# Continuation of the connection matrix for singularly perturbed hyperbolic equations

by

## Maria C. Carbinatto (São Carlos) and
## Krzysztof P. Rybakowski (Rostock)

**Abstract.** Let $\Omega \subset \mathbb{R}^N$, $N \leq 3$, be a bounded domain with smooth boundary, $\gamma \in L^2(\Omega)$ be arbitrary and $\phi \colon \mathbb{R} \to \mathbb{R}$ be a $C^1$-function satisfying a subcritical growth condition. For every $\varepsilon \in \, ]0, \infty[$ consider the semiflow $\pi_\varepsilon$ on $H_0^1(\Omega) \times L^2(\Omega)$ generated by the damped wave equation

$$\varepsilon \partial_{tt} u + \partial_t u = \Delta u + \phi(u) + \gamma(x), \quad x \in \Omega,\ t > 0,$$
$$u(x,t) = 0, \qquad\qquad\qquad\quad x \in \partial\Omega,\ t > 0.$$

Moreover, let $\pi'$ be the semiflow on $H_0^1(\Omega)$ generated by the parabolic equation

$$\partial_t u = \Delta u + \phi(u) + \gamma(x), \quad x \in \Omega,\ t > 0,$$
$$u(x,t) = 0, \qquad\qquad\qquad\ x \in \partial\Omega,\ t > 0.$$

Let $\Gamma \colon H^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ be the imbedding $u \mapsto (u, \Delta u + \phi(u) + \gamma)$. We prove that whenever $K'$ is a compact isolated $\pi'$-invariant set and $(M_p')_{p \in P}$ is a partially ordered Morse decomposition of $K'$ then the imbedded sets $K = \Gamma(K')$ and $M_{p,0} = \Gamma(M_p')$, $p \in P$, continue, for $\varepsilon > 0$ small, to an isolated $\pi_\varepsilon$-invariant set $K_\varepsilon$ a Morse decomposition $(M_{p,\varepsilon})_{p \in P}$ of $K_\varepsilon$, relative to $\pi_\varepsilon$, such that the homology index braid of $(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon})_{p \in P})$ is isomorphic to the homology index braid of $(\pi', K', (M_p')_{p \in P})$. This, in particular, implies a connection matrix continuation principle.

**1. Introduction.** Let $N \in \{1, 2, 3\}$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\gamma \in L^2(\Omega)$ be arbitrary and $\phi \colon \mathbb{R} \to \mathbb{R}$ be a $C^1$-function such that, for $N \geq 2$, there are constants $\bar{r}$ and $\overline{C} \in [0, \infty[$ with $|\phi'(u)| \leq \overline{C}(1 + |u|^{\bar{r}})$ for $u \in \mathbb{R}$. If $N = 3$ we also assume that $\bar{r} < 2$, i.e. that $\phi$ has subcritical growth.

For every $\varepsilon \in\, ]0, \infty[$ consider the following damped wave equation:

$$(\text{Hyp}_\varepsilon) \qquad \begin{aligned} \varepsilon \partial_{tt} u + \partial_t u &= \Delta u + \phi(u) + \gamma(x), & x \in \Omega,\, t > 0, \\ u(x,t) &= 0 & x \in \partial\Omega,\, t > 0. \end{aligned}$$

It is well-known that equation $(\text{Hyp}_\varepsilon)$ generates a local semiflow (actually, a local flow) $\pi_\varepsilon$ on $H_0^1(\Omega) \times L^2(\Omega)$.

Setting, formally, $\varepsilon = 0$ in equation $(\text{Hyp}_\varepsilon)$ we obtain the parabolic equation

$$(\text{Par}) \qquad \begin{aligned} \partial_t u &= \Delta u + \phi(u) + \gamma(x), & x \in \Omega,\, t > 0, \\ u(x,t) &= 0, & x \in \partial\Omega,\, t > 0. \end{aligned}$$

Again it is well-known that equation (Par) generates a local semiflow $\pi'$ on $H_0^1(\Omega)$.

It is a natural question whether, for $\varepsilon \to 0$, solutions of $\pi_\varepsilon$ converge, in some sense, to solutions of $\pi'$, properly imbedded into $H_0^1(\Omega) \times L^2(\Omega)$. This question was considered in the context of attractors by Hale and Raugel [12] and in the context of Conley index theory by the second author in [17].

In fact, in [17] the following result was proved.

THEOREM A. *Let $\Gamma \colon H^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ be the imbedding $u \mapsto (u, \Delta u + \phi(u) + \gamma)$. Moreover, let $K'$ be a compact (in $H_0^1(\Omega)$) isolated invariant set relative to $\pi'$. Then $K' \subset H^2(\Omega)$, and the set $K_0 := \Gamma(K')$ continues to a family $(K_\varepsilon)_{\varepsilon \in [0, \widetilde{\varepsilon}_0]}$, where $\widetilde{\varepsilon}_0 > 0$, such that for every $\varepsilon \in\, ]0, \widetilde{\varepsilon}_0]$, $K_\varepsilon$ is a compact isolated invariant set relative to $\pi_\varepsilon$ such that the Conley index of $(\pi_\varepsilon, K_\varepsilon)$ is equal to the Conley index of $(\pi', K')$. Furthermore, $(K_\varepsilon)_{\varepsilon \in [0, \widetilde{\varepsilon}_0]}$ is upper-semicontinuous at $\varepsilon = 0$.*

*Moreover, let $(M_p')_{p \in P}$ be a $\prec$-ordered Morse decomposition of $K'$ relative to $\pi'$. Then the family $(M_{p,0})_{p \in P} := (\Gamma(M_p'))_{p \in P}$ continues to a family $((M_{p,\varepsilon})_{p \in P})_{\varepsilon \in [0, \widetilde{\varepsilon}_0]}$, where for every $\varepsilon \in\, ]0, \widetilde{\varepsilon}_0]$, $(M_{p,\varepsilon})_{p \in P}$ is a $\prec$-ordered Morse decomposition of $K_\varepsilon$ relative to $\pi_\varepsilon$. Furthermore, $((M_{p,\varepsilon})_{p \in P})_{\varepsilon \in [0, \widetilde{\varepsilon}_0]}$ is upper-semicontinuous at $\varepsilon = 0$.*

We refer to [17, Theorems 6.1 and 7.4, and Remark 7.5] for the precise statement of a more general result.

Theorem A extends previous upper-semicontinuity results for attractors by Hale and Raugel.

The main goal of this paper is to refine Theorem A to a continuation result for homology index braids. More precisely, we shall prove

THEOREM B. *In the notation of Theorem A, for all $\varepsilon \in\, ]0, \varepsilon_0]$, the (co)homology index braids of $(\pi', K', (M_p')_{p \in P})$ and $(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon})_{p \in P})$ are isomorphic.*

We recall that if two homology index braids $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic, then, by [10, Proposition 1.5], $\mathcal{H}_1$ and $\mathcal{H}_2$ determine the same collection

of $C$-connection matrices. Thus Theorem B shows that, for small $\varepsilon > 0$, the Morse decomposition $(M_{p,\varepsilon})_{p \in P}$ of the invariant set $K_\varepsilon$ relative to the damped wave equation $(\mathrm{Hyp}_\varepsilon)$ has the same set of $C$-connection matrices as the Morse decomposition $(M'_p)_{p \in P}$ of the invariant set $K'$ relative to the reaction-diffusion equation (Par). In particular, this means that homologically nontrivial heteroclinic connections of the simpler equation (Par) survive in the more complicated equation $(\mathrm{Hyp}_\varepsilon)$.

Theorem B will be proved by an application of some abstract continuation results for regularly and singularly perturbed semiflows, recently established in our papers [5] and [7]. Our approach to the proof of Theorem B is similar to the proof of Theorem A in [17]. As in [17], the first attempt would be to make a change of variables

$$\Phi \colon (u, v) \mapsto (u, w) := (u, v - \Delta u - \phi(u) - \gamma)$$

in $(\mathrm{Hyp}_\varepsilon)$, consider the corresponding conjugate semiflows $\widetilde{\pi}_\varepsilon = \Phi^* \pi_\varepsilon$, $\varepsilon \in {]0, \infty[}$, and then apply the results of [7] to the family $\widetilde{\pi}_\varepsilon$, $\varepsilon \in [0, \infty[$, where $\widetilde{\pi}_0 = \pi'$. However, as it was pointed out in [17], there is an inherent difficulty in the present situation due to the fact that the transformation $\Phi$ is defined on the space $H^2(\Omega) \times L^2(\Omega)$, which is only a subset of the phase space $H_0^1(\Omega) \times L^2(\Omega)$ of the semiflows $\pi_\varepsilon$, so $\widetilde{\pi}_\varepsilon$ is not well-defined for $\varepsilon \in {]0, \infty[}$. That is why we first study, in Section 3, a finite-dimensional perturbation problem (equations (3.1) and (3.2) below), to which a variable transformation like $\Phi$ is applicable. Results from [7] then yield a continuation result for homology index braids for the corresponding family of finite-dimensional semiflows (cf. Theorems 3.1 and 3.6 below).

Combining the compactness and smoothing results for parabolic equations and the boundedness and smoothing results for damped wave equations established in [17] with the results from Section 3, we then obtain, in the last section of the paper, our main result, Theorem 4.2, which implies Theorem B above.

**2. Preliminaries.** The purpose of this section is to recall a few concepts from Conley index theory and some preliminary results needed later in this paper. We assume the reader's familiarity with the infinite-dimensional Conley index theory, as developed in [14], [15] and expounded in [16], and with the papers [9], [11], [5] and [6]. Moreover, we recall some results from the papers [1], [4], [7] and [17].

In this section, unless otherwise specified, $Y$ is a metric space, $\pi$ is a local semiflow on $Y$ and all concepts are defined *relative to* $\pi$.

Suppose that $Z$ is a subset of $Y$. We denote by $\mathrm{Inv}_\pi(Z)$ the largest invariant subset of $Z$.

$Z$ is called $\pi$-*admissible* if it is closed and whenever $(x_n)_n$ and $(t_n)_n$ are such that $t_n \to \infty$ and $x_n\pi\,[0,t_n] \subset Z$ for all $n \in \mathbb{N}$, then the sequence $(x_n\pi t_n)_n$ has a convergent subsequence. We say that $\pi$ *does not explode in* $Z$ if whenever $x \in Y$ and $x\pi t \in Z$ as long as $x\pi t$ is defined, then $x\pi t$ is defined for all $t \in [0, \infty[$. Moreover $Z$ is called *strongly $\pi$-admissible* if $Z$ is $\pi$-admissible and $\pi$ does not explode in $Z$.

Let $N$ and $Z$ be subsets of $Y$. The set $Z$ is called $N$-*positively invariant* if whenever $x \in Z$, $t \geq 0$ are such that $x\pi\,[0,t] \subset N$, then $x\pi\,[0,t] \subset Z$.

Let $N$, $Z_1$ and $Z_2$ be subsets of $Y$. Then $Z_2$ is called an *exit ramp for $N$ within $Z_1$* if whenever $x \in Z_1$ and $x\pi t' \notin N$ for some $t' \in [0, \infty[$, then there exists a $t_0 \in [0, t']$ such that $x\pi\,[0, t_0] \subset N$ and $x\pi t_0 \in Z_2$.

