

Regular spaces of small extent are ω -resolvable

by

István Juhász, Lajos Soukup and Zoltán Szentmiklóssy (Budapest)

Abstract. We improve some results of Pavlov and Filatova, concerning a problem of Malykhin, by showing that every regular space X that satisfies $\Delta(X) > e(X)$ is ω -resolvable. Here $\Delta(X)$, the dispersion character of X , is the smallest size of a non-empty open set in X , and $e(X)$, the extent of X , is the supremum of the sizes of all closed-and-discrete subsets of X . In particular, regular Lindelöf spaces of uncountable dispersion character are ω -resolvable.

We also prove that any regular Lindelöf space X with $|X| = \Delta(X) = \omega_1$ is even ω_1 -resolvable. The question whether regular Lindelöf spaces of uncountable dispersion character are maximally resolvable remains wide open.

1. Introduction. We start by recalling a few basic definitions and facts concerning resolvability. A topological space X is said to be λ -resolvable (λ a cardinal) if X contains λ many mutually disjoint dense subsets. A natural upper bound on the resolvability of X is

$$\Delta(X) = \min\{|G| : G \text{ is non-empty open in } X\},$$

called the *dispersion character* of X . So, X is said to be *maximally resolvable* if it is $\Delta(X)$ -resolvable. The expectation is that “nice” spaces should be maximally resolvable, as supported e.g. by the well-known facts that compact Hausdorff, or metric, or linearly ordered spaces are all maximally resolvable.

It is also well-known, however, that there is a countable regular (hence “nice”) space with no isolated points that is *irresolvable*, i.e. not even 2-resolvable. Since countable spaces are (hereditarily) Lindelöf, this prompted Malykhin [6] to ask the following natural question: Is every regular Lindelöf space of uncountable dispersion character (at least 2-)resolvable? He also noted that the answer to this question is negative if regular is weakened to Hausdorff.

2010 *Mathematics Subject Classification*: 54A35, 03E35, 54A25.

Key words and phrases: resolvable, Lindelöf, countable extent.

Pavlov [7] proved the following very deep result that gave a partial affirmative answer to Malykhin's question: If X is any regular space satisfying $\Delta(X) > e(X)^+$ then X is ω -resolvable. (In fact, he only needed the following assumption on X that we call π -regularity and that is clearly weaker than regularity: The regular closed sets in X form a π -network, i.e. every non-empty open set includes a non-empty regular closed set.)

We recall that the *extent* $e(X)$ of X is the supremum of the sizes of all closed-and-discrete subsets of X . Since Lindelöf spaces have countable extent, it follows that regular Lindelöf spaces of dispersion character $> \omega_1$ are ω -resolvable. Thus only the case $\Delta(X) = \omega_1$ of Malykhin's problem remained, and this case was settled by Filatova [2]: Any regular Lindelöf space X with $\Delta(X) = \omega_1$ is 2-resolvable. However, her method of proof did not seem to give even 3-resolvable, not to mention ω -resolvable as in Pavlov's result.

Our main result in this paper, Theorem 3.1, improves Pavlov's above-mentioned result by showing that the assumption $\Delta(X) > e(X)^+$ can be relaxed to $\Delta(X) > e(X)$. This, of course, immediately implies that Filatova's 2-resolvable can also be improved to ω -resolvable. We think moreover that the proof of our strengthening of Pavlov's result is significantly simpler than his original argument, especially in the case when $\Delta(X)$ is singular.

We do not know, however, if a regular space X satisfying $\Delta(X) > e(X)$ is always maximally resolvable, or even if regular Lindelöf spaces of uncountable dispersion character are maximally resolvable. This problem should be confronted with our result from [4] stating that any topological space X satisfying $\Delta(X) > s(X)$ is maximally resolvable. Here $s(X)$, the *spread* of X , is the supremum of the sizes of all (relatively) discrete subsets of X .

Theorem 4.1 in the present paper implies that, for any infinite cardinal κ , if all regular Lindelöf spaces of cardinality and dispersion character κ^+ are κ -resolvable then all such spaces are actually κ^+ -resolvable as well. This then, together with Theorem 3.1, implies that any regular Lindelöf space X with $|X| = \Delta(X) = \omega_1$ is even ω_1 -resolvable, i.e. maximally resolvable. Considering that after Pavlov and before Filatova this was the unsolved "hard case" of Malykhin's problem, for which even 2-resolvability was unknown, it seems not to be unreasonable to raise the question whether regular Lindelöf spaces of uncountable dispersion character are maximally resolvable.

2. Preliminary results. In this section we have collected some known and some new results that will play an essential role in the proof of our main results, Theorems 3.1 and 4.1.

First we fix a couple of important pieces of notation: For a topological space X , we denote by $\tau^+(X)$ the collection of all non-empty open sets in X and by $\text{RC}^+(X)$ the family of all non-empty regular closed subsets of X .

As we have noted above, if X is regular then $\text{RC}^+(X)$ is a π -network in X .

We shall make very frequent use of the following simple but basic result.

THEOREM 2.1 (El'kin, [1]). *If X is a topological space, κ is any cardinal, and the family*

$$\mathfrak{R}_\kappa(X) = \{Z \subset X : Z \text{ is } \kappa\text{-resolvable}\}$$

is a π -network in X then $X \in \mathfrak{R}_\kappa(X)$, i.e. X is κ -resolvable.

