of $\mathcal{F}$ there is $X \in \mathcal{X}$ such that $[X]^{<\aleph_0} \cap \mathcal{G}_n \neq \emptyset$ for each n. Although the notion of splitting will be sufficient for most of the arguments to follow, at one point a more complicated notion will be used. A collection $\mathcal{X}$ of subsets of $\mathbb{N}$ will be said to be *fully-$\mathcal{F}$-splitting* if for every sequence $\{(\mathcal{G}_n, Y_n)\}_{n=0}^{\infty}$ such that each $\mathcal{G}_n$ is an infinite set of pairwise disjoint subsets of $\mathcal{F}$ and $Y_n \subseteq \mathbb{N}$, there is $X \in \mathcal{X}$ such that for each n there is $a \in \mathcal{G}_n$ such that $\Delta_a(X \cap a) = Y_n \cap |a|$; in other words, X is a copy of Y_n on the collapse of some member of $\mathcal{G}_n$.

The notion of splitting which already exists in the literature, for example in [5], corresponds to what is here called fully-$[\mathbb{N}]^1$-splitting. It is worth noting that all the small families considered here will be of bounded cardinality, and if all elements of a small family $\mathcal{F}$ have cardinality less than k then in defining fully-$\mathcal{F}$-splitting one need not consider arbitrary sequences $\{(\mathcal{G}_n, Y_n)\}_{n=0}^{\infty}$ but only those for which $Y_n \subseteq k$. In [5] A. Dow has shown that if $\mathcal{X}$ is splitting and W is obtained by iterated Laver forcing over V then $\mathcal{X}$ is still splitting in W. Let $\mathbb{L}$ denote the Laver partial order and let $\mathbb{L}_\beta$ denote the countable support iteration of $\mathbb{L}$ of length β. A modification of the argument in [5] shows the following.

THEOREM 4.1. *If $\mathcal{F}$ is small and $\mathcal{X}$ is fully-$\mathcal{F}$-splitting in V then*

$$1 \Vdash_{\mathbb{L}_\beta} \text{“}\check{\mathcal{X}} \text{ is fully-}\check{\mathcal{F}}\text{-splitting”}$$

where β is an ordinal.

Proof. The proof is almost the same as the proof of Lemma 9 on pages 245–247 of [5]. One obvious change is that the A_n of that proof are now required to enumerate pairs $(\mathcal{G}_n, Y_n)$ such that $\mathcal{G}_n$ is an $\mathbb{L}$-name for an infinite collection of pairwise disjoint elements of $\mathcal{F}$, and Y_n is an $\mathbb{L}$-name for a subset of $\mathbb{N}$. The only other change required is that Fact 2 on page 246 must be replaced (5) by the following:

NEW FACT 2. *If $S \in \mathbb{L} \cap \mathfrak{M}$ and $n \in \omega$ there is $S' \subseteq S$ with the same root as S such that the collection $\{S'\langle t\rangle \ : \ S'\langle t\rangle \in \mathfrak{M}$ and $S'\langle t\rangle \Vdash$ “$(\exists a \in \mathcal{G}_n)$ $\Delta_a(a \cap X) = Y_n \cap |a|$”$\}$ is pre-dense below S'.*

In order to prove this the following claim is required:

CLAIM. *If T is a well founded tree such that each non-maximal node has infinitely many immediate successors and F is a function from the maximal nodes of T to $\mathcal{F}$ then there is a subtree $T' \subseteq T$ such that*

- *each maximal node of T' is a maximal node of T,*
- *each non-maximal node of T' has infinitely many immediate successors,*
- *there is $F' : T' \to \mathcal{F}$ such that*
  - *if $t \subseteq t'$ are in T' then $F'(t) \subseteq F'(t')$,*
  - *if t is a maximal element of T' then $F'(t) = F(t)$,*
  - *if $t^\frown m \in T'$ and $t^\frown k \in T'$ and $m \neq k$ then $F'(t^\frown m) \cap F'(t^\frown k) = F'(t)$.*

Proof. The claim is easily proved by induction on the rank of T using the fact that $\mathcal{F}$ is small. ∎

In order to prove New Fact 2 let r be the root of S and let $\mathring{l}$ be a name for the value of the Laver real at $|r|$. Let T be the well founded tree whose

(5) The X in [5] seems to be a typographical error and should be changed to S. Also the “$\models$” there is clearly intended to be “$\Vdash$”.

maximal nodes are minimal elements, t, of S such that there is some $S_t \subseteq S$ such that the root of S_t is t and

$$S_t \Vdash_{\mathbb{L}} \text{“}\check{F}_t \in \mathcal{G}_n \text{ and } \min(\check{F}_t) > \mathring{l} \text{ and } Y_n \cap \max(\check{F}_t) = \check{y}_t\text{”}$$

and define $F(t) = F_t$. Use the Claim to find a well founded tree $T' \subseteq T$ as well as a function $F' : T' \to \mathcal{F}$ as in the conclusion of the claim. For any $t \in T'$ which is not maximal define $y_t = y_t \cap |F'(t)|$. Let $T'' \subseteq T'$ be a subtree such that $r \in T''$ and if $s \in T''$ is not maximal in T'' then s has infinitely many immediate successors in t'' and the set of $y_{t^\frown n}$ such that $t^\frown n \in T''$ converges to $z_t \subseteq \mathbb{N}$. Let

$$T^* = \{t \in T' : \Delta_{F'(t)} F'(t) \cap X = z_t \cap |F'(t)|\}.$$

Observe that the requirement in the definition of F_t that $\min(F_t) > \mathring{l}$ guarantees that $F'(r) = \emptyset$ and hence $r \in T^*$. In fact, if $t \in T^*$ and t is not maximal in T then t has infinitely many successors in T^*. To see this, it may as well be assumed that $F'(t^\frown m) \setminus F'(t) \neq \emptyset$ for all m such that $t^\frown m \in T'$. It follows that $\{F'(t^\frown m) \setminus F'(t) : t^\frown m \in T'\}$ is a pairwise disjoint, infinite family in $\mathcal{F}$ and $z'_t = \{n - |F'(t)|\}_{n \in z_t} \subseteq \mathbb{N}$ and both are in $\mathfrak{M}$. Hence there are infinitely many m such that $\Delta_{F'(t^\frown m) \setminus F'(t)} F'(t^\frown m) \setminus F'(t) \cap X = z'_t \cap |F'(t^\frown m) \setminus F'(t)|$, and for any such m it follows that $\Delta_{F'(t^\frown m)} F'(t^\frown m) \cap X = z_t \cap |F'(t)|$. In other words, t has infinitely many successors in T^*.

Let $S' = T^* \cup \bigcup\{S_t : t \in \max(T^*)\}$. It is clear that, provided that $S' \in \mathbb{L}$, $S' \Vdash_{\mathbb{L}}$ “$(\exists a \in \mathcal{G}_n)\ \Delta_a(a \cap X) = Y_n \cap |a|$”. In order to see that $S' \in \mathbb{L}$, let $s \in S'$. If $s \geq t$ for some maximal $t \in T^*$ then $s \in S_t$, and since t is the root of S_t, it follows that s has infinitely many successors. Hence it may be assumed that s is a non-maximal element of T^*. It has already been established that s has infinitely many successors in T^* and hence s has infinitely many successors in S'. ■

The preceding result will not be used in full generality. For most of the argument the following two corollaries will suffice.

Corollary 4.1. *If $G \subseteq \mathbb{L}_\beta$ is generic over V, $B : \mathbb{N} \to \mathbb{N}$ is a function in V and $F \in V[G]$ is an infinite function from $D \subseteq \mathbb{N}$ to $\mathbb{N}$ such that $F(d) < B(d)$ for $d \in D$ then there is a function $H \in V$ such that $H(d) = F(d)$ for infinitely many $d \in D$.*

Proof. Let $\mathcal{F}_B$ be the set of singletons $\{\{(n, m)\} : n \in \mathbb{N} \text{ and } m < B(n)\}$. Observe that $\mathcal{F}_B$ is small and that the set of functions in V bounded by B is $\mathcal{F}_B$-splitting in V. Letting $\mathcal{G}_m = \{\{(d, F(d))\} : d \in D \text{ and } d > m\}$ it follows that there is a function $H \in V$ such that $H \cap \mathcal{G}_m \neq \emptyset$ for each m. Hence $H(d) = F(d)$ for infinitely many $d \in D$. ■

Corollary 4.2. *If $G \subseteq \mathbb{L}_\beta$ is generic over V, $B : \mathbb{N} \to \mathbb{N}$ is a function in V such that $B(n) > n$ for all n and $F \in V[G]$ is an infinite function from*

$D \subseteq \mathbb{N}$ to $\mathbb{N}$ such that $B(d) < F(d)$ for $d \in D$ then there is $X \subseteq \mathbb{N}$ in V such that

- *for each $x \in X$ there is no element of X between x and $B(x)$,*
- *there are infinitely many $x \in X$ such that $F(x) \in X$.*

Proof. Let $\mathcal{F}_B$ be the set of pairs $\{\{n, m\} : n \in \mathbb{N}$ and $m > B(n)\}$. Let $\mathcal{X}_B$ be the set of all infinite subsets $X \subseteq \mathbb{N}$ in V such that for each $x \in X$ there is no element of X between x and $B(x)$. Observe that $\mathcal{F}_B$ is small and that $\mathcal{X}_B$ is $\mathcal{F}_B$-splitting in V. Letting $\mathcal{G}_m$ be an infinite pairwise disjoint subset of $\{\{d, F(d)\} : d \in D$ and $d > m\}$ it follows that there is $X \in \mathcal{X}_B$ such that for each m there is $a \in \mathcal{G}_m$ such that $a \subseteq X$. Hence X satisfies the conclusion. ∎

LEMMA 4.1. *If V satisfies $2^{\aleph_0} = \aleph_1$ then there is an almost commuting family $\mathcal{P}$ of permutations such that for each $G \subseteq \mathbb{L}_{\omega_1}$ which is a generic filter over V and for each permutation h of $\mathbb{N}$ in $V[G] \setminus V$ there is some $\pi \in \mathcal{P}$ which does not almost commute with h.*

Proof. Let $\{(q_\eta, h_\eta)\}_{\eta\in\omega_1}$ enumerate all pairs (q, h) such that

$$q \Vdash_{\mathbb{L}_{\omega_1}} \text{“}h \text{ is a permutation of } \mathbb{N}\text{”}.$$

This enumeration will be used to construct involutions $\{\pi_\eta\}_{\eta\in\omega_1}$ by induction on η. It will be established that for each $\xi < \omega_1$ there is $q \le q_\xi$ such that either $q \Vdash_{\mathbb{L}_{\omega_1}}$ “$h_\xi \in V$” or there is $\beta \le \xi$ such that

$$q \Vdash_{\mathbb{L}_{\omega_1}} \text{“}|\mathrm{NC}(h_\xi, \check{\pi}_\beta)| = \aleph_0\text{”}. \tag{4.1}$$

It is immediate that $\mathcal{P} = \{\pi_\eta\}_{\eta\in\omega_1}$ will have the desired properties.

In order to describe the inductive construction, suppose that $\{\pi_\eta\}_{\eta\in\xi}$ have been constructed. Let $\{\eta_i\}_{i=0}^{\infty}$ enumerate ξ and define $p_i = \pi_{\eta_i}$. Let $\Omega_i(k) = \Omega_{\{p_n\}_{n=0}^{i}}(k)$ and note that because the $\{\pi_\eta\}_{\eta\in\xi}$ are almost commuting involutions it follows from Lemma 3.1 that $|\Omega_i(k)| \le 2^i$ for all but finitely many k. Let $\tau_m(k)$ denote the canonical isomorphism type of $\Omega_m(k)$. To be more precise, let

$$\tau_m(k) = (|\Omega_m(k)|, p_1 \circ \Delta^{-1}_{\Omega_m(k)}, p_2 \circ \Delta^{-1}_{\Omega_m(k)}, \ldots, p_m \circ \Delta^{-1}_{\Omega_m(k)}, \Delta_{\Omega_m(k)}(k), \le)$$

and note the role of the constant determined by k. In particular, if $n \le m$ and $\tau_m(i) = \tau_m(k)$ then $\tau_n(i) = \tau_n(k)$ because these can be defined from the constant using the first m permutations. Each $\tau_m(k)$ will be referred to as an *m-isomorphism type.*

Next, define $\overline{\Gamma} : \mathbb{N} \to \mathbb{N}$ such that for each j the following conditions hold:

(1) If τ is a j-isomorphism type and $\{k \in \mathbb{N} : \tau_j(k) = \tau\}$ is finite then $\tau_j(k) \ne \tau$ for each $k \ge \overline{\Gamma}(j)$.