If $Z_1$ and $Z_2$ are subsets of $Y$ then $Z_2$ is called an *exit ramp for $Z_1$* if $Z_2$ is an exit ramp for $N$ within $Z_1$, where $N = Z_1$.

DEFINITION 2.1 ([11]). A pair $(N_1, N_2)$ is called a *Franzosa–Mischaikow-index pair (or FM-index pair) for $S$* if:

(1) $N_1$ and $N_2$ are closed subsets of $Y$ with $N_2 \subset N_1$ and $N_2$ is $N_1$-positively invariant;
(2) $N_2$ is an exit ramp for $N_1$;
(3) $S$ is closed, $S \subset \mathrm{Int}_Y(N_1 \setminus N_2)$ and $S$ is the largest invariant set in $\mathrm{Cl}_Y(N_1 \setminus N_2)$.

Given an isolated invariant set $K$ having a strongly $\pi$-admissible isolating neighborhood we denote by $h(K) = h(\pi, K)$ the Conley index of $K$ and by $H(K) = H(\pi, K) = H(h(K))$ the homology Conley index, where $H$ is the singular homology functor (with coefficients in some fixed $R$-module $G$, where $R$ is a commutative ring).

For the rest of this paper let $P$ be a fixed finite set and $\prec$ be a fixed strict partial order on $P$.

A set $I \subset P$ is called a $\prec$-*interval* if whenever $i, j, k \in P$, $i, k \in I$ and $i \prec j \prec k$, then $j \in I$. We denote by $\mathcal{I}(\prec)$ the set of all $\prec$-intervals in $P$. The set $I$ is called a $\prec$-*attracting interval* if whenever $i, j \in P$, $j \in I$ and $i \prec j$, then $i \in I$. Let $\mathcal{A}(\prec)$ denote the set of all $\prec$-attracting intervals in $P$. Of course, $\mathcal{A}(\prec) \subset \mathcal{I}(\prec)$.

An *adjacent $n$-tuple of $\prec$-intervals* is a sequence $(I_j)_{j=1}^n$ of pairwise disjoint $\prec$-intervals whose union is a $\prec$-interval and such that, whenever $j < k$, $p \in I_j$ and $p' \in I_k$, then $p' \not\prec p$ (i.e. $p \prec p'$ or else $p$ and $p'$ are not related by $\prec$). We denote by $\mathcal{I}_n(\prec)$ the set of all adjacent $n$-tuples of $\prec$-intervals.

Recall the following definition.

DEFINITION 2.2 (cf. [11]). Let $\pi$ be a local semiflow on $Y$ and $S$ be a compact invariant set relative to $\pi$. A family $(M_p)_{p \in P}$ of subsets of $S$ is called a $\prec$-*ordered Morse decomposition of $S$ (relative to $\pi$)* if the following

properties hold:

(1) The sets $M_p$, $p \in P$, are closed, $\pi$-invariant and pairwise disjoint.
(2) For every full solution $\sigma$ of $\pi$ lying in $S$, either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$, or there are $k, l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

Let $\mathcal{C}$ denote the set of all continuous functions from $\mathbb{R}$ to $Y$ endowed with the metric introduced in [1]. Let $\mathcal{T}$ be a subset of $\mathcal{C}$. A set $S \subset Y$ is called $\mathcal{T}$-*invariant* if $S = \mathrm{Inv}_{\mathcal{T}}(S)$, i.e. if and only if for every $y \in S$ there is a $\sigma \in \mathcal{T}$ such that $\sigma(\mathbb{R}) \subset S$ and $y = \sigma(0)$. If $\pi$ is a local semiflow on $Y$ and $N \subset Y$ then we denote by $\mathcal{T}_{\pi,N}$ the set of all full solutions of $\pi$ lying in $N$.

The previous concept can be generalized as follows:

DEFINITION 2.3 (cf. [4]). Let $\mathcal{T}$ be a subset of $\mathcal{C}$. A family $(M_p)_{p \in P}$ of subsets of $Y$ is called a $\prec$-*ordered* $\mathcal{T}$-*Morse decomposition* if the following properties hold:

(1) The sets $M_p$, $p \in P$, are closed, $\mathcal{T}$-invariant and pairwise disjoint.
(2) For every $\sigma \in \mathcal{T}$ either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are $k, l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

It is easily proved that, for $\pi$ and $S$ as in Definition 2.2, a family $(M_p)_{p \in P}$ of subsets of $S$ is a $\prec$-ordered Morse decomposition of $S$ (relative to $\pi$) if and only if $(M_p)_{p \in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition, where $\mathcal{T} := \mathcal{T}_{\pi,S}$.

If $A, B \subset Y$ then the $\mathcal{T}$-*connection set* $\mathrm{CS}_{\mathcal{T}}(A, B)$ *from* $A$ *to* $B$ is the set of all points $y \in Y$ for which there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = y$, $\alpha(\sigma) \subset A$ and $\omega(\sigma) \subset B$. If $\pi$, $S$ are as in Definition 2.2 and $\mathcal{T} := \mathcal{T}_{\pi,S}$, then we write

$$\mathrm{CS}_{\pi,S}(A, B) := \mathrm{CS}_{\mathcal{T}}(A, B).$$

Recall also the following definitions.

DEFINITION 2.4 (cf. [17]). We say that $N$ is a $\mathcal{T}$-*isolating neighborhood* (*of a subset* $S$ *of* $Y$) if $N$ is closed in $Y$ and $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_Y(N)$ (with $S = \mathrm{Inv}_{\mathcal{T}}(N)$). If $S \subset Y$ and there exists a set $N \subset Y$ such that $N$ is a $\mathcal{T}$-isolating neighborhood of $S$ then we call $S$ a $\mathcal{T}$-*isolated invariant set*.

DEFINITION 2.5 (cf. [1]). Given $\sigma_1$ and $\sigma_2$ in $\mathcal{C}$ with $\sigma_1(0) = \sigma_2(0)$ the map $\sigma_1 \triangleright \sigma_2 \colon \mathbb{R} \to X$ defined by

$$(\sigma_1 \triangleright \sigma_2)(t) = \begin{cases} \sigma_1(t) & \text{if } t \leq 0, \\ \sigma_2(t) & \text{if } t \geq 0, \end{cases}$$

is called the *cut-and-glue of* $(\sigma_1, \sigma_2)$. A subset $\mathcal{T}$ of $\mathcal{C}$ is called *cut-and-glue-invariant* if $\sigma_1 \triangleright \sigma_2 \in \mathcal{T}$ for all $\sigma_1, \sigma_2 \in \mathcal{T}$ with $\sigma_1(0) = \sigma_2(0)$.

For the rest of this paper, unless specified otherwise, let $(X_0, d_0)$ be a metric space, $\varepsilon_0$ be a positive number and, for each $\varepsilon \in\, ]0, \varepsilon_0]$, $(Y_\varepsilon, d_\varepsilon)$ be a metric space and $\theta_\varepsilon \in Y_\varepsilon$ be a distinguished point of $Y_\varepsilon$. The open ball in $Y_\varepsilon$ with center $v$ and radius $\beta > 0$ is denoted by $B_\varepsilon(v, \beta)$.

For each $\varepsilon \in \,]0, \varepsilon_0]$ define the set $Z_\varepsilon := X_0 \times Y_\varepsilon$. Endow $Z_\varepsilon$ with the metric

$$\Gamma_\varepsilon((u,v),(u',v')) := \max\{\,d_0(u,u'), d_\varepsilon(v,v')\,\}, \quad (u,v),(u',v') \in Z_\varepsilon.$$

Given a subset $V$ of $X_0$, $\beta > 0$ and $\varepsilon \in \,]0, \varepsilon_0]$ define the "inflated" subset $[V]_{\varepsilon,\beta}$ of $Z_\varepsilon$ as follows:

$$[V]_{\varepsilon,\beta} := \{(u,v) \in Z_\varepsilon \mid u \in V \text{ and } v \in \mathrm{Cl}_\varepsilon B_\varepsilon(\theta_\varepsilon, \beta)\}.$$

Let $\pi_0$ be a local semiflow on $X_0$ and for every $\varepsilon \in \,]0, \varepsilon_0]$ let $\pi_\varepsilon$ denote a local semiflow on $Z_\varepsilon$. We say that the family $(\pi_\varepsilon)_{\varepsilon \in \,]0,\varepsilon_0]}$ of local semiflows *converges singularly* to the local semiflow $\pi_0$ if whenever $(\varepsilon_n)_n$ and $(t_n)_n$ are sequences of positive (resp. nonnegative) numbers such that $\varepsilon_n \to 0$, $t_n \to t_0$ as $n \to \infty$, for some $t_0 \in [0, \infty[$, and whenever $u_0 \in X_0$ and $w_n \in Z_{\varepsilon_n}$, $n \in \mathbb{N}$, are such that $\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$ and $u_0 \pi_0 t_0$ is defined, then there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $w_n \pi_{\varepsilon_n} t_n$ is defined and $\Gamma_{\varepsilon_n}(w_n \pi_{\varepsilon_n} t_n, (u_0 \pi_0 t_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$.

Let $\eta$ be a positive number and $N$ be a closed subset of $X_0$. We say that $N$ is a *singularly strongly admissible set with respect to $\eta$ and the family* $(\pi_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ of local semiflows if the following conditions are satisfied:

(1) $N$ is a strongly $\pi_0$-admissible set;
(2) for each $\varepsilon \in \,]0, \varepsilon_0]$ the set $[N]_{\varepsilon,\eta}$ is strongly $\pi_\varepsilon$-admissible;
(3) whenever $(\varepsilon_n)_n$ and $(t_n)_n$ are sequences of positive numbers such that $\varepsilon_n \to 0$, $t_n \to \infty$ as $n \to \infty$ and whenever $w_n \in Z_{\varepsilon_n}$, $n \in \mathbb{N}$, are such that $w_n \pi_{\varepsilon_n} [0, t_n] \subset [N]_{\varepsilon_n,\eta}$, $n \in \mathbb{N}$, then there exist a $u_0 \in N$ and a subsequence of the sequence $(w_n \pi_{\varepsilon_n} t_n)_n$ of endpoints, denoted again by $(w_n \pi_{\varepsilon_n} t_n)_n$, such that $\Gamma_{\varepsilon_n}(w_n \pi_{\varepsilon_n} t_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$.

The following singular continuation result for Morse decompositions was established in [4].

THEOREM 2.6 ([4, Corollaries 4.14 and 4.15]). *Assume* $(\pi_\varepsilon)_{\varepsilon \in \,]0,\varepsilon_0]}$ *is a family of local semiflows that converges singularly to the local semiflow* $\pi_0$, $\beta \in \,]0, \infty[$ *and* $\widetilde{N}$ *is a singularly strongly admissible set with respect to* $\beta$ *and* $(\pi_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$. *Moreover, suppose that* $S_0 := \mathrm{Inv}_{\pi_0}(\widetilde{N})$ *and* $(M_{p,0})_{p \in P}$ *is a* $\prec$-*ordered Morse decomposition of* $S_0$ *relative to* $\pi_0$. *For each* $p \in P$, *let* $V_p \subset \widetilde{N}$ *be closed in* $X_0$ *and such that* $M_{p,0} = \mathrm{Inv}_{\pi_0}(V_p) \subset \mathrm{Int}_{X_0}(V_p)$.

