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Quasi-homomorphisms

by

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Abstract. We study the stability of homomorphisms between topological (abelian) groups. Inspired by the "singular" case in the stability of Cauchy's equation and the technique of quasi-linear maps we introduce quasi-homomorphisms between topological groups, that is, maps $\omega : \mathcal{G} \to \mathcal{H}$ such that $\omega(0) = 0$ and

$$\omega(x+y) - \omega(x) - \omega(y) \to 0$$

(in $\mathcal{H}$) as $x, y \to 0$ in $\mathcal{G}$. The basic question here is whether ω is approximable by a true homomorphism a in the sense that $\omega(x) - a(x) \to 0$ in $\mathcal{H}$ as $x \to 0$ in $\mathcal{G}$. Our main result is that quasi-homomorphisms $\omega : \mathcal{G} \to \mathcal{H}$ are approximable in the following two cases:

- $\mathcal{G}$ is a product of locally compact abelian groups and $\mathcal{H}$ is either $\mathbb{R}$ or the circle group $\mathbb{T}$.
- $\mathcal{G}$ is either $\mathbb{R}$ or $\mathbb{T}$ and $\mathcal{H}$ is a Banach space.

This is proved by adapting a classical procedure in the theory of twisted sums of Banach spaces. As an application, we show that every abelian extension of a quasi-Banach space by a Banach space is a topological vector space. This implies that most classical quasi-Banach spaces have only approximable (real-valued) quasi-additive functions.

Introduction and statement of the main result. This paper deals with the stability of homomorphisms on topological abelian groups. Our methods (and results) lie on the frontiers of stability theory, extension of topological groups, and homology of topological linear spaces.

As a motivation, consider an additive function on the line, that is, a map $a : \mathbb{R} \to \mathbb{R}$ satisfying

$$a(s+t) = a(s) + a(t) \qquad (s, t \in \mathbb{R}). \tag{1}$$

Suppose ω is a *small* perturbation of a, say $\omega = a + \varepsilon$, where ε is a *small* function, in some sense to be made precise. Then ω will satisfy (1) approximately, in the sense that the Cauchy difference

$$\Delta(\omega)(s,t) = \omega(s+t) - \omega(s) - \omega(t) \qquad (s, t \in \mathbb{R})$$

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will be small. This is the obvious way of constructing a nearly additive map. A very natural question is whether that is the only way. Needless to say, there are several possibilities of understanding a *small* perturbation or a *nearly additive* map. For instance, answering a question of Ulam [25], Hyers proved in [11] that if $|\Delta(\omega)(s,t)| \leq \varepsilon$ for some fixed ε and all $s,t \in \mathbb{R}$, then there is an additive map $a : \mathbb{R} \to \mathbb{R}$ such that $|\omega(s) - a(s)| \leq \varepsilon$ for all $s \in \mathbb{R}$. This result is commonly considered as the origin of stability theory, although Pólya and Szegő [18] proved a similar result for real sequences already in 1925. The book [12] reflects the current state of that theory. The present paper is, however, closest in spirit to the dissertation [4].

There is another source that influenced our work: the theory of twisted sums and the so-called quasi-linear techniques introduced by Kalton [13]. Here, we adopt the approach of Domański as presented in [8]. There, a map $\omega : \mathfrak{Z} \to \mathfrak{Y}$ acting between topological vector spaces is called*quasi-linear* if $\omega(0) = 0$ and

- it is *quasi-additive*: $\Delta(\omega)(x,y) \to 0$ in $\mathfrak{Y}$ as $(x,y) \to (0,0)$ in $\mathfrak{Z} \times \mathfrak{Z}$;
- it is *quasi-homogeneous*: $\omega(\lambda x) - \lambda\omega(x) \to 0$ in $\mathfrak{Y}$ as $(\lambda, x) \to (0,0)$ in $\mathbb{K} \times \mathfrak{Z}$.

Each quasi-linear map $\omega : \mathfrak{Z} \to \mathfrak{Y}$ gives rise to an extension of $\mathfrak{Z}$ by $\mathfrak{Y}$, that is, another topological vector space $\mathfrak{X}$ (usually denoted by $\mathfrak{Y} \oplus_\omega \mathfrak{Z}$) containing $\mathfrak{Y}$ as a subspace and such that $\mathfrak{X}/\mathfrak{Y}$ is (isomorphic to) $\mathfrak{Z}$. Moreover, this extension is trivial (that is, $\mathfrak{Y}$ is complemented in $\mathfrak{X}$, and therefore $\mathfrak{X} = \mathfrak{Y} \oplus \mathfrak{Z}$) if and only if the associated map ω is approximable by a true linear map $L : \mathfrak{Z} \to \mathfrak{Y}$ in the sense that $\omega(x) - L(x) \to 0$ as $x \to 0$ in $\mathfrak{Z}$. Moreover, if $\mathfrak{Z}$ and $\mathfrak{Y}$ are F-spaces, then every extension comes from some quasi-linear $\omega : \mathfrak{Z} \to \mathfrak{Y}$. If in addition $\mathfrak{Z}$ and $\mathfrak{Y}$ are locally bounded (that is, quasi-Banach spaces) then ω can be chosen to be homogeneous. In this case quasi-additivity becomes

$$\|\Delta(\omega)(x,y)\|_{\mathfrak{Y}} \leq K(\|x\|_{\mathfrak{Z}} + \|y\|_{\mathfrak{Z}}) \tag{2}$$

for some constant independent of $x \in \mathfrak{Z}$, and the linear map L approximates ω if and only if one has an estimate of the form $\|\omega(x) - L(x)\|_{\mathfrak{Y}} \leq C\|x\|_{\mathfrak{Z}}$ for some C and all $x \in \mathfrak{Z}$. All this can be found in [8, 13, 14, 15, 5, 7].