Since for every $G \in \tau^+(X)$ there is $H \in \tau^+(X)$ such that $H \subset G$ and $|H| = \Delta(H)$, and moreover then $R \in \text{RC}^+(X)$ and $R \subset H$ imply $|R| = \Delta(R)$ ($= |H|$), we obtain the following simple but useful corollary.

COROLLARY 2.2. *Let \mathcal{C} be a regular closed hereditary class of regular spaces, i.e. such that $X \in \mathcal{C}$ implies $\text{RC}^+(X) \subset \mathcal{C}$. If every space $X \in \mathcal{C}$ with $|X| = \Delta(X)$ has a κ -resolvable subspace then every member of \mathcal{C} is κ -resolvable.*

In proving that certain spaces are ω -resolvable, like for Theorem 3.1, the following result comes naturally handy.

THEOREM 2.3 (Illanes, [3]). *If a topological space X is k -resolvable for each $k < \omega$ then X is ω -resolvable.*

Now we turn to formulating and proving some new results that will be needed in our reasonings. They may turn out to be of independent interest.

We begin with some, rather standard, notation: If A is any subset of a topological space then A' denotes the derived set of A , that is, the set of all accumulation points of A , while we use A° to denote the set of all *complete accumulation* points of A .

The following quite technical result is new, although it owes its basic idea to Filatova's work [2].

LEMMA 2.4. *Let X be a regular space, κ be a regular cardinal, and consider the family*

$$(2.1) \quad \mathcal{D} = \{D \in [X]^\kappa : D' = D^\circ, D \cap D^\circ = \emptyset, \text{ and } \forall E \in [D]^\kappa (E^\circ \neq \emptyset)\}.$$

If X has a dense subset Y with $|Y| \leq \kappa$ such that for each point $y \in Y$ there is a set $D \in \mathcal{D}$ with $y \in D^\circ$, then X is 2-resolvable.

Proof. First, let us fix a κ -type enumeration of Y (with repetitions permitted): $Y = \{y_\alpha : \alpha < \kappa\}$. We shall then, by induction on $\alpha < \kappa$, define $D_\alpha \in \mathcal{D}$ and $i_\alpha \in 2$ in such a way that, putting for $0 < \alpha \leq \kappa$ and $i < 2$,

$$E_{\alpha,i} = \bigcup \{D_\beta : \beta < \alpha \text{ and } i_\beta = i\} \cup \bigcup \{D_\beta^\circ : \beta < \alpha \text{ and } i_\beta = 1 - i\},$$

for any $\alpha \leq \kappa$ we have both

$$(2.2) \quad E_{\alpha,0} \cap E_{\alpha,1} = \emptyset$$

and

$$(2.3) \quad \{y_\beta : \beta < \alpha\} \subset \overline{E_{\alpha,0}} \cap \overline{E_{\alpha,1}}.$$

To start with, we pick $D_0 \in \mathcal{D}$ with $y_0 \in D_0^\circ$ and put $i_0 = 0$. Then (2.2) and (2.3) are trivially satisfied. Now, assume that $0 < \alpha < \kappa$, and that $\{D_\beta : \beta < \alpha\}$ and $\{i_\beta : \beta < \alpha\}$ have been defined and satisfy the inductive hypotheses (2.2) and (2.3). We distinguish three cases.

First, if

$$y_\alpha \in \overline{E_{\alpha,0}} \cap \overline{E_{\alpha,1}}$$

then we may simply let $D_\alpha = D_0$ and $i_\alpha = i_0 = 0$. Clearly, (2.2) and (2.3) will remain valid for $\alpha + 1$.

Next, if

$$y_\alpha \notin \overline{E_{\alpha,0}} \cup \overline{E_{\alpha,1}}$$

then, using the regularity of X , we can pick an open neighbourhood U of y_α for which $\overline{U} \cap \bigcup_{\beta < \alpha} D_\beta = \emptyset$. Now choose $D \in \mathcal{D}$ with $y_\alpha \in D^\circ$ and set $D_\alpha = U \cap D$ and $i_\alpha = 0$. Again, it is easy to check that with these choices, (2.2) and (2.3) remain valid.

If none of the above two alternatives holds then $y_\alpha \in \overline{E_{\alpha,0}} \Delta \overline{E_{\alpha,1}}$, i.e. there is $j \in 2$ such that $y_\alpha \in \overline{E_{\alpha,j}} \setminus \overline{E_{\alpha,1-j}}$. Suppose e.g. that $j = 0$; the case $j = 1$ can be handled symmetrically.

We may then choose an open neighbourhood U of y_α such that $\overline{U} \cap \overline{E_{\alpha,1}} = \emptyset$, and a set $D \in \mathcal{D}$ with $y_\alpha \in D^\circ$. For every $\beta < \alpha$ we have $|U \cap D_\beta| < \kappa$: Indeed, if $i_\beta = 0$ then $|U \cap D_\beta| = \kappa$ would imply

$$\emptyset \neq (U \cap D_\beta)^\circ \subset \overline{U} \cap D_\beta^\circ \subset \overline{U} \cap E_{\alpha,1},$$

a contradiction. And if $i_\beta = 1$ then we even have $U \cap D_\beta = \emptyset$.

Consequently, as κ is regular, we have $|\bigcup\{D \cap D_\beta : \beta < \alpha\}| < \kappa$, hence $D_\alpha = U \cap D \setminus \bigcup\{\overline{D_\beta} : \beta < \alpha\} \in \mathcal{D}$ and $y_\alpha \in D_\alpha^\circ$. Let us now put $i_\alpha = 1$. Then $y_\alpha \in D_\alpha^\circ \subset \overline{D_\alpha} \subset \overline{E_{\alpha+1,1}}$ implies $y_\alpha \in \overline{E_{\alpha+1,0}} \cap \overline{E_{\alpha+1,1}}$, hence (2.3) remains valid for $\alpha + 1$.

