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Stratified model categories

by

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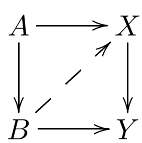
Abstract. The fourth axiom of a model category states that given a commutative square of maps, say $i : A \to B$, $g : B \to Y$, $f : A \to X$, and $p : X \to Y$ such that $gi = pf$, if i is a cofibration, p a fibration and either i or p is a weak equivalence, then a lifting (i.e. a map $h : B \to X$ such that $ph = g$ and $hi = f$) exists. We show that for many model categories the two conditions that either i or p above is a weak equivalence can be embedded in an infinite number of conditions which imply the existence of a lifting (roughly, the weak equivalence condition can be split between i and p). There is a similar modification of the fifth axiom. We call such model categories "stratified" and show that the simplest model categories have this property. Moreover, under some assumptions a category associated to the category of simplicial sets by a family of adjoint functors has this structure. Postnikov decompositions and n-types exist in any such category.

1. Introduction. In [10], Quillen introduced the notion of a model category in order to single out "those categories in which one can do homotopy theory". A model category is just an ordinary category with three classes of maps: *weak equivalences, fibrations* and *cofibrations* which are required to satisfy some axioms patterned on the properties of topological spaces.

For example, the fourth axiom (see below) combines the extension problem and the lifting problem into a single question: given a CW-inclusion $A \hookrightarrow B$, a mapping $X \to Y$, and a solid arrow diagram:



does a dashed arrow exist making the diagram commute? When Y is a single point this is the extension problem, and when A is the empty space this is

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the lifting problem. It is well known that many questions in topology can be phrased as special cases of this problem.

Of the five axioms of a model category, the last two—MC4 and MC5—play the most important role:

MC4. *Suppose that in the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array} \tag{1.1}$$

the map i is a cofibration, p is a fibration, and either i or p is a weak equivalence. Then a lifting exists (i.e. a map $h : B \to X$ such that $ph = g$ and $hi = f$).

MC5. *Each map $f : X \to Y$ can be factored as*

$$X \xrightarrow{i} Z \xrightarrow{p} Y$$

where i is a cofibration, p is a fibration, and moreover, we can choose either i or p to be a weak equivalence.

It is natural to wonder whether a lifting in MC4 exists under some different assumptions and whether some factorizations with different homotopy properties than those in MC5 exist.

The main goal of this paper is to establish that for many categories the answer to the above questions is "yes". Specifically, we show that in any stratified model category (see 3.8 for the definition) the condition that either i or p is a weak equivalence in either statement above can be replaced by any one of a countably infinite family of conditions which includes the two above as special cases.

In [4], P. Hirschhorn described D. M. Kan's notion of a cofibrantly generated model category. This is a model category in which there is a set of generating cofibrations and a set of generating acyclic cofibrations which permit a very straightforward description of the model category structure. Many cofibrantly generated model categories have some extra structure given by the fact that a map is a weak equivalence iff it is an "n-equivalence" for each nonnegative integer n. Moreover, a map is an n-equivalence iff it is an "n-epimorphism" and an "n-monomorphism" (see 3.2 for details).

This way of looking at weak equivalences leads us to introduce the main notion of this paper: a *stratified model category*, which is short for *stratified cofibrantly generated model category* (3.8). Suppose **C** is a stratified model category. In §4 we prove that the following more flexible versions of MC4 and MC5 hold.

MC4′ (see 4.8). *Suppose we have a commutative diagram in* **C** *of the form* 1.1, *where i is a cofibration and p is a fibration. Suppose there is an n such that*

(1) *i is an m-equivalence for $m < n$ and an n-epimorphism.*
(2) *p is an m-equivalence for $m > n$ and an n-monomorphism.*

Then a lifting in 1.1 *exists.*

MC5′ (see 4.6). *For every n, every map $f : X \to Y$ in* **C** *can be factored as*

$$X \xrightarrow{i} Z \xrightarrow{p} Y$$

where

(1) *i is a cofibration, an m-equivalence for $m < n$ and an n-epimorphism.*
(2) *p is a fibration, an m-equivalence for $m > n$ and an n-monomorphism.*

We recover the original conditions in MC4 and MC5 by taking $n = \infty$ and $n = -1$. The method of proof consists in an appropriate use of the small object argument of Quillen [10].

Now the obvious question arises: which model categories are stratified? We show that the basic model categories: topological spaces, simplicial sets and nonnegative chain complexes over a ring all have this structure. Moreover, in §5 we show that under some conditions a category related to the category of simplicial sets by a family of adjoint functors is a stratified model category.

As an application of MC5′, in §6 we show that Postnikov type decompositions exist in an arbitrary stratified model category. Specifically, we show that for each object X there exists a canonical tower of fibrations starting with the terminal object, whose successive terms give better and better homotopy approximations to X.

Since the n-skeleton of a CW-complex is not a homotopy invariant, J. H. C. Whitehead introduced the n-type, which depends only on the $(n+1)$-skeleton. As an application of MC4′ and MC5′, in §7 we show that canonical n-types exist in an arbitrary stratified model category and have the expected properties.

After the original version of the paper was submitted I have learned that D. Isaksen [6] has independently established that MC4′ and MC5′ hold for the specific case of simplicial sets. I thank Adam Przeździecki for drawing my attention to this. I also thank the referee for a number of very helpful comments.

2. Preliminary notions. We now recall some categorical notions which will be useful later. Let **C** be an arbitrary category. We say that X is a *retract* of Y if there exists a map $i : X \to Y$ and a map $r : Y \to X$ such that $ri = \mathrm{id}_X$. For example, in the category of modules one can show that a module P is projective iff it is a retract of a free module.

Given a category $\mathbf{C}$, the category of morphisms, $\mathrm{Mor}(\mathbf{C})$, is defined as a category whose objects are morphisms in $\mathbf{C}$, and the maps are defined as follows. A morphism from $f : X \to Y$ to $g : Z \to W$ is a pair of morphisms $k : X \to Z$ and $l : Y \to W$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{k} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{l} & W \end{array}$$

We say that a map f is a retract of a map g if it is a retract in the category $\mathrm{Mor}(\mathbf{C})$.