(2) If τ is a j-isomorphism type and $\{k \in \mathbb{N} : \tau_j(k) = \tau\}$ is infinite then there are $\{l_i\}_{i=0}^2$ such that
   (a) $\tau_j(l_i) = \tau$ for $i < 3$,
   (b) $\Omega_j(l_n) \cap \Omega_j(l_m) = \emptyset$ if $0 \le n < m < 3$,
   (c) $j < \min(\Omega_j(l_i)) \le \max(\Omega_j(l_i)) < \overline{\Gamma}(j)$ for $i < 3$.

(3) If $x < j$ then $\max(\Omega_j(x)) < \overline{\Gamma}(j)$.
(4) If m and n are less than or equal to j then $\max(\mathrm{NC}(p_n, p_m)) < \overline{\Gamma}(j)$.

Finally, define $\Gamma(j) = \overline{\Gamma}(\overline{\Gamma}(\overline{\Gamma}(j)))$.

The induction depends on considering various cases.

Case 1: $q_\xi \not\Vdash_{\mathbb{L}_{\omega_1}}$ "$(\forall \beta < \xi)\ |\mathrm{NC}(h_\xi, \pi_\beta)| < \aleph_0$". In this case let $q \le q_\xi$ and $\beta < \xi$ be such that $q \Vdash_{\mathbb{L}_{\omega_1}}$ "$|\mathrm{NC}(h_\xi, \pi_\beta)| = \aleph_0$" and note that condition (4.1) is satisfied. In this case simply let π_ξ be the identity permutation.

Case 2: Case 1 fails and $q_\xi \not\Vdash_{\mathbb{L}_{\omega_1}}$ "$h_\xi \notin V$". In this case simply let $q \le q_\xi$ and h be a permutation in V such that $q \Vdash_{\mathbb{L}_{\omega_1}}$ "$h_\xi = \check{h}$".

Case 3: Cases 1 and 2 both fail and there is some integer J such that

$$q_\xi \not\Vdash_{\mathbb{L}_{\omega_1}} \text{“}(\forall k > J)\ h_\xi \restriction \Omega_J(k) \text{ is a permutation of } \Omega_J(k)\text{”}.$$

Before continuing with this case, define a partial function F by letting $F(i)$ for each integer $i \ge J$ be the least integer such that there is some j such that

$$i < j < F(i), \tag{4.2}$$

$$\tau_i(j) = \tau_i(F(i)), \tag{4.3}$$

$$\overline{\Gamma}(i) < \min(\Omega_i(j)) \quad \text{and} \quad \overline{\Gamma}(i) < \min(\Omega_i(F(i))), \tag{4.4}$$

$$\Omega_i(j) \ne \Omega_i(F(i)), \tag{4.5}$$

$$h_\xi \circ \Delta^{-1}_{\Omega_i(F(i))} \ne h_\xi \circ \Delta^{-1}_{\Omega_i(j)}. \tag{4.6}$$

The first thing to note is that $q_\xi \Vdash_{\mathbb{L}_{\omega_1}}$ "$(\forall i \in \mathbb{N})\ F(i)$ is defined". In order to see that, suppose that $q \le q_\xi$ and i provide a counterexample; in other words, $q \Vdash_{\mathbb{L}_{\omega_1}}$ "$\check{F}(\check{i})$ is not defined". It is possible to extend q to q' such that for each i-isomorphism type τ there is a permutation h^τ such that

$$q' \Vdash_{\mathbb{L}_{\omega_1}} \text{“}(\forall k > i)\ i < \min(\Omega_i(k)) \text{ and } \tau_i(k) = \tau \Rightarrow h_\xi \restriction \Omega_i(k) = h^\tau \circ \Delta_{\Omega_i(k)}\text{”}.$$

But this means that q' forces that h_ξ is determined by $\{h^\tau : \tau \text{ is an } i\text{-isomorphism type}\}$ and the value of h_ξ on $\Omega_i(k)$ for those finitely many k such that $\overline{\Gamma}(i) \not< \min(\Omega_i(k))$. Hence, $q' \Vdash_{\mathbb{L}_{\omega_1}}$ "$h_\xi \in V$", contradicting the assumption that Case 2 fails.

Subcase 3A: $q_\xi \Vdash_{\mathbb{L}_{\omega_1}}$ "$(\exists^\infty n)\ F(n) > \Gamma(n)$". In this case if we set $D = \{n \in \mathbb{N} : \Gamma(n) < F(n)\}$ then $F \restriction D$ is an infinite function. By Corollary 4.2

it is possible to find $X \in V$ and $q \leq q_\xi$ such that

$$q \Vdash_{\mathbb{L}_{\omega_1}} \text{“}(\exists^\infty x \in X)\ F(x) \in X\text{”} \tag{4.7}$$

and for each $x \in X$ there is no element of X between x and $\Gamma(x)$. For $x \in X$ let x^* be the least element of X greater than x. Choose $L_x < x^*$ such that $\tau_x(L_x) = \tau_x(x^*)$ and $\overline{\Gamma}(x) < \min(\Omega_x(L_x))$. To see that this is possible observe that $x^* > \Gamma(x) > \overline{\Gamma}(\overline{\Gamma}(x))$ and so $\{k : \tau_x(k) = \tau_x(x^*)\}$ must be infinite by (1) in the definition of $\overline{\Gamma}$. Hence L_x can be found between $\overline{\Gamma}(x)$ and $\overline{\Gamma}(\overline{\Gamma}(x))$ using (2) in the definition of $\overline{\Gamma}$. It follows that $\Delta_{\{\Omega_x(L_x),\Omega_x(x^*)\}}$ commutes with p_m if $m \leq x$. Moreover, $\max(\Omega_x(x^*)) < \overline{\Gamma}(x^*)$. Hence, if x and y are distinct members of X then the domain of $\Delta_{\{\Omega_x(L_x),\Omega_x(x^*)\}}$ is disjoint from $\Delta_{\{\Omega_y(L_y),\Omega_y(y^*)\}}$. Hence, if π_ξ is defined by

$$\pi_\xi(n) = \begin{cases} \Delta_{\{\Omega_x(L_x),\Omega_x(x^*)\}}(n) & \text{if } n \in \Omega_x(L_x) \cup \Omega_x(x^*) \text{ for some } x \in X, \\ n & \text{otherwise,} \end{cases}$$

then π_ξ almost commutes with each p_m and hence with each π_η.

It remains to show that $q \Vdash_{\mathbb{L}_{\omega_1}}$ “$|\mathrm{NC}(h_\xi, \pi_\xi)| = \aleph_0$”. To this end let $x \in X$ be such that $F(x)$ also belongs to X. Let z be the greatest element of X below $F(x)$; in other words, $F(x) = z^*$ and so $\tau_z(L_z) = \tau_z(F(x))$. Since $x \leq z$ it follows that $\tau_x(L_z) = \tau_x(F(x))$ also. Furthermore, since $F(x)$ is, by definition, the least integer such that there some j such that conditions (4.2)–(4.6) hold with x in place of i it follows that $\tau_x(j) = \tau_x(F(x)) = \tau_x(L_z)$ and

$$h_\xi \circ \Delta^{-1}_{\Omega_x(L_z)} = h_\xi \circ \Delta^{-1}_{\Omega_x(j)}$$

for any such j. Since

$$h_\xi \circ \Delta^{-1}_{\Omega_x(F(x))} \neq h_\xi \circ \Delta^{-1}_{\Omega_x(j)}$$

it follows that $\Delta_{\{\Omega_x(L_z),\Omega_x(F(x))\}}$ does not commute with $h_\xi \restriction \Omega_x(L_z)$. Because $x \leq z$ this implies that $\Delta_{\{\Omega_z(L_z),\Omega_z(F(x))\}}$ does not commute with $h_\xi \restriction \Omega_z(L_z)$. Since

$$\Delta_{\{\Omega_z(L_z),\Omega_z(F(x))\}} \subseteq \pi_\xi$$

it must be that $q \Vdash_{\mathbb{L}_{\omega_1}}$ “$(\exists n > x)\ h_\xi(\pi_\xi(n)) \neq \pi_\xi(h_\xi(n))$”. From (4.7) it follows that (4.1) holds.

Subcase 3B: $q_\xi \not\Vdash_{\mathbb{L}_{\omega_1}}$ “$(\exists^\infty n)\ F(n) > \Gamma(n)$”. Using Corollary 4.1 let $q \leq q_\xi$ and $H \in V$ be such that

$$q \Vdash_{\mathbb{L}_{\omega_1}} \text{“}(\exists^\infty n)\ H(n) = F(\Gamma^n(0))\text{”}.$$

For each n choose, if possible, an integer L_n such that

- $\Gamma^n(0) < L_n < H(n) < \Gamma^{n+1}(0)$,
- $\overline{\Gamma}(\Gamma^n(0)) < \min(\Omega_{\Gamma^n(0)}(L_n))$,
- $\tau_{\Gamma^n(0)}(L_n) = \tau_{\Gamma^n(0)}(H(n))$,

and then define π_ξ by

$$\pi_\xi(m) = \begin{cases} \Delta_{\{\Omega_{\Gamma^n(0)}(L_n),\Omega_{\Gamma^n(0)}(H(n))\}}(m) \\ \qquad \text{if } m \in \Omega_{\Gamma^n(0)}(L_n) \cup \Omega_{\Gamma^n(0)}(H(n)), \\ m \quad \text{otherwise.} \end{cases}$$

Since $\overline{\Gamma}(\Gamma^n(0)) < \min(\Omega_{\Gamma^n(0)}(L_n))$ and $H(n) < \Gamma^{n+1}(0)$ for each n it follows, using (3) in the definition of $\overline{\Gamma}$, that the domains of

$$\Delta_{\{\Omega_{\Gamma^n(0)}(L_n),\Omega_{\Gamma^n(0)}(H(n))\}} \quad \text{and} \quad \Delta_{\{\Omega_{\Gamma^m(0)}(L_m),\Omega_{\Gamma^m(0)}(H(m))\}}$$

are disjoint if $n \neq m$; in other words, there is no contradiction in the definition of the involution π_ξ. It is immediate from (4.2) and (4.3) in the definition of F that if $H(n) = F(\Gamma^n(0))$ then L_n exists and, from (4.6), that $\Delta_{\{\Omega_{\Gamma^n(0)}(L_n),\Omega_{\Gamma^n(0)}(H(n))\}}$ does not commute with h_ξ. Hence (4.1) holds.

Case 4: Neither of Cases 1, 2 or 3 holds. In this case it may be assumed that

$$q_\xi \Vdash_{\mathbb{L}_{\omega_1}} \text{“}(\forall i)(\forall k)(\exists j > k)\ h_\xi \restriction \Omega_i(k) \text{ is not a permutation of } \Omega_i(k)\text{”}$$

because otherwise Case 3 holds for some extension of q_ξ. Since q_ξ forces h_ξ to be a permutation it follows that

$$q_\xi \Vdash_{\mathbb{L}_{\omega_1}} \text{“}(\forall i)(\forall k)(\exists j > k)\ h_\xi(j) \notin \Omega_i(j)\text{”}.$$

Let $F_0(i)$ be the least integer $j > i$ satisfying

$$h_\xi(j) \notin \Omega_i(j), \tag{4.8}$$

$$(\forall m \le i)(\forall k \ge \min(\Omega_i(j)))\ h_\xi(p_m(k)) = p_m(h_\xi(k)), \tag{4.9}$$

$$(\exists^\infty m)\ \tau_i(j) = \tau_i(m), \tag{4.10}$$

and define $F_1(i) = h_\xi(F_0(i))$.