Returning to nearly additive functions, it is quite natural to consider (nonhomogeneous) mappings $\omega : \mathbb{R} \to \mathbb{R}$ satisfying

$$|\Delta(\omega)(s,t)| \leq \varepsilon(|s| + |t|) \quad (s,t \in \mathbb{R}) \tag{3}$$

for some $\varepsilon \geq 0$. One may ask if such a function can be approximated by some additive function a in the sense that

$$|\omega(s) - a(s)| \leq K|s| \quad (s \in \mathbb{R}). \tag{4}$$

Simple examples show that this is not the case. Actually, if $\theta : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function vanishing at zero, the Kalton–Peck map

$$\omega_\theta(t) = t\theta(\log_2 |t|) \qquad (\omega_\theta(0) = 0)$$

satisfies (3) with ε being the Lipschitz constant of θ. However, ω_θ is close to an additive map in the sense of (4) if and only if θ is bounded. Moreover, Kalton and Peck proved in [14] that if ω is continuous at zero and satisfies (3) then there is a Lipschitz function θ (with Lipschitz constant at most ε) such that

$$|\omega(t) - \omega_\theta(t)| \le K\varepsilon|t| \qquad (t \in \mathbb{R}),$$

where K is an absolute constant. So, the relevant question becomes whether an arbitrary map satisfying (3) can be approximated by an additive one in the sense that $\omega - a$ is continuous at zero. This was our starting point, in spite of the abstract approach of the paper.

As often happens in mathematics, some abstraction leads to a better understanding of the problems. By mimicking quasi-additivity, we will consider mappings which approximately behave as homomorphisms in the following sense.

Definition 1. A mapping $\omega : \mathcal{G} \to \mathcal{H}$ acting between topological abelian groups is a *quasi-homomorphism* if $\omega(0) = 0$ and the map $\Delta(\omega) : \mathcal{G} \times \mathcal{G} \to \mathcal{H}$ given by $\Delta(\omega)(x, y) = \omega(x + y) - \omega(x) - \omega(y)$ is continuous at the origin of $\mathcal{G} \times \mathcal{G}$.

In this setting, the basic question is whether ω is *approximable* by a (generally discontinuous) homomorphism $a : \mathcal{G} \to \mathcal{H}$ in the sense that the difference $\omega - a$ is continuous at the origin of $\mathcal{G}$. Of course, every perturbation of a homomorphism by a mapping (vanishing and) continuous at the origin of $\mathcal{G}$ is a quasi-homomorphism.

Our main result in this line is the following.

Theorem 1. *Every quasi-homomorphism $\omega : \mathcal{G} \to \mathcal{H}$ is approximable in each of the following cases*:

(a) *$\mathcal{G}$ is an arbitrary product of locally compact abelian groups and $\mathcal{H}$ is either $\mathbb{R}$ or the circle group $\mathbb{T}$.*

(b) *$\mathcal{G}$ is either $\mathbb{R}$ or $\mathbb{T}$ and $\mathcal{H}$ is a Banach space.*

The plan of the paper is as follows. §1 contains the proof of (a). First, we prove it for a single locally compact abelian group. As in [4], the main idea borrows from the theory of twisted sums. We show that each quasi-homomorphism $\omega : \mathcal{G} \to \mathcal{H}$ gives rise to a topological extension of $\mathcal{G}$ by $\mathcal{H}$, that is, a further group $\mathcal{E}$ containing $\mathcal{H}$ as a normal subgroup in such a way that $\mathcal{E}/\mathcal{H} = \mathcal{G}$. The key point is that the induced extension is trivial (which means that one has a natural decomposition $\mathcal{E} = \mathcal{H} \oplus \mathcal{G}$) if and only if ω is approximable. The approximation of quasi-characters on locally compact abelian groups then follows from classical results by Dixmier [6] and van Kampen [17]. We complete the proof of (a) by using some standard procedures from the stability of functional equations.

The proof of (b) occupies §2. Here, we adapt some ideas from [13, 14] to show that the behavior of a quasi-homomorphism depends only on its restriction to any dense subgroup of $\mathcal{G}$ when the target group $\mathcal{H}$ is complete. In this way we show that every quasi-additive map from $\mathbb{Q}$ into any Banach space is not only approximable, but continuous at zero—which implies (b), thus completing the proof of Theorem 1.

Finally, in §3 we study quasi-additive maps between the groups underlying some infinite-dimensional (quasi-) Banach spaces. It is shown that if $\mathfrak{Z}$ and $\mathfrak{Y}$ are Banach spaces, then every abelian extension of $\mathfrak{Z}$ by $\mathfrak{Y}$ is in fact a topological vector space. We then conclude that most classical Banach spaces have only approximable (real-valued) quasi-additive functions.

1. Quasi-characters

1.1. *Extensions.* In this short subsection we present some standard facts about extensions of topological groups. All topological groups are assumed to be Hausdorff. The group operation is written additively in general (despite the fact that we do not assume commutativity), with the only exception of the circle group $\mathbb{T}$. The set of all neighborhoods of the origin in a topological group $\mathcal{G}$ will be denoted by $\mathcal{O}_{\mathcal{G}}$.

Definition 2. Let $\mathcal{G}$ and $\mathcal{H}$ be topological groups. A *topological extension* of $\mathcal{G}$ by $\mathcal{H}$ is a short exact sequence

$$0 \to \mathcal{H} \xrightarrow{\imath} \mathcal{E} \xrightarrow{\pi} \mathcal{G} \to 0 \tag{5}$$

in which $\mathcal{E}$ is a topological group and the arrows represent relatively open continuous homomorphisms.

Note that there is no open mapping theorem for topological groups, and so the requirement of having relatively open homomorphisms is not superfluous. Less technically, we can regard $\mathcal{E}$ as a topological group containing $\mathcal{H}$ as a closed normal subgroup in such a way that $\mathcal{E}/\mathcal{H}$ is (topologically isomorphic to) $\mathcal{G}$. First of all, one needs to know when two extensions are essentially the same. A topological extension $0 \to \mathcal{H} \to \mathcal{F} \to \mathcal{G} \to 0$ is *equivalent* to (5) if there is a continuous homomorphism $T : \mathcal{E} \to \mathcal{F}$ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle T} & & \Vert & & \\ 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

It follows from the five-lemma [24, Lemma 9.1.5] that such a T must be bijective. The following nice result, due to Roelcke [20], implies that T is in fact a topological isomorphism and shows that equivalence of topological extensions is an equivalence relation. For a proof, see [20] or [8, Lemma A].