To show the same for (2.2), note first that $D_\beta^\circ \subset \overline{E_{\alpha,1}}$ for all $\beta < \alpha$, hence $\overline{D_\alpha} \cap D_\beta^\circ = \emptyset$ for each $\beta < \alpha$. Moreover, $D_\alpha^\circ \subset \overline{U}$ implies $D_\alpha^\circ \cap D_\beta = \emptyset$ for any $\beta < \alpha$ with $i_\beta = 1$, which together with $D_\alpha \cap D_\alpha^\circ = \emptyset$ yields

$$(E_{\alpha,0} \cup D_\alpha^\circ) \cap (E_{\alpha,1} \cup D_\alpha) = E_{\alpha+1,0} \cap E_{\alpha+1,1} = \emptyset.$$

Of course, if $j = 1$ then we shall have $i_\alpha = 0$.

After having completed the inductive construction, it is trivial to conclude that $E_{\kappa,0}$ and $E_{\kappa,1}$ are two disjoint dense subsets of X . ■

We shall use Lemma 2.4 in the proof of our main result, in the induction step of a procedure where we move from n -resolvability to $(n+1)$ -resolvability.

In our subsequent preliminary result, rather than the extent $e(X)$, its “hat” version $\widehat{e}(X)$ will appear. We recall that $\widehat{e}(X)$ is defined as the smallest

cardinal λ such that X has no closed-and-discrete subset of size λ . Thus we clearly have $\widehat{e}(X) \leq e(X)^+$; moreover, $\widehat{e}(X) \leq \kappa$ is simply equivalent to the statement that for every set $A \in [X]^\kappa$ we have $A' \neq \emptyset$. We start by defining an auxiliary concept that will be needed in this result.

DEFINITION 2.5. Let X be a topological space and κ a cardinal. A subset $H \subset X$ is called κ -*approachable* in X if there are κ many pairwise disjoint sets $\{X_\alpha : \alpha < \kappa\} \subset [X]^\kappa$ such that

$$\forall Y \in [X_\alpha]^\kappa (Y^\circ \neq \emptyset) \quad \text{and} \quad H = X_\alpha^\circ$$

for all $\alpha < \kappa$.

The following lemma shows that this definition is not empty.

LEMMA 2.6. Assume that κ is a regular cardinal and X is a space for which $\widehat{e}(X) \leq \kappa$. Then

- (1) $A \in [X]^\kappa$ and $|A'| < \kappa$ imply $A^\circ \neq \emptyset$;
- (2) if $A \in [X]^\kappa$ is such that $|A^\circ| < \kappa$ and $B^\circ \neq \emptyset$ for all $B \in [A]^\kappa$ then there is a subset $H \subset A^\circ$ that is κ -approachable in X .

Proof. (1) Every point $x \in A' \setminus A^\circ$ has an open neighbourhood U_x such that $|U_x \cap A| < \kappa$. Then $U = \bigcup \{U_x : x \in A' \setminus A^\circ\}$ covers $A' \setminus A^\circ$ and $|A \cap U| < \kappa$ because κ is regular. So we have $|A \setminus U| = \kappa$, and hence $\emptyset \neq (A \setminus U)' = A^\circ$ by $\kappa \geq \widehat{e}(X)$.

(2) We start by fixing κ pairwise disjoint sets $\{A_\alpha : \alpha < \kappa\} \subset [A]^\kappa$, and for any $\alpha < \beta < \kappa$ we write

$$A_{\alpha,\beta} = \bigcup_{\alpha \leq \nu < \beta} A_\nu.$$

For fixed $\alpha < \kappa$ the sequence

$$\{A_{\alpha,\beta}^\circ : \beta \in \kappa \setminus \alpha\}$$

is increasing and hence must stabilize since $|A^\circ| < \kappa$. This means that there is an ordinal $f(\alpha) < \kappa$ such that

$$A_{\alpha,\beta}^\circ = A_{\alpha,f(\alpha)}^\circ$$

for all $\beta \in \kappa \setminus f(\alpha)$. Similarly, the sequence

$$\{A_{\alpha,f(\alpha)}^\circ : \alpha < \kappa\}$$

is decreasing and hence it stabilizes: There is an ordinal $\alpha^* < \kappa$ such that

$$A_{\alpha^*,f(\alpha^*)}^\circ = A_{\alpha,f(\alpha)}^\circ$$

whenever $\alpha^* \leq \alpha < \kappa$. We claim that the set $H = A_{\alpha^*,f(\alpha^*)}^\circ$ is κ -approachable in X .

To see this, choose $I \in [\kappa \setminus \alpha^*]^\kappa$ in such a way that for any $\alpha, \beta \in I$ with $\alpha < \beta$ we have $f(\alpha) < \beta$. This is possible because κ is regular. Then the

sets

$$\{A_{\alpha, f(\alpha)} : \alpha \in I\} \subset [X]^\kappa$$

are pairwise disjoint and, by definition, for all $\alpha \in I$ we have both

$$\forall B \in [A_{\alpha, f(\alpha)}]^\kappa (B^\circ \neq \emptyset) \quad \text{and} \quad A_{\alpha, f(\alpha)}^\circ = H. \quad \blacksquare$$

Our next resolvability result uses κ -approachable sets in an essential way.