Subcase 4A: $F_1(\Gamma^n(0)) < \Gamma^{n+1}(0)$ for all but finitely many n. In this case, using Conditions 4.9 and 4.10 as well as (2) in the definition of $\overline{\Gamma}$, it is possible to conclude that for each n there are j^n_0, j^n_1 and j^n_2 such that

- $\tau_{\Gamma^n(0)}(j^n_0) = \tau_{\Gamma^n(0)}(j^n_1)$,
- $h_\xi \restriction \Omega_{\Gamma^n(0)}(j^n_1)$ is a bijection from $\Omega_{\Gamma^n(0)}(j^n_1)$ onto $\Omega_{\Gamma^n(0)}(j^n_2)$,
- $\Omega_{\Gamma^n(0)}(j^n_0)$, $\Omega_{\Gamma^n(0)}(j^n_1)$ and $\Omega_{\Gamma^n(0)}(j^n_2)$ are all distinct,
- $\overline{\Gamma}(\Gamma^n(0)) < \min(\Omega_{\Gamma^n(0)}(j^n_i))$ for each $i < 3$.

Using Corollary 4.1, let $\Theta : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ belonging to V and $q \le q_\xi$ be such that

$$q \Vdash_{\mathbb{L}_{\omega_1}} \text{“}(\exists^\infty n)\ \Theta(n) = (\Theta_0(n), \Theta_1(n), \Theta_2(n)) = (j^n_0, j^n_1, j^n_2)\text{”}. \tag{4.11}$$

Note that, without loss of generality, it may be assumed that

$$\tau_{\Gamma^n(0)}(\Theta_0(n)) = \tau_{\Gamma^n(0)}(\Theta_1(n))$$

for each n. As in the previous cases, the domains of

$$\Delta_{\{\Omega_{\Gamma^n(0)}(\Theta_0(n)),\Omega_{\Gamma^n(0)}(\Theta_1(n))\}} \quad \text{and} \quad \Delta_{\{\Omega_{\Gamma^m(0)}(\Theta_0(m)),\Omega_{\Gamma^m(0)}(\Theta_1(m))\}}$$

are disjoint if $n \neq m$. Moreover, the domain of $\Delta_{\{\Omega_{\Gamma^n(0)}(\Theta_0(n)),\Omega_{\Gamma^n(0)}(\Theta_1(n))\}}$ is disjoint from $\Omega_{\Gamma^m(0)}(\Theta_2(m))$ even in the case $n = m$. Hence, letting π_ξ be defined by

$$\pi_\xi(m) = \begin{cases} \Delta_{\{\Omega_{\Gamma^n(0)}(\Theta_0(n)),\Omega_{\Gamma^n(0)}(\Theta_1(n))\}}(m) & \\ & \text{if } m \in \Omega_{\Gamma^n(0)}(\Theta_0(n)), \Omega_{\Gamma^n(0)}(\Theta_1(n)), \\ m & \text{otherwise,} \end{cases}$$

it follows that π_ξ is an involution which almost commutes with each π_η for $\eta \in \xi$. Furthermore, π_ξ is the identity on each $\Omega_{\Gamma^n(0)}(\Theta_2(n))$.

Moreover, if $\Theta(n) = (j_0^n, j_1^n, j_2^n)$ then π_ξ is the identity on $\Omega_{\Gamma^n(0)}(j_2^n)$. Hence, in this case, $\pi_\xi(h_\xi(h_\xi^{-1}(j_2^n))) = j_2^n$. On the other hand, $h_\xi^{-1}(j_2^n)$ belongs to $\Omega_{\Gamma^n(0)}(j_1^n)$ and so $\pi_\xi(h_\xi^{-1}(j_2^n))$ belongs to $\Omega_{\Gamma^n(0)}(j_0^n)$. In particular, it does not belong to $\Omega_{\Gamma^n(0)}(j_1^n)$. Since $h_\xi \restriction \Omega_{\Gamma^n(0)}(j_1^n)$ is a bijection from $\Omega_{\Gamma^n(0)}(j_1^n)$ onto $\Omega_{\Gamma^n(0)}(j_2^n)$ by (4.9), it follows that $h_\xi(\pi_\xi(h_\xi^{-1}(j_2^n)))$ does not belong to $\Omega_{\Gamma^n(0)}(j_2^n)$ and, in particular, it is not equal to j_2^n. In other words, q forces that $h_\xi(\pi_\xi(h_\xi^{-1}(j_2^n))) \neq \pi_\xi(h_\xi(h_\xi^{-1}(j_2^n)))$ and $\Gamma^n(0) < h_\xi^{-1}(j_2^n) < \Gamma^{n+1}(0)$. From (4.11) it follows that (4.1) holds.

Subcase 4B: There are infinitely many n such that $F_0(\Gamma^n(0)) > \Gamma^{n+1}(0)$. Using Corollary 4.2 find $q \leq q_\xi$ and $X \subseteq \mathbb{N}$ such that $X \in V$ and

$$q \Vdash_{\mathbb{L}_{\omega_1}} \text{“}(\exists^\infty x \in X)\ F_0(x) \in X\text{”}, \tag{4.12}$$

and for each $x \in X$ there is no element of X between x and $\Gamma(x)$. For each $x \in X$ let x^* be the least element of X greater than x. For each such x there is some j_x such that

- $x < j_x < x^*$,
- $\overline{\Gamma}(x) < \min(\Omega_x(j_x))$,
- $\Omega_x(j_x) \cap \Omega_x(x^*) = \emptyset$,
- $\tau_x(j_x) = \tau_x(x^*)$.

Let π_ξ be defined by

$$\pi_\xi(m) = \begin{cases} \Delta_{\{\Omega_x(j_x),\Omega_x(x^*)\}}(m) & \text{if } m \in \Omega_x(j_x) \cup \Omega_x(x^*) \text{ for some } x \in X, \\ m & \text{otherwise.} \end{cases}$$

Observe that if $x \in X$ and $F(x) = z^*$ for some $z \in X$ then $q \Vdash_{\mathbb{L}_{\omega_1}}$ “$h_\xi(j_z) \in \Omega_x(j_z)$” since $F_0(x) = z^*$ is defined to be the least integer k such that $h_\xi(k) \notin \Omega_x(k)$. Hence $q \Vdash_{\mathbb{L}_{\omega_1}}$ “$\pi_\xi(h_\xi(j_z)) \in \Omega_x(z^*)$”. However, $\pi_\xi(j_z) = z^*$ and so $q \Vdash_{\mathbb{L}_{\omega_1}}$ “$h_\xi(\pi_\xi(j_z)) = h_\xi(z^*) \notin \Omega_x(z^*)$”. It follows from (4.12) that (4.1) holds.

SUBCASE 4C: Neither Case 4A nor Case 4B holds. Let $\mathcal{F} = [\mathbb{N}]^2$ and let $\mathcal{X}$ be the set of all infinite subsets of $\mathbb{N}$ in V. Clearly, $\mathcal{F}$ is small and $\mathcal{X}$ is fully-$\mathcal{F}$-splitting. Let $\mathcal{G}_m$ be a maximal pairwise disjoint set of pairs $\{i, j\}$ such that $i > m$ and

$$\Gamma^i(0) < \Gamma^j(0) < F_1(\Gamma^i(0)) \leq \Gamma^{j+1}(0),$$

and let $Y_m = \{0\}$. Because Case 4A fails, each $\mathcal{G}_m$ is an infinite pairwise disjoint subset of $\mathcal{F}$. Using Theorem 4.1 applied to $\{(\mathcal{G}_m, Y_m)\}_{m=0}^{\infty}$ it is possible to find $X \in \mathcal{X}$ such that there are infinitely many $x \in X$ such that $\Gamma^j(0) < F_1(\Gamma^x(0)) \leq \Gamma^{j+1}(0)$ and $j \notin X$.

Now let $F(x) = F_0(\Gamma^x(0))$ for $x \in X$ and note that F is also bounded by a function in V because Case 4B fails. Using Corollary 4.1 it is possible to find $q \leq q_\xi$ and f in V such that

$$q \Vdash_{\mathbb{L}_{\omega_1}} \text{“}(\exists^\infty x \in X)\ f(x) = F(x)\text{”}. \tag{4.13}$$

For each $x \in X$ choose, if possible, j_x such that

- $\Gamma^x(0) < j_x < \Gamma^{x+1}(0)$,
- $\overline{\Gamma}(\Gamma^x(0)) < \min(\Omega_x(j_x))$,
- $\tau_{\Gamma^x(0)}(j_x) = \tau_{\Gamma^x(0)}(f(x))$,
- $\Omega_{\Gamma^x(0)}(j_x) \neq \Omega_{\Gamma^x(0)}(f(x))$,

and observe that if $f(x) = F(x)$ then such a j_x exists. This is so because, by the definition of F_0, there are infinitely many j such that $\tau_{\Gamma^x(0)}(j) = \tau_{\Gamma^x(0)}(F_0(\Gamma^x(0))) = \tau_{\Gamma^x(0)}(f(x))$. As in previous cases, let π_ξ be defined by

$$\pi_\xi(m) = \begin{cases} \Delta_{\{\Omega_{\Gamma^x(0)}(j_x)),\Omega_{\Gamma^x(0)}(f(x))\}}(m) \\ \qquad \text{if } m \in \Omega_{\Gamma^x(0)}(j_x)) \cup \Omega_{\Gamma^x(0)}(f(x)) \text{ for some } x \in X, \\ m \quad \text{otherwise.} \end{cases}$$

Just as in previous cases, it is easy to check that π_ξ is well defined and almost commutes with each π_η for $\eta \in \xi$. Moreover,

$$(\forall j \notin X)(\forall m \notin \Omega_{\Gamma^{j-1}(0)}(f(j-1))) \qquad \Gamma^j(0) < m \leq \Gamma^{j+1}(0) \Rightarrow \pi_\xi(m) = m. \tag{4.14}$$

Now suppose that $x \in X$ and $f(x) = F(x)$ and

$$\Gamma^j(0) < F_1(\Gamma^x(0)) \leq \Gamma^{j+1}(0)$$

and $j \notin X$. Then $\pi_\xi(j_x) = f(x) = F_0(\Gamma^x(0))$ and hence $h_\xi(f(x)) = h_\xi(F_0(\Gamma^x(0))) = F_1(\Gamma^x(0))$ and so $F_1(\Gamma^x(0)) \notin \Omega_{\Gamma^x(0)}(f(x))$. From (4.14) we conclude that $\pi_\xi(F_1(\Gamma^x(0))) = F_1(\Gamma^x(0))$. Next, note that since $j_x \neq f(x)$ it follows that $h_\xi(j_x) \neq h_\xi(f(x)) = F_1(\Gamma^x(0))$ and so $\pi_\xi(h_\xi(j_x)) \neq \pi_\xi(F_1(\Gamma^x(0)) = F_1(\Gamma^x(0)) = h_\xi(\pi_\xi(j_x))$. From (4.13) it follows that (4.1) holds. □

The following result, due to S. Shelah, is 5.31 in [12]. It will suffice to know that Laver forcing $\mathbb{L}$ is NEP without having to define the concept.