*Let* $\eta \in \,]0, \beta]$. *For* $\varepsilon \in \,]0, \varepsilon_0]$ *and* $p \in P$ *set* $S_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}([\widetilde{N}]_{\varepsilon,\eta})$ *and* $M_{p,\varepsilon} := \mathrm{Inv}_{\pi_\varepsilon}([V_p]_{\varepsilon,\eta})$. *Then there is an* $\widetilde{\varepsilon} \in \,]0, \varepsilon_0]$ *such that for every* $\varepsilon \in \,]0, \widetilde{\varepsilon}]$ *and* $p \in P$, $S_\varepsilon \subset \mathrm{Int}_{Z_\varepsilon}([\widetilde{N}]_{\varepsilon,\eta})$, $M_{p,\varepsilon} \subset \mathrm{Int}_{Z_\varepsilon}([V_p]_{\varepsilon,\eta})$, $p \in P$, *and the family* $(M_{p,\varepsilon})_{p \in P}$ *is a* $\prec$-*ordered Morse decomposition of* $S_\varepsilon$ *relative to* $\pi_\varepsilon$.

Suppose that $S$ is a compact invariant set relative to $\pi$ and let $(M_p)_{p \in P}$ be a $\prec$-ordered Morse decomposition of $S$ (relative to $\pi$). An *index filtration*

*for* $(\pi, S, (M_p)_{p \in P})$ is a family $\mathcal{N} = (N(I))_{I \in \mathcal{A}(\prec)}$ of closed subsets of $Y$ such that

(1) for each $I \in \mathcal{A}(\prec)$, the pair $(N(I), N(\emptyset))$ is an FM-index pair in $M(I)$,
(2) for any $I_1, I_2 \in \mathcal{A}(\prec)$, $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ and $N(I_1 \cup I_2) = N(I_1) \cup N(I_2)$.

Moreover, $\mathcal{N}$ is called *strongly $\pi$-admissible* if $N(P)$ is strongly $\pi$-admissible.

Let $\mathcal{N}$ be a strongly $\pi$-admissible index filtration for $(\pi, S, (M_p)_{p \in P})$. For $J \in \mathcal{I}(\prec)$ the set $M(J)$ is an isolated invariant set and we write $H(J) = H(\pi, J) := H(\pi, M(J))$. If $(I, J) \in \mathcal{I}_2(\prec)$, then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$, where $IJ := I \cup J$. Hence there is the corresponding homology index sequence

$$\longrightarrow H_q(I) \xrightarrow{i_{I,J}} H_q(IJ) \xrightarrow{p_{I,J}} H_q(J) \xrightarrow{\partial_{I,J}} H_{q-1}(I) \longrightarrow$$

of $(\pi, M(IJ), M(I), M(J))$. Using the filtration $\mathcal{N}$ one proves that for every triple $(I, J, K) \in \mathcal{I}_3(\prec)$ the following diagram, made up of the four homology index sequences defined by $(I, J, K)$, commutes:
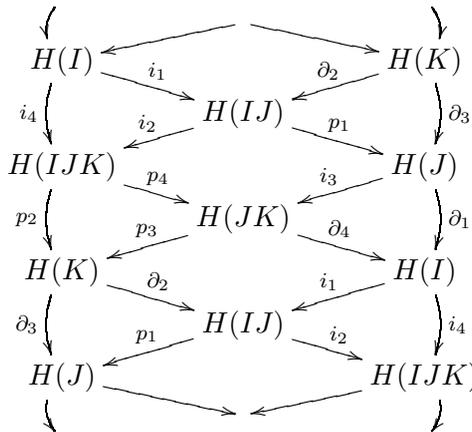


The collection of all the homology indices $H(\pi, M(J))$, $J \in \mathcal{I}(\prec)$, and all the maps $i_{I,J}$, $p_{I,J}$ and $\partial_{I,J}$, $(I, J) \in \mathcal{I}_2(\prec)$, is called the *homology index braid of* $(\pi, S, (M_p)_{p \in P})$ and is denoted by $\mathcal{H}(\pi, S, (M_p)_{p \in P})$.

For the rest of this section assume that, for $i = 1, 2$, $\pi_i$ is a local semiflow on the metric space $Y_i$, $S_i$ is an isolated invariant set and $(M_{p,i})_{p \in P}$ is a $\prec$-ordered Morse decomposition of $S_i$ relative to $\pi_i$. Write $M_i(I) = M_{\pi_i, S_i}(I)$, $H_i(I) = H(\pi_i, M_i(I))$ and $\mathcal{H}_i := \mathcal{H}(\pi_i, S_i, (M_{p,i})_{p \in P})$, for $i = 1, 2$ and $I \in \mathcal{I}(\prec)$.

Suppose $\theta := (\theta(J))_{J \in \mathcal{I}(\prec)}$ is a family $\theta(J) \colon H_1(J) \to H_2(J)$, $J \in \mathcal{I}(\prec)$, of maps such that, for all $(I, J) \in \mathcal{I}_2(\prec)$, the diagram

$$(2.1) \quad \begin{array}{ccccccccc}
\longrightarrow & H_{1,q}(I) & \xrightarrow{i_{I,J}} & H_{1,q}(IJ) & \xrightarrow{p_{I,J}} & H_{1,q}(J) & \xrightarrow{\partial_{I,J}} & H_{1,q-1}(I) & \longrightarrow \\
& \downarrow{\theta(I)} & & \downarrow{\theta(IJ)} & & \downarrow{\theta(J)} & & \downarrow{\theta(I)} & \\
\longrightarrow & H_{2,q}(I) & \xrightarrow{i_{I,J}} & H_{2,q}(IJ) & \xrightarrow{p_{I,J}} & H_{2,q}(J) & \xrightarrow{\partial_{I,J}} & H_{2,q-1}(I) & \longrightarrow
\end{array}$$

commutes. Then we say that $\theta$ is a *morphism from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ and we write $\theta \colon \mathcal{H}_1 \to \mathcal{H}_2$. If each $\theta(J)$ is an isomorphism, then we say that $\theta$ is an *isomorphism* and that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *isomorphic* homology index braids, and we write $\mathcal{H}_1 \cong \mathcal{H}_2$.

We may analogously define the concept of cohomology index braids and morphisms between them: just use cohomology instead of homology and reverse all the arrows.

PROPOSITION 2.7. *Let $X$ and $X'$ be metric spaces, $\pi$ be a local semiflow on $X$ and $\pi'$ be a local semiflow on $X'$. Let $\gamma \colon X \to X'$ be a homeomorphism which conjugates $\pi$ with $\pi'$. Let $S$ be an isolated $\pi$-invariant set which has a strongly $\pi$-admissible isolating neighborhood, and $(M_p)_{p \in P}$ be a Morse decomposition of $S$ relative to $\pi$. Set $S' = \gamma(S)$ and $M_p' = \gamma(M_p)$, $p \in P$.*

*Then $S'$ is an isolated $\pi'$-invariant set which has a strongly $\pi'$-admissible isolating neighborhood, and $(M_p')_{p \in P}$ is a Morse decomposition of $S'$ relative to $\pi'$. Moreover,*

$$\mathcal{H}(\pi, S, (M_p)_{p \in P}) \cong \mathcal{H}(\pi', S', (M_p')_{p \in P}).$$

*Proof.* Intuitively, this result is clear. A rigorous proof follows by an application of [8, Theorem 3.2] in the case of homology, and of [18, Theorem 4.2] in the case of cohomology. ∎

We will require the following singular continuation principle for homology index braids and connection matrices established in [7].

THEOREM 2.8 ([7, Theorem 3.10]). *Assume the hypotheses (and thus also the conclusions) of Theorem 2.6 hold and let $\widetilde{\varepsilon} > 0$ be as in that theorem. Fix $\widetilde{\beta}_0 \in \,]0, \beta[$. Suppose that there exists a $\beta_0 > 0$ such that for all $\varepsilon \in \,]0, \varepsilon_0]$ and all $\eta \in \,]0, \beta_0]$ the set $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon, \eta))$ is contractible. Then there exists an $\varepsilon_c \in \,]0, \widetilde{\varepsilon}]$ such that*

$$\mathcal{H}(\pi_0, S_0, (M_p)_{p \in P}) \cong \mathcal{H}(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p \in P})), \qquad \varepsilon \in \,]0, \varepsilon_c].$$

Theorem 2.8 refines the corresponding singular Conley index continuation principle established in [2].

REMARK 2.9. The above results were originally proved for singular homology. Analogous definitions and results hold for the Alexander–Spanier cohomology (with coefficients in $G$) with the obvious modifications (cf. [6]).

**3. A finite-dimensional perturbation problem.** In this section we state and prove a continuation principle for homology index braids and connection matrices for a class of singularly perturbed ordinary differential equations defined on a finite-dimensional space (cf. Theorems 3.1 and 3.6). Throughout this section, we assume the reader's familiarity with the results from [17, Section 3].

Let $(E, |\cdot|)$ be a finite-dimensional Banach space. Given a $C^1$-map $g\colon E \to E$ and $\varepsilon > 0$ let $\pi_{\varepsilon,g}$ be the local (semi)flow generated by the following ordinary differential equation on $E \times E$:

$$(3.1) \qquad \dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v + g(u)), \quad (u,v) \in E.$$

Furthermore, let $\pi_g'$ be the local (semi)flow on $E$ generated by the following ordinary differential equation on $E$:

$$(3.2) \qquad \dot{u} = g(u), \quad u \in E.$$

One of the goals of this section is to prove the following result.

THEOREM 3.1. *Let $N' \subset E$ be a compact isolating neighborhood relative to $\pi_g'$. Set $S' := \mathrm{Inv}_{\pi_g'}(N')$ and let $(M_p')_{p \in P}$ be a $\prec$-ordered Morse decomposition for $S'$ relative to $\pi_g'$. For each $p \in P$, let $V_p' \subset N'$ be closed in $E$ such that $M_p' = \mathrm{Inv}_{\pi_g'}(V_p') \subset \mathrm{Int}_E(V_p')$. For all $\alpha > 0$, $\varepsilon > 0$ and $p \in P$, set $S_{\varepsilon,\alpha} := \mathrm{Inv}_{\pi_{\varepsilon,g}}(N_\alpha')$ and $M_{p,\varepsilon,\alpha} := \mathrm{Inv}_{\pi_{\varepsilon,g}}(V_{p,\alpha}')$, where*

$$N_\alpha' = N_{\alpha,g}' := \{(u,v) \mid u \in N' \text{ and } |v - g(u)| \le \alpha\},$$
$$V_{p,\alpha}' = V_{p,\alpha,g}' := \{(u,v) \mid u \in V_p' \text{ and } |v - g(u)| \le \alpha\}.$$

*Then for every $\alpha \in \,]0, \infty[$, there exists an $\varepsilon_0 = \varepsilon_0(\alpha) \in \,]0, \infty[$ such that for every $\varepsilon \in \,]0, \varepsilon_0]$, the set $N_\alpha'$ is an isolating neighborhood of $S_{\varepsilon,\alpha}$ relative to $\pi_{\varepsilon,g}$, and for every $p \in P$, $V_{p,\alpha}'$ is an isolating neighborhood of $M_{p,\varepsilon,\alpha}$ relative to $\pi_{\varepsilon,g}$. Moreover, the family $(M_{p,\varepsilon,\alpha})_{p \in P}$ is a $\prec$-ordered Morse decomposition for $S_{\varepsilon,\alpha}$ relative to $\pi_{\varepsilon,g}$ and*

$$\mathcal{H}(\pi_{\varepsilon,g}, S_{\varepsilon,\alpha}, (M_{p,\varepsilon,\alpha})_{p \in P}) \cong \mathcal{H}(\pi_g', S', (M_p')_{p \in P}).$$

The proof of Theorem 3.1 will be based on Theorem 2.8. We proceed similarly to the proof of [17, Theorem 3.1]. Let $U$ be a bounded open neighborhood of $N'$ and $\widetilde{g}\colon E \to E$ be a $C^1$-map such that $g|U = \widetilde{g}|U$ and $\sup_{u \in E}(|\widetilde{g}(u)| + |D\widetilde{g}(u)|) < \infty$. The existence of $\widetilde{g}$ follows since $E$ is finite-dimensional.