We assume that the reader is familiar with the notion of colimit and limit in a category (these can be found in [8] or [3]).

A colimit of the diagram of the form

$$B_0 \to B_1 \to B_2 \to \ldots \tag{2.1}$$

is called a *sequential colimit*. An object A is called *sequentially small* with respect to 2.1 if each map $f : A \to \operatorname{colim} B_i$ factors as $A \to B_n \to \operatorname{colim} B_i$ for some integer n.

Given a diagram

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \ldots$$

we say that the canonical map $A_0 \to \operatorname{colim} A_i$ has been *obtained by sequential colimits* from the family $\{f_i\}$.

2.2. We say that a map g is *obtained* from a map f (or that Y is obtained from X) *by cobase change* along k if we have the following pushout diagram in $\mathbf{C}$:

$$\begin{array}{ccc} A & \xrightarrow{k} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

i.e. the object Y above is the colimit of the diagram

$$B \xleftarrow{f} A \xrightarrow{k} X.$$

We will also need the following definition:

2.3. Let $\mathcal{F}$ be a family of morphisms in $\mathbf{C}$. By the *closure* of $\mathcal{F}$ we mean the smallest class of morphisms in $\mathbf{C}$ which contains $\mathcal{F}$ and is closed under cobase change, sequential colimits and retracts.

Now, following Quillen [10], we recall the notion of the left lifting property (LLP) and the right lifting property (RLP).

2.4. Definition. A map $i : A \to B$ is said to have the *left lifting property* (LLP) with respect to another map $p : X \to Y$ and p is said to have the *right lifting property* (RLP) with respect to i if for any choice of f and g making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

commute, a lifting exists.

The examples below show that the above notion is very common in algebra and topology.

First, a very naive example. In the category of sets, injective functions have the LLP with respect to surjective functions. (And hence surjective functions have the RLP with respect to injective functions.)

Let's move to algebra. Let R be a ring. An R-module P is projective iff the map $0 \to P$ has the LLP with respect to all epimorphisms of R-modules. Dually, an R-module Q is injective iff the map $Q \to 0$ has the RLP with respect to all monomorphisms of R-modules.

Let's return to topology. A map $p : E \to B$ is a Serre fibration if it has the RLP with respect to all the inclusions $X \times 0 \hookrightarrow X \times \mathbb{I}$ where X is a CW-complex and $\mathbb{I}$ is the unit interval.

3. Stratified model categories. Let $\mathbf{C}$ be an arbitrary category. In order to define a stratified model category, we start by introducing *weak equivalences* and *stratified weak equivalences*.

3.1. Definition. A class $\mathcal{W}$ of morphisms in $\mathbf{C}$ is called a class of *weak equivalences* if it has the following properties:

W1 $\mathcal{W}$ contains all isomorphisms and is closed under sequential colimits and retracts.

W2 If f, g are morphisms in $\mathbf{C}$ such that gf is defined, and if two of f, g, gf are in $\mathcal{W}$, then so is the third.

3.2. Definition. A class $\mathcal{W}$ of weak equivalences is called *stratified* if for each nonnegative integer n there exists a class $\mathcal{W}_n$ of weak equivalences (the *n-equivalences*), and classes $\mathcal{E}_n$ (the *n-epimorphisms*) and $\mathcal{M}_n$ (the *n-monomorphisms*) of morphisms such that:

SW1 $f \in \mathcal{W}$ if and only if $f \in \mathcal{W}_n$ for every nonnegative integer n.

SW2 $f \in \mathcal{W}_n$ if and only if $f \in \mathcal{E}_n$ and $f \in \mathcal{M}_n$.

SW3 If $h = gf$ then $h \in \mathcal{M}_n$ implies $f \in \mathcal{M}_n$ and $h \in \mathcal{E}_n$ implies $g \in \mathcal{E}_n$.

SW4 The classes $\mathcal{E}_n$ and $\mathcal{M}_n$ are closed under composition, sequential colimits and arbitrary sums.

To simplify some statements which will appear later, we make the convention that any morphism is an m-equivalence, m-monomorphism and m-epimorphism for $m < 0$ (i.e. we assume that our "theory" has no negative "invariants").

3.3. Example. The weak homotopy equivalences of topological spaces are a stratified family of weak equivalences in an obvious way: call a map $f : X \to Y$ of topological spaces an *n-equivalence* (respectively *n-epimorphism, n-monomorphism*) if it induces isomorphisms (respectively epimorphisms, monomorphisms) on π_n for every choice of basepoints.

In order to motivate the key axiom of a stratified model category, we establish a certain elementary property of Serre fibrations.

3.4. Consider the following sets of inclusion maps:

$$\{j_n : \partial\mathbb{I}^n \to \mathbb{I}^n \mid n \geq 0\}, \quad \{k_n : \mathbb{I}^{n-1} \times \{0\} \to \mathbb{I}^n \mid n \geq 1\}.$$

Here $\mathbb{I}^0$ is a single point and $\partial\mathbb{I}^0$ is the empty space.

It is not hard to check (see [5, Theorem 3.1, p. 63]) that a map is a Serre fibration iff it has the RLP with respect to $\{k_n \mid n \geq 1\}$.

The following simple lemma is the main ingredient in obtaining MC4′ and MC5′ from the introduction. Given a Serre fibration, the lemma describes how close to a weak equivalence the fibration must be if it has the RLP with respect to some j_n.

3.5. Lemma. *For every $n \geq 0$ a Serre fibration $p : E \to B$ has the RLP with respect to j_n iff p is an n-epimorphism and $(n-1)$-monomorphism.*

Proof. The long homotopy sequence of the fibration shows that the stated condition is equivalent to the vanishing of $\pi_{n-1}(F, e_0)$:

$$\ldots \to \pi_n(E, e_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, e_0) \\ \to \pi_{n-1}(E, e_0) \to \pi_{n-1}(B, b_0) \to \ldots$$

It is clear that if $p : E \to B$ has the RLP with respect to j_n, then $\pi_{n-1}(F, e_0) = 0$.