THEOREM 2.7. *Assume that X is a space, $\kappa = |X|$ is a regular cardinal, and \mathcal{H} is a disjoint family of sets κ -approachable in X such that $\bigcup \mathcal{H}$ is dense in X . Then X is κ -resolvable.*

Proof. For each $H \in \mathcal{H}$ let us fix a disjoint family $\{A_{H, \alpha} : \alpha < \kappa\} \subset [X]^\kappa$ which witnesses that H is κ -approachable in X , i.e.

$$\forall B \in [A_{H, \alpha}]^\kappa (B^\circ \neq \emptyset) \quad \text{and} \quad A_{H, \alpha}^\circ = H$$

for all $\alpha < \kappa$.

Note that if $H, K \in \mathcal{H}$ and $\alpha, \beta \in \kappa$ with $\langle H, \alpha \rangle \neq \langle K, \beta \rangle$ then we have $|A_{H, \alpha} \cap A_{K, \beta}| < \kappa$. Indeed, if $H = K$ then $A_{H, \alpha} \cap A_{K, \beta} = \emptyset$. And if $H \neq K$ then $|A_{H, \alpha} \cap A_{K, \beta}| = \kappa$ would imply $\emptyset \neq (A_{H, \alpha} \cap A_{K, \beta})^\circ \subset H \cap K$, contradicting $H \cap K = \emptyset$. This means that the family

$$\mathcal{A} = \{A_{H, \alpha} : H \in \mathcal{H}, \alpha < \kappa\} \subset [X]^\kappa$$

is *almost disjoint*.

But $|\mathcal{H}| \leq |X| = \kappa$ implies $|\mathcal{A}| = \kappa$, and then by the regularity of κ it follows that \mathcal{A} is also *essentially disjoint*. This means that for every pair $\langle H, \alpha \rangle$ there is a set $F_{H, \alpha} \in [A_{H, \alpha}]^{< \kappa}$ such that the collection

$$\{B_{H, \alpha} = A_{H, \alpha} \setminus F_{H, \alpha} : \langle H, \alpha \rangle \in \mathcal{H} \times \kappa\}$$

is already disjoint. Note also that for each $\langle H, \alpha \rangle \in \mathcal{H} \times \kappa$ we have

$$B_{H, \alpha}^\circ = A_{H, \alpha}^\circ = H.$$

We claim that for every $\alpha < \kappa$ the set

$$D_\alpha = \bigcup \{B_{H, \alpha} : H \in \mathcal{H}\}$$

is dense in X . Indeed, for any $U \in \tau^+(X)$ there is a set $H \in \mathcal{H}$ with $H \cap U \neq \emptyset$, so we may pick a point $x \in H \cap U$. But then $x \in H = B_{H, \alpha}^\circ$ implies $|U \cap B_{H, \alpha}| = \kappa$, so that $U \cap D_\alpha \neq \emptyset$. Thus $\{D_\alpha : \alpha < \kappa\}$ is a family of κ many pairwise disjoint dense sets in X . \blacksquare

From Lemma 2.6 and Theorem 2.7 we may immediately deduce the following corollary that will be needed in the proof of Theorem 3.1. Maybe ironically, it does not even mention κ -approachable sets, but its proof does.

COROLLARY 2.8. *Assume that κ is a regular cardinal and X is a space for which*

$$\widehat{e}(X) \leq |X| = \kappa.$$

If the family

$$\mathcal{A}' = \{A' : A \in [X]^\kappa \text{ and } |A'| < \kappa\}$$

is a π -network in X then X is κ -resolvable.

Proof. By Lemma 2.6 every member A' of \mathcal{A}' includes a set that is κ -approachable in X , hence if \mathcal{A}' is a π -network in X then so is the family \mathcal{G} of all the sets that are κ -approachable in X . But then the union of any maximal disjoint subfamily $\mathcal{H} \subset \mathcal{G}$ is clearly dense in X , hence all the assumptions of Theorem 2.7 are satisfied. ■

We now turn to another circle of preliminary results that will be used in the proof of our main Theorem 3.1. Again, we have to start with some definitions.

DEFINITION 2.9. If X is a topological space, $Y \subset X$, and κ is an infinite cardinal, then we call

$$\overline{Y}^{<\kappa} = \bigcup \{\overline{S} : S \in [Y]^{<\kappa}\}$$

the $<\kappa$ -closure of Y in X . We say that Y is $<\kappa$ -closed in X if $Y = \overline{Y}^{<\kappa}$. If $\kappa = \mu^+$ then instead of $<\mu^+$ -closure (resp. $<\mu^+$ -closed) we simply say μ -closure (resp. μ -closed).

DEFINITION 2.10. A chain decomposition of length β (for some ordinal β) of a set X is an increasing and continuous sequence $s = \langle X_\alpha : \alpha < \beta \rangle$ such that $X = \bigcup \{X_\alpha : \alpha < \beta\}$. (Continuity means that $X_\delta = \bigcup \{X_\alpha : \alpha < \delta\}$ for any limit ordinal $\delta < \beta$. Since we also consider 0 a limit ordinal, this implies $X_0 = \emptyset$.)

