Selivanovski hard sets are hard

by

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Abstract. Let $H \subseteq Z \subseteq 2^\omega$. For $n \geq 2$, we prove that if Selivanovski measurable functions from $2^\omega$ to $Z$ give as preimages of $H$ all $\Sigma^1_n$ subsets of $2^\omega$, then so do continuous injections.

Let $H \subseteq Z$ be subsets of the Cantor space $C = 2^\omega$. Say that $(H, Z)$ is $\Sigma^1_n$-hard if for any $\Sigma^1_n$ set $Q \subseteq C$ there is a continuous function $f : C \to Z$ with $Q = f^{-1}[H]$.

Kechris [1] proved that using here Borel rather than continuous functions we get the same family of pairs. For $n \geq 2$ Sabok [4] improved this by replacing Borel functions with functions such that preimages of all sets from the canonical subbasis of $C$ are in $\Sigma^1_1 \cup \Pi^1_1$.

We show for $n \geq 2$ that by changing in the definition of $\Sigma^1_n$-hardness “continuous” to “Selivanovski measurable” we do not get more pairs, and by changing “continuous” to “continuous injective” we do not get fewer pairs.

Recall that a function is Selivanovski measurable if preimages of open sets belong to the $\sigma$-field of Selivanovski sets (also called $C$-sets), which is the least $\sigma$-field that contains all Borel sets and is closed under the Suslin operation.

Kechris and Sabok use effective descriptive set theory, and Kechris asked about a classical proof of his theorem. Our proof is classical and can be adapted to give Kechris’s theorem (see [3] for a direct classical proof of Kechris’s theorem).

Theorem. Let $n \geq 2$ and $H \subseteq Z \subseteq C$. If Selivanovski measurable functions from $C$ to $Z$ give as preimages of $H$ all $\Sigma^1_n$ subsets of $2^\omega$, then so do continuous injections.

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(1) Kechris formulated his result for $n = 1$, but his proof works for any $n \geq 1$. DOI: 10.4064/fm228-1-2 [17] © Instytut Matematyczny PAN, 2015
Note that since for any separable metrizable space $S$ there exists a Borel injection $e: C \to S$ whose inverse is continuous (e.g., $e(s)(i) = 1 \iff s \in O_i$, where $\{O_i\}_{i \in \omega}$ is a basis of $S$), and $e$ can be chosen to be continuous if $S$ is zero-dimensional, we can change in the Theorem the range space $Z$ to any separable metrizable space and the domain space $C$ to any zero-dimensional uncountable Polish space.

Note also that the Theorem cannot be extended to $n = 1$: pick distinct points $z_0$ and $z_1$ in $C$ and let $Z = \{z_0, z_1\}$; if $Q \subseteq C$ is $\Sigma^1_1$, then the map sending $Q$ to $z_0$ and $C \setminus Q$ to $z_1$ is Selivanovski measurable; however, no non-clopen $Q \subseteq C$ is a continuous preimage of $H = \{z_0\}$.

1. Spaces, pointclasses, functions. All our spaces are separable and metrizable; let $X$, $Y$, and $Z$ range over such spaces. We identify the Baire space $N = \omega^\omega$ with

$$\{x \in C: \forall i \exists j > i \ x(j) = 1\}. $$

For $Q \subseteq X \times Y$, $f: X \times Y \to Z$, and $x \in X$, define the sections $Q_x \subseteq Y$ and $f_x: Y \to Z$ by $y \in Q_x \iff (x, y) \in Q$ and $f_x(y) = f(x, y)$.

A pointclass is a map $\Phi$ that assigns to any space $X$ a family $\Phi_X = \Phi(X)$ of subsets of $X$; we often drop $X$ if context permits. Let $\Phi_{XY} = \Phi(X, Y)$ be the family of all $\Phi$ measurable functions from $X$ to $Y$, i.e., functions such that preimages of open subsets of $Y$ are in $\Phi(X)$.

Let $B$ and $S$ be the pointclasses of Borel and Selivanovski sets. Selivanovski sets have the Baire property, and thus Selivanovski measurable functions are Baire measurable.