Since the differential equations defining $\pi_{\varepsilon,g}$ and $\pi_{\varepsilon,\widetilde{g}}$ coincide on the open neighborhood $U \times E$ of $N_\alpha'$ in $E \times E$ it follows that $N_\alpha'$ (resp. $V_{p,\alpha}'$) is an isolating neighborhood relative to $\pi_{\varepsilon,g}$ if and only if $N_\alpha'$ (resp. $V_{p,\alpha}'$) is an isolating neighborhood relative to $\pi_{\varepsilon,\widetilde{g}}$ and

$$\mathrm{Inv}_{\pi_{\varepsilon,g}}(N_\alpha') = S_{\varepsilon,\alpha} = \mathrm{Inv}_{\pi_{\varepsilon,\widetilde{g}}}(N_\alpha'),$$
$$\mathrm{Inv}_{\pi_{\varepsilon,g}}(V_{p,\alpha}') = M_{p,\varepsilon,\alpha} = \mathrm{Inv}_{\pi_{\varepsilon,\widetilde{g}}}(V_{p,\alpha}'), \quad p \in P.$$

Moreover, $(M_{p,\varepsilon,\alpha})_{p\in P}$ is a $\prec$-ordered Morse decomposition for $S_{\varepsilon,\alpha}$ relative to $\pi_{\varepsilon,g}$ if and only if $(M_{p,\varepsilon,\alpha})_{p\in P}$ is a $\prec$-ordered Morse decomposition for $S_{\varepsilon,\alpha}$ relative to $\pi_{\varepsilon,\widetilde{g}}$.

Similarly, $N'$ (resp. $V_p'$) is an isolating neighborhood relative to $\pi_g'$ if and only if $N'$ (resp. $V_p'$) is an isolating neighborhood relative to $\pi_{\widetilde{g}}'$ and

$$\mathrm{Inv}_{\pi_g'}(N') = S' = \mathrm{Inv}_{\pi_{\widetilde{g}}'}(N'), \quad \mathrm{Inv}_{\pi_g'}(V_p') = M_p' = \mathrm{Inv}_{\pi_{\widetilde{g}}'}(V_p'), \quad p \in P.$$

Furthermore, $(M_p')_{p\in P}$ is a $\prec$-ordered Morse decomposition for $S'$ relative to $\pi_g'$ if and only if $(M_p')_{p\in P}$ is a $\prec$-ordered Morse decomposition for $S'$ relative to $\pi_{\widetilde{g}}'$. Therefore,

$$\mathcal{H}(\pi_{\varepsilon,g}, S_{\varepsilon,\alpha}, (M_{p,\varepsilon,\alpha})_{p\in P}) \cong \mathcal{H}(\pi_g', S', (M_p')_{p\in P})$$

if and only if

$$\mathcal{H}(\pi_{\varepsilon,\widetilde{g}}, S_{\varepsilon,\alpha}, (M_{p,\varepsilon,\alpha})_{p\in P}) \cong \mathcal{H}(\pi_{\widetilde{g}}', S', (M_p')_{p\in P}).$$

Hence we may assume, without loss of generality, that

$$C := \sup_{u\in E}(|g(u)| + |Dg(u)|) < \infty.$$

In particular, $g$ is globally Lipschitzian and so both $\pi_{\varepsilon,g}$, $\varepsilon > 0$, and $\pi_g'$ are global semiflows. We write $\pi_\varepsilon := \pi_{\varepsilon,g}$, $\varepsilon > 0$, and $\pi' := \pi_g'$ for short.

Notice that the map $\Phi\colon E\times E \to E\times E$, $\Phi(u,v) = (u,w) := (u, v - g(u))$ is a $C^1$-diffeomorphism with inverse $\Phi^{-1}$ given by $\Phi^{-1}(u,w) = (u,v) := (u, w + g(u))$. Let $\widetilde{\pi}_\varepsilon$ be the conjugate of $\pi_\varepsilon$ via $\Phi$, i.e. $(u,w)\widetilde{\pi}_\varepsilon t := \Phi((\Phi^{-1}(u,w))\pi_\varepsilon t)$, $(u,w) \in E \times E$, $t \in [0,\infty[$. Note that $\widetilde{\pi}_\varepsilon$ is the semiflow generated by the equation

$$(3.3) \qquad \dot{u} = w + g(u), \quad \dot{w} = -(1/\varepsilon)w - Dg(u)(w + g(u)).$$

Let $B_\alpha$ be the closed ball in $E$ with radius $\alpha$ centered at zero. It follows that $\Phi(N_\alpha') = N' \times B_\alpha$ (resp. $\Phi(V_{p,\alpha}') = V_p' \times B_\alpha$, $p \in P$).

We have the following

PROPOSITION 3.2. *For every* $\alpha \in ]0,\infty[$, *there exists an* $\varepsilon_0 = \varepsilon_0(\alpha) \in ]0,\infty[$ *such that for all* $\varepsilon \in ]0,\varepsilon_0]$ *and* $p \in P$, *the set* $N' \times B_\alpha$ (*resp.* $V_p' \times B_\alpha$) *is an isolating neighborhood of* $\widetilde{S}_{\varepsilon,\alpha}$ (*resp.* $\widetilde{M}_{p,\varepsilon,\alpha}$) *relative to* $\widetilde{\pi}_\varepsilon$, *where*

$$\widetilde{S}_{\varepsilon,\alpha} = \mathrm{Inv}_{\widetilde{\pi}_\varepsilon}(N' \times B_\alpha), \quad \widetilde{M}_{p,\varepsilon,\alpha} = \mathrm{Inv}_{\widetilde{\pi}_\varepsilon}(V_p' \times B_\alpha), \quad p \in P.$$

*Moreover, the family* $(\widetilde{M}_{p,\varepsilon,\alpha})_{p\in P}$ *is a* $\prec$-*ordered Morse decomposition for* $\widetilde{S}_{\varepsilon,\alpha}$ *relative to* $\widetilde{\pi}_\varepsilon$ *and*

$$\mathcal{H}(\widetilde{\pi}_\varepsilon, \widetilde{S}_{\varepsilon,\alpha}, (\widetilde{M}_{p,\varepsilon,\alpha})_{p\in P}) \cong \mathcal{H}(\pi', S', (M_p')_{p\in P}).$$

*Proof.* Define $X_0 = Y_\varepsilon = E$, $\theta_\varepsilon = 0$ and $d_0(u,u') = d_\varepsilon(u,u') = |u - u'|$ for all $\varepsilon > 0$ and $u, u' \in E$. It follows that $N' \times B_\alpha = [N']_{\varepsilon,\alpha}$ for all $\varepsilon > 0$ and $\alpha > 0$. Notice that $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon, \alpha)) = B_\alpha$ for all $\varepsilon > 0$ and

$\alpha > 0$, and so $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon, \alpha))$ is contractible. The proof of [17, Lemma 3.6], shows that the set $N'$ is singularly strongly admissible with respect to $\alpha$ and the family $(\widetilde{\pi}_\varepsilon)_{\varepsilon > 0}$ singularly converges to $\pi'$. Now an application of Theorems 2.6 and 2.8 completes the proof. ∎

PROPOSITION 3.3. *Let $\alpha \in \,]0, \infty[$ and let $\varepsilon_0 = \varepsilon_0(\alpha) \in \,]0, \infty[$ be as in Proposition 3.2. Then for every $\varepsilon \in \,]0, \varepsilon_0]$ and $p \in P$, the set $N'_\alpha$ (resp. $V'_{p,\alpha}$) is an isolating neighborhood of $S_{\varepsilon,\alpha}$ (resp. $M_{p,\varepsilon,\alpha}$) relative to $\pi_\varepsilon$, the family $(M_{p,\varepsilon,\alpha})_{p \in P}$ is a $\prec$-ordered Morse decomposition for $S_{\varepsilon,\alpha}$ relative to $\pi_\varepsilon$ and*

$$\mathcal{H}(\pi_\varepsilon, S_{\varepsilon,\alpha}, (M_{p,\varepsilon,\alpha})_{p \in P}) \cong \mathcal{H}(\widetilde{\pi}_\varepsilon, \widetilde{S}_{\varepsilon,\alpha}, (\widetilde{M}_{p,\varepsilon,\alpha})_{p \in P}).$$

*Proof.* This follows from Proposition 2.7. ∎

*Proof of Theorem 3.1.* This follows from Propositions 3.3 and 3.2. ∎

We shall now generalize Theorem 3.1 to comprise isolating neighborhoods which are more general than $N'_\alpha$. As in [17], define $\mathcal{T}_g$ to be the set of all functions $z \colon \mathbb{R} \to E \times E$ such that there is a full bounded solution $u \colon \mathbb{R} \to E$ of $\pi'_g$ so that $z(t) = (u(t), g(u(t)))$ for all $t \in \mathbb{R}$. Thus, defining the map $\Gamma_g \colon E \to E \times E$ by $\Gamma_g(\xi) = (\xi, g(\xi))$, $\xi \in E$, we see that $\mathcal{T}_g$ is the set of all functions $z \colon \mathbb{R} \to E \times E$ such that there is a full bounded solution $u \colon \mathbb{R} \to E$ of $\pi'_g$ with $z = \Gamma_g \circ u$.