LEMMA 1 (Roelcke). *Let* $0 \to \mathcal{H} \to \mathcal{E} \to \mathcal{G} \to 0$ *be a topological extension. Suppose* τ *is a group topology on* $\mathcal{E}$ *weaker than the original topology. If* $0 \to \mathcal{H} \to (\mathcal{E}, \tau) \to \mathcal{G} \to 0$ *is topologically exact, then* τ *is the original topology of* $\mathcal{E}$. ∎

The extension (5) is said to be *trivial* (or to *split*) if it is equivalent to the direct sum extension $0 \to \mathcal{H} \to \mathcal{H} \oplus \mathcal{G} \to \mathcal{G} \to 0$. This happens if and only if there exists a continuous homomorphism $P : \mathcal{E} \to \mathcal{H}$ such that $P \circ \imath = \mathrm{Id}_{\mathcal{H}}$. For abelian extensions (that is, those in which $\mathcal{E}$ is abelian) this is still equivalent to the existence of a continuous homomorphism $S : \mathcal{G} \to \mathcal{E}$ such that $\pi \circ S = \mathrm{Id}_{\mathcal{G}}$. In general, the latter condition is strictly weaker than triviality and leads to the notion of a semi-direct product. See [24, §9.2] for basic information in the algebraic setting.

Let us say that a property $\mathcal{P}$ is a *three-group property* if whenever (5) is a topological extension in which $\mathcal{H}$ and $\mathcal{G}$ have $\mathcal{P}$ then $\mathcal{E}$ also has $\mathcal{P}$. For instance, local compactness [10, Theorem 5.25], metrizability [21] and completeness [22, Theorem 12.1] are three-group properties, while commutativity is not [24, Exercise 9.6.11].

1.2. *Quasi-homomorphisms versus extensions.* The basic connection between quasi-homomorphisms and extensions is: every quasi-homomorphism $\omega : \mathcal{G} \to \mathcal{H}$ induces a topological extension of $\mathcal{G}$ by $\mathcal{H}$ which splits if and only if ω is approximable.

The proof of the following result is contained in that given for topological vector spaces in [8, Proposition 3.1].

LEMMA 2. *Let* $\omega : \mathcal{G} \to \mathcal{H}$ *be a quasi-homomorphism acting between abelian groups. Then the sets*

$$W(V, U) = \{(y, z) : y - \omega(z) \in V,\ z \in U\} \qquad (U \in \mathcal{O}_{\mathcal{G}},\ V \in \mathcal{O}_{\mathcal{H}})$$

form a neighborhood base at the origin for a group topology on $\mathcal{H} \times \mathcal{G}$. ∎

Let $\mathcal{H} \oplus_\omega \mathcal{G}$ denote the group $\mathcal{H} \times \mathcal{G}$ equipped with the above topology. Clearly, the homomorphism $\mathcal{H} \to \mathcal{H} \oplus_\omega \mathcal{G}$ given by $y \mapsto (y, 0)$ is injective, continuous, and relatively open, so that $\mathcal{H}$ can be regarded as a closed subgroup of $\mathcal{H} \oplus_\omega \mathcal{G}$. Moreover, the map $\mathcal{H} \oplus_\omega \mathcal{G} \to \mathcal{G}$ given by $(y, z) \mapsto z$ is a continuous open homomorphism onto $\mathcal{G}$ whose kernel is the image of the preceding one. Thus, the sequence $0 \to \mathcal{H} \to \mathcal{H} \oplus_\omega \mathcal{G} \to \mathcal{G} \to 0$ is a topological extension of $\mathcal{G}$ by $\mathcal{H}$.

Note that a homomorphism $\mathcal{H} \oplus_\omega \mathcal{G} \to \mathcal{H}$ is a projection onto $\mathcal{H}$ (in the purely algebraic sense) if and only if it has the form $P(x, y) = x - a(y)$, where $a : \mathcal{G} \to \mathcal{H}$ is an algebraic homomorphism. As in the linear case [8, Corollary 3.1], it turns out that P is continuous (equivalently, continuous at the origin) if and only if $\omega - a$ is continuous at the origin of $\mathcal{G}$. Thus one has:

Lemma 3. *A quasi-homomorphism is approximable if and only if the induced extension splits.* ∎

1.3. *Proof of* (a). We are ready to prove the first part of the main theorem for a single locally compact abelian group.

Suppose $\omega : \mathcal{G} \to \mathbb{L}$ is a quasi-homomorphism, where $\mathbb{L}$ is either $\mathbb{R}$ or $\mathbb{T}$ and $\mathcal{G}$ is a locally compact abelian group. Consider the induced extension

$$0 \to \mathbb{L} \to \mathbb{L} \oplus_\omega \mathcal{G} \to \mathcal{G} \to 0.$$

Clearly, $\mathbb{L} \oplus_\omega \mathcal{G}$ is also locally compact and abelian. By [10, Theorem 24.36] (if $\mathbb{L} = \mathbb{R}$) and [10, Theorem 24.11] (for $\mathbb{L} = \mathbb{T}$), there is a continuous character $\chi : \mathbb{L} \oplus_\omega \mathcal{G} \to \mathbb{L}$ extending the identity on $\mathbb{L}$. Clearly, χ is a continuous homomorphism projecting $\mathbb{L} \oplus_\omega \mathcal{G}$ onto $\mathbb{L}$. An appeal to Lemma 3 shows that ω is approximable.

The case where $\mathcal{G}$ is a product of locally compact abelian groups requires some preparations. First, we need a classical result by Hyers (see [11] or [12, Theorem 1.1 and Corollary 1.2]. Although only the case $\mathfrak{Y} = \mathbb{R}$ is necessary here, the general case will be used later.

Lemma 4 (Hyers). *Let $\mathcal{G}$ be an abelian group* (*no topology is assumed*) *and $\mathfrak{Y}$ a Banach space. Suppose $\omega : \mathcal{G} \to \mathfrak{Y}$ is a mapping such that*

$$\|\omega(x + y) - \omega(x) - \omega(y)\|_{\mathfrak{Y}} \le \varepsilon$$

for all $x, y \in \mathcal{G}$. Then there exists an additive mapping $a : \mathcal{G} \to \mathfrak{Y}$ such that $\|\omega(x) - a(x)\|_{\mathfrak{Y}} \le \varepsilon$ for all $x \in \mathcal{G}$. If $\mathcal{G}$ is torsion-free, *then a is unique.* ∎

Lemma 5. *Let $\omega : \mathcal{G} \to \mathfrak{Y}$ be a quasi-additive mapping*, *where $\mathcal{G}$ is an abelian topological group and $\mathfrak{Y}$ a Banach space. If ω maps a neighborhood of the origin in $\mathcal{G}$ into a bounded set in $\mathfrak{Y}$*, *then ω is continuous at the origin of $\mathcal{G}$.*