Clearly, if $s = \langle X_\alpha : \alpha < \beta \rangle$ is a chain decomposition of X and $Y \subset X$ then $s|Y = \langle Y \cap X_\alpha : \alpha < \beta \rangle$ is a chain decomposition of Y . Moreover, if $C \subset \beta$ is a cub (closed and unbounded) subset of β and $C = \{\gamma_i : i < \delta\}$ is the increasing enumeration of C then $s[C] = \langle X_{\gamma_i} : i < \delta \rangle$ is again a chain decomposition of X .

LEMMA 2.11. Assume that $\kappa = \text{cf}(\kappa) \leq \lambda$ are infinite cardinals and X is a topological space with $|X| = \lambda$. Then X has a chain decomposition $s = \langle X_\alpha : \alpha < \text{cf}(\lambda) \rangle$ such that $\{X_\alpha : \alpha < \text{cf}(\lambda)\} \subset [X]^{<\lambda}$, moreover,

$$(C1) \quad X_\alpha \cap \overline{X \setminus X_\alpha}^{<\kappa} \subset \overline{X_{\alpha+1} \setminus X_\alpha}^{<\kappa} \text{ for each } \alpha < \text{cf}(\lambda).$$

If, in addition, X is π -regular and not $\text{cf}(\lambda)$ -resolvable then there are a cub set $C \subset \text{cf}(\lambda)$ and a regular closed set $Y \in \text{RC}^+(X)$ such that we also have

$$(C2) \quad Y \setminus X_\alpha \text{ is } <\kappa\text{-closed for each } \alpha \in C.$$

Hence Y has a chain-decomposition $\{Y_\alpha : \alpha < \text{cf}(\lambda)\} \subset [Y]^{<\lambda}$ such that

$$(2.4) \quad Y \setminus Y_\alpha \text{ is } <\kappa\text{-closed for all } \alpha < \kappa.$$

Proof. Let us consider first every pair $\langle x, S \rangle$ such that $x \notin S$ but $x \in \overline{S}^{<\kappa}$, and assign to $\langle x, S \rangle$ a set $A(x, S) \in [S]^{<\kappa}$ with $x \in \overline{A(x, S)}$. Moreover, choose a chain decomposition $\{Z_\alpha : \alpha < \text{cf}(\lambda)\} \subset [X]^{<\lambda}$ of X in an arbitrary manner. Then we define the sequence $\langle X_\alpha : \alpha < \text{cf}(\lambda) \rangle$ by transfinite recursion on α as follows:

- (i) $X_0 = \emptyset$;
- (ii) $X_\alpha = \bigcup \{X_\beta : \beta < \alpha\}$ if $\alpha > 0$ is limit;
- (iii) $X_{\alpha+1} = X_\alpha \cup Z_\alpha \cup \bigcup \{A(x, X \setminus X_\alpha) : x \in X_\alpha \cap \overline{X \setminus X_\alpha}^{<\kappa}\}$.

Since $\kappa \leq \lambda$ is regular, we can show by an easy transfinite induction that $|X_\alpha| < \lambda$ for all $\alpha < \text{cf}(\lambda)$, moreover condition (C1) obviously follows from case (iii) of our definition. This proves the first half of the lemma.

Now assume, in addition, that X is not $\text{cf}(\lambda)$ -resolvable. For each $A \subset \text{cf}(\lambda)$ let us set

$$R_A = \bigcup_{\alpha \in A} (X_{\alpha+1} \setminus X_\alpha).$$

If R_A were dense in X for all $A \in [\text{cf}(\lambda)]^{\text{cf}(\lambda)}$, clearly X would be $\text{cf}(\lambda)$ -resolvable, hence there is a cofinal $A \subset \text{cf}(\lambda)$ and an open set $U \in \tau^+(X)$ with $U \cap R_A = \emptyset$.

We claim that then for every closed set $F \subset U$ and for every $\alpha \in A$ the set $F \setminus X_\alpha$ is $<\kappa$ -closed. Indeed, assume on the contrary that $x \in \overline{F \setminus X_\alpha}^{<\kappa} \cap X_\alpha$. Then, by (C1), there is a set $S \in [X_{\alpha+1} \setminus X_\alpha]^{<\kappa}$ with $x \in \overline{S}$. Since $x \in F \subset U$, this implies $U \cap S \neq \emptyset$, which contradicts $U \cap R_A = \emptyset$, as $\alpha \in A$ and $S \subset X_{\alpha+1} \setminus X_\alpha$.

Now, if X is also π -regular then there is a regular closed set $Y \in \text{RC}^+(X)$ with $Y \subset U$. Let us consider

$$C = \{\alpha < \text{cf}(\lambda) : Y \setminus X_\alpha \text{ is } <\kappa\text{-closed}\}.$$

Then C is clearly closed in $\text{cf}(\lambda)$ and $A \subset C$ by the above, hence C is cub in $\text{cf}(\lambda)$. Consequently, $s[C] \upharpoonright Y$ is a chain decomposition of Y that satisfies (2.4). ■

We have one more preparatory result involving chain decompositions that will be used in the proof of our main theorem.

LEMMA 2.12. *Assume that Y is a π -regular space that is not ω -resolvable. Then for every chain decomposition $\{Y_\alpha : \alpha < \mu\}$ of Y there are $T \in \text{RC}^+(Y)$ and a dense subset $Z \subset T$ such that*

$$(2.5) \quad \overline{Y_\alpha \cap Z} \subset Y_\alpha \quad \text{for all } \alpha < \mu.$$

Proof. By the continuity of chain decompositions, for each point $x \in Y$ there is a unique ordinal $\alpha(x) < \mu$ such that

$$x \in Y_{\alpha(x)+1} \setminus Y_{\alpha(x)}.$$

For any subset $A \subset Y$ let us define

$$A^* = \{x \in A : x \notin \overline{A \cap Y_{\alpha(x)}}\}.$$

We claim that A^* is dense in A for every $A \subset Y$.