LEMMA 4.2. *Let $\{B_\alpha\}_{\alpha\in\omega_1}$ be a family of Borel sets in a model of set theory V and suppose that $V \models \bigcap_{\alpha\in\omega_1} B_\alpha = \emptyset$. Let $\mathbb{P}$ be a NEP partial order with definition in V and suppose that $\{\mathbb{P}_\alpha\}_{\alpha\in\omega_2}$ is a countable support iteration such that $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{P}$ for every $\alpha \in \omega_2$. If*

$$1 \Vdash_{\mathbb{P}_{\omega_1}} \text{“} \bigcap_{\alpha\in\omega_1} B_\alpha = \emptyset \text{”}$$

then

$$1 \Vdash_{\mathbb{P}_{\omega_2}} \text{“} \bigcap_{\alpha\in\omega_1} B_\alpha = \emptyset \text{”}.$$

THEOREM 4.2. *It is consistent that $A([\mathbb{N}]^{<\aleph_0}) = \aleph_1 < \mathfrak{a}$.*

Proof. The model witnessing this is the one obtained by forcing with $\mathbb{L}_{\omega_2}$ over a model V satisfying $2^{\aleph_0} = \aleph_1$. From Lemma 4.1 it follows that there is an almost commuting family $\mathcal{P}$ of permutations such that for each $G \subseteq \mathbb{L}_{\omega_1}$ which is a generic filter over V and for each permutation h of $\mathbb{N}$ in $V[G] \setminus V$ there is some $\pi \in \mathcal{P}$ which does not almost commute with h. Then if $\mathcal{Q}$ is any maximal almost commuting family of permutations in V containing $\mathcal{P}$ it follows that if $G \subseteq \mathbb{L}_{\omega_1}$ is a generic filter over V then for each permutation h of $\mathbb{N}$ in $V[G]$ there is some $\pi \in \mathcal{Q}$ such that either $\pi = h$ or π does not almost commute with h. In other words, letting B_π be the Borel set of all permutations of the integers which almost commute with π but are not equal to π, we have

$$1 \Vdash_{\mathbb{L}_{\omega_1}} \text{“} \bigcap_{\pi\in\mathcal{Q}} B_\pi = \emptyset\text{”}.$$

It follows from Lemma 4.2 that

$$1 \Vdash_{\mathbb{L}_{\omega_2}} \text{“} \bigcap_{\pi\in\mathcal{Q}} B_\pi = \emptyset\text{”}$$

or, in other words, $1 \Vdash_{\mathbb{L}_{\omega_2}}$ “$A([\mathbb{N}]^{<\aleph_0}) = \aleph_1$”. The fact that $\mathfrak{a} = \aleph_2$ in this model is well known and can be found, for example, in [2]. ∎

5. The Cohen model and the summable ideals. The remaining results will deal with quotients of groups of permutations of the integers with respect to ideals other than the ideal of finite sets. Since Section 6 will be devoted to establishing that $\mathfrak{a}(\mathcal{I}) < A(\mathcal{I}) = 2^{\aleph_0}$ for a certain ideal, it is natural to ask whether the phenomenon $A(\mathcal{I}) < 2^{\aleph_0}$ might not be a peculiarity of the finite ideal. This section will show that this is not the case. The ideal $\mathcal{I}_{1/x}$ is defined to be the set of all $X \subseteq \mathbb{N}$ such that $\sum_{x\in X} 1/x < \infty$. It will be shown that $\mathbb{S}(\mathcal{I}_{1/x})/\mathbb{F}(\mathcal{I}_{1/x})$ has a maximal abelian subgroup of size $\aleph_1$ in any model obtained by adding uncountably many Cohen reals. The basic scheme of the argument is that in a model of the form $V[\{c_\xi\}_{\xi\in\omega_1}]$ where $\{c_\xi\}_{\xi\in\omega_1}$ are Cohen reals, it is possible to define permutations $\{\pi_\xi\}_{\xi\in\omega_1}$ with $\pi_\xi \in V[\{c_\eta\}_{\eta\in\xi+1}]$ which almost commute and are close to maximal in the following sense: Given any permutation π which is not first order definable using elements of $\{\pi_\xi\}_{\xi\in\omega_1}$ as parameters, there is some $\xi \in \omega_1$ such that $\mathrm{NC}(\pi,\pi_\xi) \notin \mathcal{I}_{1/x}$. In other words, if G is the group generated by $\{\pi_\xi\}_{\xi\in\omega_1}$, and $G' \supseteq G/\mathbb{F}(\mathcal{I}_{1/x})$ is any maximal abelian group, each element of which is the equivalence class of some permutation which is first order definable using elements of $\{\pi_\xi\}_{\xi\in\omega_1}$ as parameters, then the cardinality of G' is the same as that of G, and moreover, G' is a maximal abelian subgroup of $\mathbb{S}(\mathcal{I}_{1/x})/\mathbb{F}(\mathcal{I}_{1/x})$. The rest of the section will concentrate on the construction of $\{\pi_\xi\}_{\xi\in\omega_1}$. Consequently, many of the results of this section will assume as a hypothesis a family of permutations with certain properties. These can profitably be thought of as the permutations obtained from the Cohen reals at some stage of the transfinite induction.

Definition 5.1. A family of permutations $\mathcal{F}$ will be said to be *tame* if

(1) each $\pi\in\mathcal{F}$ is an involution,
(2) each pair of permutations in $\mathcal{F}$ almost commute modulo the ideal $\mathcal{I}_{1/x}$,
(3) if $a\subseteq\mathcal{F}$ is finite then Ω_a consists of finite sets and
$$F(a)=\bigcup\{x\in\Omega_a : (\exists\pi\in a)(\exists n\in x)\ \pi(n)=n\}$$
belongs to the ideal $\mathcal{I}_{1/x}$, and Ω^*_a will denote $\{x\in\Omega_a : x\not\subseteq F(a)\}$,
(4) for every finite $a\subseteq\mathcal{F}$ there is $\kappa(a)\in[\Omega^*_a]^{<\aleph_0}$ and a family $\{\Phi^a_{x,y}\}_{x,y\in\Omega^*_a\setminus\kappa(a)}$ such that
   (a) each mapping $\Phi^a_{x,y}$ is an a-isomorphism from x to y,
   (b) $\Phi^a_{y,z}\circ\Phi^a_{x,y}=\Phi^a_{x,z}$,
   (c) if $a\subseteq b$ then, except for a finite set of exceptions, if x_0 and x_1 belong to Ω^*_b then for all $y_0\in\Omega^*_a$ such that $y_0\subseteq x_0$ there is some $y_1\in\Omega^*_a$ such that $\Phi^b_{x_0,x_1}\restriction y_0=\Phi^a_{y_0,y_1}$.

The notation $\Phi^a_{\{x,y\}}$ will be used to represent $\Phi^a_{x,y}\cup\Phi^a_{y,x}$.

The following lemma provides a method for enlarging tame families. This will be used in the transfinite induction mentioned in the introduction to this section.

Lemma 5.1. *If $\mathcal{F}$ is a tame family of permutations and this is witnessed by*
$$\{\Phi^a_{x,y} : a\in[\mathcal{F}]^{<\aleph_0} \text{ and } x,y\in\Omega^*_a\setminus\kappa(a)\}$$
and $\{a_n\}_{n=0}^\infty$ is a family of finite subsets of $\mathcal{F}$ such that

- *$a_n \subseteq a_{n+1}$ for each n,*
- *$\bigcup_{n=0}^{\infty} a_n = \mathcal{F}$,*
- *$\{x_n, y_n\} \in [\Omega^*_{a_n} \setminus \kappa(a_n)]^2$ for each n,*
- *$(x_n \cup y_n) \cap (x_m \cup y_m) = \emptyset$ unless $n = m$,*
- *$\mathbb{N} \setminus \bigcup_{n=0}^{\infty}(x_n \cup y_n) \in \mathcal{I}_{1/x}$,*
- *for each finite subset $a \subseteq \mathcal{F}$ and for all but finitely many n, if $x \in \Omega^*_a \setminus \kappa(a)$ and $x \subseteq x_n$ then there is some $x' \in \Omega^*_a \setminus \kappa(a)$ such that $\Phi^{a_n}_{x_n,y_n} \restriction x = \Phi^a_{x,x'}$,*

and $\pi = \bigcup_{n=0}^{\infty} \Phi^{a_n}_{\{x_n,y_n\}}$, then $\mathcal{F} \cup \{\pi\}$ is also tame.

Proof. The fact that π almost commutes with each member of $\mathcal{F}$ is immediate from the fact that each $\Phi^a_{\{x,y\}}$ is an a-isomorphism. Since each $\Phi^a_{\{x,y\}}$ is an involution, so is π. This also implies that $\Omega_{a\cup\{\pi\}}$ consists of finite sets. For any finite $a \subseteq \mathcal{F}$ it is immediate that

$$F(a \cup \{\pi\}) \subseteq F(a) \cup \left(\mathbb{N} \setminus \bigcup_{n=m}^{\infty} x_n \cup y_n\right) \in \mathcal{I}_{1/x}$$

where m is chosen large enough that $a \subseteq a_m$.

The finite sets $\lambda(a) \subseteq \Omega^*_a$ for $a \in [\mathcal{F} \cup \{\pi\}]^{<\aleph_0}$ and the family

$$\left\{\Psi^a_{x,y} : a \in [\mathcal{F} \cup \{\pi\}]^{<\aleph_0} \text{ and } x, y \in \Omega^*_a \setminus \lambda(a)\right\}$$

witnessing that $\mathcal{F} \cup \{\pi\}$ is tame must be defined. If $a \subseteq \mathcal{F}$ define $\lambda(a) = \kappa(a)$ and $\Psi^a_{x,y} = \Phi^a_{x,y}$ for $x, y \in \Omega^*_a \setminus \lambda(a)$. In order to define $\lambda(a \cup \{\pi\})$, start by using the hypothesis of the lemma to find an integer $\overline{\lambda}(a \cup \{\pi\})$ so great that if $n \geq \overline{\lambda}(a \cup \{\pi\})$ and $x \in \Omega^*_a \setminus \kappa(a)$ and $x \subseteq x_n$ then there is some $x' \in \Omega^*_a$ such that $\Phi^{a_n}_{x_n,y_n} \restriction x = \Phi^a_{x,x'}$. Then define

$$\lambda(a \cup \{\pi\}) = \kappa(a) \cup \bigcup_{i<\overline{\lambda}(a\cup\{\pi\})} \{x \in \Omega^*_a : x \subseteq x_i \cup y_i\}$$

and note that it is finite since the x_i are finite. Observe that if $z \in \Omega^*_{a\cup\{\pi\}} \setminus \lambda(a \cup \{\pi\})$ then $z \subseteq x_m \cup y_m$ for some $m \geq \overline{\lambda}(a \cup \{\pi\})$. Denote this integer m by $m(z)$. Consequently, if $z \in \Omega^*_{a\cup\{\pi\}} \setminus \lambda(a \cup \{\pi\})$ then $z = z^x \cup z^y$ where $z^x \subseteq x_{m(z)}$ and $z^y \subseteq y_{m(z)}$ and both z^x and z^y belong to Ω^*_a. Given z and z in $\Omega^*_{a\cup\{\pi\}} \setminus \lambda(a \cup \{\pi\})$ define

$$\Psi^{a\cup\{\pi\}}_{z,w} = \Phi^a_{z^x,w^x} \cup \Phi^a_{z^y,w^y}$$

and note that $\Psi^{a\cup\{\pi\}}_{z,w}$ is an $a \cup \{\pi\}$-isomorphism. It is also routine to verify that condition (4)(c) of Definition 5.1 holds. To see that (4)(c) is satisfied let $a \subseteq b \subseteq \mathcal{F} \cup \{\pi\}$ be finite. Let $a' = a \setminus \{\pi\}$ and $b' = b \setminus \{\pi\}$. It may as well be assumed that $\pi \in b$ since otherwise the hypothesis that $\mathcal{F}$ is tame can be applied directly. Let z_0 and z_1 in $\Omega_b^* \setminus \lambda(b)$ be arbitrary and let $w_0 \in \Omega_a^*$ be such that $w_0 \subseteq z_0$. Then

$$\begin{aligned}\Psi^b_{z_0,z_1}\restriction w_0 &= (\Phi^{b'}_{z_0^x,z_1^x} \cup \Phi^{b'}_{z_0^y,z_1^y})\restriction w_0 = \Phi^{b'}_{z_0^x,z_1^x}\restriction(w_0 \cap z_0^x) \cup \Phi^{b'}_{z_0^y,z_1^y}\restriction(w_0 \cap z_0^y)\\ &= \Phi^{a'}_{w_0\cap z_0^x,w_1^x} \cup \Phi^{a'}_{w_0\cap z_0^y,w_1^y}\end{aligned}$$

for some w_1^x and w_1^y provided that neither $w_0 \cap z_0^x$ nor $w_0 \cap z_0^y$ come from the finite set of exceptions to condition (4)(c) for a' and b'. There are then two cases to consider. If $a' = a$ then, because $w_0 \in \Omega_a$, either $w_0 \cap z_0^x = \emptyset$ or $w_0 \cap z_0^y = \emptyset$. Hence either

$$\Phi^{a'}_{w_0\cap z_0^x,w_1^x} \cup \Phi^{a'}_{w_0\cap z_0^y,w_1^y} = \Phi^a_{w_0,w_1^x}$$

or

$$\Phi^{a'}_{w_0\cap z_0^x,w_1^x} \cup \Phi^{a'}_{w_0\cap z_0^y,w_1^y} = \Phi^a_{w_0,w_1^y},$$

and in either case the result is established. On the other hand, if $a \neq a'$ then $\pi \in a$ and since $w_0 \in \Omega_a$ it follows that $\pi(w_0 \cap z_0^x) = w_0 \cap z_0^y$. Hence $\pi(w_1^x) = w_1^y$ and so, letting $w_1 = w_1^x \cup w_1^y$,

$$\Phi^{a'}_{w_0\cap z_0^x,w_1^x} \cup \Phi^{a'}_{w_0\cap z_0^y,w_1^y} = \Psi^a_{w_0,w_1}$$

as required. ∎

For the rest of this section some simplifying notation will be introduced to refer to closed sets of orbits associated with families of permutations. The elements of Ω_a^* will be enumerated as $\{\Omega_a^i\}_{i=0}^{\infty}$ in such a way that $i < j$ if and only if $\min(\Omega_a^i) < \min(\Omega_a^j)$. If $\mathcal{F}$ is a tame family and a is a finite subset of $\mathcal{F}$ then, in this context, $\kappa(a)$ will be an integer such that condition (4) in Definition 5.1 holds for all Ω_a^i with $i \geq \kappa(a)$; in other words, the finite set of exceptions to condition (4) is contained in $\{\Omega_a^i\}_{i=0}^{\kappa(a)}$. The following definition describes a partial order which can be used to create a permutation satisfying the hypothesis of Lemma 5.1.