PROPOSITION 3.4. *Set $\pi'_g := \pi'$ and $\Gamma := \Gamma_g$. Let $K'$ be a compact isolated invariant set relative to $\pi'$. Then $\Gamma(K')$ is a $\mathcal{T}_g$-isolated invariant set. Let $N$ be a bounded $\mathcal{T}_g$-isolating neighborhood of $\Gamma(K')$. Let $\beta \in \,]0, \infty[$ and $B'_\beta$ be the closed $\beta$-neighborhood of $K'$ in $E$. For each $\alpha \in \,]0, \infty[$, define*

$$B'_{\beta,\alpha} := \{(u, v) \mid u \in B'_\beta, \, |v - g(u)| \leq \alpha\}.$$

*Then there exist an $\alpha \in \,]0, \infty[$ and a $\beta \in \,]0, \infty[$ such that $B'_\beta$ is an isolating neighborhood of $K'$ relative to $\pi'$ and $B'_{\beta,\alpha} \subset N$. Moreover, there exists an $\varepsilon_1 \in \,]0, \infty[$ such that for every $\varepsilon \in \,]0, \varepsilon_1]$, $\mathrm{Inv}_{\pi_\varepsilon}(B'_{\beta,\alpha}) = \mathrm{Inv}_{\pi_\varepsilon}(N)$.*

*Proof.* This follows from [17, Lemma 3.8 and proof of Theorem 3.9]. ∎

We also need the following auxiliary result:

PROPOSITION 3.5. *Set $\pi'_g := \pi'$ and $\Gamma := \Gamma_g$. Let $K'$ be a compact isolated invariant set relative to $\pi'$ and $K := \Gamma(K')$. Let $\mathcal{T}$ be the set of all $(u, v) \in \mathcal{T}_g$ such that $(u(t), v(t)) \in K$ for all $t \in \mathbb{R}$. Moreover, let $(M'_p)_{p \in P}$ be a $\prec$-ordered Morse decomposition of $K'$ relative to $\pi'$ and define $M_{p,\Gamma} := \Gamma(M'_p)$, $p \in P$. For every $I \in \mathcal{I}(\prec)$ set*

$$M'(I) := \bigcup_{p,q \in I} \mathrm{CS}_{\pi',K'}(M'_p, M'_q)$$

and $M_\Gamma(I) := \Gamma(M'(I))$. Then $(M_{p,\Gamma})_{p\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition and

$$(3.4) \qquad M_\Gamma(I) = \bigcup_{p,q\in I} \mathrm{CS}_\mathcal{T}(M_p, M_q), \quad I \in \mathcal{I}(\prec).$$

*Moreover, the sets $K$ and $M_{p,\Gamma}$, $p \in P$, and $M_\Gamma(I)$, $I \in \mathcal{I}(\prec)$, are $\mathcal{T}_g$-isolated invariant sets.*

*Proof.* Write $M_p = M_{p,\Gamma}$ and $M(I) = M_\Gamma(I)$ for short. We see either directly or using [4, Corollary 3.6] that $M'_p$, $p \in P$, and $M'(I)$, $I \in \mathcal{I}(\prec)$, are compact isolated invariant sets relative to $\pi'$. By [17, Lemma 3.8], the sets $K$, $M_p$, $p \in P$, and $M(I)$, $I \in \mathcal{I}(\prec)$, are $\mathcal{T}_g$-isolated invariant sets. Moreover, the sets $M_p$ are closed in $E \times E$ (being compact in $E \times E$) and pairwise disjoint ($\Gamma$ being one-to-one). If $p \in P$ and $(\overline{u}, \overline{v}) \in M_p$ are arbitrary, then $\overline{u} \in M'_p$ so there is a full solution $u$ of $\pi'$ lying in $M'_p$ with $\overline{u} = u(0)$. Thus $u$ lies in $K'$ and so $z := \Gamma \circ u$ lies in $M_p \subset K$ and is an element of $\mathcal{T}_g$. Hence $z \in \mathcal{T}$, $z$ lies in $M_p$ and $z(0) = (\overline{u}, \overline{v})$. It follows that $M_p$ is $\mathcal{T}$-invariant.

Let $(u, v) \in \mathcal{T}$ be arbitrary. Then $u$ is a full solution of $\pi'$ lying in $K'$. Thus either $u$ lies in $M'_k$ for some $k \in P$, which implies that $(u, v)$ lies in $M_k$, or else there are $k, l \in P$ with $k \prec l$, $\alpha(u) \subset M'_l$ and $\omega(u) \subset M'_k$. In the latter case it is clear from the continuity of $\Gamma$ that $\alpha(u, v) \subset M_l$ and $\omega(u, v) \subset M_k$. We have proved that $(M_p)_{p\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition.

Now let $p, q \in P$ be arbitrary. We claim that

$$\Gamma(\mathrm{CS}_{\pi',K'}(M'_p, M'_q)) = \mathrm{CS}_\mathcal{T}(M_p, M_q).$$

If $u$ is a full solution of $\pi'$ lying in $K'$ with $\alpha(u) \subset M'_p$ and $\omega(u) \subset M'_q$ then, clearly, $z := \Gamma \circ u \in \mathcal{T}_g$, $z$ lies in $K$, $\alpha(z) \subset M_p$ and $\omega(z) \subset M_q$. Thus

$$\Gamma(\mathrm{CS}_{\pi',K'}(M'_p, M'_q)) \subset \mathrm{CS}_\mathcal{T}(M_p, M_q).$$

Conversely, if $z \in \mathcal{T}$ with $\alpha(z) \subset M_p$ and $\omega(z) \subset M_q$ then $z = \Gamma \circ u$, where $u$ is a full solution of $\pi'$ lying in $K'$. Clearly, $\alpha(u) \subset M'_p$ and $\omega(u) \subset M'_q$. Hence

$$\Gamma(\mathrm{CS}_{\pi',K'}(M'_p, M'_q)) \supset \mathrm{CS}_\mathcal{T}(M_p, M_q).$$

This proves our claim, which, in turn, immediately implies 3.4. ∎

The generalization of Theorem 3.1 reads as follows:

THEOREM 3.6. *Assume the hypotheses (and hence the conclusions) of Proposition 3.5. Let $N$ be a bounded $\mathcal{T}_g$-isolating neighborhood of $K$ and $N_p \subset N$ be a $\mathcal{T}_g$-isolating neighborhood of $M_{p,\Gamma}$, $p \in P$. For $\varepsilon \in \,]0, \infty[$, set $\pi_{\varepsilon,g} := \pi_\varepsilon$, $K_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}(N)$, and $M_{p,\varepsilon,\Gamma} := \mathrm{Inv}_{\pi_\varepsilon}(N_p)$, $p \in P$.*

*Then there is an $\widetilde{\varepsilon}_0 \in \,]0, \infty[$ such that for every $\varepsilon \in \,]0, \widetilde{\varepsilon}_0]$, $N$ is an isolating neighborhood of $K_\varepsilon$ relative to $\pi_\varepsilon$, and for every $p \in P$, $N_p$ is an isolating neighborhood of $M_{p,\varepsilon,\Gamma}$ relative to $\pi_\varepsilon$. Moreover, the family $(M_{p,\varepsilon,\Gamma})_{p\in P}$ is a*

$\prec$-ordered Morse decomposition of $K_\varepsilon$ and

$$\mathcal{H}(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon,\Gamma})_{p\in P}) \cong \mathcal{H}(\pi', K', (M'_p)_{p\in P}).$$

*Proof.* Let $\beta \in \,]0,\infty[$. Let $B'_\beta$ be the closed $\beta$-neighborhood of $K'$ in $E$. For $p \in P$, let $B'_{p,\beta}$ be the closed $\beta$-neighborhood of $M'_p$ in $E$. For each $\alpha \in \,]0,\infty[$, define

$$B'_{\beta,\alpha} := \{(u,v) \mid u \in B'_\beta, \, |v - g(u)| \leq \alpha\},$$
$$B'_{p,\beta,\alpha} := \{(u,v) \mid u \in B'_{p,\beta}, \, |v - g(u)| \leq \alpha\}, \quad p \in P.$$

Proposition 3.4 implies that there exist an $\alpha \in \,]0,\infty[$ and a $\beta \in \,]0,\infty[$ such that $B'_\beta$ is an isolating neighborhood of $K'$ relative to $\pi'$ and $B'_{p,\beta}$ is an isolating neighborhood of $M'_p$ relative to $\pi'$ for $p \in P$. Moreover, $B'_{\beta,\alpha} \subset N$ and $B'_{p,\beta,\alpha} \subset N_p$ for $p \in P$. Furthermore, there exists an $\varepsilon_1 \in \,]0,\infty[$ such that for every $\varepsilon \in \,]0,\varepsilon_1]$, $\mathrm{Inv}_{\pi_\varepsilon}(B'_{\beta,\alpha}) = \mathrm{Inv}_{\pi_\varepsilon}(N)$ and $\mathrm{Inv}_{\pi_\varepsilon}(B'_{p,\beta,\alpha}) = \mathrm{Inv}_{\pi_\varepsilon}(N_p)$, $p \in P$.

Define $N' := B'_\beta$ and $V'_p := B'_{p,\beta}$, $p \in P$. Hence for every $\varepsilon \in \,]0,\varepsilon_1]$, we have $N'_\alpha = B'_{\beta,\alpha}$ and $V'_{p,\alpha} = B'_{p,\beta,\alpha}$, $p \in P$. Thus

(3.5) $\quad K' = \mathrm{Inv}_{\pi'}(N') \subset \mathrm{Int}_E(N'), \quad M'_p = \mathrm{Inv}_{\pi'}(V'_p) \subset \mathrm{Int}_E(V'_p), \quad p \in P.$

Moreover, for all $\varepsilon \in \,]0,\varepsilon_1]$,

(3.6)
$$K_\varepsilon = \mathrm{Inv}_{\pi_\varepsilon}(N) = \mathrm{Inv}_{\pi_\varepsilon}(B'_{\beta,\alpha}) = \mathrm{Inv}_{\pi_\varepsilon}(N'_\alpha),$$
$$M_{p,\varepsilon,\Gamma} = \mathrm{Inv}_{\pi_\varepsilon}(N_p) = \mathrm{Inv}_{\pi_\varepsilon}(B'_{p,\beta,\alpha}) = \mathrm{Inv}_{\pi_\varepsilon}(V'_{p,\alpha}), \quad p \in P.$$

Now an application of Theorem 3.1 together with (3.5) and (3.6) implies that there exists an $\widetilde{\varepsilon}_0 \in \,]0,\varepsilon_1]$ such that for all $\varepsilon \in \,]0,\widetilde{\varepsilon}_0]$, the family $(M_{p,\varepsilon,\Gamma})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $K_\varepsilon$ relative to $\pi_\varepsilon$ and

$$\mathcal{H}(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon,\Gamma})_{p\in P}) \cong \mathcal{H}(\pi', K', (M'_p)_{p\in P}). \quad \blacksquare$$

**4. Continuation of homology index braids for singularly perturbed hyperbolic equations.** In this section we will state and prove the main result of this paper (cf. Theorem 4.2). After recalling some notations from [17] we state a continuation result for Morse decomposition proved in [17] (cf. Theorem 4.1). Then we state our continuation principle for homology index braids and connection matrices for singularly perturbed hyperbolic equations (cf. Theorem 4.2). We assume the reader's familiarity with the results from [17].

Let $X$ be a real Hilbert space and $A \colon D(A) \subset X \to X$ be a positive selfadjoint operator with compact resolvent. Then $A$ is sectorial on $X$ and so it generates the corresponding family $(X^\alpha)_{\alpha\in[0,\infty[}$ of fractional power spaces (cf. [13]). Let $(\phi_\nu)_{\nu\in\mathbb{N}}$ be a complete $X$-orthonormal basis of $X$ consisting of eigenfunctions of $A$. Let $P_n \colon X \to X$ be the orthogonal projection of

$X$ onto the subspace spanned by the first $n$ eigenfunctions. Moreover, set $Q_n := I - P_n$ where $I$ is the identity map on $X$.