We shall also use the pointclasses $\Sigma^1_n, \Pi^1_n,$ and $\Delta^1_n$, $n \geq 1$. For an arbitrary space $X$, the families $\Sigma^1_n(X), \Pi^1_n(X),$ and $\Delta^1_n(X)$ are defined in the same way as for a Polish space (see [2, 25.A]): the $\Pi^1_n(X)$ sets are the complements of $\Sigma^1_n(X)$ sets, and the $\Sigma^1_n(X)$ sets are the projections of $\Pi^1_{n-1}(X \times N)$ sets, if $n > 1$, and of closed subsets of $X \times N$, if $n = 1$; also, $\Delta^1_n(X) = \Sigma^1_n(X) \cap \Pi^1_n(X)$.

We have

$$B(X) \subseteq \Delta^1_1(X) \subseteq S(X) \subseteq \Delta^2_2(X);$$

if $X$ is an uncountable Polish space, then the first inclusion is improper, and the next two are proper (see [2]; for $S \neq \Delta^1_2$ see Section 4).

**Lemma 1.** Let $\Phi \in \{B, S, \Sigma^1_n, \Pi^1_n, \Delta^1_n\}$.

1. If $X \subseteq X'$, then:
   (a) $Q' \in \Phi_{X'} \Rightarrow X \cap Q' \in \Phi_X$,
   (b) $Q \in \Phi_X \Rightarrow \exists Q' \in \Phi_{X'}$, $Q = X \cap Q'$, if $\Phi \neq \Delta^1_n$,
   (c) $Q \in \Phi_X \cap X \in \Phi_{X'} \Rightarrow Q \in \Phi_{X'}$. 


(2) If $Y$ is $\Sigma^1_n$ in a Polish space, then projections along $Y$ of $\Sigma^1_n$ subsets of $X \times Y$ are $\Sigma^1_n(X)$.

(3) $\Phi$ is closed under countable unions, countable intersections, and sections. The class of $\Phi$ measurable functions is closed under sections.

(4) If $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ are $\Phi$ measurable, then the Cartesian product function $(x_0, x_1) \mapsto (f_0(x_0), f_1(x_1))$ is $\Phi$ measurable.

(5) A function is $\Phi$ measurable iff preimages of closed sets are $\Phi$ sets. For any function, the notions of $\Sigma^1_n$, $\Pi^1_n$, and $\Delta^1_n$ measurability coincide.

(6) The graph of a $\Phi$ measurable function is a $\Phi$ set.

(7) Preimages of $\Phi$ sets under $\Phi$ measurable functions are $\Phi$ sets.

(8) If the domain of a $\Delta^1_n$ measurable function is $\Sigma^1_n$ in a Polish space, then images of $\Sigma^1_n$ sets are $\Sigma^1_n$ sets.

(9) If $Y$ is $\Sigma^1_n$ in a Polish space and the graph of $f : X \to Y$ is $\Sigma^1_n$, then $f \in \Delta^1_n(X, Y)$.

Proof. (4) The open subsets of $Y_0 \times Y_1$ are the countable unions of products $V_0 \times V_1$, with $V_0$ and $V_1$ open; the preimage of $V_0 \times V_1$ is $f_0^{-1}(V_0) \times f_1^{-1}(V_1) \in \Phi_{X_0 \times X_1}$.

(5) Closed sets are $G_\delta$, and open sets are $F_\sigma$.

(6) If $f \in \Phi_{XY}$, then graph $f$ is the preimage of the diagonal of $Y^2$ under the $\Phi$ measurable function $(x, y) \mapsto (f(x), y)$.

(7) We give a proof for $\Phi = \Sigma^1_n$. Embed $Y$ into a Polish space $Y'$; given any $Q \in \Sigma^1_n(Y)$, get $Q' \in \Sigma^1_n(Y')$ with $Q' \cap Y = Q$; then

$$f^{-1}(Q) = \{x \in X : \exists y \in Y' \ y \in Q' \land f(x) = y \}$$

is the projection along $Y'$ of the intersection of $\Sigma^1_n(X \times Y')$ sets: $X \times Q'$ and graph $f$.

