

Injective comodules and Landweber exact homology theories

by

Mark Hovey (Middletown, CT)

Abstract. We classify the indecomposable injective $E(n)_*E(n)$ -comodules, where $E(n)$ is the Johnson–Wilson homology theory. They are suspensions of the $J_{n,r} = E(n)_*(M_r E(r))$, where $0 \leq r \leq n$, with the endomorphism ring of $J_{n,r}$ being $\widehat{E(r)}^* \widehat{E(r)}$, where $\widehat{E(r)}$ denotes the completion of $E(r)$.

Introduction. Perhaps the most important homology theories in algebraic topology are complex bordism MU and the many theories derived from it. In particular, if p is a prime, and we localize at p as we do throughout this paper, then MU splits as a coproduct of suspended copies of a spectrum known as BP , with $BP_*(*) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where v_i has degree $2(p^i - 1)$. Derived from BP are the many Landweber exact homology theories [Lan76], such as the Johnson–Wilson theory $E(n)$, with $E(n)_*(*) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]$ and Morava E -theory E_n (see [DH95]).

As is well-known [Rav86, Theorem 2.2.8], for all of these homology theories h , the homology groups h_*X form a graded comodule over the graded Hopf algebroid (h_*, h_*h) . It therefore behooves us to learn as much as possible about the category of h_*h -comodules. In particular, products are not exact in the category of h_*h -comodules, and their derived functors form the E_2 -term of a spectral sequence [Hov07] converging to the h_* -homology of a product of spectra. This spectral sequence offers one possible approach to the chromatic splitting conjecture of Hopkins [Hov95].

The author has thus been engaged in attempting to understand as much of the structure of the category of h_*h -comodules as possible, following in the footsteps of Peter Landweber, whose papers [Lan76], [Lan73], and the less well-known but excellent [Lan79] are the basis for our understanding of BP_*BP -comodules. In particular, the author and Strickland, in [HS05a]

2000 *Mathematics Subject Classification*: 55N22, 55S25, 13C11, 16W30.

Key words and phrases: injective comodule, Landweber exact homology theory.

and [HS05b], have shown that analogues of Landweber’s theorems hold in the category of E_*E -comodules, where E denotes, as it will throughout the paper, a Landweber exact homology theory of height n such as $E(n)$ or E_n . Recall that this means that there is a ring homomorphism $BP_* \rightarrow E_*$ so that the sequence (p, v_1, \dots, v_n) is regular in E_* and v_n is a unit in $E_*/(p, v_1, \dots, v_{n-1})$.

The present paper is devoted to the study of injective BP_*BP -comodules and E_*E -comodules. Injective comodules have not been studied before, except briefly in [HS05b], because one can almost always use relatively injective comodules to compute the derived functors of interest in topology. This includes the Ext groups $\text{Ext}_{E_*E}(E_*, E_*X)$ and even the derived functors of product.

However, we will show in this paper that the injective comodules have a very rigid structure, in analogy to the Matlis theory of injective modules over a Noetherian commutative ring (described in [Lam99, Section 3I]). There are only $n + 1$ different isomorphism classes of indecomposable injective E_*E -comodules, where E is, as always, a Landweber exact homology theory of height n . They are the injective hulls J_r of E_*/I_r , where $0 \leq r \leq n$ and $I_r = (p, v_1, \dots, v_{r-1})$. Furthermore, if E_* is evenly graded, $J_r \cong E_*(M_r E(r))$ up to suspension, where M_r is the fiber of the map of Bousfield localizations $L_r \rightarrow L_{r-1}$. Also, the endomorphism ring of J_r as an E_*E -comodule is $\widehat{E(r)}^* \widehat{E(r)}$, where $\widehat{E(r)} = L_{K(r)} E(r)$ is the completion of $E(r)$ at I_r .

Unfortunately, a good algebraic description of $\widehat{E(r)}^* \widehat{E(r)}$ is not known. We know that $E_r^* E_r$ is the twisted completed group ring $E_{r*}[[S_r]]$ on the large Morava stabilizer group (this is an old result of Hopkins and Ravenel; see [Bak89] or [Hov04b] for proofs of it). It therefore seems likely that $\widehat{E(r)}^* \widehat{E(r)}$ is very closely related to the stabilizer group S_r . If we accept this, then we are seeing all of the stabilizer groups S_r for $0 \leq r \leq n$ in the category of E_*E -comodules. Since the chromatic splitting conjecture is in some sense about how the apparently unrelated stabilizer groups actually are related to each other, seeing all of the stabilizer groups together like this is a good sign and might be useful. We can also find a decomposable injective comodule whose endomorphism ring is $E_r^* E_r$.

As a corollary of our work, we rediscover the folklore isomorphisms

$$\begin{aligned} \widehat{E(n)}^*(X) &\cong \text{Hom}_{E(n)_*}(E(n)_*(M_n X), \Sigma^{-n} E(n)_*/I_n^\infty), \\ E_n^* X &\cong \text{Hom}_{E_n^*}(E_n^*(M_n X), \Sigma^{-n} E_n^*/I_n^\infty). \end{aligned}$$

As far as the author knows, these isomorphisms have not been written down before, though they were certainly known to Hopkins, Greenlees, Sadofsky, and others.

Throughout this paper, p will be a fixed prime integer, all spectra will be localized at p , and $n > 0$ will be a fixed positive integer. The symbol E will denote a Noetherian Landweber exact homology theory of height n , like $E(n)$ or E_n . The symbol (A, Γ) will denote a flat Hopf algebroid.

The author thanks Andy Baker for sharing his insights into $\widehat{E(r)}^* \widehat{E(r)}$.

1. BP_*BP -comodules and E_*E -comodules. The purpose of this section is to remind the reader of some of the results on the structure of BP_*BP -comodules, E_*E -comodules, and the relation between them.