Definition 5.2. Given a tame family $\mathcal{F}$ such that this is witnessed by

$$\{\Phi^a_{i,j} : a \in [\mathcal{F}]^{<\aleph_0} \text{ and } i,j \geq \kappa(a)\}$$

and which also satisfies

$$(\forall \pi \in \mathcal{F})\ \lim_{n\to\infty} \frac{\pi(n)}{n} = 1, \tag{5.1}$$

define the partial order $\mathbb{P}(\mathcal{F})$ to consist of all pairs

$$p = (\{(a_m^p, i_m^p, j_m^p)\}_{m=0}^{k^p}, \varepsilon_p) = (\{(a_m, i_m, j_m)\}_{m=0}^{k}, \varepsilon)$$

such that, letting $D(p) = \bigcup_{m=0}^{k} \Omega_{a_m}^{i_m} \cup \Omega_{a_m}^{j_m}$,

(1) $k \in \mathbb{N}$,
(2) each a_m is a finite subset of $\mathcal{F}$,

(3) $a_m \subseteq a_{m+1}$,
(4) i_m and j_m are distinct integers greater than $\kappa(a_m)$,
(5) $(\Omega^{i_m}_{a_m} \cup \Omega^{j_m}_{a_m}) \cap (\Omega^{i_n}_{a_n} \cup \Omega^{j_n}_{a_n}) = \emptyset$ unless $n = m$,
(6) if $a \subseteq a_k$ and
   - $i, j > j_k$,
   - $\Omega^{i'}_a \subseteq \Omega^i_{a_k}$,

   then $\Phi^{a_k}_{i,j} \restriction \Omega^{i'}_a = \Phi^a_{i',j'}$ for some j',
(7) $\varepsilon > 0$,

and if

$$\delta = \sup_{u \in \mathbb{N} \setminus D(p)} \sup_{\pi \in a_k} \left| 1 - \frac{\pi(u)}{u} \right|$$

then

$$\left(1 + 2^\alpha \left(1 - \frac{1}{(1+\delta)^\alpha} + \frac{j_k + 1}{\min(\Omega^k_{a_k})}\right)\right)(1+\delta)^\alpha < 1 + \varepsilon \tag{5.2}$$

where $\alpha = |a_{k^p}|$. Also,

$$\sum_{n < j_k \text{ and } n \notin D(p)} 1/n + \sum_{n > j_k \text{ and } y \in F(a)} 1/n < 1. \tag{5.3}$$

Define $p \leq q$ if $\varepsilon_p \leq \varepsilon_q$ and $(a^p_m, i^p_m, j^p_m) = (a^q_m, i^q_m, j^q_m)$ for $m \leq k^q$ and

$$1 - \varepsilon < \frac{\Phi^{a^p_m}_{i^p_m, j^p_m}(u)}{u} < 1 + \varepsilon \tag{5.4}$$

for $m > k^q$ and $u \in \Omega^{i^p_m}_{a^p_m}$.

The following technical lemma will be useful in applying the partial order $\mathbb{P}(\mathcal{F})$.

Lemma 5.2. *Suppose that $\mathcal{F}$ is a tame family and a is a finite subset of $\mathcal{F}$ and $\alpha = |a|$. Suppose also that $\varepsilon > 0$ and $m \in \mathbb{N}$, and furthermore that*

$$(\forall i \geq m)(\forall x \in \Omega^i_a)(\forall \pi \in a) \left| 1 - \frac{\pi(x)}{x} \right| < \varepsilon, \tag{5.5}$$

$$\sum_{k \in F(a)} 1/k < \log\left(\frac{1 + 2\beta}{1 + \beta}\right). \tag{5.6}$$

Then the following inequalities hold for any $i \geq m$ and any k:

$$\frac{\max(\Omega^i_a)}{\min(\Omega^i_a)} < (1+\varepsilon)^\alpha, \tag{5.7}$$

$$\frac{\min(\Omega^{i+1}_a)}{\min(\Omega^i_a)} < 1 + \beta + \frac{m 2^\alpha}{\min(\Omega^i_a)}, \tag{5.8}$$

$$\frac{\min(\Omega_a^{i+k})}{\min(\Omega_a^i)} < \left(1 + \beta + \frac{m2^\alpha}{\min(\Omega_a^i)}\right)^k, \tag{5.9}$$

$$(1-\varepsilon)^\alpha < \frac{\Phi^a_{i,i+k}(n)}{n} < \left(1 + \beta + \frac{m2^\alpha}{\min(\Omega_a^i)}\right)^k (1+\varepsilon)^\alpha. \tag{5.10}$$

Proof. To prove (5.7) the first thing to note is that if $\{x, y\} \subseteq \Omega_a^i$ then there is $k \leq \alpha$ and a sequence $(\pi_1, \dots, \pi_k) \in a^k$ such that

$$x = \pi_1 \circ \cdots \circ \pi_k(y).$$

Given $x \in \Omega_a^i$ let $k(x)$ be the least integer such that

$$x = \pi_1 \circ \cdots \circ \pi_{k(x)}(\min(\Omega_a^i))$$

and proceed by induction on $k(x)$ to show that

$$\frac{x}{\min(\Omega_a^i)} < (1+\varepsilon)^\alpha$$

for every $x \in \Omega_a^i$. If $k(x) = 0$ then $x = \min(\Omega_a^i)$ and the result is clear. Suppose that the lemma has been established for all x such that $k(x) = n$. Given x such that $k(x) = n+1$ it is possible to find x' such that $k(x') = n$ and $x = \pi(x')$ for some $\pi \in a$. From (5.5) it follows that

$$\frac{\pi(x')}{x'} < 1 + \varepsilon$$

and from the induction hypothesis it follows that

$$\frac{x'}{\min(\Omega_a^i)} < (1+\varepsilon)^n,$$

and hence

$$\frac{x}{\min(\Omega_a^i)} < (1+\varepsilon)^{n+1}$$

as desired.

To see that (5.8) holds begin by observing that if $m \leq i' \leq i$ and $\Omega_a^{i'} \setminus \min(\Omega_a^i) \neq \emptyset$ then, by (5.7),

$$\min(\Omega_a^i) < \max(\Omega_a^{i'}) < \min(\Omega_a^{i'})(1+\varepsilon)^\alpha,$$

and hence

$$\min(\Omega_a^{i'}) > \frac{\min(\Omega_a^i)}{(1+\varepsilon)^\alpha}.$$

Therefore, the cardinality of

$$\bigcup_{m \leq i' \leq i} \Omega_a^{i'} \setminus \min(\Omega_a^i)$$

is no greater than

$$\left(\min(\Omega_a^i) - \frac{\min(\Omega_a^i)}{(1+\varepsilon)^\alpha}\right) 2^\alpha = \min(\Omega_a^i)\beta.$$

Using the fact that
$$\sum_{n=\min(\Omega_a^i)+\min(\Omega_a^i)\beta}^{\min(\Omega_a^i)+\min(\Omega_a^i)2\beta} \frac{1}{n} < \log\left(\frac{1+2\beta}{1+\beta}\right)$$
it follows that $F(a) \cup \bigcup_{m \le i' \le i} \Omega_a^{i'}$ does not cover the interval of integers between $\min(\Omega_a^i)$ and $\min(\Omega_a^i) + \min(\Omega_a^i)2\beta$. Hence
$$\min(\Omega_a^{i+1}) < \min(\Omega_a^i) + \min(\Omega_a^i)2\beta + m2^\alpha$$
as required.

The general statement (5.9) follows by repeated application of (5.8).

To prove (5.10) let $n \in \Omega_a^i$. Combining (5.7) and (5.9) yields
$$\frac{\Phi_{i,i+k}^a(n)}{n} \le \frac{\max(\Omega_a^{i+k})}{\min(\Omega_a^i)} < \left(1 + \beta + \frac{m}{\min(\Omega_a^i)}\right)^k (1+\varepsilon)^\alpha,$$
establishing the last half of the inequality. For the first half, note that $\min(\Omega_a^i) \le \min(\Omega_a^{i+k})$, and hence from (5.7) and (5.9) it follows that
$$\frac{\Phi_{i,i+k}^a(n)}{n} \ge \frac{\min(\Omega_a^{i+k})}{\max(\Omega_a^i)} \ge \frac{\min(\Omega_a^i)}{\max(\Omega_a^i)} > \frac{1}{(1+\varepsilon)^\alpha} > (1-\varepsilon)^\alpha. \ \blacksquare$$

Lemma 5.3. *If $p \in \mathbb{P}(\mathcal{F})$ and $j \ge j_{k^p}^p$ then there is $q \le p$ such that $\Omega_a^j \subseteq D(q)$.*

Proof. It suffices to prove this for the case that $j = 1 + j_{k^p}^p$. Let
$$q = (\{(a_m^q, i_m^q, j_m^q)\}_{m=0}^{k^p+1}, \varepsilon_p)$$
where $(a_m^q, i_m^q, j_m^q) = (a_m^p, i_m^p, j_m^p)$ if $m \le k^p$ and $i_{k^p+1}^q = j$, $j_{k^p+1}^q = j+1$ and $a_{k^p+1}^q = a_{k^p}$. From condition (5.2) of Definition 5.2 and conclusion (5.10) of Lemma 5.2, with $k = 1$ it follows that requirement (5.4) is satisfied and so $q \in \mathbb{P}(\mathcal{F})$ and $q \le p$. $\blacksquare$

Lemma 5.4. *If $p \in \mathbb{P}(\mathcal{F})$ and $\varepsilon > 0$ and $a \subseteq \mathbb{N}$ is finite then there is $q \le p$ such that $\varepsilon_q \le \varepsilon$ and $a_{k^q}^q \supseteq a$.*

Proof. Let $b = a \cup a_{k^p}^p$. First apply Lemma 5.3 to extend p to q' so that the domain of $D(q')$ is sufficiently large that condition (6) of Definition 5.2 holds as well as condition (5.2) with
$$\delta = \sup_{u \in \mathbb{N} \setminus D(q')} \sup_{\pi \in b} \left|1 - \frac{\pi(u)}{u}\right|$$
and ε_p replaced by ε. It can also be arranged that $\kappa(b) < \max(D(q'))$ and that
$$\sum_{n < j_k \text{ and } n \notin D(q')} 1/n + \sum_{n > j_k \text{ and } y \in F(b)} 1/n < 1.$$

Then let l be the least integer j such that $\Omega_b^j \cap D(q') = \emptyset$ and let

$$q = (\{(a_m^q, i_m^q, j_m^q)\}_{m=0}^{k^{q'}+1}, \varepsilon)$$

where $(a_m^q, i_m^q, j_m^q) = (a_m^{q'}, i_m^{q'}, j_m^{q'})$ if $m \leq k^{q'}$ and $i_{k^{q'}+1}^q = l$, $j_{k^{q'}+1}^q = l+1$ and $a_{k^{q'}+1}^q = b$. Note that just as in the proof of Lemma 5.3 it follows that $q \in \mathbb{P}(\mathcal{F})$ and $q \leq q' \leq p$. ∎

COROLLARY 5.1. *If $G \subseteq \mathbb{P}(\mathcal{F})$ is generic define $(a_n, i_n, j_n) = (a_n^p, i_n^p, j_n^p)$ for some* (*or*, *equivalently*, *any*) *$p \in G$. If π_G is defined by*

$$\pi_G = \bigcup_{n=0}^{\infty} \Phi_{\{i_n, j_n\}}^{a_n}$$

then π_G almost commutes with each member of $\mathcal{F}$ and the family $\mathcal{F} \cup \{\pi_G\}$ is tame.