(8) For $Q \subseteq X$, $f(Q)$ is the projection of $(Q \times Y) \cap \text{graph } f$ along $X$.

(9) For $Q \subseteq Y$, $f^{-1}(Q)$ is the projection of $(X \times Q) \cap \text{graph } f$ along $Y$. □

Denote by $\mathcal{P}(X)$ the family of all Cantor (i.e., homeomorphic to $C$) subsets of $X$ endowed with the Vietoris topology. Note that if $G$ is $G_\delta$ in $X$ then $\mathcal{P}(G)$ is $G_\delta$ in $\mathcal{P}(X)$. Also, if $X$ is a perfect Polish space, then so is $\mathcal{P}(X)$, and if $G$ is comeager in such an $X$, then $\mathcal{P}(G)$ is comeager in $\mathcal{P}(X)$.

Recall that if $g : X \to Y$ is Baire measurable, then there is a comeager set $G \subseteq X$ such that $g|G$ is continuous. So, if $X$ is a perfect Polish space, then $g$ is continuous on comeagerly many $p \in \mathcal{P}(X)$ (on any $p \in \mathcal{P}(G)$). Equivalently, if sets $Q^n \subseteq X$, $n \in \omega$, have the Baire property, then there is a comeager set $G \subseteq X$ such that the sets $G \cap Q^n$, $n \in \omega$, are clopen in $G$. So, if $X$ is a perfect Polish space, then for comeagerly many $p \in \mathcal{P}(X)$, the sets $p \cap Q^n$, $n \in \omega$, are clopen in $p$. 

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Let $\mathcal{P} = \mathcal{P}(\mathcal{C})$, and let $\pi : \mathcal{P} \times \mathcal{C} \to \mathcal{C}$ be a continuous function such that each section $\pi_p$, $p \in \mathcal{P}$, is a homeomorphism from $\mathcal{C}$ onto $p$ (e.g., let $\pi_p$ be induced by the unique bijection from $2^{<\omega}$ onto the split nodes of the tree $\{s|l : s \in p, l \in \omega\}$ which preserves the lexicographic ordering).

For $z \in \mathcal{C}$, define $z^* \in \mathcal{C}$ by $z^*(i) = z(2i)$, and write

$$C_x = \{z \in \mathcal{C} : z^* = x\}, \quad \mathcal{P}_x = \mathcal{P}(C_x), \quad x \in \mathcal{C}.$$

Fix also a list $\{I_n\}_{n \in \omega}$ all of clopen subsets of $\mathcal{C}$, with $I_0 = \emptyset$.

Finally, the main notion: if $n \geq 1$ and $H \subseteq Z \subseteq \mathcal{C}$, we say that $(H, Z)$ is

- $\Sigma_n^1$-hard if $\forall Q \in \Sigma_n^1(C) \exists$ continuous $f : \mathcal{C} \to Z$ with $Q = f^{-1}[H]$,
- $\mathcal{S}\Sigma_n^1$-hard if $\forall Q \in \Sigma_n^1(C) \exists$ Selivanovski measurable $f : \mathcal{C} \to Z$ with $Q = f^{-1}[H]$.

2. Injections. We first show how hardness can be realized via injections.

**Lemma 2.** Suppose that $(H, Z)$ is $\Sigma_n^1$-hard for some $n \geq 1$. Then any $\Sigma_n^1$ subset of $\mathcal{C}$ can be obtained as the preimage of $H$ under a continuous injection from $\mathcal{C}$ into $Z$.

**Proof.** Define $c : N \times \mathcal{C} \to \mathcal{C}$ by

$$c(s, y)(i) = 1 \iff y \in I_{s(i)}.$$

Then $c$ is continuous, and $\{c_s\}_{s \in N}$ is the family of all continuous functions from $\mathcal{C}$ to $\mathcal{C}$.

**Claim.** $\exists p \in \mathcal{B}_N \mathcal{P} \forall s \in N \ p(s) \subseteq C_s \land c_s|p(s)$ is injective or constant.