Indeed, if U is open and $U \cap A \neq \emptyset$ then pick $a \in U \cap A$ such that $\alpha(a)$ is minimal. Then $a \in A^*$ because by the minimality of $\alpha(a)$ we have $U \cap Y_{\alpha(a)} = \emptyset$.

Consider now the recursive formula

$$D_j = \left(Y \setminus \bigcup_{i < j} D_i \right)^*.$$

The pairwise disjoint sets $\{D_j : j < \omega\}$ cannot be all dense in Y because Y is not ω -resolvable, but $D_0 = Y^*$ is dense. So there is $m \in \omega$ such that D_m is dense but D_{m+1} is not, hence $U \cap D_{m+1} = \emptyset$ for some $U \in \tau^+(X)$. Now, pick $T \in \text{RC}^+(Y)$ with $T \subset U$. Then $U \cap D_{m+1} = U \cap (Y \setminus \bigcup_{i \leq m} D_i)^* = \emptyset$ implies

$$(2.6) \quad T \subset U \subset \bigcup_{j \leq m} D_j,$$

and clearly $Z = T \cap D_m$ is dense in T .

Now it remains to show that

$$(2.7) \quad \overline{Z \cap Y_\alpha} \subset Y_\alpha \quad \text{for all } \alpha < \mu.$$

To see this, fix $\alpha < \mu$ and consider first any point $x \in T$. Then $x \in D_j$ for some $j \leq m$ by (2.6). This means that $x \in (Y \setminus \bigcup_{i < j} D_i)^*$, i.e. $x \notin \overline{(Y \setminus \bigcup_{i < j} D_i) \cap Y_{\alpha(x)}}$. But $D_m \subset Y \setminus \bigcup_{i < j} D_i$, hence $x \notin \overline{D_m \cap Y_{\alpha(x)}}$.

On the other hand, for every $x \in \overline{Z \cap Y_\alpha} (\subset T)$ we have $x \in \overline{D_m \cap Y_\alpha}$ because $Z \subset D_m$. This together with $x \notin \overline{D_m \cap Y_{\alpha(x)}}$ implies $\alpha(x) < \alpha$ because the sets Y_β are increasing. So, by the definition of $\alpha(x)$ we have $x \in Y_{\alpha(x)+1} \subset Y_\alpha$, and this means that $\overline{Z \cap Y_\alpha} \subset Y_\alpha$. ■

Our next preliminary results will be used in the proof of Theorem 4.1, a stepping-up result concerning resolvability of certain spaces.

LEMMA 2.13. *Assume that κ is an infinite cardinal, X a topological space, and we have a disjoint subfamily $\mathcal{H} \subset \mathfrak{R}_\kappa(X)$ such that, for each $U \in \tau^+(X)$,*

$$|\{H \in \mathcal{H} : H \cap U \neq \emptyset\}| = \kappa^+.$$

Then X is κ^+ -resolvable.

Proof. Obviously, $|\mathcal{H}| = \kappa^+$, so we can fix a one-one enumeration $\mathcal{H} = \{H_\xi : \xi < \kappa^+\}$ of \mathcal{H} . Every H_ξ is κ -resolvable, and so has a partition

$$(2.8) \quad H_\xi = \bigcup \{H_\xi^i : i < \xi\}$$

into dense subsets. Then for every $i < \kappa^+$ the set

$$(2.9) \quad D^i = \bigcup \{H_\xi^i : i < \xi < \kappa^+\}$$

is dense in X . Indeed, for each $U \in \tau^+(X)$ by our assumption there is $\xi > i$ with $U \cap H_\xi \neq \emptyset$. But H_ξ^i is dense in H_ξ , so we have $U \cap D^i \supset U \cap H_\xi^i \neq \emptyset$ as well. As the dense sets $\{D^i : i < \kappa^+\}$ are pairwise disjoint, our proof is complete. ■

To formulate the following corollary of Lemma 2.13, we need one more definition.

DEFINITION 2.14. Let X be any topological space and κ an infinite cardinal. A (necessarily closed) subset $F \subset X$ is called κ -nice in X if there is a disjoint family $\{A_\alpha : \alpha < \kappa^+\} \subset \mathfrak{R}_\kappa(X)$ such that

$$F = \bigcap_{\alpha < \kappa^+} \overline{\bigcup \{A_\beta : \beta \in \kappa^+ \setminus \alpha\}}.$$

Following the terminology of [5], we call a space λ -compact if every subset of it of cardinality λ has a complete accumulation point.

COROLLARY 2.15. Let κ be an infinite cardinal and X be a κ^+ -compact space. If there is a disjoint family \mathcal{F} of both κ -resolvable and κ -nice subsets of X such that $|\mathcal{F}| \leq \kappa^+$ and $\bigcup \mathcal{F}$ is dense in X , then X has a κ^+ -resolvable open subset.

Proof. If for every $U \in \tau^+(X)$ we have $|\{F \in \mathcal{F} : F \cap U \neq \emptyset\}| = \kappa^+$ then X itself is κ^+ -resolvable by Lemma 2.13. So assume that $U \in \tau^+(X)$ is such that $\mathfrak{F}^* = \{F \in \mathcal{F} : U \cap F \neq \emptyset\}$ has cardinality $\leq \kappa$.