Proof. From Definition 5.2 it follows that π_G is an involution and from Lemma 5.4 that it almost commutes with each member of $\mathcal{F}$. From Lemmas 5.3 and 5.4 it follows that $\bigcup_{n=0}^{\infty} a_n = \mathcal{F}$ and $\mathbb{N} \setminus \bigcup_{n=0}^{\infty}(i_n \cup j_n) \in \mathcal{I}_{1/x}$. Now Lemma 5.1 shows that the family $\mathcal{F} \cup \{\pi_G\}$ is tame. ∎

LEMMA 5.5. *If $p \in \mathbb{P}(\mathcal{F})$ and $\theta \in \mathbb{S}(\mathcal{I}_{1/x})$ but π is not first order definable using finitely many parameters from $\mathcal{F}$ and $k \in \mathbb{N}$ then there is $q \leq p$ such that*

$$q \Vdash_{\mathbb{P}(\mathcal{F})} \text{“}\sum_{i=k}^{\infty}\{1/i : \pi_G(\theta(i)) \neq \theta(\pi_G(i))\} > 1\text{”}$$

where π_G is as defined in Corollary 5.1.

Proof. As a convenience, let $a = a_{k^p}^p$, $\alpha = |a|$. Let t^* be such that hypothesis (5.6) of Lemma 5.2 holds with $m = t^*$ and then choose $t \geq t^*$ to be some integer such that the inequalities

$$1 - \varepsilon^p < \frac{\pi(x)}{x} < 1 + \varepsilon^p$$

hold for any $j > t$ and $x \in \Omega_a^j$. By appealing to Lemma 5.3 it may assumed that $t \leq k^p$. A final application of Lemma 5.3 will allow the assumption that if

$$\delta = \sup_{j \in \mathbb{N} \setminus D(p)} \sup_{\pi \in a} \left| 1 - \frac{\pi(j)}{j} \right|$$

then

$$\left(1 + 2^{\alpha}\left(1 - \frac{1}{(1+\delta)^{\alpha}} + \frac{t+1}{\min(\Omega_a^J)}\right)\right)^6 (1+\delta)^{\alpha} < 1 + \varepsilon_p \tag{5.11}$$

where $J = k^p + 1$.

For use later on, let $\overline{\varepsilon}$ be so small that

$$\frac{2}{1-2^\alpha 6\overline{\varepsilon}} < 1+\varepsilon^p$$

and choose ζ small enough that

$$\frac{2\,(1+2^\alpha((1+\zeta)^\alpha-1))\,(1+\zeta)^\alpha}{1-2^\alpha 6(\overline{\varepsilon}+(1+\zeta)^\alpha-1)} < 1+\varepsilon^p. \tag{5.12}$$

Using Lemma 5.3, is may be assumed ([6]) that

$$1-\zeta < \frac{\pi(x)}{x} < 1+\zeta$$

for all $x \notin D(p)$ and $\pi \in a$. A final application of Lemma 5.3 shows that it may also be assumed that

$$\sum_{n \in F(a)\setminus t} 1/n < \log(2). \tag{5.13}$$

The following fact will play a role later in the proof but is included here to explain the significance of the exponent 6 in inequality (5.11) as well as in the indexing to follow.

Claim 4. *Given any $\pi \in \mathrm{Sym}(6)$ other than the identity there is $\sigma \in \mathrm{Sym}(6)$ without fixed points such that σ is an involution and σ does not commute with π.*

Proof. The proof is elementary. ∎

Define $E_i = \bigcup_{w=0}^{5} \Omega_a^{J+6i+w}$ for each $i \in \mathbb{N}$. Given a fixed point free involution H of some interval of integers $[J, J+2K]$ let

$$p^H = (\{(a_m, i_m, j_m)\}_{m=0}^{k^p+K}, \varepsilon^p)$$

where

$$(a_m, i_m, j_m) = \begin{cases} (a_m^p, i_m^p, j_m^p) & \text{if } m \le k^p, \\ (a, i_m, H(i_m)) & \text{if } m > k^p, \end{cases}$$

and $\{i_m\}_{m=k^p+1}^{k^p+K}$ enumerates a maximal subset of the domain of H which is disjoint from its image under H. Note that it may turn out that $p^H \notin \mathbb{P}(\mathcal{F})$ because it is possible that, for example,

$$\frac{\Phi^a_{\{i_m, H(i_m)\}}(n)}{n} > 1+\varepsilon^p$$

([6]) Actually, it must also be observed that Lemma 5.3 does not require changing the value of ε^p.

for some $n \in \Omega_a^{i_m}$. However,

$$\text{(5.14)}\qquad \text{if } |i_m - H(i_m)| < 6 \text{ then } 1-\varepsilon^p < \frac{\Phi^a_{\{i_m,H(i_m)\}}(n)}{n} < 1+\varepsilon^p$$

by condition (5.11) and conclusion (5.10) of Lemma 5.2.

Observe that if $X \subseteq \mathbb{N}$ then by conclusion (5.9) of Lemma 5.2,

$$\text{(5.15)}\qquad X \in \mathcal{I}_{1/x} \quad \text{if and only if} \quad E(X) \in \mathcal{I}_{1/x}$$

where $E(X) = \bigcup_{X\cap E_j \neq \emptyset} E_j$. This will be used repeatedly in order to restrict the possible structure of θ.

To begin, let $W : \mathbb{N} \to \mathbb{N}$ be defined so that if $x \in \Omega_a^i$ then $\theta(x) \in \Omega_a^{W(x)}$. First note that if there exist x_1 and x_2 in Ω_a^u such that $W(x_1) \neq W(x_2)$ then, because Ω_a is a minimal set closed under the permutations in a, there must be some permutation π in the group generated by a such that $\pi(x_1) = x_2$, and hence $\theta(\pi(x_1)) \neq \pi(\theta(x_1))$. Therefore, by (5.15), it may be assumed that if Z is the set of all z such that there is $y \in \Omega_a(z)$ such that $W(z) \neq W(y)$ then $E(Z) \in \mathcal{I}_{1/x}$. Let $\sum_{z\in Z} 1/z = s^Z$.

Next, let W' be defined for $x \notin E(Z)$ such that if $x \in E_i$ then $\Omega_a^{W(x)} \in E_{W'(x)}$. Let

$$X = \{x \in \mathbb{N} : (\exists i \in \mathbb{N})(\exists y \in E_i)\ x \in E_i \text{ and } W'(y) \neq W'(x)\}$$

and suppose that $E(X) \notin \mathcal{I}_{1/x}$. For each i such that $E_i \cap Z = \emptyset$ and $E_i \cap X \neq \emptyset$ choose $y_i \in E_i$ and $x_i \in E_i$ such that $W'(y_i) \neq W'(x_i)$ and let σ_i be a fixed point free involution of $\{J+6i+n\}_{n=0}^5$ such that if $y_i \in \Omega_a^{w_y}$ and $x_i \in \Omega_a^{w_x}$ then $\sigma_i(w_y) = w_x$. (Note that $w_y \neq w_x$ since $E_i \cap Z = \emptyset$.). Then, using (5.15), it is possible to choose $K \in \mathbb{N}$ such that $\sum_{i=0}^K 1/y_i > 1 + s^Z$. If $E_i \cap Z \neq \emptyset$ and $0 \leq i \leq K$ let σ_i be any fixed point free involution of $\{J+6i+n\}_{n=0}^5$. Let $H = \bigcup_{i=0}^k \sigma_i$ and note that $q = p^H \in \mathbb{P}(\mathcal{F})$ by (5.14) and $q \leq p$. It follows from the choice of K and the definition of X that q satisfies the requirements of the lemma.

Hence, it may be assumed that $E(X) \in \mathcal{I}_{1/x}$ and W' is constant on E_i provided that $E_i \subseteq \mathbb{N} \setminus E(X)$. Let $Y = \{i \in \mathbb{N} : E_i \subseteq \mathbb{N} \setminus E(X)\}$. Let W'' be defined on Y such that if $x \in E_i$ then $\Omega_a^{W'(x)} \subseteq E_{W''(i)}$. Therefore there is a partition $Y = Y_0 \cup Y_1 \cup Y_2 \cup Y_3$ such that $W''(Y_i) \cap Y_i = \emptyset$ for each $i \in 3$ and W'' is the identity on Y_3. Let $j \in 4$ be such that $\bigcup_{i\in Y_j} E_i \notin \mathcal{I}_{1/x}$. Observe that for each $i \in Y_j$ there is a permutation ϱ_i of 6 such that if $z \in \Omega_a^{J+6i+u}$ then $W(z) = J + 6W''(i) + \varrho_i(u)$.

First assume that $j \in 3$. Choose σ_i and β_i to be any involutions of 6 without fixed points such that $\varrho_i(\sigma_i(u)) \neq \beta_i(\varrho_i(u))$ for some $u < 6$. It follows that if $z \in \Omega_a^{J+6i+u}$ then $\Phi^a_{\{J+6i+u,J+6i+\sigma_i(u)\}}(z) \in \Omega_a^{J+6i+\sigma_i(u)}$. Moreover, $\theta(z) \in \Omega_a^{J+6W''(i)+\varrho_i(u)}$ and $\theta(\Phi^a_{\{J+6i+u,J+6i+\sigma_i(u)\}}(z)) \in \Omega_a^{J+6W''(i)+\varrho_i(\sigma_i(u))}$.

However,

$$\Phi^a_{\{J+6W''(i)+\varrho_i(u),J+6W''(i)+\beta_i(\varrho_i(u))\}}(\theta(z)) \in \Omega_a^{J+6W''(i)+\beta_i(\varrho_i(u))} \neq \Omega_a^{J+6W''(i)+\varrho_i(\sigma_i(u))}.$$

Therefore if q forces π_G to contain both

$$\Phi^a_{\{J+6i+u,J+6i+\sigma_i(u)\}} \quad \text{and} \quad \Phi^a_{\{J+6W''(i)+\varrho_i(u),J+6W''(i)+\beta_i(\varrho_i(u))\}}$$

this will guarantee that $\pi_G(\theta(z)) \neq \theta(\pi_G(z))$ for $z \in \Omega_a^{J+6i+u}$.

With this in mind, let K be such that

$$\sum_{i\in K\cap Y_j} \frac{1}{\max(E_i)} > 1,$$

let

$$H = \bigcup_{i\in K\cap Y_j} \sigma_i \cup \beta_{W''(i)}$$

and note that $p^H \le p$ by (5.14). The choice of K guarantees that $q = p^H$ is as required by the lemma.

Hence, assume that $j = 3$. If the set of $i \in Y_3$ such that ϱ_i is not the identity is not in $\mathcal{I}_{1/x}$ then using Claim 4 it is possible to choose, for each $i \in Y_3$, an involution σ_i of 6 without fixed points which does not commute with ϱ_i. Using an argument very similar to the previous case it is possible to find a sufficiently large K so that if

$$H = \bigcup_{i\in K\cap Y_3} \sigma_i$$

then p^H is as required by the lemma. Notice that since there are no β_i in this case, Claim 4 must be used in this argument.