**Proof of Claim.** Let

$$Q = \{(s, p) \in N \times \mathcal{P} : p \subseteq C_s \land c_s|p \text{ is injective or constant}\}.$$

We claim that (1) $Q$ is $G_\delta$, and (2) $\forall s \in N \ Q_s$ is nonmeager in $\mathcal{P}_s$. Once this is established, we can use the uniformization theorem for Borel sets with “large sections” [2] 18.6] to get the desired $p$.

(1) Consider in $N \times C^2$ the open set $\nabla$ and the closed set $\Delta$ defined by

$$\nabla = \{(s, y_0, y_1) \in N \times C^2 : c_s(y_0) \neq c_s(y_1)\},$$

$$\Delta = \{(s, y_0, y_1) \in N \times C^2 : y_0 = y_1\}.$$

Note that $(s, p) \in Q$ iff $p \subseteq C_s$ and

$$\{s\} \times p^2 \subseteq \nabla \cup \Delta \lor \{s\} \times p^2 \subseteq (N \times C^2) \setminus \nabla.$$

Now, “$p \subseteq C_s$” defines a closed set in $N \times \mathcal{P}$. The displayed line defines, in turn, a $G_\delta$ set: the map $(s, p) \mapsto \{s\} \times p^2$ is continuous, and the set

$$\mathcal{P}(\nabla \cup \Delta) \cup \mathcal{P}((N \times C^2) \setminus \nabla)$$

is $G_\delta$ in $\mathcal{P}(N \times C^2)$ because the sets $\nabla \cup \Delta$ and $(N \times C^2) \setminus \nabla$ are $G_\delta$ in $N \times C^2$. 
(2) Fix $s \in \mathcal{N}$. Either $c_s$ is constant on a nonempty open set $U \subseteq C_s$ — then $\mathcal{P}(U)$ is nonempty open in $\mathcal{P}_s$, and $p^2 \subseteq C^2 \setminus \nabla_s$ for $p \in \mathcal{P}(U)$; or else $C^2_s \cap \nabla_s$ is dense open in $C^2_s$ — then there are comeagerly many $p \in \mathcal{P}_s$ with $p^2 \subseteq \nabla_s \cup \Delta_s$ by the Kuratowski–Mycielski theorem \cite{2} 19.1. □

Claim

Now, consider the following Borel injection from $\mathcal{N} \times C$ into $C$:

$$h(s, y) = \pi(p(s), y).$$

If $Q \in \Sigma_n^1(C)$, then $h[\mathcal{N} \times Q] \in \Sigma_n^1(C)$. As $(H, Z)$ is $\Sigma_n^1$-hard, for some continuous $f : C \to Z$,

$$h[C \times Q] = f^{-1}[H].$$

Hence, since $h$ is injective,

$$C \times Q = h^{-1}[f^{-1}[H]].$$

Pick $s$ with $f = c_s$. Then

$$Q = h^{-1}_s[c^{-1}_s[H]] = (c_s h_s)^{-1}[H].$$

But $c_s h_s$ is injective or constant, as $h_s$ is a bijection onto $p(s)$, and $c_s p(s)$ is injective or constant.

If $c_s h_s$ is injective, we are done. Otherwise, it must be the case that $Q \in \{C, \varnothing\}$. Then there is also a continuous injective $g : C \to Z$ with $Q = g^{-1}[H]$ since both $H$ and $Z \setminus H$ contain copies of $C$ □.

3. Suslin operation. For any set $A$, a set $T \subseteq A^{<\omega}$ is a tree if it is closed under initial segments. A tree $T$ is well-founded if $\neg \exists t \in A^{\omega} \forall l \in \omega \ t|l \in T$.

Henceforth let $A = \omega^{<\omega}$, and let $E$ be the set of all nonempty well-founded subtrees of $A^{<\omega}$. Identifying $\text{Pow}(A^{<\omega})$ with $C$, we view $E$ as a $\Pi_1^1$ subset of $C$.