Now suppose that $\pi_{n-1}(F, e_0) = 0$ and we wish to construct a lift in the diagram

$$\begin{array}{ccc} \partial\mathbb{I}^n & \xrightarrow{f} & E \\ {\scriptstyle j_n}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{I}^n & \xrightarrow{g} & B \end{array}$$

First, let $H : \mathbb{I}^n \times \mathbb{I} \to B$ be the homotopy between g and the constant map to b_0, such that $H(x,0) = g$ and $H(x,1) = b_0$. So we have a diagram

$$\begin{array}{ccc} \partial\mathbb{I}^n \times \{0\} & \xrightarrow{f} & E \\ \downarrow & & \downarrow p \\ \partial\mathbb{I}^n \times \mathbb{I} & \xrightarrow{H|} & B \end{array}$$

where $H|$ means the restriction of H to $\partial\mathbb{I}^n \times \mathbb{I}$. Since p is a Serre fibration, there exists a lift $\widetilde{H} : \partial\mathbb{I}^n \times \mathbb{I} \to E$ covering $H|$. In particular, $\widetilde{H}(x,1) \in F$. This gives a map $\partial\mathbb{I}^n \times \{1\} \to F$. By assumption, this map extends to a map $K : \mathbb{I}^n \times \{1\} \to F$. So we can form a diagram

$$\begin{array}{ccc} \partial\mathbb{I}^n \times \mathbb{I} \amalg \mathbb{I}^n \times \{1\} & \longrightarrow & E \\ \downarrow & & \downarrow p \\ \mathbb{I}^n \times \mathbb{I} & \xrightarrow{H} & B \end{array}$$

Since p is a Serre fibration, a lift $\widehat{H} : \mathbb{I}^n \times \mathbb{I} \to E$ exists. Then $h(x) = \widehat{H}(x,0)$ is the desired lift in the original diagram.

To recall the notion of a cofibrantly generated model category, we need to recall the small object argument.

3.6. Definition. We say that a set $\mathcal{F}$ of maps permits the *small object argument* if the domain of each map in $\mathcal{F}$ is sequentially small with respect to sequential colimits involving diagrams where each object is obtained from the previous one by cobase change along the maps in $\mathcal{F}$.

Following [4], we recall D. M. Kan's notion of a cofibrantly generated model category.

3.7. Definition. A *cofibrantly generated model category* is a model category **C** with arbitrary colimits such that:

(1) There exists a set I of *generating cofibrations* that permits the small object argument and has the property that a map is an acyclic fibration if and only if it has the RLP with respect to every generating cofibration.

(2) There exists a set J of *generating acyclic cofibrations* that permits the small object argument and has the property that a map is a fibration if and only if it has the RLP with respect to every generating acyclic cofibration.

We are now ready to define a stratified model category.

3.8. Definition. A *stratified model category* **C** is a cofibrantly generated model category with the following structure:

(1) A class $\mathcal{W}$ of stratified weak equivalences.

(2) A set I of generating cofibrations such that $I = \bigcup_{n=0}^{\infty} I_n$ and each $f \in I_n$ is an m-equivalence for all $m < n-1$ and an $(n-1)$-epimorphism.

(3) A class J of generating acyclic cofibrations. (Hence, in particular, every element in the closure of J is a weak equivalence.)

These are required to satisfy the following axioms:

CC1 For every integer $n \geq 0$, if p is a fibration, then p has the RLP with respect to I_n if and only if $p \in \mathcal{E}_n$ and $p \in \mathcal{M}_{n-1}$.

CC2 For every integer $n \geq -1$, every map in the closure of $J \cup \bigcup_{m=n+1}^{\infty} I_m$ is an m-equivalence for all $m < n$ and an n-epimorphism.

CC3 The set $I \cup J$ permits the small object argument.

3.9. Example (topological spaces). For $m < 0$ or $m = \infty$ by $\pi_m(A)$ we mean the trivial group. Here we take 3.3 as the stratified weak equivalences. Moreover, we let

$$I_n = \{j_n : \partial\mathbb{I}^n \to \mathbb{I}^n\}, \quad n \geq 0,$$
$$J = \{k_n : \mathbb{I}^{n-1} \times \{0\} \to \mathbb{I}^n \mid n = 1, 2, 3, \ldots\}.$$

These choices give the category **Top** of topological spaces the structure of a stratified model category. CC1 follows from 3.5. For CC2, a map obtained by cobase change from $\partial\mathbb{I}^n \to \mathbb{I}^n$, $n \geq 1$, is simply the inclusion of a space X into $X' = X \cup \mathbb{I}^n$, that is, X with an n-cell attached via the boundary. Such a map induces isomorphisms on π_m for all $m < n-1$ and an epimorphism on π_{n-1}. Now note that the set of m-equivalences is closed under sequential colimits and retracts. CC3 follows from the compactness of $\mathbb{I}^n$ and $\partial\mathbb{I}^n$.

3.10. Example (nonnegative chain complexes over a ring). Let R be a commutative ring. Consider the category $\mathbf{Ch}_R$ of nonnegatively graded chain complexes over a commutative ring R. Let 0 denote either the zero R-module or the chain complex which has the zero R-module in every dimension.

The class of weak equivalences is the class of maps which induce isomorphisms in homology in every dimension. It is stratified in the obvious way: a map is an n-equivalence (respectively n-epimorphism, n-monomorphism) if it induces an isomorphism (resp. epimorphism, monomorphism) in homology in dimension n.

For $n \geq 1$, let $D^n(R)$ denote the chain complex which has R in dimensions n and $n-1$ and zeros elsewhere, with identity as the nonzero boundary map. Finally, for $n \geq 1$, let $S^{n-1}(R)$ be the chain complex which has R in dimension $n-1$ and zeros elsewhere. Moreover, let $D^0 = S^0$ and S^{-1} denote the zero chain complex. We let $j_n : S^{n-1}(R) \to D^n(R)$ equal the identity in $(n-1)$-st grading and zero elsewhere. Finally, we let $k_n : 0 \to D^n(R)$.