Recall that the Hopf algebroid (A, Γ) is called an *Adams Hopf algebroid* when Γ is the colimit of a filtered system of comodules Γ_i , where each Γ_i is finitely generated and projective over A . As explained in [Hov04a, Section 1.4] (though originally due to Hopkins), both (BP_*, BP_*BP) and (E_*, E_*E) are Adams Hopf algebroids. When (A, Γ) is an Adams Hopf algebroid, the category of Γ -comodules is a Grothendieck category [Ste75, Chapter V]. This means that it is an abelian category in which filtered colimits are exact, and that there is a family of generators $\{P_i\}$, in the sense that $\Gamma\text{-comod}(\bigoplus P_i, -)$ is a faithful functor. The most natural collection of generators consists of the dualizable comodules; these are the comodules that are finitely generated and projective over A . These are studied and proved to be generators in Sections 1.3 and 1.4 of [Hov04a]. Note that BP_* and its suspensions do not generate the category of BP_*BP -comodules, and we know of no naturally defined proper subcollection of the dualizable comodules which does so.

As is well-known, the category of Γ -comodules is a closed symmetric monoidal category [Hov04a, Section 1.3]. The tensor product is denoted $M \wedge N$; it is isomorphic as an A -module to the tensor product $M \otimes_A N$ of left A -modules (a different notation is used because the usual tensor product symbol is reserved for the tensor product of A -bimodules, which occurs frequently in the theory). The internal Hom object is denoted $F(M, N)$; there is a natural map $F(M, N) \rightarrow \text{Hom}_A(M, N)$ of A -modules that is an isomorphism when M is finitely presented over A .

There are enough injectives in any Grothendieck category. One can see this directly for Γ -comodules by noting that $\Gamma \otimes_A J$ is an injective Γ -comodule whenever J is an injective A -module. Note that this is a bimodule tensor product, and all the Γ -coaction is concentrated on the Γ tensor factor; however, it is well-known that $\Gamma \otimes_A J$ is isomorphic as a comodule to $\Gamma \wedge J$ (see [Hov04a, Lemma 1.1.5]). Moreover, if A is Noetherian, as it is for $(A, \Gamma) = (E_*, E_*E)$, the category of Γ -comodules is locally Noetherian as well. This means that subcomodules of the generators satisfy the ascending chain condition; this is obvious because dualizable comodules are

finitely generated over A . In any locally Noetherian Grothendieck category, direct sums of injectives are injective and every injective is a direct sum of indecomposable injectives in an essentially unique way (see [Ste75, Sections V.4, V.5]).

There is a functor Φ_* from BP_*BP -comodules to E_*E -comodules defined by $\Phi_*M = E_* \otimes_{BP_*} M$. This functor is much studied in [HS05a] and [HS05b]. The functor Φ_* is exact and has a right adjoint Φ^* . The composite $\Phi_*\Phi^*$ is naturally equivalent to the identity, and the composite $\Phi^*\Phi_*$ is the localization functor L_n with respect to the hereditary torsion theory of v_n -torsion comodules. In particular, Φ^* defines an equivalence of categories between E_*E -comodules and L_n -local BP_*BP -comodules. This result is valid even if E_* is not Noetherian, so the results in this paper about injective E_*E -comodules also apply to the case when E_* is not Noetherian. It is proved in [HS05b, Section 2] that Φ_* and Φ^* preserve injectives, filtered colimits, and arbitrary direct sums. In fact, if T_nM denotes the subcomodule of v_n -torsion elements in M , then T_n also preserves injectives, and $L_nJ = J/T_nJ$ for injective BP_*BP -comodules J .

2. Indecomposable injectives. We have remarked above that every injective E_*E -comodule is a direct sum of indecomposable injectives. Matlis has a well-known theory of indecomposable injectives over Noetherian rings (see, for example, [Lam99, Section 3I]), and we will mimic his theory for indecomposable injective E_*E -comodules and BP_*BP -comodules. The object of this section is to classify all the indecomposable injective E_*E -comodules.

We first point out that indecomposable injectives are relevant even for BP_*BP -comodules.

PROPOSITION 2.1. *Suppose J is an injective BP_*BP -comodule for which there exists an n such that J has no v_n -torsion. Then J is a direct sum of indecomposable injectives in an essentially unique way.*

Before proving this proposition, we point out the following lemma.

LEMMA 2.2. *If N is an E_*E -comodule, then N is indecomposable if and only if Φ^*N is indecomposable.*

Proof. In general, an object M in an abelian category \mathcal{C} is indecomposable if and only if the ring $\mathcal{C}(M, M)$ has no nontrivial idempotents. Since Φ^* is fully faithful, the result follows. ■

Proof of Proposition 2.1. Since J has no v_n -torsion and J is injective, J is L_n -local. Hence $J = \Phi^*\Phi_*J$. Now Φ_*J is an injective object in the category of E_*E -comodules, and hence $\Phi_*J \cong \bigoplus J_\alpha$, where each J_α is an indecomposable injective E_*E -comodule. Since Φ^* preserves direct sums,

we get

$$J \cong \bigoplus \Phi^* J_\alpha.$$

Since Φ^* preserves injectives, and also indecomposables by Lemma 2.2, we see that J is a direct sum of indecomposable injectives. The uniqueness of this direct sum decomposition follows from the Krull–Remak–Schmidt–Azumaya theorem as in Section V.5 of [Ste75], using the fact that the endomorphism ring of an indecomposable injective is always local [Ste75, Proposition V.5.1]. ■

The next thing to do is to enumerate all the indecomposable injectives. Recall that the indecomposable injective modules over a commutative Noetherian ring R are the injective hulls of the R/\mathfrak{p} , where \mathfrak{p} is a prime ideal in R . The ideals $I_r = (p, v_1, \dots, v_{r-1})$ are the prime invariant ideals in $E(n)_*$ for $0 \leq r \leq n$ and in BP_* for $0 \leq r \leq \infty$ (see [HS05a, Theorem 5.6] for the $E(n)_*$ case). For an arbitrary Landweber exact theory of height n , it is possible that the I_r are not actually prime, but they remain the “categorically prime” invariant ideals in E_* , as explained in [HS05a, Theorem 5.6]. Hence we let $J_{n,r}$ for $0 \leq r \leq n$ denote the injective hull of E_*/I_r in the category of E_*E -comodules, and let J_r for $0 \leq r \leq \infty$ denote the injective hull of BP_*/I_r in the category of BP_*BP -comodules.