Proof. The hypothesis implies that there is $V \in \mathcal{O}_\mathcal{G}$ such that $\|\omega(x)\| \le K$ for some K and all $x \in V$. Fix $\varepsilon > 0$. Choose n such that $K/2^n < \varepsilon/2$. Now, take $W \in \mathcal{O}_\mathcal{G}$, with $W \subset V$, such that

$$\|\omega(x + y) - \omega(x) - \omega(y)\| \le \frac{\varepsilon}{2} \quad (x, y \in W)$$

and let $U \in \mathcal{O}_\mathcal{G}$ be such that $U + \overset{2^n \text{ times}}{\cdots\cdots} + U \subset W$. A straightforward induction on $k = 1, \dots, n$ yields

$$\|\omega(2^k x) - 2^k \omega(x)\| \le (2^k - 1)\,\frac{\varepsilon}{2} \qquad (x \in U).$$

Taking $k = n$ and dividing by 2^n, we obtain

$$\left\|\frac{\omega(2^n x)}{2^n} - \omega(x)\right\| \le \frac{\varepsilon}{2} \qquad (x \in U),$$

and so

$$\|\omega(x)\| \le \frac{\varepsilon}{2} + \frac{K}{2^n} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad (x \in U). \ \blacksquare$$

When the target group is the circle, it will be convenient to introduce the following invariant metric on $\mathbb{T}$:

$$d(z, w) = |\mathrm{Arg}(z/w)| \quad (-\pi < \mathrm{Arg}(\zeta) \le \pi).$$

Note that $d(z, w)$ represents the arc length between z and w. The following lemma is a stability result of "Hyers–Ulam" type proved in [2] for discrete amenable groups. Note that every abelian group is amenable.

Lemma 6. *Let $\mathcal{G}$ be an abelian group* (*no topology is assumed*) *and let $\omega : \mathcal{G} \to \mathbb{T}$ be any mapping such that for some $\varepsilon < \pi/3$ the estimate*

$$d(\omega(x + y), \omega(x)\omega(y)) \le \varepsilon$$

holds for all $x, y \in \mathcal{G}$. Then there is a unique character $a : \mathcal{G} \to \mathbb{T}$ such that $d(\omega(x), a(x)) \le \varepsilon$ for all $x \in \mathcal{G}$. $\blacksquare$

Lemma 7. *Suppose that $\omega : \mathcal{G} \to \mathbb{T}$ is a quasi-character such that $d(\omega(x), 1) < \pi/3$ for all $x \in \mathcal{G}$. Then ω is continuous at the origin.*

Proof. Define $f : \mathcal{G} \to \mathbb{R}$ by $f(x) = \mathrm{Arg}(\omega(x))$. We see that f is quasi-additive. Since $-\pi/3 < \mathrm{Arg}(\omega(x)) < \pi/3$ for all $x \in \mathcal{G}$, one has

$$\begin{aligned} f(x + y) - f(x) - f(y) &= \mathrm{Arg}(\omega(x + y)) - \mathrm{Arg}(\omega(x)) - \mathrm{Arg}(\omega(y)) \\ &= \mathrm{Arg}\left(\frac{\omega(x + y)}{\omega(x)\omega(y)}\right) \to 0 \end{aligned}$$

as $x, y \to 0$ in $\mathcal{G}$. On the other hand, f has bounded range. According to Lemma 5, f (and so ω) is continuous at the origin of $\mathcal{G}$. $\blacksquare$

End of the proof of (a). Write $\mathcal{G} = \prod_{i \in I} \mathcal{G}_i$, with $\mathcal{G}_i$ locally compact abelian groups, and let $\mathbb{L}$ denote either $\mathbb{R}$ or $\mathbb{T}$. We use additive notation for $\mathbb{L}$. Let $\omega : \mathcal{G} \to \mathbb{L}$ be a quasi-homomorphism. By the very definition there is $U \in \mathcal{O}_{\mathcal{G}}$ such that

$$d(\omega(x + y), \omega(x) + \omega(y)) \le 1 \qquad (x, y \in U). \tag{6}$$

Clearly, $U = (\prod_{j \in J} U_j) \times (\prod_{i \in I \setminus J} \mathcal{G}_i)$, where J is a finite subset of I and $U_j \in \mathcal{O}_{\mathcal{G}_j}$ for all $j \in J$.

Write $\mathcal{G} = \mathcal{G}^0 \times \mathcal{G}^1$, where $\mathcal{G}^0 = \prod_{j\in J} \mathcal{G}_j$ and $\mathcal{G}^1 = \prod_{i\in I\setminus J} \mathcal{G}_i$. Since (6) holds for all $x, y \in \mathcal{G}^1$, we can apply Lemma 6 (for $\mathbb{L} = \mathbb{T}$; note that $1 < \pi/3$) or Lemma 4 (for $\mathbb{L} = \mathbb{R}$) to get a unique homomorphism $a_1 : \mathcal{G}^1 \to \mathbb{L}$ with

$$d(\omega(x), a_1(x)) \leq 1 \qquad (x \in \mathcal{G}^1).$$

It follows from Lemma 7 (for $\mathbb{L} = \mathbb{T}$) and Lemma 5 (for $\mathbb{L} = \mathbb{R}$) that $\omega_1 - a_1$ is continuous at the origin of $\mathcal{G}^1$.

On the other hand, $\mathcal{G}^0$ is locally compact, and so there is a homomorphism $a_0 : \mathcal{G}^0 \to \mathbb{L}$ such that $\omega_0 - a_0$ is continuous at the origin of $\mathcal{G}^0$. Clearly, the homomorphism $a : \mathcal{G} = \mathcal{G}^0 \times \mathcal{G}^1 \to \mathbb{L}$ given by $a(x, y) = a_0(x) + a_1(y)$ approximates ω near the origin of $\mathcal{G}$. ■

2. Vector-valued maps

2.1. *Restriction to dense subgroups.* In this subsection we quickly adapt the corresponding results from the linear theory to show that if the target group $\mathcal{H}$ is complete, the behavior of a quasi-homomorphism $\omega : \mathcal{G} \to \mathcal{H}$ depends only on its restriction to any dense subgroup of $\mathcal{G}$. This is the crucial step in the proof of the second part of Theorem 1. Since topological groups tend to exhibit some pathologies with regard to completeness, let us remark that:

- If $\mathcal{E}$ is a metrizable, complete abelian group and $\mathcal{H}$ a closed subgroup of $\mathcal{E}$, then $\mathcal{E}/\mathcal{H}$ is complete under the quotient topology [22, Theorem 11.18].
- The completion of a topological abelian group always exists, as a complete abelian group, and has the usual uniqueness property. Arbitrary topological groups need not admit completions. This is due to the fact that a noncommutative topological group has two uniform structures rather than one. See [1, chapitre III, §3].