In the following:

- $(\uparrow)$ is the one-term sequence consisting of $\uparrow$;
- $i \in \omega$;
- $\sigma, \varsigma, \tau \in A$; $\theta, \vartheta \in A^{<\omega}$;
- $\varnothing$, resp. $\emptyset$, is the empty sequence in $A$, resp. $A^{<\omega}$;
- for $\theta \neq \emptyset$, last $\theta$ is the last term of $\theta$;
- $\varsigma^* \sigma$ and $\vartheta^* \theta$ denote the concatenations of the respective sequences; but $\sigma^* \mathcal{i} = \sigma^* (i)$ and $\vartheta^* \sigma = \vartheta^* (\sigma)$; so last $\vartheta^* \emptyset = \text{last } \vartheta$ and last $\vartheta^* \emptyset = \emptyset$;
- $\varepsilon \in E$; $\theta^* \varepsilon = \{\theta^* \vartheta : \vartheta \in \varepsilon\}$; $\varepsilon_{\theta} = \{\vartheta : \theta^* \vartheta \in \varepsilon\}$;
- $s, t \in \mathcal{N}$; $s \leq t$ iff $\forall l \ s(l) \leq t(l)$.

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(2) Fix $G \in G_*(C) \setminus F_*(C)$. Let $g : C \to Z$ be continuous with $G = g^{-1}[H]$. Then $g[G]$ is uncountable, as otherwise $G = g^{-1}[g[G]]$ would be $F_\sigma$. Being an uncountable $\Sigma^1_3$ set, $g[G]$ contains a copy of $C$. The same argument works for $g$ with $Q = g^{-1}[Z \setminus H]$. 

We use the symbol \( \bigwedge \) for the Suslin operation: given sets \( \{ Q^\sigma \}_{\sigma \in A} \),
\[
\bigwedge_{\sigma} Q^\sigma = \bigcup_{\sigma \subseteq s} \bigcap_{s} Q^\sigma.
\]

Note that if a family \( F \subseteq \text{Pow}(X) \) is closed under the operation \( X \setminus \bigwedge_{\sigma} Q^\sigma \),
then \( F \) is a \( \sigma \)-field closed under the Suslin operation; so, if \( F \) also contains
a basis of \( X \), then \( F \supseteq \mathcal{S}_X \).

**Lemma 3.** Suppose that \( X \) is compact and \( \{ Q^\sigma \}_{\sigma \in A} \subseteq \text{Pow}(X) \).
If each \( \bigwedge_{\tau} Q^\hat{\tau}, \sigma \in A \), is clopen, then there exists \( t \in \mathbb{N} \) such that
\[
\bigwedge_{\sigma} Q^\sigma = \bigcup_{s \leq t} \bigcap_{\sigma \subseteq s} Q^\sigma.
\]

*Proof.* Let \( \hat{Q}^\sigma = \bigwedge_{\tau} Q^\hat{\tau} \). Note that \( \hat{Q}^\emptyset = \bigwedge_{\sigma} Q^\sigma \), and for each \( \sigma \),
\[
\hat{Q}^\sigma = \bigcup_{i \in \omega} \hat{Q}^{\sigma i}.
\]
Since the tilded sets above are compact and clopen, there exist \( k_{\sigma} \in \omega \) such that
if \( k \geq k_{\sigma} \) and if “\( i \in \omega \)” is changed to “\( i \leq k \)”,
then the equality is preserved. It follows that \( t \in \mathbb{N} \) given by
\[
t(\ell) = \max\{k_{\sigma} : |\sigma| = \ell \land \forall l < \ell \; \sigma(l) \leq t(l)\}
\]
works.  

4. Coding. We construct a \( \Delta^1_2 \) measurable function that is universal
for \( \mathcal{S}_{\text{CC}} \). Define \( U^\theta_{\varepsilon} \subseteq C \) by
\[
U^\theta_{\varepsilon} = \begin{cases}
I_{|\text{last } \theta|}, & \theta \notin \varepsilon, \\
C \setminus \bigwedge_{\sigma} U^\theta_{\varepsilon \sigma}, & \theta \in \varepsilon,
\end{cases}
\]
and then define \( u : \mathcal{E} \times C \to C \) by
\[
u(\varepsilon, x)(i) = 1 \iff x \in U^{|\langle i \rangle}_{\varepsilon i}.
\]

**Lemma 4.** \( u \in \Delta^1_2(\mathcal{E} \times C, C) \) and \( \{ u_{\varepsilon} \}_{\varepsilon \in \mathcal{E}} = \mathcal{S}_{\text{CC}} \).