LEMMA 2.3. *The injective comodules $J_{n,r}$ for $0 \leq r \leq n$ and J_r for $0 \leq r \leq \infty$ are indecomposable.*

Proof. According to [Ste75, Proposition V.2.8], it suffices to show that E_*/I_r and BP_*/I_r are *coirreducible*, which means that any two nontrivial subcomodules M, N have nontrivial intersection. This is obvious for $BP_*/I_\infty = \mathbb{F}_p$. For the other cases, we use the fact that every nontrivial BP_*BP -comodule or E_*E -comodule has a nonzero primitive [HS05a, Theorem 5.1]. For example, assume $0 < r < n$. Then the primitives in E_*/I_r are isomorphic to $\mathbb{F}_p[v_r]$ by [HS05a, Theorem 5.2]. Hence M contains all sufficiently high powers of v_r , as does N , and so $M \cap N \neq 0$. The other cases are similar, using the computations of the primitives in BP_*/I_r [Rav86, Theorem 4.3.2] and in E_*/I_r [HS05a, Theorem 5.2]. ■

We also note that J_r has no v_r -torsion in view of [HS05b, Theorem 2.7]. (We will see below that $J_{n,r}$ also has no v_r -torsion.)

THEOREM 2.4. *An E_*E -comodule J is an indecomposable injective comodule if and only if $J \cong \Sigma^t J_{n,r}$ for some $0 \leq r \leq n$ and some t . Similarly, a BP_*BP -comodule with no v_n -torsion is an indecomposable injective comodule if and only if $J \cong \Sigma^t J_r$ for some $0 \leq r \leq \infty$ and some t .*

There are probably many different indecomposable injective BP_*BP -comodules that are v_n -torsion for all n .

Proof. Suppose J is an indecomposable injective E_*E -comodule. Then J has a nonzero subcomodule that is finitely generated over E_* (see [Hov04a, Proposition 1.4.1]). By the filtration theorem [HS05a, Theorem 5.7], any nontrivial finitely generated E_*E -comodule contains an isomorphic copy of $\Sigma^t E_*/I_r$ for some t and some r with $0 \leq r \leq n$. Since J is injective, we see that $\Sigma^t J_r \subseteq J$; since J is also indecomposable, $\Sigma^t J_r \cong J$.

Similarly, suppose J is an indecomposable injective BP_*BP -comodule with no v_n -torsion. It is once again true that J contains a nonzero subcomodule that is finitely generated over BP_* . The new ingredient is [Lan79, Corollary 7], which implies that, since J has no v_n -torsion, its finitely generated subcomodules are in fact finitely presented over BP_* . Therefore we can use the usual Landweber filtration theorem [Lan76] to find a subcomodule of the form $\Sigma^t BP_*/I_r$ for $0 \leq r \leq n$. As before, this implies that $J \cong \Sigma^t J_r$. ■

Note that the obvious inclusions

$$BP_*/I_n \rightarrow \Sigma^t BP_*/(p^\infty, \dots, v_{n-1}^\infty) \rightarrow v_n^{-1} \Sigma^t BP_*/(p^\infty, \dots, v_{n-1}^\infty)$$

are essential extensions, so these important comodules (defined more precisely after Theorem 3.1) have the same injective hull J_n . Here $t = |v_1| + \dots + |v_{n-1}|$, and the inclusion sends $1 \in BP_*/I_n$ to $1/pv_1 \cdots v_{n-1}$.

It is also useful to know that Φ_* and Φ^* map these indecomposable injectives as one would expect.

LEMMA 2.5. *We have $\Phi_* J_r \cong J_{n,r}$ and $\Phi^* J_{n,r} \cong J_r$ for $0 \leq r \leq n$.*

Proof. Recall that both Φ_* and Φ^* preserve injectives by [HS05b, Corollary 2.5], and Φ^* preserves indecomposables by Lemma 2.2. Hence $\Phi^* J_{n,r}$ is an indecomposable injective, necessarily without v_n -torsion. On the other hand, since

$$\Phi_* BP_*/I_r = E_*/I_r \subseteq J_{n,r},$$

we conclude that

$$L_n(BP_*/I_r) \subseteq \Phi^* J_{n,r}.$$

But $L_n(BP_*/I_r)$ is either BP_*/I_r itself if $r < n$, or $v_n^{-1} BP_*/I_n$ if $r = n$ [HS05a, Lemma 5.3]. In either case we see that $J_r \subseteq \Phi^* J_{n,r}$. Equality must hold since $\Phi^* J_{n,r}$ is an indecomposable injective.

It then follows that $\Phi_* J_r = \Phi_* \Phi^* J_{n,r} \cong J_{n,r}$, as required. ■

This lemma implies that $J_{n,r}$ and J_r share much of the same structure. For example, we have the following corollary.

COROLLARY 2.6. *Suppose $0 \leq r \leq n$. Then J_r and $J_{n,r}$ are I_r -torsion and v_r -periodic.*

Proof. For J_r , this follows from Proposition 2.2, Proposition 2.6, and Theorem 2.7 of [HS05b]. It is clear that if M is I_r -torsion, so is $\Phi_* M$, and so

$J_{n,r}$ is I_r -torsion. In particular, since v_r is a primitive modulo I_r , this means that $v_r^{-1}J_{n,r}$ is a well-defined comodule. The kernel K of $J_{n,r} \rightarrow v_r^{-1}J_{n,r}$ is the v_r -torsion in J_r and K intersects E_*/I_r trivially. Since $J_{n,r}$ is an essential extension of E_*/I_r , we conclude that $J_{n,r}$ has no v_r -torsion. This means that $v_r^{-1}J_{n,r}$ is an essential extension of $J_{n,r}$, so $v_r^{-1}J_{n,r} = J_{n,r}$. ■

3. Structure of indecomposable injectives. We would like to know more of the structure of the indecomposable injectives J_r and $J_{n,r}$, in analogy to Matlis' theory described in [Lam99, Section 3I].