Let $\mathcal{G}$ and $\mathcal{H}$ be metrizable complete topological abelian groups. Suppose $\omega : \mathcal{G}_0 \to \mathcal{H}$ is a quasi-homomorphism, where $\mathcal{G}_0$ is a dense subgroup of $\mathcal{G}$. Consider the induced extension $0 \to \mathcal{H} \to \mathcal{H} \oplus_\omega \mathcal{G}_0 \to \mathcal{G}_0 \to 0$. Let $\mathcal{E}(\omega)$ denote the completion of $\mathcal{H} \oplus_\omega \mathcal{G}_0$. One has a commutative diagram of continuous homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{H} \oplus_\omega \mathcal{G}_0 & \longrightarrow & \mathcal{G}_0 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E}(\omega) & \longrightarrow & \mathcal{E}(\omega)/\mathcal{H} & \longrightarrow & 0 \end{array} \tag{7}$$

It is clear that the vertical arrows are injective and also that $\mathcal{G}_0 \to \mathcal{E}(\omega)/\mathcal{H}$ is relatively open, with dense range. Therefore, $\mathcal{E}(\omega)/\mathcal{H}$ (which is complete) turns out to be topologically isomorphic to the completion of $\mathcal{G}_0$, that is, to $\mathcal{G}$. Thus, the lower row in (7) can be regarded as a topological extension of $\mathcal{G}$ by $\mathcal{H}$, the homomorphism $\mathcal{E}(\omega) \to \mathcal{E}(\omega)/\mathcal{H}$ being the extension of $\mathcal{H}\oplus_\omega\mathcal{G}_0 \to \mathcal{G}_0$ to the completions. Moreover, the upper row in (7) splits if and only if so does the lower one. Indeed, let P be a continuous homomorphism projecting $\mathcal{H}\oplus_\omega\mathcal{G}_0$ onto $\mathcal{H}$. Then the extension of P to $\mathcal{E}(\omega)$ is a projection onto $\mathcal{H}$. Conversely, if $P : \mathcal{E}(\omega) \to \mathcal{H}$ is a splitting homomorphism for the lower row, then the restriction of P to $\mathcal{H}\oplus_\omega\mathcal{G}_0$ is a projection onto $\mathcal{H}$. We have proved the following.

LEMMA 8. *Let $\mathcal{G}$ and $\mathcal{H}$ be metrizable complete topological abelian groups and $\mathcal{G}_0$ a dense subgroup of $\mathcal{G}$. Every quasi-homomorphism $\omega : \mathcal{G}_0 \to \mathcal{H}$ gives rise to a topological extension $0 \to \mathcal{H} \to \mathcal{E}(\omega) \to \mathcal{G} \to 0$. That extension splits if and only if ω is approximable by a homomorphism $a : \mathcal{G}_0 \to \mathcal{H}$.* ∎

COROLLARY 1. *Let $\mathcal{G}$ be a subgroup of $\mathbb{Q}$. Then every quasi-additive map $\omega : \mathcal{G} \to \mathbb{R}$ is continuous at zero.*

Proof. If $\mathcal{G}$ is discrete this is obvious. Otherwise $\mathcal{G}$ is dense in $\mathbb{R}$ and so there is an additive map $a : \mathcal{G} \to \mathbb{R}$ such that $\omega - a$ is continuous at zero. Since $\mathcal{G} \subset \mathbb{Q}$ we see that $a(x) = tx$, for some fixed $t \in \mathbb{R}$ and all $x \in \mathcal{G}$, and so a is continuous at zero. Hence ω is itself continuous at zero. ∎

A direct proof of this result for some dense subgroup of $\mathbb{Q}$ would be interesting in view of the following result. (For mappings satisfying the estimate (3) this was done in [3].)

PROPOSITION 1 (cf. [3, Lemma 1] and [4, Theorem 5.14]). *Let $\omega : \mathcal{G} \to \mathcal{H}$ be a quasi-homomorphism between abelian groups, where $\mathcal{H}$ is complete. If ω is approximable on some dense subgroup of $\mathcal{G}$, then it is approximable on the whole of $\mathcal{G}$.*

Proof. One has a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{H}\oplus_\omega\mathcal{G} \arrow[r] & \mathcal{G} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] \arrow[u, equal] & \mathcal{H}\oplus_\omega\mathcal{G}_0 \arrow[r] \arrow[u] & \mathcal{G}_0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

It is clear that the middle vertical arrow embeds $\mathcal{H}\oplus_\omega\mathcal{G}_0$ as a dense subgroup of $\mathcal{H}\oplus_\omega\mathcal{G}$. Since $\mathcal{H}$ is complete, we conclude that the upper row splits if and only if so does the lower one. ∎

2.2. *Proof of* (b). We are now ready to complete the proof of our main result. In view of Proposition 1, if the domain group is $\mathbb{R}$, this is a straightforward consequence of

PROPOSITION 2. *Let $\mathfrak{Y}$ be a Banach space. Every quasi-additive map $\omega : \mathbb{Q} \to \mathfrak{Y}$ is continuous at zero.*

Proof. According to Lemma 5 all that remains is to show that there is a neighborhood of zero in $\mathbb{Q}$ where ω is bounded. Otherwise there is a sequence $(q_n)_n$ converging to zero in $\mathbb{Q}$ such that $\omega(q_n)$ is unbounded in $\mathfrak{Y}$. The uniform boundedness principle of Banach and Steinhaus yields a continuous linear functional $y^* \in \mathfrak{Y}^*$ such that $y^*(\omega(q_n))$ is an unbounded sequence of real numbers. This implies that the quasi-additive map $y^* \circ \omega : \mathbb{Q} \to \mathbb{R}$ is discontinuous at zero, which contradicts Corollary 1. ∎

The following result allows one to "transfer" approximability from $\mathbb{R}$ to $\mathbb{T}$ and completes the proof of Theorem 1. Note that there are plenty of (necessarily discontinuous) homomorphisms from $\mathbb{T}$ into any topological vector space.