We now let

$$I_n = \{j_n : S^{n-1}(R) \to D^n(R)\}, \qquad n \geq 0,$$
$$J = \{k_n : 0 \to D^n(R) \mid n = 1, 2, 3, \ldots\}.$$

Below we prove CC1 for the category $\mathbf{Ch}_R$. CC2 follows from the fact that attaching copies of $D^n(R)$ along $S^{n-1}(R)$ does not alter homology in dimensions below $n-1$. Finally, CC3 follows because $S^{n-1}(R)$ is finitely generated. Therefore $\mathbf{Ch}_R$ is a stratified model category.

3.11. Lemma. *Let $p : X \to Y$ in $\mathbf{Ch}_R$ be a fibration, and n be a nonnegative integer. A lift exists in the diagram*

$$\begin{array}{ccc} S^{n-1}(R) & \longrightarrow & X \\ \downarrow & & \downarrow p \\ D^n(R) & \longrightarrow & Y \end{array} \tag{3.12}$$

if and only if p is an $(n-1)$-monomorphism and n-epimorphism.

Proof. The result is immediate for $n = 0$ (recall that $S^{-1}(R)$ denotes the complex consisting of zeros only, and $D^0(R)$ the complex with R in dimension 0 and zeros elsewhere).

So assume that $n \geq 1$. Note first that the diagrams above are in one-to-one correspondence with sets of elements $x_{n-1} \in X_{n-1}$, $y_n \in Y_n$, and $y_{n-1} \in Y_{n-1}$ such that

$$\begin{array}{ccccc} & & x_{n-1} & \xrightarrow{d_{n-1}} & 0 \\ & & \downarrow p_{n-1} & & \\ y_n & \xrightarrow{d_n} & y_{n-1} & & \end{array} \tag{3.13}$$

Finding a lift in the diagram 3.12 is equivalent to finding an element $x_n \in X_n$ such that $d_n(x_n) = x_{n-1}$ and $p_n(x_n) = y_n$ in the diagram above.

We will first show that if a lift exists then p is an n-epimorphism and $(n-1)$-monomorphism.

Suppose that $\sigma \in H_n(Y)$. So $\sigma = [y_n]$ for some $y_n \in Y_n$ which is a cycle. Consider the diagram 3.13 determined by the data: the y_n which represents σ, $y_{n-1} = 0$ and $x_{n-1} = 0$. The lift determines an element $x_n \in X_n$ such that $p_n(x_n) = y_n$ and $d_n(x_n) = 0$. So p_* is an n-epimorphism.

Now suppose that $\tau \in H_{n-1}(X)$ and $p_*(\tau) = 0$. Let $\tau = [x_{n-1}]$ for some $x_{n-1} \in X_{n-1}$ which is a cycle. Let $y_{n-1} = p_{n-1}(x_{n-1})$. Since $p_*(\tau) = 0$, there exists an element $y_n \in Y_n$ such that $d_n(y_n) = y_{n-1}$. Again, this determines a diagram 3.13. The lift produces an element $x_n \in X_n$ such that $d_n(x_n) = x_{n-1}$. So in fact $\tau = 0$.

Now suppose that p_* is an n-epimorphism and $(n-1)$-monomorphism. We wish to show that a lifting exists in diagram 3.12.

Note first that since p is a fibration, p_n is onto. So, there exists an $z_n \in X_n$ such that $p_n(z_n) = y_n$. However, it is not necessarily true that $d_n(z_n) = x_{n-1}$. So let $w_{n-1} = d_n(z_n) - x_{n-1}$. The above properties of z_n and x_{n-1} imply that $p_{n-1}(w_{n-1}) = 0$ and $d_{n-1}(w_{n-1}) = 0$. So w_{n-1} determines a homology class which lies in the kernel of p_*. But since p_* is an $(n-1)$-monomorphism, this homology class must vanish. So there exists an element $u_n \in X_n$ such that $d_n(u_n) = w_{n-1}$. If $p_n(u_n)$ were equal to zero, we could set $x_n = z_n - u_n$ as our desired element. However, $p_n(u_n)$ is not necessarily zero. Let $v_n = p_n(u_n)$. Since $d_n(v_n) = 0$, v_n represents a homology class in $H_n(Y)$. But we assumed that p_* is an n-epimorphism, so there exists an element $t_n \in X_n$ such that $d_n(t_n) = 0$ and $p_n(t_n) = v_n$. The element $x_n = z_n - u_n - t_n$ has the desired properties.

3.14. Example (simplicial sets). Let Δ^{op} be the simplicial category (a good textbook reference is [9]), so that a simplicial set is a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Sets}$. The category of simplicial sets will be denoted by $\mathbf{S}$. We now describe a stratified model category structure on $\mathbf{S}$.

For $n \geq 0$, let $\Delta[n]$ be the standard n-simplex. Moreover, let ι_n be the nondegenerate simplex of dimension n in $\Delta[n]$. Let $\dot{\Delta}[n]$ be the simplicial subset of $\Delta[n]$ generated by its nondegenerate simplices of dimension less than n, and $j_n : \dot{\Delta}[n] \to \Delta[n]$ be the boundary inclusion.

For $n \geq 1$ and $0 \leq m \leq n$ let $V[n,m]$ be the simplicial subset of $\Delta[n]$ generated by the simplices $d_q(\iota_n)$ for $0 \leq q \leq n$, $q \neq m$, and $k_{n,m} : V[n,m] \to \Delta[n]$ be the natural inclusion.

We now let

$$I_n = \{j_n : \dot{\Delta}[n] \to \Delta[n]\}, \qquad n \geq 0,$$
$$J = \{k_{n,m} : V[n,m] \to \Delta[n] \mid n = 1,2,\ldots,\ m = 0,1,\ldots,n\}.$$

A map $f : X \to Y$ of simplicial sets is an n-equivalence (n-epimorphism, n-monomorphism) if $|f|$ is an n-equivalence (n-epimorphism, n-monomorphism) of the geometric realizations in the sense of Example 3.9.