For each $F \in \mathcal{F}^*$ let us fix a disjoint family

$$\{A_\alpha^F : \alpha < \kappa^+\} \subset \mathfrak{R}_\kappa(X)$$

witnessing that F is nice, as required in Definition 2.14. We claim that for every pair $\{F, G\} \in [\mathcal{F}^*]^2$ there is an $\alpha = \alpha(F, G) < \kappa^+$ such that

$$\bigcup \{A_\beta^F : \beta \in \kappa^+ \setminus \alpha\} \cap \bigcup \{A_\beta^G : \beta \in \kappa^+ \setminus \alpha\} = \emptyset.$$

Indeed, otherwise we could select a set $I \in [\kappa^+]^{\kappa^+}$ and distinct points $\{x_\alpha : \alpha \in I\}$ such that

$$x_\alpha \in \bigcup \{A_\beta^F : \beta \in \kappa^+ \setminus \alpha\} \cap \bigcup \{A_\beta^G : \beta \in \kappa^+ \setminus \alpha\} \neq \emptyset$$

whenever $\alpha \in I$. But then $\{x_\alpha : \alpha \in I\}^\circ \neq \emptyset$ would be a subset of

$$\bigcap_{\alpha \in I} \overline{\bigcup \{A_\beta^F : \beta \in \kappa^+ \setminus \alpha\}} \cap \bigcap_{\alpha \in I} \overline{\bigcup \{A_\beta^G : \beta \in \kappa^+ \setminus \alpha\}} = F \cap G,$$

contradicting $F \cap G = \emptyset$.

Now, $|\mathcal{F}^*| \leq \kappa$ implies that there is an ordinal $\gamma < \kappa^+$ such that $\alpha(F, G) < \gamma$ for all pairs $\{F, G\} \in [\mathcal{F}^*]^2$. Consequently, the elements of the family

$$\mathcal{J} = \{A_\alpha^F : F \in \mathcal{F}^* \text{ and } \alpha \in \kappa^+ \setminus \gamma\} \subset \mathfrak{R}_\kappa(X)$$

are pairwise disjoint and, by our assumptions, both $\bigcup \mathcal{F}^* \cap U$ and $\bigcup \mathcal{J} \cap U$ are dense in U .

Thus, for every $V \in \tau^+(U)$ there is $F \in \mathcal{F}^*$ for which $V \cap F \neq \emptyset$. But this clearly implies that $|\{\alpha \in \kappa^+ \setminus \gamma : V \cap A_\alpha^F \neq \emptyset\}| = \kappa^+$, hence U and the family

$$\mathcal{H} = \mathcal{J} \upharpoonright U = \{U \cap A_\alpha^F : F \in \mathcal{F}^* \text{ and } \alpha \in \kappa^+ \setminus \gamma\}$$

satisfy the assumptions of Lemma 2.13, consequently U is κ^+ -resolvable. ■

3. The main result. We are now ready to formulate and prove our main result.

THEOREM 3.1. *Let κ be an uncountable regular cardinal. Then every regular space X satisfying*

$$\Delta(X) \geq \kappa \geq \widehat{e}(X)$$

is ω -resolvable. Consequently, every regular space X that satisfies $\Delta(X) > e(X)$ is ω -resolvable.

Proof. For convenience, after fixing κ , we denote by \mathcal{C} the class of all regular spaces X that satisfy $\Delta(X) \geq \kappa \geq \widehat{e}(X)$. Clearly, \mathcal{C} is regular closed hereditary, that is, for every $X \in \mathcal{C}$ we have $\text{RC}^+(X) \subset \mathcal{C}$. By Corollary 2.2, to prove that all members of \mathcal{C} are ω -resolvable it suffices to show that every $X \in \mathcal{C}$ with $|X| = \Delta(X)$ is ω -resolvable.

To achieve this, for any cardinal $\lambda \geq \kappa$ we set

$$\mathcal{C}_\lambda = \{X \in \mathcal{C} : |X| = \Delta(X) = \lambda\},$$

and then we prove by induction on $\lambda \geq \kappa$ that

($*_\lambda$) every member of \mathcal{C}_λ is ω -resolvable.

So let us assume now that $\lambda \geq \kappa$ and ($*_\mu$) holds whenever $\kappa \leq \mu < \lambda$. Clearly, this implies that every space $X \in \mathcal{C}$ with $|X| < \lambda$ is ω -resolvable.

To deduce ($*_\lambda$) from this, by Theorem 2.3 it suffices to show that every member of \mathcal{C}_λ is n -resolvable for all $n \in \omega \setminus \{0\}$. This, in turn, will be proved by a subinduction on $n \in \omega \setminus \{0\}$. Therefore we assume from here on that for some $n > 0$,

(\circ_n) every member of \mathcal{C}_λ is n -resolvable,

and we want to deduce (\circ_{n+1}) from this. (Of course, (\circ_1) holds trivially.)

To prove (\circ_{n+1}) , we observe first that the class \mathcal{C}_λ is also regular closed hereditary, hence by Corollary 2.2 again, (\circ_{n+1}) is implied by the following seemingly weaker statement:

(\circ'_{n+1}) every member of \mathcal{C}_λ has an $(n+1)$ -resolvable subspace.

Now, the proof of (\circ'_{n+1}) branches into two: the initial case $\lambda = \kappa$ and the case $\lambda > \kappa$ of the induction on λ . These are handled differently.