Therefore, by omitting a set in $\mathcal{I}_{1/x}$, it may be assumed that ϱ_i is the identity for all $i \in Y_3$. For any permutation ϱ of Ω_a^J and $z \in 6$ let $Y(\varrho, z)$ be the set of all $i \in Y_3$ such that

$$\Phi_{J+6i+z,J} \circ \theta \circ \Phi_{J,J+6i+z} \restriction \Omega_a^J = \varrho.$$

If for each $z \in 6$ there is only one permutation ϱ_z of Ω_a^J such that $\bigcup_{i\in Y(\varrho_z,z)} \Omega_a^{J+6i+z} \notin \mathcal{I}_{1/x}$ then $\theta/\mathbb{F}(\mathcal{I}_{1/x})$ can be defined from a and $\{\varrho_z\}_{z\in 6}$. So it may be assumed that it is possible to choose $z \in 6$ and a permutation ϱ_z of Ω_a^J such that if $U_0 = Y(\varrho_z, z)$ and $U_1 = \mathbb{N} \setminus U_0$ then $\bigcup_{i\in U_0} \Omega_a^{J+6i+z} \notin \mathcal{I}_{1/x}$ and $\bigcup_{i\in U_1} \Omega_a^{J+6i+z} \notin \mathcal{I}_{1/x}$. For $j \in U_1$ let $\varrho_j = \Phi_{J+6j+z,J} \circ \theta \circ \Phi_{J,J+6+z}$ and note that $\varrho_j \neq \varrho$. The key point to keep in mind is that if $i \in U_0$ and $j \in U_1$ then

$$\begin{aligned}\Phi^a_{J+6j+z,J} &\circ (\theta \circ \Phi^a_{J+6i+z,J+6j+z} \circ \theta^{-1} \circ (\Phi^a_{J+6i+z,J+6j+z})^{-1}) \circ \Phi^a_{J,J+6j+z} \\ &= (\Phi^a_{J+6j+z,J} \circ \theta \circ \Phi^a_{J,J+6j+z}) \circ (\Phi^a_{J+6i+z,J} \circ \theta^{-1} \circ \Phi^a_{J,J+6i+z}) = \varrho \circ \varrho_j^{-1}\end{aligned}$$

and it follows that $\theta \circ \Phi^a_{\{J+6i+z,J+6j+z\}} \circ \theta^{-1} \circ (\Phi^a_{\{J+6i+z,J+6j+z\}})^{-1}$ is different from the identity. Therefore if H is any involution such that $H(J + 6i + z) = J + 6j + z$ for $i \in U_0$ and $j \in U_1$ then then there is $x \in E_i$ such that the inequality $\pi_G(\theta(x)) \neq \theta(\pi(x))$ is forced by p^H. However, notice that, unlike all previous cases, i and j are not equal, and so if $|i - j|$ is too large then requirement (5.4) may fail for p^H. The remainder of the argument is devoted to showing that there are sufficiently many pairs $(i, j) \in U_0 \times U_1$ such that p^H satisfies (5.4) as well as the conclusion of the lemma.

To this end, let $U = \{n \in U_0 : n + 1 \in U_1\}$. For $n \in U$ let n_0 be the greatest integer such that the interval $[n - n_0, n]$ is contained in U_0 and let n_1 be the largest integer such that $[n + 1, n + 1 + n_1] \subseteq U_1$. Let U_0^* be the set of all $n \in U$ such that $n_0 \leq n_1$ and U_1^* be the set of all $n \in U$ such that $n_0 > n_1$. Define $U_i' = \bigcup_{n \in U_i^*} [n - n_0, n + n_1 + 1]$ and observe that $Y_3 = U_0' \cup U_1'$. Hence, either U_0' or U_1' fails to belong to $\mathcal{I}_{1/x}$. In either case the following argument is similar so assume that $U_0' \notin \mathcal{I}_{1/x}$.

Recall the definitions of $\overline{\varepsilon}$ and ζ at the beginning of the proof. It will first be shown that if

$$m > n > m(1 - \overline{\varepsilon})$$

then

$$1 - \varepsilon^p < \frac{\Phi^a_{J+6n+u,J+6m+u}(i)}{i} < 1 + \varepsilon^p \tag{5.16}$$

for any $i \in \Omega_a^{J+6n+u}$. Keep in mind that $\min(E_i) = \min(\Omega_a^{J+6i})$ for any i. Begin by observing, using conclusion (5.7) of Lemma 5.2, that if Ω_a^j intersects the interval $[\min(E_n)(1 + \zeta)^\alpha, \min(E_m)]$ then $J + 6n \leq j \leq J + 6m$. Hence,

$$\mathbb{N} \cap [\min(E_n)(1 + \zeta)^\alpha, \min(E_m)] \subseteq \bigcup_{z=n}^{m} E_z \cup F(a). \tag{5.17}$$

Note that

$$|F(a) \cap [\min(E_n)(1 + \zeta)^\alpha, \min(E_m)]| \leq \min(E_m)/2$$

because of condition (5.13). Therefore,

$$\min(E_m) - \min(E_n)(1 + \zeta)^\alpha \leq 2^\alpha 6(m - n) + \min(E_m)/2. \tag{5.18}$$

It follows that

$$\begin{aligned}
&\min(E_m)/2 - \min(E_n) \\
&\qquad \le 2^\alpha 6(m-n) + (\min(E_n)(1+\zeta)^\alpha - \min(E_n)) \\
&\qquad \le 2^\alpha 6(m-n) + 2^\alpha(J+6n)((1+\zeta)^\alpha - 1) \\
&\qquad \le 2^\alpha(6(m-n) + 6n((1+\zeta)^\alpha - 1)) + 2^\alpha J((1+\zeta)^\alpha - 1) \\
&\qquad \le 2^\alpha(6(m-n) + 6m((1+\zeta)^\alpha - 1)) + 2^\alpha J((1+\zeta)^\alpha - 1) \\
&\qquad \le 2^\alpha 6m(\overline{\varepsilon} + (1+\zeta)^\alpha - 1) + 2^\alpha J((1+\zeta)^\alpha - 1),
\end{aligned}$$

and hence

$$\frac{\min(E_m)}{2\min(E_n)} - 1 \le \frac{m}{\min(E_n)}\left(2^\alpha 6(\overline{\varepsilon} + (1+\zeta)^\alpha - 1)\right) + 2^\alpha J((1+\zeta)^\alpha - 1).$$

Therefore, using the fact that $m \le \min(E_m)$ and $J \le \min(E_n)$,

$$\frac{\min(E_m)}{\min(E_n)} \le \frac{2\,(1 + 2^\alpha((1+\zeta)^\alpha - 1))}{1 - 2^\alpha 6(\overline{\varepsilon} + (1+\zeta)^\alpha - 1)} \le \frac{1+\varepsilon^p}{(1+\zeta)^\alpha}$$

by inequality (5.12). Therefore, using conclusion (5.7) of Lemma 5.2,

$$\frac{\varPhi^a_{J+6n+u,J+6m+u}(i)}{i} \le \frac{\max(E_m)}{\min E_n} \le 1+\varepsilon^p$$

for any $u < 6$ and $i \in E_n$. Similar reasoning shows that both inequalities (5.16) hold. Consequently, defining H so that $H(J+6n+u) = J+6m+u$ will not conflict with condition (5.4) holding for p^H.

The only question which remains is whether it is possible to add enough of these extensions to provide a large witness to π_G not commuting with θ. In case there is some K such that $n_0 \ge n(1-\overline{\varepsilon})$ for all $n \ge K$ it follows that $\sum_{n\in U_0'\setminus K}\sum_{j=n_0}^{n} 1/j = \infty$. Moreover, for each $n \ge K$ and j such that $n_0 \le j \le n$ there is some $x_j \in E_j$ such that

$$\theta(\varPhi^a_{\{J+6j+z,J+6(j+n_0)+z\}}(x_j)) \ne \varPhi^a_{\{J+6j+z,J+6(j+n_0)+z\}}(\theta(x_j)) \tag{5.19}$$

since $j+n_0 < n+n_1$. Hence, by (5.15), it follows that $\sum_{j\in U_0'\setminus K} 1/x_j = \infty$ and so it is possible to choose M so that $\sum_{j\in U_0'\setminus K}^{M} 1/x_j \ge 1$. Defining H so that $H(J+6j+z) = J+6(j+n_0)+z$ for $n \in U_0'$ such that $K \le n \le M$ and $n_0 \le j \le n$ will satisfy the lemma because, in this case, $n(1-\overline{\varepsilon}) \le n_0 \le j \le n$ and so $p^H \in \mathbb{P}(\mathcal{F})$ by (5.16).

In the other case, there is an infinite set $U'' \subseteq U_0'$ such that $n_0 < n(1-\overline{\varepsilon})$ for each $n \in U''$. It follows that if $n \in U''$ and $n(1-\overline{\varepsilon}) \le j \le n$ then there is some $x_j \in E_j$ such that

$$\theta(\varPhi^a_{\{J+6j+z,J+6(j+\lceil n(1-\overline{\varepsilon})\rceil)+z\}}(x_j)) \ne \varPhi^a_{\{J+6j+z,J+6(j+\lceil n(1-\overline{\varepsilon})\rceil)+z\}}(\theta(x_j)).$$

Using Lemma 5.2 it follows that for $n \in U''$,

$$\sum_{i=\lceil n(1-\overline{\varepsilon})\rceil}^{n} 1/x_i \geq \sum_{i=\lceil n(1-\overline{\varepsilon})\rceil}^{n} \frac{1}{\max(E_i)} \geq \sum_{i=\lceil n(1-\overline{\varepsilon})\rceil}^{n} \frac{1}{\min(\Omega_a^{J+6i+5})(1+\zeta)^a}$$
$$\geq \sum_{i=\lceil n(1-\overline{\varepsilon})\rceil}^{n} \frac{1}{2^\alpha (J+6i+5)2(1+\zeta)^a},$$

and elementary calculations using condition (5.13) show that the limit as n increases to infinity of the last term of the inequality is

$$\frac{1}{2^\alpha 12(1+\zeta)^a} \ln\left(\frac{1}{1-\overline{\varepsilon}}\right) = \gamma > 0.$$

Now it suffices to choose a finite subset $T \subseteq U''$ such that

$$|T| > 2/\gamma$$

and

$$\sum_{i=\lceil n(1-\overline{\varepsilon})\rceil}^{n} \frac{1}{2^\alpha (J+6i+5)2(1+\zeta)^a} > \frac{\gamma}{2}$$

for all $n \in T$. Then define $H(J+6i+z) = J+6(i+\lceil n(1-\overline{\varepsilon})\rceil)+z$ for $n \in T$ and $n(1-\overline{\varepsilon}) \leq i \leq n$ and note that setting $q = p^H \leq p$ as before satisfies the requirements of the lemma. ∎

THEOREM 5.1. *It is consistent that $A(\mathcal{I}_{1/x}) = \aleph_1 < 2^{\aleph_0}$.*

Proof. Let V be a model where $2^{\aleph_0} > \aleph_1$ and let V' be obtained from V by adding $\aleph_1$ Cohen reals. To be precise, $V' = \bigcup_{\alpha\in\omega_1} V_\alpha$ where $\{\pi_\beta\}_{\beta\in\alpha} \in V_\alpha$ and $V_{\alpha+1} = V_\alpha[G_\alpha]$ where G_α is Cohen generic over V_α for the partial order $\mathbb{P}(\{\pi_\beta\}_{\beta\in\alpha})$. Moreover, $\pi_\alpha = \pi_{G_\alpha}$. Using Lemmas 5.3 and 5.4 and Lemma 5.1 it follows that $\{\pi_\beta\}_{\beta\in\alpha}$ is a tame family whose elements almost commute for each $\alpha \leq \omega_1$. Let $\Gamma \supseteq \{\pi_\alpha\}_{\alpha\in\omega_1}$ be a maximal almost abelian subgroup of the subgroup of all $\pi \in \mathbb{S}(\mathcal{I}_{1/x})/\mathbb{F}(\mathcal{I}_{1/x})$ which are first order definable from some finite subset of $\{\pi_\alpha\}_{\alpha\in\omega_1}$. To see that Γ is maximal in $\mathbb{S}(\mathcal{I}_{1/x})/\mathbb{F}(\mathcal{I}_{1/x})$ suppose that $\pi \in V[\{\pi_\beta\}_{\beta\in\alpha}]$. If π is first order definable from some finite subset of $\{\pi_\beta\}_{\beta\in\alpha}$ then either $\pi \in \Gamma$ or there is some $\theta \in \Gamma$ such that $\mathrm{NC}(\pi,\theta) \notin \mathcal{I}_{1/x}$. On the other hand, if π is not first order definable from some finite subset of $\{\pi_\beta\}_{\beta\in\alpha}$ then by Lemma 5.5 and genericity it follows that $\mathrm{NC}(\pi,\pi_\alpha) \notin \mathcal{I}_{1/x}$. ∎

6. It is possible that $\mathfrak{a}(\mathcal{I}) < A(\mathcal{I})$. Since it has been shown in Proposition 2.1 that $A([\mathbb{N}]^{<\aleph_0}) \leq \mathfrak{a}$ it is natural to wonder whether there might be a more general result asserting that $A(\mathcal{I})$ is bounded by $\mathfrak{a}(\mathcal{I})$ as defined in Definition 1.2. It will be shown that no such result holds, at least not in the generality indicated.