LEMMA 9 (see [9, Theorem III.4.1] and [15, Chapter 5, §§1–2]). *Let $\mathcal{S}$ be a closed subgroup of $\mathcal{G}$ and let $\mathcal{H}$ be another abelian group. Suppose that all quasi-homomorphisms $\mathcal{G} \to \mathcal{H}$ are approximable and that every continuous homomorphism $\mathcal{S} \to \mathcal{H}$ can be extended to a continuous homomorphism $\mathcal{G} \to \mathcal{H}$. Then every quasi-homomorphism $\omega : \mathcal{G}/\mathcal{S} \to \mathcal{H}$ is approximable. In particular, so is every quasi-homomorphism from $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ into a Banach space.*

Proof. Let $\omega : \mathcal{G}/\mathcal{S} \to \mathcal{H}$ be a quasi-homomorphism. If $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{S}$ denotes the natural quotient homomorphism, it is clear that $\omega \circ \pi : \mathcal{G} \to \mathcal{H}$ is a quasi-homomorphism. Let $a : \mathcal{G} \to \mathcal{H}$ be any homomorphism approximating $\omega \circ \pi$. Since $\omega \circ \pi$ vanishes on $\mathcal{S}$, it is clear that the restriction of a to $\mathcal{S}$ is a continuous homomorphism $\mathcal{S} \to \mathcal{H}$. Let $a^* : \mathcal{G} \to \mathcal{H}$ be a continuous homomorphism extending it. Then $a - a^* : \mathcal{G} \to \mathcal{H}$ is a (generally discontinuous) homomorphism vanishing on $\mathcal{S}$. It follows that there is a unique homomorphism $b : \mathcal{G}/\mathcal{S} \to \mathcal{H}$ such that $a - a^* = b \circ \pi$. It is easily seen that b approximates ω. ∎

2.3. *A uniform approximation principle for quasi-additive functions on the line.* We conclude this section by showing that a family of functions $\mathbb{R} \to \mathbb{R}$ which are "uniformly quasi-additive" can be "uniformly approximated" by additive functions. Precisely:

COROLLARY 2. *Let Ω be a family of quasi-additive mappings $\omega : \mathbb{R} \to \mathfrak{Y}_\omega$, where $\mathfrak{Y}_\omega$ are possibly different Banach spaces. Suppose Ω is uniformly quasi-additive in the sense that for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\|\omega(s+t) - \omega(s) - \omega(t)\|_\omega \le \varepsilon \qquad (|s|, |t| \le \delta) \tag{8}$$

for all $\omega \in \Omega$, where $\|\cdot\|_\omega$ denotes the norm of $\mathfrak{Y}_\omega$. Then Ω is uniformly approximable in the sense that to each $\omega \in \Omega$ there corresponds an additive map $a_\omega : \mathbb{R} \to \mathfrak{Y}_\omega$ in such a way that the family $\{\omega - a_\omega : \omega \in \Omega\}$ is *equicontinuous at zero: that is, for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$\|\omega(s) - a_\omega(s)\|_\omega \le \varepsilon \quad (|s| \le \delta)$$

for all $\omega \in \Omega$.

This can be deduced from Theorem 1(b), as in the proof of the implication (d)⇒(e) of Theorem 2.1 in [7], but we give the details for the sake of completeness.

Proof. We can regard the whole family Ω as a map $\mathbb{R} \to \prod_{\omega\in\Omega} \mathfrak{Y}_\omega$, taking $\Omega(s) = (\omega(s))_{\omega\in\Omega}$. Let $\ell_\infty(\Omega, \mathfrak{Y}_\omega)$ denote the Banach space of all families $(y_\omega)_{\omega\in\Omega} \in \prod_{\omega\in\Omega} \mathfrak{Y}_\omega$ for which the norm

$$\|(y_\omega)_\omega\|_\infty = \sup_{\omega\in\Omega} \|y_\omega\|_\omega$$

is finite. Choose δ_0 such that (8) holds for $\varepsilon = 1$. It is clear that $\Omega(s+t) - \Omega(s) - \Omega(t) \in \ell_\infty(\Omega, \mathfrak{Y}_\omega)$ whenever $|s|, |t| \le \delta_0$. Let Π be any linear projection of $\prod_{\omega\in\Omega} \mathfrak{Y}_\omega$ onto $\ell_\infty(\Omega, \mathfrak{Y}_\omega)$ and define $\widetilde{\Omega} : \mathbb{R} \to \ell_\infty(\Omega, \mathfrak{Y}_\omega)$ as $\widetilde{\Omega} = \Pi \circ \Omega$. Since

$$\widetilde{\Omega}(s+t) - \widetilde{\Omega}(s) - \widetilde{\Omega}(t) = \Omega(s+t) - \Omega(s) - \Omega(t) \quad (|s|, |t| \le \delta_0)$$

it follows from (8) that $\widetilde{\Omega} : \mathbb{R} \to \ell_\infty(\Omega, \mathfrak{Y}_\omega)$ is quasi-additive, and also that $\Omega - \widetilde{\Omega}$ is additive on $[-\delta_0/2, \delta_0/2]$, as a map $\mathbb{R} \to \prod_{\omega\in\Omega} \mathfrak{Y}_\omega$. It is clear that there is a unique additive map $A' : \mathbb{R} \to \prod_{\omega\in\Omega} \mathfrak{Y}_\omega$ extending $\Omega - \widetilde{\Omega}$. Finally, by Theorem 1, there is an additive mapping $A'' : \mathbb{R} \to \ell_\infty(\Omega, \mathfrak{Y}_\omega)$ such that

$$\|\widetilde{\Omega}(s) - A''(s)\|_\infty \to 0 \quad \text{as } s \to 0.$$

Since $\Omega = A' + A'' + \widetilde{\Omega} - A''$ on $[-\delta_0/2, \delta_0/2]$, we see that $\Omega - A' - A'' : [-\delta_0/2, \delta_0/2] \to \ell_\infty(\Omega, \mathfrak{Y}_\omega)$ is well defined and continuous at zero. Putting $a_\omega(s) = (A'(s) + A''(s))_\omega$, one concludes that $\{\omega - a_\omega : \omega \in \Omega\}$ is equicontinuous at zero. ∎

3. Quasi-additive functions on locally bounded spaces. The purpose of this section is to show that quasi-additive functions on most classical (quasi-) Banach spaces are approximable. This will be done by connecting extensions of topological groups to F-space extensions. By an *F-space extension* we mean a short exact sequence of (real) F-spaces and (continuous linear) operators

$$0 \to \mathfrak{Y} \xrightarrow{\imath} \mathfrak{X} \xrightarrow{\pi} \mathfrak{Z} \to 0. \tag{9}$$

The open mapping theorem [15] guarantees that π is open and also that $\imath$ is an isomorphic embedding, so that F-space extensions are topological extensions with respect to the underlying additive structures. On the other hand, any continuous additive map between topological linear spaces is linear (over $\mathbb{R}$). Hence (9) splits in the category of topological linear spaces (that is, there is a continuous operator projecting $\mathfrak{X}$ onto $\mathfrak{Y}$) if and only if it does as an extension of topological groups.