*Proof.* For the first part it is enough to see that \( x \in U^\theta_{\varepsilon} \) is \( \Delta^1_2 \). We have
\[
x \in U^\theta_{\varepsilon} \iff \exists d \subseteq \varepsilon \; \varphi \land \theta \in d \iff \forall d \subseteq \varepsilon \; \varphi \Rightarrow \theta \in d,
\]
where \( \varphi \) is
\[
\forall \theta \left( (\theta \notin \varepsilon \Rightarrow x \in I_{|\text{last } \theta|}) \land (\theta \in \varepsilon \Rightarrow \neg \exists s \; \sigma \subseteq s \; \sigma \cdot \sigma \in d) \right).
\]

For the second part it is enough to see that \( \{ U^\theta_{\varepsilon} \}_{\varepsilon \in \mathcal{E}} = \mathcal{S}_C \) whenever
\( \theta = \langle i \rangle \). In fact, this is true for any \( \theta \).

\[\text{(3)} \]

\[
\DeclareMathOperator*{\mathpalette\souslin}{\souslin}\DeclareMathOperator*{\mathpalette\suslin}{\suslin}
\]
The \( \subseteq \) inclusion is clear. To see the \( \supseteq \) inclusion note first that \( U^\theta \vartheta = U^\vartheta \) for all \( \theta \) and \( \vartheta \). Also, for any \( \theta \) and any \( \{\varepsilon^\sigma\}_{\sigma \in A} \), if

\[
\varepsilon = \{\theta|l: \ l \leq |\theta|\} \cup \bigcup_{\sigma}(\theta^\sigma)^*\varepsilon^\sigma,
\]

then

\[
U^\theta = C \setminus \bigcap_{\sigma} U^\theta_{\varepsilon^\sigma}.
\]

It follows that \( \{U^\theta_{\varepsilon}\}_{\varepsilon \in E} \) is a \( \sigma \)-field closed under the Suslin operation.

We still need to see that \( \{U^\theta_{\varepsilon}\}_{\varepsilon \in E} \) contains a basis of \( C \). First, if \( \theta \) is terminal in \( \varepsilon \), then \( U^\theta_{\varepsilon,\omega} = I_{|\varepsilon|} = \emptyset \), so \( \bigcap_{\sigma} U^\theta_{\varepsilon^\sigma} = \emptyset \), hence \( U^\theta = C \). Next, given any \( n \neq 0 \), let

\[
\varepsilon = \{\theta|l: \ l \leq |\theta|\} \cup \{\theta^\sigma: |\sigma| \neq n\}.
\]

Now, if \( |\sigma| \neq n \) then \( \theta^\sigma \) is terminal in \( \varepsilon \), so \( U^\theta_{\varepsilon^\sigma} = C \), and if \( |\sigma| = n \) then \( \theta^\sigma \notin \varepsilon \), so \( U^\theta_{\varepsilon^\sigma} = I_{|\sigma|} = I_n \). Altogether this gives \( \bigcap_{\sigma} U^\theta_{\varepsilon^\sigma} = I_n \), hence \( U^\theta = C \setminus I_n \). \( \blacksquare \)

5. Uniformization

**Lemma 5.** There is \( p \in \Delta^1_2(E, \mathcal{P}) \) such that for each \( \varepsilon \),

\[
p(\varepsilon) \in \mathcal{P}_\varepsilon \quad \text{and} \quad u_\varepsilon|p(\varepsilon) \text{ is continuous}.
\]