The verification of the axioms of a stratified model category is very similar to the two examples considered earlier. Below we indicate the proof of CC1.

3.15. Lemma. *Let $p : E \to B$ be a Kan fibration. Then for each $n \geq 0$ a lift exists in*

$$\begin{array}{ccc} \dot{\Delta}[n] & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta[n] & \longrightarrow & B \end{array}$$

if and only if $(p_* : \pi_{n-1}(E, e_0) \to \pi_{n-1}(B, b_0)) \in \mathcal{M}_{n-1}$ *and* $(p_* : \pi_n(E, e_0) \to \pi_n(B, b_0)) \in \mathcal{E}_n$ *for every choice of basepoints* $e_0 \in E$, $b_0 \in B$ *such that* $p(e_0) = b_0$.

Proof. By Proposition 7.3 in [9], $F = p^{-1}(b_0)$ is a Kan complex. The portion

$$\pi_n(F, e_0) \to \pi_n(E, e_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, e_0) \\ \to \pi_{n-1}(E, e_0) \to \pi_{n-1}(B, b_0)$$

of the long exact sequence of the fibration p shows that the condition on the map p_* is equivalent to the vanishing of the group $\pi_{n-1}(F, e_0)$. The proof is now a simplicial argument which follows the topological case very closely.

In Section 5 we show that under certain conditions a category related to the category of simplicial sets by a pair of adjoint functors has the structure of a stratified model category.

4. Main theorems. The main goal of this section is to prove MC4′ and MC5′ from the introduction.

We need the so-called infinite gluing construction, which is due to Quillen [10]. The description below is based on [3].

4.1. *The infinite gluing construction.* Let $\mathcal{F} = \{f_\lambda : A_\lambda \to B_\lambda\}_{\lambda \in \Lambda}$ be a set of morphisms in $\mathbf{C}$. (For $\mathcal{F}$ we will take the union of certain sets I_n and J.) Suppose $p : X \to Y$ is a morphism in $\mathbf{C}$; we want to factor p as $X \to X' \to Y$ where the first morphism is a cofibration and the second has the RLP with respect to $\mathcal{F}$.

For each $\lambda \in \Lambda$ let $S(\lambda)$ consist of all pairs (h, g) of maps making the following diagram commute:

$$\begin{array}{ccc} A_\lambda & \xrightarrow{g} & X \\ {\scriptstyle f_\lambda}\downarrow & & \downarrow{\scriptstyle p} \\ B_\lambda & \xrightarrow{h} & Y \end{array} \tag{4.2}$$

We define the *gluing construction* $G^1(\mathcal{F}, p)$ to be the object of $\mathbf{C}$ given by the pushout diagram

$$\begin{array}{ccc} \coprod_{\lambda \in \Lambda} \coprod_{(h,g) \in S(\lambda)} A_\lambda & \longrightarrow & X \\ {\scriptstyle \coprod f_\lambda}\downarrow & & \downarrow{\scriptstyle i_1} \\ \coprod_{\lambda \in \Lambda} \coprod_{(h,g) \in S(\lambda)} B_\lambda & \longrightarrow & G^1(\mathcal{F}, p) \end{array}$$

As indicated, there is a natural map $i_1 : X \to G^1(\mathcal{F}, p)$. By the universal property of colimits, the commutative diagram 4.2 induces a map

$p_1 : G^1(\mathcal{F}, p) \to Y$ such that $p_1 i_1 = p$. Now repeat the process: for $n > 1$ define objects $G^n(\mathcal{F}, p)$ and maps $p_n : G^n(\mathcal{F}, p) \to Y$ inductively by setting $G^n(\mathcal{F}, p) = G^1(\mathcal{F}, p_{n-1})$ and $p_n = (p_{n-1})_1$. This gives an infinite commutative diagram

$$\begin{array}{ccccccccccc}
X & \xrightarrow{i_1} & G^1(\mathcal{F},p) & \xrightarrow{i_2} & G^2(\mathcal{F},p) & \xrightarrow{i_3} & \cdots & \xrightarrow{i_n} & G^n(\mathcal{F},p) & \longrightarrow & \cdots \\
\Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle p_2} & & & & \Big\downarrow{\scriptstyle p_n} & & \\
Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & \cdots & \xrightarrow{=} & Y & \xrightarrow{=} & \cdots
\end{array}$$

Let $G^\infty(\mathcal{F}, p)$, the *infinite gluing construction*, denote the colimit of the upper row in the above diagram; there are natural maps $i_\infty : X \to G^\infty(\mathcal{F}, p)$ and $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ such that $p_\infty i_\infty = p$. By construction, the map i_∞ is in the closure of $\mathcal{F}$.

4.3. Proposition. *Suppose that for each $\lambda \in \Lambda$ the object A_λ is sequentially small with respect to the top row in the diagram above. Then the map $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ has the RLP with respect to each map in the family $\mathcal{F}$.*

Proof. Suppose we want to find a lift in a commutative diagram

$$\begin{array}{ccc}
A_\lambda & \xrightarrow{g} & G^\infty(\mathcal{F},p) \\
\Big\downarrow{\scriptstyle f_\lambda} & & \Big\downarrow{\scriptstyle p_\infty} \\
B_\lambda & \xrightarrow{h} & Y
\end{array}$$

Since A_λ is sequentially small in the appropriate sense, there exists an integer n such that the map g is the composition of a map $g' : A_\lambda \to G^n(\mathcal{F}, p)$ with the natural map $G^n(\mathcal{F}, p) \to G^\infty(\mathcal{F}, p)$. Therefore, we can enlarge the above diagram to

$$\begin{array}{ccccccc}
A_\lambda & \xrightarrow{g'} & G^n(\mathcal{F},p) & \xrightarrow{i_{n+1}} & G^{n+1}(\mathcal{F},p) & \longrightarrow & G^\infty(\mathcal{F},p) \\
\Big\downarrow{\scriptstyle f_\lambda} & & \Big\downarrow{\scriptstyle p_n} & & \Big\downarrow{\scriptstyle p_{n+1}} & & \Big\downarrow{\scriptstyle p_\infty} \\
B_\lambda & \xrightarrow{h} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y
\end{array}$$

in which the composition of the maps in the top row is g. The pair (g', h) appears as an index in the construction of $G^{n+1}(\mathcal{F}, p)$ from $G^n(\mathcal{F}, p)$—it indexes the gluing of B_λ to $G^n(\mathcal{F}, p)$ along g'. By the universal properties of colimits there exists a map $B_\lambda \to G^{n+1}(\mathcal{F}, p)$ making the appropriate diagram commute. Composing with the canonical map $G^{n+1}(\mathcal{F}, p) \to G^\infty(\mathcal{F}, p)$ gives a lifting in the original square.