As in [12] and [17], we will assume the following Standing Hypothesis:

(4.1)    $N \in \{1, 2, 3\}$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and such that $X = L^2(\Omega)$ and $X^1$ is continuously included in $H^2(\Omega)$; $\gamma \in L^2(\Omega)$ and $\phi \colon \mathbb{R} \to \mathbb{R}$ is a $C^1$-function such that, for $N \geq 2$, there are constants $\bar{r}$ and $\overline{C} \in [0, \infty[$ with $|\phi'(\xi)| \leq \overline{C}(1+|\xi|^{\bar{r}})$ for all $\xi \in \mathbb{R}$. If $N = 3$ then $\bar{r} < 2$.

Define the map $f \colon X^{1/2} \to X$ by

$$f(u) = \widehat{\phi}(u) + \gamma, \quad u \in X^{1/2}.$$

We denote by $\pi'_f$ the local semiflow on $X^{1/2}$ generated by the abstract parabolic equation (see [13])

$$\dot{u} = -Au + f(u), \quad u \in X^{1/2}.$$

For $\varepsilon \in ]0, \infty[$ we also consider the local semiflow $\pi_{\varepsilon,f}$ on $X^{1/2} \times X$ generated by the damped hyperbolic equation

$$\dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v - Au + f(u)), \quad (u, v) \in X^{1/2} \times X.$$

Let $\Gamma_f \colon X^1 \to X^{1/2} \times X$ be the map defined by

$$\Gamma_f(u) := (u, -Au + f(u)), \quad u \in X^1.$$

Moreover, we denote by $\mathcal{T}_f$ the set of all maps $z \colon \mathbb{R} \to X^{1/2} \times X$ for which there is a full bounded solution $u$ of $\pi'_f$ such that $z(t) = \Gamma_f(u(t))$, $t \in \mathbb{R}$. In view of [17, Corollary 4.7], every full bounded solution of $\pi'_f$ lies in $X^1$, so the definition of $\mathcal{T}_f$ makes sense.

The following result has been proved in [17].

THEOREM 4.1 ([17, Theorem 7.4]). *For* $\varepsilon \in ]0, \infty[$, *set* $\pi' := \pi'_f$, $\pi_\varepsilon := \pi_{\varepsilon,f}$ *and* $\Gamma := \Gamma_f$. *Let* $K'$ *be a compact isolated invariant set relative to* $\pi'$ *and* $K := \Gamma(K')$. *Let* $\mathcal{T}$ *be the set of all* $(u, v) \in \mathcal{T}_f$ *such that* $(u(t), v(t)) \in K$ *for all* $t \in \mathbb{R}$. *Moreover, let* $(M'_p)_{p \in P}$ *a family of subsets of* $K'$ *which is a Morse decomposition of* $K'$ *relative to* $\pi'$ *and let* $M_p := \Gamma(M'_p)$, $p \in P$. *For every* $I \in \mathcal{I}(\prec)$ *set*

$$M'(I) := \bigcup_{p,q \in I} \mathrm{CS}_{\pi',K'}(M'_p, M'_q)$$

*and* $M(I) := \Gamma(M'(I))$. *Then* $(M_p)_{p \in P}$ *is a* $\prec$-*ordered* $\mathcal{T}$-*Morse decomposition and*

$$M(I) = \bigcup_{p,q \in I} \mathrm{CS}_{\mathcal{T}}(M_p, M_q), \quad I \in \mathcal{I}(\prec).$$

*Moreover, the sets* $K$, $M_p$, $p \in P$ *and* $M(I)$, $I \in \mathcal{I}(\prec)$, *are* $\mathcal{T}_f$-*isolated invariant sets.*

Let $N$ be a bounded $\mathcal{T}_f$-isolating neighborhood of $K$, $V_p \subset N$ be a $\mathcal{T}_f$-isolating neighborhood of $M_p$, $p \in P$, and $V_I \subset N$ be a $\mathcal{T}_f$-isolating neighborhood of $M(I)$, $I \in \mathcal{I}(\prec)$. For $\varepsilon \in ]0,\infty[$ set $K_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}(N)$, $M_{p,\varepsilon} := \mathrm{Inv}_{\pi_\varepsilon}(V_p)$, $p \in P$, and $M_\varepsilon(I) := \mathrm{Inv}_{\pi_\varepsilon}(V_I)$, $I \in \mathcal{I}(\prec)$. Then there is an $\widetilde{\varepsilon} \in ]0,\infty[$ such that for every $\varepsilon \in ]0,\widetilde{\varepsilon}]$, $N$ (resp. $V_p$, resp. $V_I$) is an isolating neighborhood of $K_\varepsilon$ (resp. $M_{p,\varepsilon}$, resp. $M_\varepsilon(I)$) relative to $\pi_\varepsilon$, for all $p \in P$ and all $I \in \mathcal{I}(\prec)$, the family $(M_{p,\varepsilon})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $K_\varepsilon$ and

$$M_\varepsilon(I) = \bigcup_{p,q\in I} \mathrm{CS}_{\pi_\varepsilon, K_\varepsilon}(M_{p,\varepsilon}, M_{q,\varepsilon}), \quad I \in \mathcal{I}(\prec).$$

We can now state our main result.

THEOREM 4.2. *Assume the hypotheses* (*and thus also the conclusions*) *of Theorem 4.1 hold and let $\widetilde{\varepsilon} \in ]0,\infty[$ be as in that theorem. Then there is an $\varepsilon_0 \in ]0,\widetilde{\varepsilon}]$ such that for every $\varepsilon \in ]0,\varepsilon_0]$,*

$$\mathcal{H}(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon})_{p\in P}) \cong \mathcal{H}(\pi', K', (M'_p)_{p\in P}).$$

Theorem 4.2 refines the corresponding Conley index continuation principle established in [17].

The rest of this section is devoted to the proof of our main theorem. We give the proof for the singular homology case. The Alexander–Spanier cohomology case is analogous.

Theorem 4.2 follows from a series of lemmas. For the rest of this section, for $n \in \mathbb{N}$ and $\theta \in [0,1]$, let $f_{n,\theta} \colon X^{1/2} \to X$ be defined by

$$\begin{aligned} f_{n,\theta} &= (1-\theta)f(u) + \theta P_n f(P_n u) \\ &= (1-\theta)(\widehat{\phi}(u) + \gamma) + \theta P_n(\widehat{\phi}(P_n u) + \gamma), \quad u \in X^{1/2}. \end{aligned}$$

The following lemma has been proved in [17].

LEMMA 4.3 ([17, Theorem 4.3 and Proposition 4.4]). *Let $N' \subset X^{1/2}$ be bounded and closed. Furthermore, let $(n_m)_m$ be a sequence in $\mathbb{N}$ with $n_m \to \infty$ and $(\theta_m)_m$ be an arbitrary sequence in $[0,1]$. For every $m \in \mathbb{N}$, let $u_m$ be a full solution of $\pi'_{f_{n_m,\theta_m}}$ lying in $N'$. Then there is a sequence $(m_k)_k$ with $m_k \to \infty$ and there is a full solution $u$ of $\pi'$ lying in $N'$ such that $u_{m_k}(t) \to u(t)$ in $X^{1/2}$, uniformly for $t$ lying in compact subsets of $\mathbb{R}$.*

LEMMA 4.4. *Let $N'$ be bounded and closed in $X^{1/2}$ with $K' := \mathrm{Inv}_{\pi'}(N') \subset \mathrm{Int}_{X^{1/2}}(N')$. For each $p \in P$, let $V'_p \subset N'$ be closed in $X^{1/2}$ such that $M'_p = \mathrm{Inv}_{\pi'}(V'_p) \subset \mathrm{Int}_{X^{1/2}}(V'_p)$. For $n \in \mathbb{N}$, $\theta \in [0,1]$ and $p \in P$ define $K'_{n,\theta} = \mathrm{Inv}_{\pi'_{f_{n,\theta}}}(N')$ and $M'_{p,n,\theta} = \mathrm{Inv}_{\pi'_{f_{n,\theta}}}(V'_p)$. Then there is an $n_0 \in \mathbb{N}$ so that whenever $n \geq n_0$ and $\theta \in [0,1]$, then $K'_{n,\theta} \subset \mathrm{Int}_{X^{1/2}}(N')$, $M'_{p,n,\theta} \subset$*

$\operatorname{Int}_{X^{1/2}}(V'_p)$, $p \in P$, and the family $(M'_{p,n,\theta})_{p \in P}$ is a $\prec$-ordered Morse decomposition of $K'_{n,\theta}$ relative to $\pi'_{f_{n,\theta}}$. Moreover, for all $n \geq n_0$ and all $\theta \in [0,1]$,

$$\mathcal{H}(\pi'_{f_{n,\theta}}, K'_{n,\theta}, (M'_{p,n,\theta})_{p \in P}) \cong \mathcal{H}(\pi', K', (M'_p)_{p \in P}).$$

*Proof.* This follows from [5, Corollaries 3.9 and 3.10]. ■

Given $n \in \mathbb{N}$ we consider the local semiflow $\pi'_n$ generated on the finite-dimensional space $Y_n := P_n(X^{1/2}) = P_n(X)$ by the ordinary differential equation

$$(4.2) \qquad \dot{u} = -Au + P_n f(P_n u), \qquad u \in Y_n.$$

The local semiflow $\pi'_n$ is the *n-Galerkin approximation of $\pi'$*.

Moreover, let $\pi''_n$ be the semiflow generated on $Z_n := Q_n(X^{1/2})$ by the evolution equation

$$(4.3) \qquad \dot{u} = -Au, \qquad u \in Z_n.$$

By [17, Proposition 4.2], and its proof, the space $Y_n$ is positively invariant relative to the local semiflow $\pi'_{f_{n,1}}$ and every bounded $\pi'_{f_{n,1}}$-invariant set is included in $Y_n$ and is $\pi'_n$-invariant. Moreover, every $\pi'_n$-invariant set is $\pi'_{f_{n,1}}$-invariant. Setting $K'_n := K'_{n,1}$ and $M'_{p,n} := M'_{p,n,1}$, $p \in P$, we thus see that, whenever $n \geq n_0$, where $n_0 \in \mathbb{N}$ is as in Lemma 4.4, then $K'_n$ is a compact $\pi'_n$-invariant set and the family $(M'_{p,n})_{p \in P}$ is a $\prec$-ordered Morse decomposition of $K'_n$ relative to $\pi'_n$. Moreover, the following continuation result has been proved in [5].

LEMMA 4.5. *Let $n_0 \in \mathbb{N}$ be as in Lemma 4.4. Then for all $n \geq n_0$,*

$$\mathcal{H}(\pi'_{f_{n,1}}, K'_n, (M'_{p,n})_{p \in P}) \cong \mathcal{H}(\pi'_n, K'_n, (M'_{p,n})_{p \in P}).$$

Now, Lemmas 4.4 and 4.5 imply the following result.