The object of this section is to prove the following theorem.

THEOREM 3.1. *If $E = E(n)$, then there is an isomorphism of comodules*

$$J_{n,n} \cong E(n)_*E(n) \otimes_{E(n)_*} \Sigma^t E(n)_*/I_n^\infty,$$

where $t = |v_1| + \dots + |v_{n-1}|$.

Note that in this theorem, we are not free to use any Noetherian Landweber exact homology theory of height n , but must use $E(n)$ itself. In particular, we cannot use Morava E -theory.

In this theorem, $E(n)_*/I_n^\infty$ is thought of as an $E(n)_*$ -module, not as a comodule, and it is the usual construction used in algebraic topology. That is, we inductively define $E(n)_*/I_r^\infty$ via the short exact sequence

$$0 \rightarrow E(n)_*/I_r^\infty \rightarrow v_r^{-1}E(n)_*/I_r^\infty \rightarrow E(n)_*/I_{r+1}^\infty \rightarrow 0.$$

Additive generators of $E(n)_*/I_n^\infty$ are given by

$$\frac{\lambda v_n^r}{p^{i_0} v_1^{i_1} \dots v_{n-1}^{i_{n-1}}}$$

where $\lambda \in \mathbb{F}_p$, $r \in \mathbb{Z}$ and i_0, i_1, \dots, i_{n-1} are positive integers. The action of p, v_1, \dots, v_n is the obvious one, with a product being 0 if it ever removes any of p, v_1, \dots, v_{n-1} from the denominator. For example, $E(n)_*/I_n$ is the submodule of $\Sigma^t E(n)_*/I_n^\infty$ generated by $1/pv_1 \dots v_{n-1}$.

We can make a similar construction to form E_{n*}/I_n^∞ , with

$$E_{n*} = W\mathbb{F}_q[[u_1, \dots, u_{n-1}]] [u, u^{-1}]$$

the coefficient ring of Morava E -theory, where $q = p^n$, $W\mathbb{F}_q$ is the Witt vectors of the Galois field \mathbb{F}_q , the u_i have degree 0, and u has degree -2 . There is a ring homomorphism $E(n)_* \rightarrow E_{n*}$ that takes v_r to $u_r u^{1-p^r}$ for $1 \leq r < n$ and v_n to u^{1-p^n} . Hence $I_r = (p, u_1, \dots, u_{r-1})$ as an ideal of E_{n*} . The elements

$$\frac{\lambda u^r}{p^{i_0} u_1^{i_1} \dots u_{n-1}^{i_{n-1}}},$$

where $\lambda \in \mathbb{F}_q$, $r \in \mathbb{Z}$ and i_0, i_1, \dots, i_{n-1} are all positive integers, are additive generators for E_{n*}/I_n^∞ .

It is important to note that $E(n)_*/I_n^\infty$ is **not** the increasing union of the $E(n)_*/I_n^r$ for $n > 1$. Indeed, $E(n)_*/I_n^2$ has distinct elements p and v_1 that are both killed by I_n , whereas the only elements of $E(n)_*/I_n^\infty$ killed by I_n are $\alpha v_n^r/pv_1 \cdots v_{n-1}$ for $\alpha \in \mathbb{F}_p$ and $r \in \mathbb{Z}$. Thus no shifted copy of $E(n)_*/I_n^2$ can sit inside $E(n)_*/I_n^\infty$ for $n > 1$.

Theorem 3.1 has the following corollary.

COROLLARY 3.2. *We have*

$$J_r \cong BP_*BP \otimes_{BP_*} \Sigma^t E(r)_*/I_r^\infty$$

and, for any Noetherian Landweber exact E of height n ,

$$J_{n,r} \cong E_*BP \otimes_{BP_*} \Sigma^t E(r)_*/I_r^\infty$$

for $0 \leq r \leq n$. Here $t = |v_1| + \cdots + |v_{r-1}|$.

In this corollary, E is, as always, a Noetherian Landweber exact homology theory of height n .

Proof. The corollary follows from the fact that

$$\Phi^*(E_*E \otimes_{E_*} M) = BP_*BP \otimes_{BP_*} M$$

(see Lemma 2.4 of [HS05a]), and Lemma 2.5. ■

We will prove Theorem 3.1 by first using Matlis theory to show that $\Sigma^t E(n)_*/I_n^\infty$ is the injective hull of $E(n)_*/I_n$ in the category of $E(n)_*$ -modules. This shows that $E(n)_*E(n) \otimes_{E(n)_*} \Sigma^t E(n)_*/I_n^\infty$ is an injective comodule. We then show that both of the maps in the composition

$$E(n)_*/I_n \xrightarrow{i} E(n)_*E(n) \otimes_{E(n)_*} E(n)_*/I_n \xrightarrow{1 \otimes j} E(n)_*E(n) \otimes_{E(n)_*} \Sigma^t E(n)_*/I_n^\infty$$

are essential extensions. Here i is the unit of the adjunction between the forgetful functor and the extended comodule functor, with $i(a) = \eta_L(a) \otimes 1$, and j is the embedding of the $E(n)_*$ -module $E(n)_*/I_n$ into its injective hull.