To state our results it is useful to introduce the ordered set $\overline{N}$ defined as follows:

$$\overline{N} = \{n \in \mathbb{Z} \mid n \geq -1\} \cup \{\infty\}. \tag{4.4}$$

4.5. Definition. Let $n \in \overline{N}$. We call a pair of maps $i : A \to B$ and $p : X \to Y$ an *n-admissible pair* if

- i is a cofibration, an m-equivalence for all $m<n$ and an n-epimorphism,
- p is a fibration, an m-equivalence for all $m>n$ and an n-monomorphism.

4.6. Theorem. *For every $n \in \overline{N}$, every map $p : X \to Y$ can be factored in a canonical way as*

$$X \xrightarrow{i_\infty} Z \xrightarrow{p_\infty} Y$$

where (i_∞, p_∞) is an n-admissible pair, and moreover, every fibration which is an m-equivalence for all $m > n$ and an n-monomorphism has the RLP with respect to i_∞.

Proof. In 4.1, take for $\mathcal{F}$ the set

$$J \cup \bigcup_{m=n+1}^{\infty} I_m.$$

Let i_∞, p_∞ and $Z = G(\mathcal{F}, p)$ be the result of making the infinite gluing construction. By CC3, the domains of the maps in the family $\mathcal{F}$ are sequentially small with respect to the particular sequential colimits that arise. By definition, the map i_∞ is a cofibration and by CC2 it is an m-equivalence for $m < n$ and an n-epimorphism. The map p_∞ has the RLP with respect to J by 4.3, hence it is a fibration. Moreover, again by 4.3, p_∞ has the RLP with respect to $\bigcup_{m=n+1}^{\infty} I_m$. Therefore, by CC1, p_∞ is an m-equivalence for $m > n$ and an n-monomorphism. The construction is canonical, as no choices were made along the way.

Suppose that the map p is a fibration which is an m-equivalence for all $m > n$ and an n-monomorphism. As a fibration, it has the RLP with respect to J. By CC1, it has the RLP with respect to $\bigcup_{m=n+1}^{\infty} I_m$. Hence, by the lemma below it has the RLP with respect to i_∞.

4.7. Lemma. *If a map $p : X \to Y$ has the RLP with respect to a family $\mathcal{F}$ of morphisms, then it has the RLP with respect to the closure of $\mathcal{F}$.*

Proof. Suppose that p has the RLP with respect to $f : A \to B$, and g is obtained by cobase change from f. We wish to show that p has the RLP with respect to g, i.e. that there is a lift $h : D \to X$ in the right hand square of the diagram

$$\begin{array}{ccccc} A & \longrightarrow & C & \xrightarrow{k} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{i} & D & \xrightarrow{j} & Y \end{array}$$

By assumption, there is a lift $l : B \to X$ in the rectangle $ABYX$. By the universal property of pushouts, the maps l and k induce a map $h : D \to X$.

Immediately we have $hg = k$. Moreover, $phi = pl = ji$ and $phg = pk = jg$, hence, by the universal properties of pushouts, $ph = j$. So the map h is the desired lift.

Let $A_0 \to A_1 \to A_2 \to \ldots$ be a family of morphisms in $\mathcal{F}$. We need to show that there is a lift in every commutative diagram of the form

$$\begin{array}{ccc} A_0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \operatorname{colim} A_\lambda & \longrightarrow & Y \end{array}$$

We can enlarge the above diagram to

$$\begin{array}{ccccc} A_0 & & \longrightarrow & & X \\ \downarrow & & & & \downarrow{\scriptstyle p} \\ A_1 & \longrightarrow & \operatorname{colim} A_i & \longrightarrow & Y \end{array}$$

Since $A_0 \to A_1$ is an element of $\mathcal{F}$, there is a lift $A_1 \to X$ in the diagram above. Using this lift we can form the diagram

$$\begin{array}{ccccc} A_1 & & \longrightarrow & & X \\ \downarrow & & & & \downarrow{\scriptstyle p} \\ A_2 & \longrightarrow & \operatorname{colim} A_i & \longrightarrow & Y \end{array}$$

The collection of such lifts for all integers n, by definition of colimit, determines a lift in the original diagram.

Suppose now that $p : X \to Y$ has the RLP with respect to $i : A \to B$ and $i' : A' \to B'$ is a retract of i. We want to construct a lift in the diagram

$$\begin{array}{ccc} A' & \xrightarrow{f} & X \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{g} & Y \end{array}$$

Since i' is a retract of i, we can enlarge the above diagram to

$$\begin{array}{ccccccc} A' & \longrightarrow & A & \longrightarrow & A' & \xrightarrow{f} & X \\ {\scriptstyle i'}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{s} & B & \xrightarrow{r} & B' & \xrightarrow{g} & Y \end{array}$$

By assumption there exists a lift $h : B \to X$ of the map $gr : B \to Y$. The composite map $hs : B' \to X$ is a lift in the original diagram.

4.8. THEOREM. *A lift exists in every commutative diagram of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

where (i, p) *is an* n*-admissible pair for some* $n \in \overline{N}$.