THEOREM 2. *Let $\mathfrak{Y}$ be a Banach space and $\mathfrak{Z}$ a locally bounded F-space. Then every topological abelian extension of $\mathfrak{Z}$ by $\mathfrak{Y}$ is equivalent to an F-space extension.*

That is, if $0 \to \mathfrak{Y} \to \mathcal{E} \to \mathfrak{Z} \to 0$ is an abelian extension of topological groups, then there is an F-space extension (9) and a topological isomorphism of groups $\mathcal{E} \to \mathfrak{X}$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{Y} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathfrak{Z} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathfrak{Y} & \longrightarrow & \mathfrak{X} & \longrightarrow & \mathfrak{Z} & \longrightarrow & 0 \end{array}$$

is commutative. The following example shows that the hypothesis on $\mathcal{E}$ cannot be removed:

EXAMPLE 1. There is an extension $0 \to \mathbb{R} \to \mathcal{E} \to \mathbb{R} \to 0$ in which $\mathcal{E}$ is not abelian.

Proof. Let $\mathcal{E} = A^+(\mathbb{R})$ be the group of all orientation preserving affine automorphisms of the real line endowed with the topology of convergence on compact subsets. A typical member of $A^+(\mathbb{R})$ can be regarded as (a, b), with $a, b \in \mathbb{R}$, $a > 0$, where $(a, b)(t) = at + b$. The composition law in $A^+(\mathbb{R})$ is then given by $(a, b) \circ (c, d) = (ac, ad + b)$, and the unit is $(1, 0)$. A neighborhood base at $(1, 0)$ is given by

$$\{(a, b) : |a - 1| + |b| < \varepsilon\} \quad (\varepsilon > 0).$$

Let $\mathcal{H}$ be the subgroup of translations: $\mathcal{H} = \{(1, b) : b \in \mathbb{R}\}$. Clearly, $\mathcal{H}$ is a normal subgroup of $A^+(\mathbb{R})$ isomorphic to $(\mathbb{R}, +)$. It is not hard to see that $A^+(\mathbb{R})/\mathcal{H}$ is topologically isomorphic to the multiplicative group $(\mathbb{R}^+, \cdot)$, so that we have an extension $0 \to (\mathbb{R}, +) \to (\mathcal{E}, \circ) \to (\mathbb{R}^+, \cdot) \to 0$. The observation that the logarithm defines a topological isomorphism between $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$ completes the proof. ∎

The proof of Theorem 2 requires the knowledge of when an extension is induced by a quasi-homomorphism. The next two lemmas are basically due to Domański.

LEMMA 10. *A topological abelian extension* (5) *is* (*equivalent to one*) *induced by a quasi-homomorphism if and only if it splits algebraically* (*there is an algebraic homomorphism $P : \mathcal{E} \to \mathcal{H}$ such that $P \circ \imath = \mathrm{Id}_{\mathcal{H}}$*) *and π admits a section continuous at the origin* (*a section for π is a map $\varrho : \mathcal{G} \to \mathcal{E}$ such that $\pi \circ \varrho = \mathrm{Id}_{\mathcal{G}}$, with $\varrho(0) = 0$*).

Sketch of the proof. Both conditions are obviously necessary: the required maps $P : \mathcal{H} \oplus_\omega \mathcal{G} \to \mathcal{H}$ and $\varrho : \mathcal{G} \to \mathcal{H} \oplus_\omega \mathcal{G}$ are given by $P(y, z) = y$ and $\varrho(z) = (\omega(z), z)$.

The sufficiency is proved as in the linear case (in which the first condition is automatic): the composition $\omega = P \circ \varrho$ defines a quasi-homomorphism between $\mathcal{G}$ and $\mathcal{H}$ and the starting extension (5) is equivalent to the extension induced by ω: the map $T : \mathcal{E} \to \mathcal{H} \oplus_\omega \mathcal{G}$ given by $T(x) = (P(x), \pi(x))$ is a continuous homomorphism making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{H} \oplus_\omega \mathcal{G} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

The details are as in [8, proof of Lemma 3.2] (see also [7, Lemma 1.1]). ∎

Actually, the second condition is always satisfied for metrizable groups:

LEMMA 11. *Let $\pi : \mathcal{E} \to \mathcal{G}$ be a quotient homomorphism of topological groups, that is, a continuous homomorphism which is surjective and open. If $\mathcal{E}$ is metrizable, then π admits a* (*not generally additive*) *section which is continuous at the origin of $\mathcal{G}$.*

The proof is contained in that given for the linear case in [8, proof of Lemma 2.2(a)].

Our immediate aim is to show that every abelian extension of two linear spaces splits algebraically. To this end let us call an abelian group $\mathcal{G}$ *divisible* if for every $x \in \mathcal{G}$ and $n \in \mathbb{N}$ there is a unique $x' \in \mathcal{G}$ such that $nx' = x$.

It is clear that divisible groups are torsion-free (since $nx = 0$ implies $x = 0$) and also that every divisible group can be regarded in the obvious way as a linear space over $\mathbb{Q}$. We now prove that divisibility is a three-group property for abelian groups.

LEMMA 12. *Let $0 \to \mathcal{H} \to \mathcal{E} \to \mathcal{G} \to 0$ be an abelian* (*algebraic*) *extension. If both $\mathcal{G}$ and $\mathcal{H}$ are divisible, then so is $\mathcal{E}$. Consequently, the extension is linear over $\mathbb{Q}$ and so it splits algebraically.*

Proof. Let $x \in \mathcal{E}$ and $n \geq 2$ be fixed. We prove there is a unique $x' \in \mathcal{E}$ such that $nx' = x$.

Let $q : \mathcal{E} \to \mathcal{H}$ denote the natural quotient homomorphism. Put $z = q(x)$. Then there is $z' \in \mathcal{G}$ such that $nz' = z$. Let x'' be any element of $\mathcal{E}$ such that $q(x'') = z'$. Clearly, $x - nx'' \in \mathcal{H}$. Hence $x - nx'' = ny$ for some $y \in \mathcal{H}$. So, $n(y + x'') = x$ is the required decomposition.