**Proof.** The desired \( p \) is obtained by the \( \Sigma^1_2 \) uniformization theorem applied to the set \( Q \) of all \( (\varepsilon, p) \in E \times \mathcal{P} \) with \( p \in \mathcal{P}_\varepsilon \) for which there exist \( \bar{n} \in \omega^{<\omega} \) and \( s \in \mathcal{N}^{<\omega} \) such that \( \forall \theta \in \varepsilon \ |\text{last } \theta| = \bar{n}(\theta) \) and

\[
\forall \theta \in \varepsilon \left( \bigcap_{\sigma} p \cap I_{\bar{n}(\theta, \sigma)} \subseteq p \setminus I_{\bar{n}(\theta)} \subseteq \bigcup_{s \leq \bar{s}(\theta)} \bigcap_{\sigma \leq s} p \cap I_{\bar{n}(\theta, \sigma)} \right).
\]

Note that \( u_\varepsilon \) is continuous on any \( p \in Q_\varepsilon \) since \( \forall \theta \ p \cap U^\theta = p \cap I_{\bar{n}(\theta)} \).

We will show: (1) \( Q \in \Sigma^1_2(E \times \mathcal{P}) \), and (2) \( \forall \varepsilon \ Q_\varepsilon \neq \emptyset \).

(1) The conditions “\( p \in \mathcal{P}_\varepsilon \)” and “\( \forall \theta \notin \varepsilon \ |\text{last } \theta| = \bar{n}(\theta) \)” define closed sets in \( E \times \mathcal{P} \) and \( \mathcal{E} \times \omega^{<\omega} \). The displayed condition, in turn, defines a \( \Pi^1_1 \) set in

\[
E \times \mathcal{P} \times \omega^{<\omega} \times \mathcal{N}^{<\omega}.
\]

Its first inclusion gives clearly a \( \Pi^1_1 \) set in \( \mathcal{P} \times \omega^{<\omega} \). Its second inclusion gives a closed set in \( \mathcal{P} \times \omega^{<\omega} \times \mathcal{N}^{<\omega} \), as it says that the compact set \( p \setminus I_{\bar{n}(\theta)} \) is contained in the projection of the compact set

\[
\{(x, s) \in \mathcal{C} \times \mathcal{N}: s \leq \bar{s}(\theta) \land \forall \sigma \leq s \ x \in p \cap I_{\bar{n}(\theta, \sigma)}\}.
\]

(2) Since for any \( \theta \) and \( \varsigma \), the set \( C_\varepsilon \cap \bigcap_{\sigma} U^\theta_{\varepsilon^\varsigma^\sigma} \) has the Baire property in \( C_\varepsilon \), we can choose \( p \in \mathcal{P}_\varepsilon \) such that for any \( \theta \) and \( \varsigma \), the set \( p \cap \bigcap_{\sigma} U^\theta_{\varepsilon^\varsigma^\sigma} \) is clopen in \( p \). In particular, for any \( \theta \in \varepsilon \), the set \( p \cap U^\theta = p \setminus \bigcap_{\sigma} U^\theta_{\varepsilon^\varsigma^\sigma} \) is clopen in \( p \).
To get $\tilde{n} \in \omega^{A^{<\omega}}$, if $\theta \notin \varepsilon$ then let $\tilde{n}(\theta) = |\text{last } \theta|$, and if $\theta \in \varepsilon$ then let $\tilde{n}(\theta)$ be any $n$ such that
\[ p \cap U_\varepsilon^\theta = p \cap I_n. \]

To get $\tilde{s} \in \mathcal{N}^{A^{<\omega}}$, if $\theta \notin \varepsilon$ then let $\tilde{s}(\theta)$ be any element of $\mathcal{N}$, and if $\theta \in \varepsilon$ then let $\tilde{s}(\theta)$ be the $t$ of Lemma 3 applied to $p$ and $\{p \cap I_{n(\theta, \sigma)}\}_{\sigma \in A}$ so that
\[ \bigcap_{\sigma}p \cap I_{\tilde{n}(\theta, \sigma)} = \bigcup_{s \leq \tilde{s}(\theta)} \bigcap_{\sigma \leq s} p \cap I_{\tilde{n}(\theta, \sigma)}. \]