Proof. By 4.6, we can factor $i : A \to B$ as

$$A \xrightarrow{i_\infty} B' \xrightarrow{p_\infty} B,$$

where (i_∞, p_∞) is an n-admissible pair and the map p_∞ has the RLP with respect to $\bigcup_{m=n+1}^{\infty} I_m$. But since both i and i_∞ are m-equivalences for all $m < n$, it follows that p_∞ is also an m-equivalence in that range. Moreover, since i is an n-epimorphism, by SW3, so is p_∞. Together, these properties imply that p_∞ is a weak equivalence. We can form a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i_\infty} & B' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_\infty} \\ B & \xrightarrow{\mathrm{Id}} & B \end{array}$$

Since p_∞ is a fibration and a weak equivalence, it has the RLP with respect to i. Let $h : B \to B'$ be the lifting in the above diagram. The map h allows us to express i as a retract of i_∞:

$$\begin{array}{ccccc} A & \xrightarrow{\mathrm{Id}} & A & \xrightarrow{\mathrm{Id}} & A \\ {\scriptstyle i}\downarrow & & {\scriptstyle i_\infty}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow{h} & B' & \xrightarrow{p_\infty} & B \end{array}$$

We now turn to what we set out to prove. By 4.6, since p is an m-equivalence for $m > n$ and an n-monomorphism, it has the RLP with respect to i_∞. Hence, by 4.7, it has the RLP with respect to i.

5. A method for establishing stratified model category structures. In this section we show that a method for establishing model category structures described in [11] can easily be extended to establish stratified model category structures.

Let Λ be an arbitrary index set. Given a category $\mathbf{D}$ closed under coproducts and a family $\mathcal{H} = \{(\Psi_\lambda, \Phi_\lambda) : \lambda \in \Lambda\}$ of adjoint functors

$$\Psi_\lambda : \mathbf{S} \rightleftarrows \mathbf{D} : \Phi_\lambda, \quad \lambda \in \Lambda, \tag{5.1}$$

under some mild hypotheses there is a natural way to define a stratified category structure on $\mathbf{D}$. Before stating the theorem, we need some preliminaries:

5.2. Definition. Let $n \geq 0$, $n \geq m \geq 0$, and f be a map of simplicial sets either of the form $\dot{\Delta}[n] \to \Delta[n]$ or $V[n,m] \to \Delta[n]$. Moreover, let $X_f = \mathrm{Dm}(f)$ and $Y_f = \mathrm{Rg}(f)$. Let E be the set of elements (e, μ, f, g) where e is an index, $\mu \in \Lambda$ and $g : \Psi_\mu(X_f) \to Z$. A *Ψ_*-regular pushout* is a pushout of the form

$$\begin{array}{ccc} \coprod_{(e,\mu,f,g)\in E} \Psi_\mu(X_f) & \xrightarrow{\coprod g} & Z \\ \Big\downarrow{\scriptstyle \coprod \Psi_\mu(f)} & & \Big\downarrow{\scriptstyle h} \\ \coprod_{(e,\mu,f,g)\in E} \Psi_\mu(Y_f) & \longrightarrow & W \end{array}$$

The morphism h is said to be *Ψ_*-induced.*

5.3. Definition. An object A of the category $\mathbf{D}$ is called *Ψ_*-sequentially small* if $\mathrm{Hom}_{\mathbf{D}}(A, -)$ commutes with sequential colimits of diagrams in which all morphisms are Ψ_*-induced as above.

Consider the following assumptions on 5.1:

5.4. Assumptions. (1) $\mathbf{D}$ has finite limits and arbitrary small colimits.

(2) For every $f : V[n,m] \to \Delta[n]$ and $\lambda, \mu \in \Lambda$, $\Phi_\lambda \Psi_\mu(f)$ is an acyclic cofibration of simplicial sets, and for every $n \geq 0$, and boundary inclusion $f : \dot{\Delta}[n] \to \Delta[n]$, the morphism $\Phi_\lambda \Psi_\mu(f)$ is an m-equivalence for every $m < n-1$ and an $(n-1)$-epimorphism.

(3) For X the domain of any f above and for each $\mu \in \Lambda$, the object $\Psi_\mu(X)$ is Ψ_*-sequentially small.

(4) For all $\lambda \in \Lambda$ the functor Φ_λ: (a) preserves coproducts; (b) takes Ψ_*-regular pushouts to homotopy pushout diagrams; (c) preserves sequential colimits in $\mathbf{D}$ in which the morphisms are Ψ_*-induced.

5.5. Theorem. *Suppose we have a category* $\mathbf{D}$ *and a family* 5.1 *of adjoint functors satisfying* 5.4. *The category* $\mathbf{D}$ *has the structure of a stratified model category given as follows*:

(1) *A map $g : X \to Y$ is an n-equivalence* (*resp. n-epimorphism*, *n-monomorphism*) *iff for all $\lambda \in \Lambda$ the map $\Phi_\lambda(g) : \Phi_\lambda(X) \to \Phi_\lambda(Y)$ is an n-equivalence* (*resp. n-epimorphism*, *n-monomorphism*) *in* $\mathbf{S}$.

(2) *The set of generating cofibrations for D is given by $I_n = \{\Psi_\lambda(f) \mid f : \dot{\Delta}[n] \to \Delta[n],\ \lambda \in \Lambda\}$.*

(3) *The generating acyclic cofibrations for D are given by $J = \{\Psi_\lambda(f) \mid f : V[n,m] \to \Delta[n],\ n \geq 1,\ 0 \leq m \leq n,\ \lambda \in \Lambda\}$.*

Proof. The fact that the weak equivalences in $\mathbf{D}$ form a stratified family follows directly from the definition. To prove CC1, consider a diagram of the form

$$\begin{array}{ccc} \Psi_i(\dot{\Delta}[n]) & \longrightarrow & E \\ \downarrow & & \downarrow p \\ \Psi_i(\Delta[n]) & \longrightarrow & B \end{array}$$

Since the functors $\Psi_\lambda, \Phi_\lambda$ form an adjoint pair, such diagrams are in one-to-one correspondence with

$$\begin{array}{ccc} \dot{\Delta}[n] & \longrightarrow & \Phi_\lambda(E) \\ \downarrow & & \downarrow \Phi_\lambda(p) \\ \Delta[n] & \longrightarrow & \Phi_\lambda(B) \end{array}$$

Now since the simplicial sets have the structure of a stratified model category, a lift exists in the above diagram precisely when

$$\Phi_\lambda(p) \in \mathcal{E}_n, \quad \Phi_\lambda(p) \in \mathcal{M}_{n-1}.$$

This in turn means that p is an n-epimorphism and $(n-1)$-monomorphism.