PROPOSITION 3.3. *The injective hull of $E(n)_*/I_n$ in the category of $E(n)_*$ -modules is $\Sigma^t E(n)_*/I_n^\infty$, where $t = |v_1| + \cdots + |v_{n-1}|$. Similarly, the injective hull of E_{n*}/I_n in the category of E_{n*} -modules is E_{n*}/I_n^∞ .*

Proof. The proof is a computation using Matlis theory, modeled on [Lam99, Section 3J]. We first note that $\mathbb{Q}/\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ is an injective object in the category of graded $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ -modules and graded homomorphisms (not necessarily of degree 0). To see this, one can just use the same proof as the proof that shows $\mathbb{Q}/\mathbb{Z}_{(p)}$ is an injective $\mathbb{Z}_{(p)}$ -module. This proof works for discrete valuation rings such as $W\mathbb{F}_q$ as well, so that $(W\mathbb{F}_q \otimes \mathbb{Q})/\mathbb{Q}[u, u^{-1}]$ is an injective object in the category of graded $W\mathbb{F}_q[u, u^{-1}]$ -modules and graded homomorphisms. The rest of the proof for E_{n*} is the same as the proof for $E(n)_*$ given below.

Now let $M = \text{Hom}_{\mathbb{Z}_{(p)}[v_n, v_n^{-1}]}^*(E(n)_*, \mathbb{Q}/\mathbb{Z}_{(p)}[v_n, v_n^{-1}])$; then M is an injective graded $E(n)_*$ -module by the generalization of [Lam99, Lemma 3.5] to the graded case. We define the element

$$\frac{v_n^r}{p^{i_0} v_1^{i_1} \cdots v_{n-1}^{i_{n-1}}} = \frac{1}{p^{i_0} v_1^{i_1} \cdots v_{n-1}^{i_{n-1}} v_n^{-r}}$$

of M to be the element that takes the monomial

$$v_1^{i_1} \cdots v_{n-1}^{i_{n-1}}$$

to $v_n^r/p^{i_0} \in \mathbb{Q}/\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$, and takes the complementary $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ -summand of $E(n)_*$ to 0. Then the submodule of M generated by these elements is $E(n)_*/I_n^\infty$. An arbitrary element of M can be written as an infinite sum of these elements.

By Proposition 3.88 of [Lam99] (again, in the graded case), the I_n -torsion in M is still an injective $E(n)_*$ -module. We claim that the I_n -torsion in M is just $E(n)_*/I_n^\infty$. Indeed, it is clear that each of the elements

$$\frac{v_n^r}{p^{i_0} v_1^{i_1} \cdots v_{n-1}^{i_{n-1}}}$$

is I_n -torsion. On the other hand, if $f \in M$ is I_n -torsion, then $I_n^k f = 0$ for large k , so $f(I_n^k) = 0$. Thus f kills all but finitely many monomials $p^{i_0} v_1^{i_1} \cdots v_{n-1}^{i_{n-1}}$ in $E(n)_*$. It follows that f is a finite sum of our generating elements, so is in $E(n)_*/I_n^\infty$.

Now it is clear that the extension $\Sigma^{-t} E(n)_*/I_n \rightarrow E(n)_*/I_n^\infty$, where 1 goes to $1/pv_1 \cdots v_{n-1}$, is essential, completing the proof. ■

We need a simple test for essential extensions of comodules.

LEMMA 3.4. *Suppose (A, Γ) is a flat Hopf algebroid for which every Γ -comodule has a primitive. Then an extension $M \rightarrow N$ of Γ -comodules is essential if and only if $Ax \cap M \neq 0$ for all primitives $x \in N$.*

Note that every E_*E -comodule has a primitive by [HS05a, Theorem 5.1].

Proof. If N' is an arbitrary nonzero subcomodule of N , it must contain a primitive x of N . The subcomodule Ax generated by x is then inside N' . The result follows easily. ■

LEMMA 3.5. *If (A, Γ) is a flat Hopf algebroid and M is an A -module, then the primitives in the extended comodule $\Gamma \otimes_A M$ are the elements $1 \otimes m$.*

Proof. Note that x is a primitive if and only if $x = f(1)$ for some comodule map $f: A \rightarrow \Gamma \otimes_A M$. But then adjointness implies f is induced by a map $g: A \rightarrow M$ of A -modules. This means that $f(a) = \eta_L(a) \otimes g(a)$ for all $a \in A$. In particular, $x = f(1) = 1 \otimes g(1)$. The converse is clear. ■

PROPOSITION 3.6. *The map*

$$E(n)_*/I_n \xrightarrow{i} E(n)_*E(n) \otimes_{E(n)_*} E(n)_*/I_n$$

defined by $i(a) = \eta_L(a) \otimes 1$ is an essential extension of $E(n)_*E(n)$ -comodules.

We note that this proposition is also true for Morava E -theory E_n , but the proof requires more care.

Proof. The primitives in $E(n)_*E(n) \otimes_{E(n)_*} E(n)_*/I_n$ are the elements $1 \otimes \bar{\lambda}v_n^k$ for $\lambda \in \mathbb{Z}_{(p)}$ (so $\bar{\lambda}$ is the reduction of λ in \mathbb{F}_p) and $k \in \mathbb{Z}$ by Lemma 3.5. But we have

$$1 \otimes \bar{\lambda}v_n^k = \eta_R(\lambda)\eta_R(v_n)^k \otimes 1.$$

Of course $\eta_R(\lambda) = \lambda$, and also $\eta_R(v_n) \equiv v_n \pmod{I_n}$. Hence

$$1 \otimes \bar{\lambda}v_n^k = \eta_L(\lambda v_n^k) \otimes 1.$$

Lemma 3.4 now completes the proof. ■

PROPOSITION 3.7. *The map*

$$E(n)_*E(n) \otimes_{E(n)_*} E(n)_*/I_n \xrightarrow{1 \otimes j} E(n)_*E(n) \otimes_{E(n)_*} \Sigma^t E(n)_*/I_n^\infty$$

is an essential extension of $E(n)_*E(n)$ -comodules. Here $t = |v_1| + \dots + |v_{n-1}|$.