For the uniqueness, suppose $nx' = nx'' = x$. Then $n(x' - x'') = 0$. Since $\mathcal{G}$ is torsion-free, we see that $x' - x'' \in \mathcal{H}$. But $\mathcal{H}$ is torsion-free too, and so $x' = x''$. ∎

Proof of Theorem 2. We may assume that $\mathcal{E} = \mathfrak{Y} \oplus_\omega \mathfrak{Z}$, where $\omega : \mathfrak{Z} \to \mathfrak{Y}$ is a suitable quasi-additive map. Clearly, $\mathfrak{Y} \oplus_\omega \mathfrak{Z}$ is a vector space over $\mathbb{Q}$. We claim it is a *topological* vector space over $\mathbb{Q}$, that is, the multiplication

$$\mathbb{Q} \times (\mathfrak{Y} \oplus_\omega \mathfrak{Z}) \to \mathfrak{Y} \oplus_\omega \mathfrak{Z}, \qquad (q, (y, z)) \mapsto (qy, qz), \tag{10}$$

is (jointly) continuous. By the argument in [8, proof of Proposition 3.1 and Lemma 3.1] this amounts to verifying that

$$\|q\omega(z) - \omega(qz)\|_{\mathfrak{Y}} \to 0 \quad \text{as } (q, z) \to (0, 0) \in \mathbb{Q} \times \mathfrak{Z}.$$

Let U be the unit ball of $\mathfrak{Z}$. For $z \in U$, define $\omega_z : \mathbb{R} \to \mathfrak{Y}$ by $\omega_z(t) = \omega(tz)$. Since U is bounded, it is straightforward to check that the family $\{\omega_z : z \in U\}$ is uniformly quasi-additive. By Corollary 2, there are additive maps $a_z : \mathbb{R} \to \mathfrak{Y}$ such that $\{\omega_z - a_z : z \in U\}$ is equicontinuous at zero. Fix $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\|\omega_z(t) - a_z(t)\|_{\mathfrak{Y}} \le \varepsilon \quad (t \in \mathbb{R},\ |t| \le \delta,\ z \in U).$$

Then, for $q \in \mathbb{Q}$, we have

$$\begin{aligned}
\|q\omega(tz) - \omega(qtz)\|_{\mathfrak{Y}} &= \|q\omega_z(t) - \omega_z(qt)\|_{\mathfrak{Y}} \\
&= \|q\omega_z(t) - a_z(qt) + a_z(qt) - \omega_z(qt)\|_{\mathfrak{Y}} \\
&\le \|q(\omega_z(t) - a_z(t))\|_{\mathfrak{Y}} + \|a_z(qt) - \omega_z(qt)\|_{\mathfrak{Y}} \\
&\le |q|\varepsilon + \varepsilon \quad (t \in \mathbb{R},\ |t| \le \delta,\ z \in U).
\end{aligned}$$

Hence $\|q\omega(z) - \omega(qz)\|_{\mathfrak{Y}} \le (1 + |q|)\varepsilon$ whenever $\|z\|_{\mathfrak{Z}} \le \delta$, which proves our claim.

Since $\mathfrak{Y} \oplus_\omega \mathfrak{Z}$ is complete (a three-group property) it is clear (see, e.g., [1, chapitre III, §6.6 et le théorème 1 du §6.5]) that the outer multiplication in (10) extends to a (jointly continuous) multiplication

$$\star : \mathbb{R} \times (\mathfrak{Y} \oplus_\omega \mathfrak{Z}) \to \mathfrak{Y} \oplus_\omega \mathfrak{Z}$$

which makes $\mathfrak{Y} \oplus_\omega \mathfrak{Z}$ into an F-space—we remark that this product by real numbers in $\mathfrak{Y} \oplus_\omega \mathfrak{Z}$ need not be the "obvious one" $t(y, z) = (ty, tz)$, which is generally discontinuous. Finally, it is easily seen that

$$0 \to \mathfrak{Y} \to (\mathfrak{Y} \oplus_\omega \mathfrak{Z}, \star) \to \mathfrak{Z} \to 0$$

is an extension of F-spaces. This completes the proof. ∎

Since every extension of locally bounded spaces comes from a *homogeneous* quasi-linear map (which forces the estimate (2) for some constant K) the real meaning of Theorem 2 is that every quasi-additive mapping $\omega : \mathfrak{Z} \to \mathfrak{Y}$ is "equivalent" to a homogeneous quasi-linear map η in the sense that $\omega - \eta$ is approximable. So, one has a representation $\omega = \eta + a + \varepsilon$, where η is homogeneous, a additive, and ε continuous at the origin.

COROLLARY 3. *Let $\mathfrak{Z}$ be a locally bounded F-space and $\mathfrak{Y}$ a Banach space. The following statements are equivalent*:

- *Every quasi-additive map $\omega : \mathfrak{Z} \to \mathfrak{Y}$ is approximable.*
- *Every F-space extension $0 \to \mathfrak{Y} \to \mathfrak{X} \to \mathfrak{Z} \to 0$ splits.* ∎

An F-space satisfying the latter condition for $\mathfrak{Y} = \mathbb{R}$ is said to be a *K-space* (see [13, 15]). The main examples of K-spaces have been supplied by Kalton and co-workers: for instance, infinite-dimensional L_p spaces ($0 \le p \le \infty$) are K-spaces if and only if $p \ne 1$ [13, 16, 15, 19]. Also, B-convex spaces (that is, Banach spaces having nontrivial type $p > 1$) are K-spaces and so are quotients of Banach K-spaces. This provides us with many examples of Banach spaces on which every quasi-additive (real-valued) function is approximable. Also, Theorem 2 can be used to read many classical results about extensions in terms of quasi-additive mappings. A sample: if $\mathfrak{Z}$ is a separable Banach K-space, then every quasi-additive map $\omega : \mathfrak{Z} \to c_0$ is approximable, and so on. We refrain from going into details here. We take our leave of the reader with the following:

PROBLEM. Is "being topologically isomorphic to an F-space" a three-group property for abelian groups? And "being a locally bounded space"? Is (at least) every quasi-additive function from the line into a locally bounded F-space approximable? What if the target space is ℓ_p or L_p, for $0 < p < 1$?

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