CC2 follows from assumptions (2) and (4). CC3 is just assumption (3).

5.6. Example. A cyclic set is a simplicial set with an extra structure given by the action of the cyclic group of order $n+1$ on the set of n-simplices. The category of cyclic sets will be denoted by $\mathbf{S}^c$. These objects were introduced by A. Connes [1]; a good description for our purposes is either in W. G. Dwyer *et al.* [2] or J. Jones [7].

There is a pair of adjoint functors relating the simplicial and cyclic categories

$$F : \mathbf{S} \rightleftarrows \mathbf{S}^c : G.$$

The functor G is the forgetful functor. The functor F associates to a simplicial set X the cyclic set obtained by replacing each $\Delta[n]$ by $\Lambda[n] = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(n, -)$ (the free cyclic set in dimension n) and performing the same gluings as for X.

It is easy to see that the assumptions of the above theorem are satisfied.

5.7. Example. Let G be a compact Lie group, and let $\mathbf{Top}^G$ be the category of spaces with a G-action.

We have a family of adjoint functors:

$$G/H \times |?| : \mathbf{S} \rightleftarrows \mathbf{Top}^G : S_*(?^H),$$

where H denotes an arbitrary closed subgroup of G and S_* is the singular functor.

It is not hard to see that this family also satisfies the assumptions of the above theorem.

6. Postnikov type decompositions. In this section we show how to introduce natural Postnikov type decompositions in a stratified model category **C**. Let $*$ denote the terminal object in **C**.

6.1. Proposition. *For any object X we have a natural diagram of spaces and maps*

$$\begin{array}{ccccccccccc} \ldots \longrightarrow & X & \xrightarrow{=} & X & \xrightarrow{=} & X & \xrightarrow{=} & X & \xrightarrow{=} & X \\ & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_{-1}} & & \downarrow \\ \ldots \longrightarrow & Y_2 & \xrightarrow{p_2} & Y_1 & \xrightarrow{p_1} & Y_0 & \xrightarrow{p_0} & Y_{-1} & \xrightarrow{p_{-1}} & * \end{array}$$

such that (i_n, p_n) is an n-admissible pair.

Proof. The result follows by iterative application of Theorem 4.6 to the canonical map $X \to *$. First, we apply 4.6 with $n = -1$ to obtain the factorization

$$X \xrightarrow{i_{-1}} Y_{-1} \xrightarrow{p_{-1}} *.$$

Next, we apply 4.6 with $n = 0$ to i_{-1}. In general, for $n \geq 0$, we apply 4.6 with $n+1$ to i_n.

The objects $\{Y_n\}_{n=-1}^{\infty}$ form a "homotopy approximation to X". For example, Y_1 is 0-equivalent to X, Y_2 is 0,1-equivalent to X, etc. On the other hand, each Y_n is not m-equivalent to $*$ only for finitely many m.

7. The homotopy category of n-types. It is natural to consider the "homotopy type of a CW-complex up to dimension n". The problem is that the n-skeleton of a CW-complex is not a homotopy type invariant. To overcome this, J. H. C. Whitehead introduced the n-type, which is a homotopy type invariant of a CW-complex which depends only on the $(n+1)$-skeleton. In this section we show that n-types exist for any stratified model category.

First, we make a definition.

7.1. Definition. An object X in a stratified model category is called *connected* if the canonical map $X \to *$ (where $*$ is the terminal object) is a 0-equivalence.

Let **C** be a stratified model category and $\mathbf{C}_0$ be the full subcategory of connected objects. Given any object X in $\mathbf{C}_0$ and integer $n \geq 1$ apply 4.6 with $n+1$ to the morphism $X \to *$ to obtain

$$X \xrightarrow{i} X' \xrightarrow{p} *,$$

where (i,p) is an $(n+1)$-admissible pair of maps.

7.2. Definition. We call the object X' above the *n-type* of X and denote it by $P_n(X)$.

The next result shows in what sense the n-type of X depends "only on the homotopy of X up to dimension $n+1$".

7.3. THEOREM. *Suppose a map $f : X \to Y$ in $\mathbf{C}_0$ is an m-equivalence for $m = 0, 1, \ldots, n+1$. Then there exists a map $\widetilde{f} : P_n(X) \to P_n(Y)$ which is a weak equivalence.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & P_n(Y) \\ \big\downarrow & & & & \big\downarrow \\ P_n(X) & \longrightarrow & * & \longrightarrow & * \end{array}$$

Since the outer vertical arrows form an $(n+1)$-admissible pair, a lift $\widetilde{f} : P_n(X) \to P_n(Y)$ exists by 4.8.

We can rewrite the above diagram in the form

$$\begin{array}{ccccc} X & \longrightarrow & P_n(X) & \longrightarrow & * \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle \widetilde{f}}\big\downarrow & & \big\downarrow \\ Y & \longrightarrow & P_n(Y) & \longrightarrow & * \end{array}$$

Since the left vertical map is an m-equivalence for $m = 0, 1, \ldots, n$, and the left horizontal maps are m-equivalences for $m = 0, 1, \ldots, n$, it follows that the middle vertical map is an m-equivalence for $m = 0, 1, \ldots, n$. Moreover, since the left horizontal maps are $(n+1)$-epimorphisms (and so is f), it follows by SW3 that so is the middle vertical map. Applying an analogous argument to the right hand square, we see that $\widetilde{f}$ is an $(n+1)$-monomorphism and m-equivalence for $m > n+1$. We conclude that it is a weak equivalence.

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