Note that this proposition completes the proof of Theorem 3.1. It is this proposition that we believe to be false for Morava E -theory E_n .

Proof. In view of Lemmas 3.4 and 3.5, it suffices to show that there is an $a \in E(n)_*$ such that

$$\eta_L(a) \otimes \frac{\lambda v_n^k}{p^{i_0} v_1^{i_1} \dots v_{n-1}^{i_{n-1}}} = 1 \otimes \frac{\lambda v_n^k}{pv_1 \dots v_{n-1}},$$

ignoring suspensions. But then it is clear that we should take

$$a = p^{i_0-1} v_1^{i_1-1} \dots v_{n-1}^{i_{n-1}-1}.$$

There is a slight subtlety, since $\eta_L(v_j) \neq \eta_R(v_j)$. We get around this by going in order, from p to v_1 to v_2 , *et cetera*, using the fact that v_j is primitive modulo I_j . More precisely, for the inductive step, we have

$$\begin{aligned} x\eta_L(v_j^{i_j-1}) \otimes \frac{v_n^k}{pv_1 \dots v_{j-1} v_j^{i_j} \dots v_{n-1}^{i_{n-1}}} &= x\eta_R(v_j^{i_j-1}) \otimes \frac{v_n^k}{pv_1 \dots v_{j-1} v_j^{i_j} \dots v_{n-1}^{i_{n-1}}} \\ &= x \otimes \frac{v_n^k}{pv_1 \dots v_j v_{j+1}^{i_{j+1}} \dots v_{n-1}^{i_{n-1}}}, \end{aligned}$$

as required. ■

4. Indecomposable injectives and duality. In this section, we study the dualities in the stable homotopy category that arise from the indecomposable injectives studied in this paper.

Fix a p -local spectrum X , and integers $n \geq r$, and consider the functor that takes a p -local spectrum Y to the abelian group $\text{Hom}_{E_*E}(E_*(X \wedge Y), J_{n,r})$, where E is a Landweber exact homology theory of height n , as usual. Since $J_{n,r}$ is an injective E_*E -comodule, this functor is exact and so is a cohomology theory of Y . Hence it is representable by some spectrum $\Delta_{n,r}X$, for which we have the natural isomorphism

$$[Y, \Delta_{n,r}X] \cong \text{Hom}_{E_*E}(E_*(X \wedge Y), J_{n,r}).$$

Using the injective BP_*BP -comodule J_r , we can also define Δ_rX with the property that

$$[Y, \Delta_rX] \cong \text{Hom}_{BP_*BP}(BP_*(X \wedge Y), J_r).$$

We have the following basic results about these duality functors.

PROPOSITION 4.1. *For a p -local spectrum X , define $\Delta_{n,r}X$ and Δ_rX as above. Denote $\Delta_{n,r}S^0$ and Δ_rS^0 by $\Delta_{n,r}$ and Δ_r . Then:*

- (1) $\Delta_{n,r}X \cong \Delta_rX$ for all $n \geq r$, and so $\Delta_{n,r}$ is independent of the choice of E .
- (2) $[Y, \Delta_rX] \cong \text{Hom}_{E(r)_*}(E(r)_*(X \wedge Y), \Sigma^t E(r)_*/I_r^\infty)$, where $t = |v_1| + \dots + |v_{n-1}|$.
- (3) $\Delta_rX \cong F(X, \Delta_r)$.
- (4) Δ_r is a BP -injective spectrum with $BP_*\Delta_r \cong J_r$, and the isomorphism

$$[X, \Delta_r] \cong \text{Hom}_{BP_*BP}(BP_*X, J_r)$$

is induced by taking BP -homology. If E_* is concentrated in even dimensions, then Δ_r is also an E -injective spectrum for all $n \geq r$, with $E_*\Delta_r \cong J_{n,r}$, and the isomorphism

$$[X, \Delta_r] \cong \text{Hom}_{E_*E}(E_*X, J_{n,r})$$

is induced by taking E -homology.

- (5) Δ_rX is L_r -local. In particular, $\Delta_rX = F(L_rX, \Delta_r)$.
- (6) $\Delta_rX = 0$ if and only if $L_rX = 0$.

Proof. For part (1), we have

$$\begin{aligned} [Y, \Delta_{n,r}X] &= \text{Hom}_{E(n)_*E(n)}(E(n)_*(X \wedge Y), J_{n,r}) \\ &\cong \text{Hom}_{BP_*BP}(BP_*(X \wedge Y), \Phi^*J_{n,r}) \\ &\cong \text{Hom}_{BP_*BP}(BP_*(X \wedge Y), J_r) = [Y, \Delta_rX], \end{aligned}$$

using the adjointness between Φ_* and Φ^* and the isomorphism $\Phi^*J_{n,r} = J_r$ of Lemma 2.5.

Part (2) follows from part (1) and the isomorphisms

$$\begin{aligned} [Y, \Delta_{r,r}X] &= \text{Hom}_{E(r)_*E(r)}(E(r)_*(X \wedge Y), J_{r,r}) \\ &\cong \text{Hom}_{E(r)_*}(E(r)_*(X \wedge Y), E(r)_*/I_r^\infty), \end{aligned}$$

the last of which follows from Theorem 3.1.

Part (3) is an exercise in adjointness, and part (4) follows from [Dev97, Proposition 1.3 and Theorem 1.5].

Part (5) is clear, since if Y is L_r -acyclic, then $E(r)_*(X \wedge Y) = 0$ and so $[Y, \Delta_r X] = 0$.