PROPOSITION 4.6 ([5, Theorem 3.11]). *Let $n_0 \in \mathbb{N}$ be as in Lemma 4.4. Then for all $n \geq n_0$,*

$$\mathcal{H}(\pi', K', (M'_p)_{p \in P}) \cong \mathcal{H}(\pi'_n, K'_n, (M'_{p,n})_{p \in P}).$$

LEMMA 4.7. *Let $(\varepsilon_m)_m$, $(n_m)_m$ and $(\theta_m)_m$ be sequences in $]0, \infty[$, $\mathbb{N}$ and $[0,1]$, respectively. Suppose that $\varepsilon_m \to 0$ and $n_m \to \infty$. For each $m \in \mathbb{N}$, let $(u_m, v_m)$ be a full solution of $\pi_{\varepsilon_m, f_{n_m, \theta_m}}$ lying in $N$. Then there is a subsequence of $((u_m, v_m))_m$, denoted again by $((u_m, v_m))_m$, and there is a full bounded solution $u$ of $\pi'$ such that $(u_m, v_m) \to (u, v)$ in $X^{1/2} \times X$, uniformly on compact subsets of $\mathbb{R}$. Here, $(u, v) = \Gamma \circ u$.*

*Proof.* This follows from [17, Theorem 5.13]. ■

LEMMA 4.8. *For $\varepsilon \in ]0, \infty[$, $n \in \mathbb{N}$, $\theta \in [0,1]$, $p \in P$ and $I \in \mathcal{I}(\prec)$, define $K_{\varepsilon, n, \theta} = \operatorname{Inv}_{\pi_{\varepsilon, f_{n, \theta}}}(N)$, $M_{p, \varepsilon, n, \theta} = \operatorname{Inv}_{\pi_{\varepsilon, f_{n, \theta}}}(V_p)$ and $M_{\varepsilon, n, \theta}(I) = \operatorname{Inv}_{\pi_{\varepsilon, f_{n, \theta}}}(V_I)$. Let $\widetilde{\varepsilon} \in ]0, \infty[$ be as in Theorem 4.1. Then there are $n' \in \mathbb{N}$*

and $\varepsilon' \in ]0, \widetilde{\varepsilon}]$ so that whenever $n \geq n'$, $\varepsilon \in ]0, \varepsilon']$ and $\theta \in [0,1]$, then $K_{\varepsilon,n,\theta} \subset \mathrm{Int}_{X^{1/2} \times X}(N)$, $M_{\varepsilon,p,n,\theta} \subset \mathrm{Int}_{X^{1/2} \times X}(V_p)$, $p \in P$, and the family $(M_{p,\varepsilon,n,\theta})_{p \in P}$ is a $\prec$-ordered Morse decomposition of $K_{\varepsilon,n,\theta}$ relative to $\pi_{\varepsilon,f_{n,\theta}}$.

*Proof.* It is well-known (cf. [3, Theorems 5.3 and 5.5]) that $N$ is strongly $\pi_{\varepsilon,f_{n,\theta}}$-admissible for all $\varepsilon \in ]0, \infty[$, $n \in \mathbb{N}$ and $\theta \in [0,1]$.

It follows from [17, Lemma 6.4] that there exist an $n_0' \in \mathbb{N}$ and an $\varepsilon_0' > 0$ such that for all $n \geq n_0'$, $\varepsilon \in ]0, \varepsilon_0']$ and $\theta \in [0,1]$ the sets $N$, $V_p$, $p \in P$, and $V_I$, $I \in \mathcal{I}(\prec)$, are isolating neighborhoods relative to $\pi_{\varepsilon,f_{n,\theta}}$.

Notice that

(4.4) $\qquad W \subset N$ implies that $\mathrm{Inv}_{\mathcal{T}_f}(W) = \mathrm{Inv}_{\mathcal{T}}(W)$.

Suppose the conclusion of the lemma does not hold. Then for all $n' \in \mathbb{N}$ and all $\varepsilon' \in ]0, \widetilde{\varepsilon}]$ there exist an $n \geq n'$, an $\varepsilon \in ]0, \varepsilon']$ and a $\theta \in [0,1]$ such that the family $(M_{p,\varepsilon,n,\theta})_{p \in P}$ is not a $\prec$-ordered Morse decomposition of $K_{\varepsilon,n,\theta}$ relative to $\pi_{\varepsilon,f_{n,\theta}}$. Hence there are sequences $(\varepsilon_m)_m$ in $]0, \infty[$, $(n_m)_m$ in $\mathbb{N}$ and $(\theta_m)_m$ in $[0,1]$ such that $\varepsilon_m \to 0$, $n_m \to \infty$ as $m \to \infty$ and, for all $m \in \mathbb{N}$, the family $(M_{p,\varepsilon_m,n_m,\theta_m})_{p \in P}$ is not a $\prec$-ordered Morse decomposition of $K_{\varepsilon_m,n_m,\theta_m}$ relative to $\pi_{\varepsilon_m,f_{n_m,\theta_m}}$.

For each $m \in \mathbb{N}$, let $\mathcal{T}_{\varepsilon_m,n_m,\theta_m}$ denote the set of all full solutions of $\pi_{\varepsilon_m,f_{n_m,\theta_m}}$ lying in $N$. It follows that

$$M_{p,\varepsilon_m,n_m,\theta_m} = \mathrm{Inv}_{\mathcal{T}_{\varepsilon_m,n_m,\theta_m}}(V_p), \quad p \in P, \, m \in \mathbb{N},$$
$$M_{\varepsilon_m,n_m,\theta_m}(I) = \mathrm{Inv}_{\mathcal{T}_{\varepsilon_m,n_m,\theta_m}}(V_I), \quad I \in I(\prec), \, m \in \mathbb{N}.$$

We claim that

(4.5) $\qquad\qquad \mathcal{T}_{\varepsilon_m,n_m,\theta_m} \to \mathcal{T} \quad$ as $m \to \infty$.

In fact, let $(m_k)_k$ be an arbitrary sequence in $\mathbb{N}$ with $m_k \to \infty$ and for every $k \in \mathbb{N}$ let $(u_k, v_k)$ be a full solution of $\pi_{\varepsilon_{m_k},f_{n_{m_k},\theta_{m_k}}}$ lying in $N$. Thus

$$\sup_{k \in \mathbb{N}} \sup_{t \in \mathbb{R}} \|(u_k(t), v_k(t))\|_{X^{1/2} \times X} < \infty$$

so, by Lemma 4.7, there is a subsequence of $((u_k, v_k))_k$, denoted again by $((u_k, v_k))_k$, and there is a full bounded solution $u$ of $\pi'$ such that $(u_k, v_k) \to (u,v)$ in $X^{1/2} \times X$, uniformly on compact subsets of $\mathbb{R}$. Here, $(u,v) = \Gamma \circ u$. Since $N$ is closed in $X^{1/2} \times X$, we see that $(u,v)$ lies in $N$, so that $(u,v)$ actually lies in $K$. Thus $(u,v) \in \mathcal{T}$ and claim (4.5) is proved.

We also claim that

(4.6) $\quad \mathcal{T}$ and $\mathcal{T}_{\varepsilon_m,n_m,\theta_m}$, $m \in \mathbb{N}$, are compact in $C(\mathbb{R} \to X^{1/2} \times X)$, translation and cut-and-glue invariant.

Recall that

(4.7) $\qquad\qquad \mathcal{T}_{\varepsilon_m,n_m,\theta_m} = \mathcal{T}_{\pi_{\varepsilon_m,f_{n_m,\theta_m}},N} \quad$ for all $m \in \mathbb{N}$.

Now (4.7) and [4, Proposition 2.7] imply the part of claim (4.6) concerning $\mathcal{T}_{\varepsilon_m,n_m,\theta_m}$, $m \in \mathbb{N}$.

In order to prove the compactness of $\mathcal{T}$, let $((u_\kappa, v_\kappa))_\kappa$ be an arbitrary sequence in $\mathcal{T}$. Thus, for every $\kappa \in \mathbb{N}$, $u_\kappa$ is a full solution of $\pi'$ lying in $K'$. Now an application of [17, Theorem 4.6] (with $f_\kappa \equiv f$) shows that there is a sequence $(\kappa_n)_n$ in $\mathbb{N}$ with $\kappa_n \to \infty$ and there is a full solution $u$ of $\pi'$ lying in $K'$ such that $u_{\kappa_n} \to u$ in $X^1$, uniformly on compact subsets of $\mathbb{R}$. Thus $(u_{\kappa_n}, v_{\kappa_n}) \to (u, v)$ in $X^{1/2} \times X$, uniformly on compact subsets of $\mathbb{R}$, where $(u, v) = \Gamma \circ u$. It follows that $(u, v) \in \mathcal{T}$ and so $\mathcal{T}$ is compact in $C(\mathbb{R} \to X^{1/2} \times X)$, as claimed. The translation and cut-and-glue invariance of $\mathcal{T}$ is obvious. This completes the proof of (4.6).

An application of [4, Theorem 3.3] together with (4.4)–(4.6) shows that there is an $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$, the family $(M_{p,\varepsilon_m,n_m,\theta_m})_{p\in P}$ is a $\prec$-ordered $\mathcal{T}_{\varepsilon_m,n_m,\theta_m}$-Morse decomposition. This is a contradiction. The lemma is proved. ∎

LEMMA 4.9. *Let* $n' \in \mathbb{N}$ *and* $\varepsilon' \in\, ]0, \infty[$ *be as in Lemma* 4.8. *Then for all* $n \geq n'$, $\varepsilon \in\, ]0, \varepsilon']$ *and* $\theta \in [0, 1]$,

$$\mathcal{H}(\pi_{\varepsilon,f_{n,\theta}}, K_{\varepsilon,n,\theta}, (M_{p,\varepsilon,n,\theta})_{p\in P}) \cong \mathcal{H}(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon})_{p\in P}).$$

*Proof.* Let $n \geq n'$ and $\varepsilon \in\, ]0, \varepsilon']$ be fixed. Let $(\theta_k)_k$ be a sequence in $[0, 1]$ such that $\theta_k \to \theta_0$ in $[0, 1]$. Then $f_{n,\theta_k}(u) \to f_{n,\theta_0}(u)$ in $X$, locally uniformly in $u \in X^{1/2}$. This implies, by the results in [16], that $\pi_{\varepsilon,f_{n,\theta_k}} \to \pi_{\varepsilon,f_{n,\theta_0}}$. Moreover, the results in [3] imply that $N$ is $(\pi_{f_{n,\theta_k}})_k$-admissible. Now Lemma 4.8 and the abstract regular continuation principle [5, Theorem 3.7] complete the proof. ∎

Given $\varepsilon \in\, ]0, \infty[$ and $n \in \mathbb{N}$ we consider the local semiflow $\pi_{\varepsilon,n}$ generated on the finite-dimensional space $X_n := P_n(X^{1/2}) \times P_n(X)$ by the ordinary differential equation

$$\dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v - Au + P_n f(P_n u)), \quad (u, v) \in X_n.$$

The local semiflow $\pi_{\varepsilon,n}$ is the *n-Galerkin approximation* of $\pi_\varepsilon$.