6. Proof of the Theorem. In view of Lemma 2, we just need to get $\Sigma^1_{\tilde{n}}$-hardness from $\mathbb{B}\Sigma^1_{\tilde{n}}$-hardness. Consider the following $\Delta^1_2$ measurable injection from $\mathcal{E} \times \mathcal{C}$ to $\mathcal{C}$:
\[ g(\varepsilon, x) = \pi(p(\varepsilon), x), \]
where $p$ is from Lemma 5. If $Q \in \Sigma^1_{\tilde{n}}(\mathcal{C})$, then $g[\mathcal{E} \times Q] \in \Sigma^1_{\tilde{n}}(\mathcal{C})$ by Lemma 1(8). So, if $(H, Z)$ is $\mathbb{B}\Sigma^1_{\tilde{n}}$-hard, then for some $f \in \mathbb{S}_{\mathcal{C}Z}$,
\[ g[\mathcal{E} \times Q] = f^{-1}[H]. \]
Hence, since $g$ is injective,
\[ \mathcal{E} \times Q = g^{-1}[f^{-1}[H]]. \]
Pick $\varepsilon$ with $f = u_\varepsilon$. Then
\[ Q = g_\varepsilon^{-1}[u_\varepsilon^{-1}[H]] = (u_\varepsilon g_\varepsilon)^{-1}[H]. \]
But $u_\varepsilon g_\varepsilon$ is continuous because $g_\varepsilon$ is a homeomorphism onto $p(\varepsilon)$ and $u_\varepsilon | p(\varepsilon)$ is continuous.

7. Kechris’s Theorem. Change $A$ to $\omega$, $A^{<\omega}$ to $\omega^{<\omega}$, $\mathbb{S}$ to $\mathcal{B}$, $\Sigma^1_2$ to $\Sigma^1_1$, $\Delta^1_2$ to $\Delta^1_1$, and $\bigcup$ to $\mathcal{U}$. So, $\mathcal{E}$ is now the set of all nonempty well-founded subtrees of $\omega^{<\omega}$, and $u$ is $\Delta^1_1$ measurable.

In Lemma 5, let $Q_\varepsilon$ consist of all $p \in \mathcal{P}_\varepsilon$ on which $u_\varepsilon$ is continuous. Then $Q_\varepsilon$ is comeager in $\mathcal{P}_\varepsilon$. Also, $Q$ is $\Pi^1_1$, since $u_\varepsilon | p$ is continuous iff
\[ \forall n \exists m \forall x \in p x \in I_m \iff u(\varepsilon, x) \in I_n, \]
and “$u(\varepsilon, x) \in I_n$” gives a $\Delta^1_1(\mathcal{E} \times \mathcal{C} \times \omega)$ set. To get $p$, use the uniformization theorem for $\Pi^1_1$ sets with “large sections” [36.23] that provides here a $\Delta^1_1$ measurable uniformization. (The $\Pi^1_1$ uniformization theorem may fail to give a $\Delta^1_1$ measurable function.)

Now, if $g$ is as in Section 6 and $Q \in \Sigma^1_1(\mathcal{C})$, then
\[ g[\mathcal{E} \times Q] = \{z \in \mathcal{C} : \exists y \in Q g(z^*, y) = z\} \in \Sigma^1_1(g[\mathcal{E} \times \mathcal{C}]), \]
as we have here the projection along $Q \in \Sigma^1_1(\mathcal{C})$ of the $\Delta^1_1(g[\mathcal{E} \times \mathcal{C}] \times \mathcal{C})$ set given by the preimage of $\{(z, z) : z \in \mathcal{C}\}$ via the $\Delta^1_1$ measurable function
\[ g[\mathcal{E} \times \mathcal{C}] \times \mathcal{C} \ni (z, y) \mapsto (g(z^*, y), z). \]
So, for some \( \varepsilon \), \( g[\mathcal{E} \times Q] = g[\mathcal{E} \times C] \cap u_{\varepsilon}^{-1}[H] \), hence \( \mathcal{E} \times Q = g^{-1}[u_{\varepsilon}^{-1}[H]] \), and, as before, \( Q = (u_{\varepsilon}g_{\varepsilon})^{-1}[H] \).

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**References**