For part (6), suppose $L_r X \neq 0$. Then $E(r)_*X \neq 0$, and so $E(r)_*X$ contains a subcomodule M isomorphic to $\Sigma^r E(r)_*/I_j$ for some $0 \leq j \leq r$ and some r by [HS05a, Theorem 5.7]. There is obviously a nonzero map (of some degree) $M \rightarrow E(r)_*/I_r^\infty$ of $E(r)_*$ -modules, obtained by modding out by I_r and including. Since $E(r)_*/I_r^\infty$ is injective by Proposition 3.3, this map extends to a nonzero map (of some degree) $E(r)_*X \rightarrow E(r)_*/I_r^\infty$. This map corresponds to a nonzero homotopy class in $\Delta_r X$, and so $\Delta_r X$ is nonzero. ■

We also point out that, as usual, there is a natural map $X \rightarrow \Delta_r^2 X$ obtained by taking the image of the identity under the isomorphisms

$$[\Delta_r X, \Delta_r X] \cong \text{Hom}_{E(r)_*}(E(r)_*(X \wedge \Delta_r X), \Sigma^t E(r)_*/I_r^\infty) \cong [X, \Delta_r^2 X].$$

The spectrum Δ_r has in fact been studied before in stable homotopy theory. Recall that L_n denotes Bousfield localization with respect to $E(n)$, or any Landweber exact theory of height n . There is a natural map $L_n X \rightarrow L_{n-1} X$, and the homotopy fiber of this map is traditionally denoted $M_n X$.

THEOREM 4.2. *Fix an integer $r \geq 1$. Then $\Delta_r \cong \Sigma^{t+r} M_r E(r)$, where $t = |v_1| + \cdots + |v_{r-1}|$. In particular, if E_* is an evenly graded Landweber exact homology theory of height $n \geq r$, then*

$$J_{n,r} \cong \Sigma^{t+r} E_*(M_r E(r)).$$

Proof. Recall from [Rav84, Theorem 6.1] that

$$\pi_* M_r BP \cong \Sigma^{-r} v_r^{-1}(BP_*/I_r^\infty).$$

In view of the localization theorem [Rav92, Theorem 7.5.2], we see that

$$BP_* M_r S^0 \cong \Sigma^{-r} v_r^{-1}(BP_*/I_r^\infty).$$

By Landweber exactness, we conclude that

$$E(r)_* M_r S^0 \cong \Sigma^{-r} E(r)_*/I_r^\infty.$$

But $M_r E(r) \cong E(r) \wedge M_r S^0$ because of the smash product theorem [Rav92, Theorem 7.5.6], and the flatness of $E(r)_*E(r)$ implies that

$$E(r)_*(E(r) \wedge X) \cong E(r)_*E(r) \otimes_{E(r)_*} E(r)_*X$$

(see Lemma 2.2.7 of [Rav86]). Some diagram chasing is necessary to show that this isomorphism is an isomorphism of comodules, where the right side is given the extended comodule structure, so that all the coaction is on the $E(r)_*E(r)$ tensor factor. Theorem 3.1 then implies that

$$E(r)_*(M_r E(r)) \cong \Sigma^{-t-r} J_{r,r}$$

as $E(r)_*E(r)$ -comodules. This isomorphism then defines a map

$$M_r E(r) \rightarrow \Sigma^{-t-r} \Delta_r$$

that is an isomorphism in $E(r)$ -homology. Since both $M_r E(r)$ and Δ_r are $E(r)$ -local, the theorem follows. ■

This theorem has the following corollary, apparently not written down before but known to several mathematicians, including Hopkins, Greenlees, and Sadofsky.

COROLLARY 4.3. *Let $\widehat{E(n)}$ denote $L_{K(n)}E(n)$, the completion of $E(n)$ at I_n . Then*

$$\widehat{E(n)}^*(X) \cong \text{Hom}_{E(n)_*}(E(n)_*(M_n X), \Sigma^{-n} E(n)_*/I_n^\infty).$$

On the simplest level, we can think of this corollary as analogous to the fact that $\text{Hom}_{\mathbb{Z}(p)}(\mathbb{Q}/\mathbb{Z}(p), \mathbb{Q}/\mathbb{Z}(p)) \cong \mathbb{Z}(p)$. As pointed out by Greenlees, one can look at this corollary as a reflection of the fact that completion (represented by $\widehat{E(n)}$) should be maps out of local cohomology (represented by $M_n E(n)$). This is the viewpoint of [GM95, Section 4].

Proof. Theorem 6.19 of [HS99] tells us that

$$\widehat{E(n)}^*(X) \cong [L_{K(n)}X, L_{K(n)}E(n)]^* \cong [M_n X, M_n E(n)]^*.$$

But Theorem 4.2 implies that

$$\begin{aligned} [M_n X, M_n E(n)]^* &\cong [\Sigma^{t+n} M_n X, \Delta_n]^* \\ &\cong \text{Hom}_{E(n)_*}(E(n)_*(\Sigma^{t+n} M_n X), \Sigma^t E(n)_*/I_n^\infty) \\ &\cong \text{Hom}_{E(n)_*}(E(n)_* M_n X, \Sigma^{-n} E(n)_*/I_n^\infty). \blacksquare \end{aligned}$$

We can also determine the endomorphism rings of the indecomposable injectives J_r and $J_{n,r}$.

COROLLARY 4.4. *The endomorphism rings $\text{End}_{BP_*BP}(J_r)$ and $\text{End}_{E(n)_*E(n)}(J_{n,r})$ for $0 \leq r \leq n$ are isomorphic to $\widehat{E(r)}^* \widehat{E(r)}$.*

Proof. Both of these endomorphism rings are isomorphic to $[\Delta_r, \Delta_r]^*$ in view of Proposition 4.1(4). But Theorem 4.2 implies that

$$[\Delta_r, \Delta_r]^* \cong [M_r E(r), M_r E(r)]^*,$$

which is in turn isomorphic to $\widehat{E(r)}^* \widehat{E(r)}$ by [HS99, Theorem 6.19]. ■

It is unfortunate that these endomorphism rings of indecomposable injectives turn out to be $\widehat{E(r)}^* \widehat{E(r)}$ instead of $E_r^* E_r$, where E_r is Morava E -theory. The ring $E_r^* E_r$ of operations in Morava E -theory is the twisted completed group ring $E_{r*}[[\Gamma]]$, where Γ is the semidirect product of the automorphism group of the height r Honda formal group law over \mathbb{F}_{p^r} with the Galois group of \mathbb{F}_{p^r} . This is an old result of Hopkins and Ravenel; see [Bak95] or [Hov04b] for a proof. Since E_r is a finite free module over $\widehat{E(r)}$, there is probably some Galois theory relating $\widehat{E(r)}^* \widehat{E(r)}$ to $E_{r*}[[\Gamma]]$, but the author does not know any details.