Moreover, let $\pi''_{\varepsilon,n}$ be the semiflow generated on $W_n := Q_n(X^{1/2}) \times Q_n(X)$ by the evolution equation

(4.8) $$\dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v - Au), \quad (u, v) \in W_n.$$

By [17, Proposition 5.3 and its proof], the space $X_n$ is positively invariant relative to the local semiflow $\pi_{\varepsilon,f_{n,1}}$, and every bounded $\pi_{\varepsilon,f_{n,1}}$-invariant set is included in $X_n$ and is $\pi_{\varepsilon,n}$-invariant. Moreover, every $\pi_{\varepsilon,n}$-invariant set is $\pi_{\varepsilon,f_{n,1}}$-invariant. Setting $K_{\varepsilon,n} := K_{\varepsilon,n,1}$ and $M_{p,\varepsilon,n} := M_{p,\varepsilon,n,1}$, $p \in P$, we thus see that, whenever $n \geq n'$, $\varepsilon \in\, ]0, \varepsilon']$, where $n' \in \mathbb{N}$ and $\varepsilon' \in\, ]0, \infty[$ are as in Lemma 4.8, then $K_{\varepsilon,n}$ is a compact $\pi_{\varepsilon,n}$-invariant set and $(M_{p,\varepsilon,n})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $K_{\varepsilon,n}$ relative to $\pi_{\varepsilon,n}$. Furthermore:

LEMMA 4.10. *Let $n' \in \mathbb{N}$ and $\varepsilon' \in \,]0, \infty[$ be as in Lemma 4.8. Then for $n \geq n'$ and $\varepsilon \in \,]0, \varepsilon']$,*

$$\mathcal{H}(\pi_{\varepsilon,n}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P}) \cong \mathcal{H}(\pi_{\varepsilon,f_{n,1}}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P}).$$

*Proof.* Fix $\varepsilon \in \,]0, \varepsilon']$ and $n \geq n'$. Let $\mathcal{N}'_{\varepsilon,n} = (N'_{\varepsilon,n}(I))_{I \in \mathcal{A}(\prec)}$ be a strongly $\pi_{\varepsilon,n}$-admissible index filtration for $(\pi_{\varepsilon,n}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P})$. (Strong $\pi_{\varepsilon,n}$-admissibility means, in this finite-dimensional case, simply that $N'_{\varepsilon,n}(P)$ is bounded in $X_n$.) Let $B = B_n$ be the unit ball in $W_n$. Since $|u\pi''_{\varepsilon,n}t|_{W_n} \leq e^{-\beta_n t}|u|_{W_n}$ for some $\beta_n \in \,]0, \infty[$ and all $u \in W_n$ and $t \in [0, \infty[$ it follows that, relative to $\pi''_{\varepsilon,n}$, $B$ is an isolating block for $\{0\}$ with empty exit set, so in particular, $B$ is positively invariant.

We define $N_{\varepsilon,n}(I) := N'_{\varepsilon,n}(I) + B \cong N'_{\varepsilon,n}(I) \times B$, $I \in \mathcal{A}(\prec)$. It is now a simple exercise to show that $\mathcal{N}_{\varepsilon,n} = (N_{\varepsilon,n}(I))_{I \in \mathcal{A}(\prec)}$ is a strongly $\pi_{\varepsilon,f_{n,1}}$-admissible index filtration for $(\pi_{\varepsilon,f_{n,1}}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P})$. We claim that

$$(4.9) \qquad \mathcal{H}(\pi_{\varepsilon,n}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P}) \cong \mathcal{H}(\pi_{\varepsilon,f_{n,1}}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P}).$$

In fact, let $J \in \mathcal{I}(\prec)$ be arbitrary. Choose $I, K \in \mathcal{A}(\prec)$ with $(I, J) \in \mathcal{I}_2(\prec)$ and $K = IJ$. Let $\phi \colon N'_{\varepsilon,n}(K)/N'_{\varepsilon,n}(I) \to N_{\varepsilon,n}(K)/N_{\varepsilon,n}(I)$ be inclusion induced and $\psi \colon N_{\varepsilon,n}(K)/N_{\varepsilon,n}(I) \to N'_{\varepsilon,n}(K)/N'_{\varepsilon,n}(I)$ be induced by the canonical projection $y + z \mapsto y$ of $X^{1/2} \times X = X_n \oplus W_n$ onto $X_n$. It follows that $\psi \circ \phi$ is the identity on $N'_{\varepsilon,n}(K)/N'_{\varepsilon,n}(I)$ while $\phi \circ \psi$ is homotopic to the identity on $N_{\varepsilon,n}(K)/N_{\varepsilon,n}(I)$ via the homotopy $N_{\varepsilon,n}(K)/N_{\varepsilon,n}(I) \times [0, 1] \to N_{\varepsilon,n}(K)/N_{\varepsilon,n}(I)$ induced by the homotopy $(y + z, \tau) \mapsto y + \tau z$ on $X^{1/2} \times X = X_n \oplus W_n$. The homotopy axiom for singular homology now implies that the map

$$\theta_{\mathcal{N}'_{\varepsilon,n}, \mathcal{N}_{\varepsilon,n}}(J) \colon H(\pi_{\varepsilon,n}, M_{\varepsilon,n}(J)) \to H(\pi_{\varepsilon,f_{n,1}}, M_{\varepsilon,n}(J))$$

induced by $\phi$ is an isomorphism. Claim (4.9) is proved, which in turn concludes the proof of the lemma. ∎

Now, Lemmas 4.9 and 4.10 imply the following proposition.

PROPOSITION 4.11. *Let $n' \in \mathbb{N}$ and $\varepsilon' \in \,]0, \infty[$ be as in Lemma 4.8. Then for $n \geq n'$ and $\varepsilon \in \,]0, \varepsilon']$,*

$$\mathcal{H}(\pi_{\varepsilon,n}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P}) \cong \mathcal{H}(\pi_{\varepsilon}, K_{\varepsilon}, (M_{p,\varepsilon})_{p \in P}).$$

LEMMA 4.12. *There exists an $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, the sets $N$ and $V_p$, $p \in P$, are $\mathcal{T}_{f_{n,1}}$-isolating neighborhoods of $K_n := \Gamma_{f_{n,1}}(K'_n)$ and $M_{p,n} := \Gamma_{f_{n,1}}(M'_{p,n})$, $p \in P$, respectively, where $N'$ and $V'_p$, $p \in P$, are as in Theorem 4.1.*

*Proof.* This follows from [17, Theorem 4.11]. ∎

We can now give the proof of Theorem 4.2.

*Proof of Theorem 4.2.* Let $n_0 \in \mathbb{N}$ be as in Lemma 4.4, $n' \in \mathbb{N}$ be as in Lemma 4.8 and $n_1 \in \mathbb{N}$ be as in Lemma 4.12. Define $\widetilde{n} := \max\{n_0, n', n_1\}$. Fix $n \geq \widetilde{n}$ arbitrarily. Proposition 4.6 implies that

$$(4.10) \qquad \mathcal{H}(\pi'_n, K'_n, (M'_{p,n})_{p \in P}) \cong \mathcal{H}(\pi', K', (M'_p)_{p \in P}).$$

Notice that

$$K'_n = \mathrm{Inv}_{\pi'_{f_{n,1}}}(N') = \mathrm{Inv}_{\pi'_n}(N' \cap Y_n) \subset \mathrm{Int}_{Y_n}(N' \cap Y_n),$$

$$M'_{p,n} = \mathrm{Inv}_{\pi'_{f_{n,1}}}(V'_p) = \mathrm{Inv}_{\pi'_n}(V'_p \cap Y_n) \subset \mathrm{Int}_{Y_n}(V'_p \cap Y_n), \quad p \in P.$$

Moreover, with the notation introduced in Section 3, it follows from Lemma 4.12 that

$$\mathrm{Inv}_{\mathcal{T}_{g_n}}(N \cap X_n) = \mathrm{Inv}_{\mathcal{T}_{f_{n,1}}}(N) = K_n \subset \mathrm{Int}_{X^{1/2} \times X}(N),$$

$$\mathrm{Inv}_{\mathcal{T}_{g_n}}(V_p \cap X_n) = \mathrm{Inv}_{\mathcal{T}_{f_{n,1}}}(V_p) = M_{p,n} \subset \mathrm{Int}_{X^{1/2} \times X}(V_p), \quad p \in P,$$

where $g_n \colon Y_n \to Y_n$ is defined by

$$g_n(u) = -Au + P_n(\widehat{\phi}(u) + \gamma), \quad u \in Y_n.$$

Hence

$$\mathrm{Inv}_{\mathcal{T}_{g_n}}(N \cap X_n) \subset \mathrm{Int}_{X_n}(N \cap X_n),$$

$$\mathrm{Inv}_{\mathcal{T}_{g_n}}(V_p \cap X_n) \subset \mathrm{Int}_{X_n}(V_p \cap X_n), \quad p \in P.$$

It follows from Theorem 3.6 that there exists an $\widetilde{\varepsilon}_0 \in\, ]0, \infty[$ such that for all $\varepsilon \in\, ]0, \widetilde{\varepsilon}_0]$, the family $(M_{p,\varepsilon,\Gamma_{g_n}})_{p \in P}$ is a $\prec$-ordered Morse decomposition of $\widetilde{K}_{\varepsilon,n}$ and $\mathcal{H}(\pi_{\varepsilon,n}, \widetilde{K}_{\varepsilon,n}, (M_{p,\varepsilon,\Gamma_{g_n}})_{p \in P}) \cong \mathcal{H}(\pi'_n, K'_n, (M'_{p,n})_{p \in P})$, where

$$\widetilde{K}_{\varepsilon,n} = \mathrm{Inv}_{\pi_{\varepsilon,n}}(N \cap X_n) = \mathrm{Inv}_{\pi_{\varepsilon,f_{n,1}}}(N) = K_{\varepsilon,n},$$

$$M_{p,\varepsilon,\Gamma_{g_n}} = \mathrm{Inv}_{\pi_{\varepsilon,n}}(V_p \cap X_n) = \mathrm{Inv}_{\pi_{\varepsilon,f_{n,1}}}(V_p) = M_{p,\varepsilon,n}, \quad p \in P.$$

Hence

$$(4.11) \qquad \mathcal{H}(\pi_{\varepsilon,n}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P}) \cong \mathcal{H}(\pi'_n, K'_n, (M'_{p,n})_{p \in P}), \quad \varepsilon \in\, ]0, \widetilde{\varepsilon}_0].$$

Define $\varepsilon_0 := \min\{\varepsilon', \widetilde{\varepsilon}_0\}$, where $\varepsilon' \in\, ]0, \infty[$ is as in Lemma 4.8, and let $\varepsilon \in\, ]0, \varepsilon_0]$. Proposition 4.11 implies that

$$(4.12) \qquad \mathcal{H}(\pi_{\varepsilon,n}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P}) \cong \mathcal{H}(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon})_{p \in P}).$$

Now (4.10)–(4.12) imply that

$$\mathcal{H}(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon})_{p \in P}) \cong \mathcal{H}(\pi', K', (M'_p)_{p \in P}), \quad \varepsilon \in\, ]0, \varepsilon_0],$$

and the proof is complete. ∎

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Departamento de Matemática, ICMC-USP
Caixa Postal 668
13.560-970 São Carlos SP, Brazil
E-mail: mdccarbi@icmc.usp.br

Institut für Mathematik
Universität Rostock
Universitätsplatz 1
18055 Rostock, Germany
E-mail: krzysztof.rybakowski@mathematik.uni-rostock.de