Fix an increasing sequence $\mathcal{N} = \{n_i\}_{i=0}^{\infty}$ of integers such that

$$\lim_{i\to\infty} \frac{n_{i+1} - n_i}{n_{i+2} - n_{i+1}} = 0$$

and define

$$\mathcal{I}(\mathcal{N}) = \left\{ A \subseteq \mathbb{N} : \lim_{i\to\infty} \frac{|A \cap [n_i, n_{i+1})|}{n_{i+1} - n_i} = 0 \right\}.$$

Theorem 6.1. *$A(\mathcal{I}(\mathcal{N})) = 2^{\aleph_0}$.*

Proof. To begin, the following claim will be established:

Claim 5. *If $g \in \mathbb{S}(\mathcal{I}(\mathcal{N}))$ then there is $B \in \mathcal{I}(\mathcal{N})$ such that if $j \in [n_i, n_{i+1}) \setminus B$ then $g(j) \in [n_i, n_{i+1})$.*

Proof. Let

$$B^+ = \bigcup_{i=0}^{\infty} \{n \in \mathbb{N} : n_i \le n < n_{i+1},\ g(n) \ge n_{i+1}\},$$
$$B^- = \bigcup_{i=0}^{\infty} \{n \in \mathbb{N} : n_i \le n < n_{i+1},\ g(n) < n_i\}.$$

If $B^+ \cup B^- \in \mathcal{I}(\mathcal{N})$ then the claim is proved. To begin, suppose $B^+ \notin \mathcal{I}(\mathcal{N})$. Choose $\varepsilon > 0$ and an infinite $Y \subseteq \mathbb{N}$ such that

$$\frac{|B^+ \cap [n_i, n_{i+1})|}{n_{i+1} - n_i} \ge \varepsilon$$

for each $i \in Y$. By thinning out Y it may also be assumed that if i and j belong to Y and $i < j$ and $m \in B^+ \cap [n_i, n_{i+1})$ then $g(m) < n_j$. It follows that

$$g(B^+) \cap [n_{i+1}, n_j) = g(B^+ \cap [n_i, n_{i+1})) \cap [n_{i+1}, n_j).$$

Therefore, if $i < k < j$ then

$$\frac{|g(B^+) \cap [n_k, n_{k+1})|}{n_{k+1} - n_k} \le \frac{n_{i+1} - n_i}{n_{k+1} - n_k}$$

and so $g(B^+) \in \mathcal{I}(\mathcal{N})$, contradicting the fact that $g \in \mathbb{S}(\mathcal{I}(\mathcal{N}))$. A similar argument applied to g^{-1} deals with B^-. ∎

Now suppose that $G \subseteq \mathbb{S}(\mathcal{I}(\mathcal{N}))$ is a maximal subset whose elements almost commute modulo $\mathcal{I}(\mathcal{N})$ and let $g \in G \setminus \mathbb{F}(\mathcal{I}(\mathcal{N}))$. By Claim 5, there is $B \in \mathcal{I}(\mathcal{N})$ such that if $j \in [n_i, n_{i+1}) \setminus B$ then $j \ne g(j) \in [n_i, n_{i+1})$. Let g' be a permutation such that $g' \restriction \mathbb{N} \setminus B = g \restriction \mathbb{N} \setminus B$ and $g' \restriction [n_i, n_{i+1})$ is a permutation of $[n_i, n_{i+1})$ for each i. (This is possible since $g : [n_i, n_{i+1}) \setminus B \to [n_i, n_{i+1})$

is one-to-one.) Note that g' belongs to the same coset of $\mathbb{F}(\mathcal{I}(\mathcal{N}))$ as g. For any $Z \subseteq \mathbb{N}$ let g_Z be defined by

$$g_Z(j) = \begin{cases} g'(j) & \text{if } j \in [n_i, n_{i+1}) \text{ and } i \in Z, \\ j & \text{if } j \in [n_i, n_{i+1}) \setminus B \text{ and } i \notin Z, \end{cases}$$

and note that $g_Z \in \mathbb{S}(\mathcal{I}(\mathcal{N}))$ for each Z. Moreover, there is some $\varepsilon > 0$ and an infinite $X \subseteq \mathbb{N}$ such that

$$\frac{|\{m \in [n_i, n_{i+1}) : g'(m) \neq m\}|}{n_{i+1} - n_i}$$

is greater than ε for each $i \in X$. It follows that if Z and W are subsets of X and $|Z \triangle W| = \aleph_0$ then g_Z and g_W do not belong to the same coset of $\mathbb{F}(\mathcal{I}(\mathcal{N}))$. Hence it will suffice to show that each g_Z commutes with each member of G.

To this end, let $Z \subseteq \mathbb{N}$ and suppose that $h \in G$ and use Claim 5 to find $C \in \mathcal{I}(\mathcal{N})$ such that $h(j) \in [n_i, n_{i+1})$ for each i and each $j \in [n_i, n_{i+1}) \setminus C$. Since h and g almost commute modulo $\mathcal{I}(\mathcal{N})$ let $D \in \mathcal{I}(\mathcal{N})$ be such that $h(g(j)) = g(h(j))$ for $j \in \mathbb{N} \setminus D$. Then let $E = B \cup h^{-1}(B) \cup C \cup D$ and note that $E \in \mathcal{I}(\mathcal{N})$ since $h \in \mathbb{S}(\mathcal{I}(\mathcal{N}))$. It will be shown that $g_Z(h(j)) = h(g_Z(j))$ for each $j \in \mathbb{N} \setminus E$. To see this, let $j \in [n_i, n_{i+1}) \setminus E$ and suppose first that $i \in Z$. In this case $g_Z(j) = g'(j) = g(j)$ because $j \notin B$. Furthermore, since $j \notin C$, $h(j) \in [n_i, n_{i+1})$ and $h(j) \notin B$, and hence $g(h(j)) = g_Z(h(j))$. Since $j \notin D$ it follows that $h(g(j)) = g(h(j))$, and hence in this case $h(g_Z(j)) = g_Z(h(j))$. If $i \notin Z$ then $g_Z(j) = j$ because $j \notin B$, and since $j \notin C$, $h(j) \in [n_i, n_{i+1})$. Therefore, since $h(j) \notin B$, $g_Z(h(j)) = h(j) = h(g_Z(j))$. ∎

THEOREM 6.2. $\mathfrak{a}(\mathcal{I}(\mathcal{N})) \leq \mathfrak{a}$.

Proof. Let $\mathcal{A}$ be a maximal almost disjoint family of size $\mathfrak{a}$. For $A \in \mathcal{A}$ define $A^* = \bigcup_{i \in A} [n_i, n_{i+1})$ and let $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$. Then $\mathcal{A}^*$ is maximal in $\mathcal{P}(\mathbb{N})/\mathcal{I}(\mathcal{N})$. ∎

It follows that $\mathfrak{a}(\mathcal{I}(\mathcal{N})) < A(\mathcal{I}(\mathcal{N}))$ in any model of set theory where $\mathfrak{a} \neq 2^{\aleph_0}$.

7. Questions. It is well known that maximal almost disjoint families of subsets of $\mathbb{N}$ cannot have any nice definition; indeed, Mathias has shown that they cannot be analytic [7]. The results of §2 which establish some similarity between $A([\mathbb{N}]^{<\aleph_0})$ and $\mathfrak{a}$ raise the following question.

QUESTION 7.1. Is there an analytic, maximal, almost commuting subgroup of $\mathbb{S}$?

The lower bound of §3 is probably not optimal.

QUESTION 7.2. Can the lower bound $A(\mathbb{S}/\mathbb{F}) \geq \mathfrak{p}$ of Theorem 3.2 be improved? Can the tower invariant $\mathfrak{t}$ serve as a lower bound?

For any function $h : \mathbb{N} \to \mathbb{R}$ one can define the *summable* ideal $\mathcal{I}_h$ to be the set of all $X \subseteq \mathbb{N}$ such that $\sum_{x\in X} h(x) < \infty$. Observe that it is possible to modify the proof of Theorem 6.1 in order to replace the ideal $\mathcal{I}(\mathcal{N})$ by a summable ideal. In particular, let $\{n_i\}_{i=0}^{\infty}$ be an increasing sequence of integers defined by $n_{i+1} - n_i = n_i^3$ and let h be defined by $h(j) = n_i^{-3}$ if $n_i \leq j < n_{i+1}$. If $g \in \mathbb{S}(\mathcal{I}_h)$ and the sets B^+ and B^- are defined as in the proof of Theorem 6.1 then it is easy to see that $\sum_{j\in B^+\cap n_i} h(g(j)) \leq |B^+ \cap n_i| n_i^{-3} \leq n_i^{-2}$ and hence Claim 5 still holds, as does the remainder of the argument of Theorem 6.1. Hence $A(\mathcal{I}_h) = 2^{\aleph_0}$. This motivates the following question.

QUESTION 7.3. For which functions h is it possible to improve Theorem 5.1 to show that $A(\mathcal{I}_h) = \aleph_1 < 2^{\aleph_0}$ in the model obtained by adding $\aleph_1$ Cohen reals?

QUESTION 7.4. Are there functions h and g such that it is consistent that $A(\mathcal{I}_h) < A(\mathcal{I}_g) < 2^{\aleph_0}$?

QUESTION 7.5. Is it possible to characterize the summable ideals $\mathcal{I}_h$ such that $A(\mathcal{I}_h) = 2^{\aleph_0}$? Can the same be done for the F_σ ideals? What can be said of the Borel or analytic ideals?

At some points in the proof of §5 the permutations constructed can be taken to be almost commuting rather than just almost commuting modulo $\mathcal{I}_{1/x}$. The following question asks whether the argument can be strengthened throughout.

QUESTION 7.6. Can Theorem 5.1 be improved to show that it is consistent with set theory that $2^{\aleph_0} > \aleph_1$ yet there is an almost commuting subgroup of $\mathbb{S}$ of cardinality $\aleph_1$ which is maximal with respect to commuting modulo $\mathcal{I}_{1/x}$? Does this hold in the Cohen model of Theorem 5.1?

The methods of §4 and §5 require that the subgroups constructed contain many involutions. While the methods can be modified to produce groups with no elements of order k for a fixed k, the following questions seem more subtle.

QUESTION 7.7. Can Theorem 4.2 be modified to assert that it is consistent that there is a maximal, abelian, torsion free subgroup of $\mathbb{S}/\mathbb{F}$ of size $\aleph_1$ and $\aleph_1 < \mathfrak{a}$?

QUESTION 7.8. Can Theorem 5.1 be modified to assert that it is consistent that there is a maximal, abelian, torsion free subgroup of $\mathbb{S}(\mathcal{I}_{1/x})/\mathbb{F}(\mathcal{I}_{1/x})$ of size $\aleph_1$ and $\aleph_1 < 2^{\aleph_0}$?

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Department of Mathematics
Rutgers University
Hill Center, Piscataway
NJ 08854-8019, U.S.A.
E-mail: shelah@math.rutgers.edu

Current address:
Institute of Mathematics
Hebrew University
Givat Ram, Jerusalem 91904, Israel

Department of Mathematics
York University
4700 Keele Street
North York, Ontario, Canada M3J 1P3

Current address:
Fields Institute
222 College Street
Toronto, Canada M5T 3J1
E-mail: steprans@fields.utoronto.ca

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