We can, however, make a decomposable injective comodule whose endomorphism ring is $E_r^* E_r \cong E_{r*}[[\Gamma]]$. Indeed, we can simply take the comodule

$$J'_{r,r} = E_{r*} E_r \otimes_{E_{r*}} E_{r*} / I_r^\infty.$$

This is an $E_{r*} E_r$ -comodule that is injective because E_{r*} / I_r^∞ is an injective E_{r*} -module by Proposition 3.3. Under the equivalence of categories between $E(r)_* E(r)$ -comodules and $E_{r*} E_r$ -comodules of [HS05a], $J'_{r,r}$ corresponds to the direct sum of $r(p^r - 1)$ copies of $J_{r,r}$, with r copies in every even dimension from 2 to $2(p^r - 1)$; this is the $E(r)_* E(r)$ -comodule

$$E(r)_* E(r) \otimes_{E(r)_*} E_{r*} / I_r^\infty.$$

There is a spectrum Δ'_r with

$$[X, \Delta'_r] \cong \text{Hom}_{E_{r*} E_r}(E_{r*} X, J'_{r,r}) \cong \text{Hom}_{E_{r*}}(E_{r*} X, E_{r*} / I_r^\infty).$$

The analogue to Theorem 4.2 tells us that $\Delta'_r \cong \Sigma^r M_r E_r$. We then get isomorphisms

$$E_n^* X \cong \text{Hom}_{E_n*}(E_n*(M_n X), \Sigma^{-n} E_n* / I_n^\infty),$$

again known before by Hopkins, Greenlees, Sadofsky, and others, and

$$\text{End}_{E_{r*} E_r}(J'_{r,r}) \cong E_r^* E_r \cong E_{r*}[[\Gamma]].$$

References

[Bak89] A. Baker, *Some families of operations in Morava K-theory*, Amer. J. Math. 111 (1989), 95–109.
 [Bak95] —, *A version of Landweber’s filtration theorem for v_n -periodic Hopf algebroids*, Osaka J. Math. 32 (1995), 689–699.
 [Dev97] E. S. Devinatz, *Morava modules and Brown–Comenetz duality*, Amer. J. Math. 119 (1997), 741–770.
 [DH95] E. S. Devinatz and M. J. Hopkins, *The action of the Morava stabilizer group on the Lubin–Tate moduli space of lifts*, *ibid.* 117 (1995), 669–710.
 [GM95] J. P. C. Greenlees and J. P. May, *Completions in algebra and topology*, in: Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995, 255–276.

- [Hov95] M. Hovey, *Bousfield localization functors and Hopkins' chromatic splitting conjecture*, in: The Čech Centennial (Boston, MA, 1993), *Contemp. Math.* 181, Amer. Math. Soc., Providence, RI, 1995, 225–250.
- [Hov04a] —, *Homotopy theory of comodules over a Hopf algebroid*, in: *Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory* (Evanston, IL, 2002), *Contemp. Math.* 346, Amer. Math. Soc., Providence, RI, 2004, 261–304.
- [Hov04b] —, *Operations and co-operations in Morava E-theory*, *Homology Homotopy Appl.* 6 (2004), 201–236.
- [Hov07] —, *The generalized homology of products*, *Glasg. Math. J.* 49 (2007), 1–10.
- [HS99] M. Hovey and N. P. Strickland, *Morava K-theories and localisation*, *Mem. Amer. Math. Soc.* 139 (1999), no. 666.
- [HS05a] —, —, *Comodules and Landweber exact homology theories*, *Adv. Math.* 192 (2005), 427–456.
- [HS05b] —, —, *Local cohomology of BP_*BP -comodules*, *Proc. London Math. Soc.* 90 (2005), 521–544.
- [Lam99] T. Y. Lam, *Lectures on Modules and Rings*, Springer, New York, 1999.
- [Lan73] P. S. Landweber, *Annihilator ideals and primitive elements in complex bordism*, *Illinois J. Math.* 17 (1973), 273–284.
- [Lan76] —, *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$* , *Amer. J. Math.* 98 (1976), 591–610.
- [Lan79] —, *New applications of commutative algebra to Brown–Peterson homology*, in: *Algebraic Topology* (Waterloo, ON, 1978), *Lecture Notes in Math.* 741, Springer, Berlin, 1979, 449–460.
- [Rav84] D. C. Ravenel, *Localization with respect to certain periodic homology theories*, *Amer. J. Math.* 106 (1984), 351–414.
- [Rav86] —, *Complex Cobordism and Stable Homotopy Groups of Spheres*, *Pure Appl. Math.* 121, Academic Press, Orlando, FL, 1986.
- [Rav92] —, *Nilpotence and periodicity in stable homotopy theory*, *Ann. of Math. Stud.* 128, Princeton Univ. Press, Princeton, NJ, 1992.
- [Ste75] B. Stenström, *Rings of Quotients. An Introduction to Methods of Ring Theory*, *Grundlehren Math. Wiss.* 217, Springer, Berlin, 1975.

Department of Mathematics
 Wesleyan University
 Middletown, CT 06459, U.S.A.
 E-mail: hovey@member.ams.org

*Received 27 March 2006;
 in revised form 23 May 2007*