CASE 1: $\lambda = \kappa$. Consider any $X \in \mathcal{C}_\kappa$ and recall that our aim is to show that X has an $(n+1)$ -resolvable subspace. If X is κ -resolvable then we are done. Otherwise by Corollary 2.8 there is $Q \in \text{RC}^+(X)$ such that

$$(3.1) \quad |A'| = \kappa \quad \text{for all } A \in [Q]^\kappa.$$

It easily follows from (3.1) that for every $Y \in [Q]^{<\kappa}$ and $B \in \text{RC}^+(Q)$ ($\subset \text{RC}^+(X)$) we have $B \setminus Y \in \mathcal{C}_\kappa$. Consequently, if $Y \cap B$ is also dense in B then B is $(n+1)$ -resolvable because $B \setminus Y$ is n -resolvable by (\circ_n) . So from here on we can assume that

$$(3.2) \quad \text{every set in } [Q]^{<\kappa} \text{ is nowhere dense.}$$

Let us now apply Lemma 2.11 to the space Q , with the choice $\kappa = \lambda$. This gives us some $Y \in \text{RC}^+(Q)$ and a chain decomposition $\{Y_\alpha : \alpha < \kappa\} \subset [Y]^{<\kappa}$ of Y such that

$$(3.3) \quad Y \setminus Y_\alpha \text{ is } <\kappa\text{-closed for each } \alpha < \kappa.$$

If Y happens to be ω -resolvable (or just $(n+1)$ -resolvable) then, of course, we are done. Otherwise we may apply Lemma 2.12 to the chain decomposition $\{Y_\alpha : \alpha < \kappa\}$ of Y to obtain $T \in \text{RC}^+(Y)$ with a dense subset $Z \subset T$ such that

$$(3.4) \quad \overline{Y_\alpha \cap Z} \subset Y_\alpha \quad \text{for all } \alpha < \kappa.$$

Write $R_\alpha = Y_{\alpha+1} \setminus Y_\alpha$ for $\alpha < \kappa$. For each $x \in Y$ we let $\alpha(x) \in \kappa$ be the unique ordinal with $x \in R_{\alpha(x)}$. We call a subset $E \subset Y$ *rare* if $|E \cap R_\alpha| \leq 1$ for all $\alpha < \kappa$. It is immediate from (3.3) and (3.4) that every rare subset E of Z of size $< \kappa$ is closed-and-discrete, i.e. satisfies $E' = \emptyset$.

Let us now consider the family

$$\mathcal{D} = \{D \in [Z]^\kappa : D \text{ is discrete and } \forall E \in [D]^{<\kappa} (E' = \emptyset)\}.$$

The derived set E' of E above is always meant to be taken in T (or equivalently in X), not in Z . It is obvious from the definition that for every $D \in \mathcal{D}$ we have $D' = D^\circ$ and $[D]^\kappa \subset \mathcal{D}$.

CLAIM 3.1.1. *For every $D \in \mathcal{D}$ we have $\Delta(D^\circ) = \kappa$, consequently $D^\circ \in \mathcal{C}_\kappa$.*

Proof. Assume that G is any open set with $G \cap D^\circ \neq \emptyset$, and pick a point $x \in G \cap D^\circ$. By the regularity of the space X there is an open set H such

that $x \in H \subset \overline{H} \subset G$. Then we have $|H \cap D| = \kappa$, as $x \in D^\circ$, and hence $|(H \cap D)^\circ| = \kappa$ by (3.1). But clearly

$$(H \cap D)^\circ \subset \overline{H} \cap D^\circ \subset G \cap D^\circ,$$

hence $|G \cap D^\circ| = \kappa$ and therefore $\Delta(D^\circ) = \kappa$. Since D° is closed in X , it is obvious that $\widehat{e}(D^\circ) \leq \widehat{e}(X) \leq \kappa$, and so $D^\circ \in \mathcal{C}_\kappa$. ■

We note that this proof used the full force of the regularity of our space.

CLAIM 3.1.2. *Assume that $V \in \text{RC}^+(T)$ and the set $S \subset V \cap Z$ is dense in V . If S is not κ -resolvable then there is some $D \in \mathcal{D}$ such that $D \subset S$.*

Proof. Since every member of $\tau^+(S)$ is somewhere dense in V and hence in Q , it follows from (3.2) that $\Delta(S) = \kappa = \text{cf}(\kappa) > \omega$. If S is not κ -resolvable then [4, Theorem 2.2] implies that S must have a (relatively) discrete subset J of size κ . Clearly, there is $D \in [J]^\kappa$ that is rare. But then $D \in \mathcal{D}$ because, as was pointed out above, $E' = \emptyset$ for all rare sets E of size $< \kappa$. ■

By the (sub)inductive assumption (\circ_n) we have a partition

$$T = \bigcup_{i < n} Z_i$$

of T into n pairwise disjoint dense subsets Z_i . Since $Z \subset T$ is also dense, it is not possible that $Z \cap Z_i$ is nowhere dense for all $i < n$. Thus we can assume, without loss of generality, that $Z \cap Z_0$ is somewhere dense, say it is dense in $V \in \text{RC}^+(T)$.

If there is some $W \in \text{RC}^+(V)$ for which $W \cap Z \cap Z_0$ is κ -resolvable (or just $(n+1)$ -resolvable) then again we are done. Otherwise, by Claim 3.1.2, for each $W \in \text{RC}^+(V)$ the set $W \cap Z \cap Z_0$ includes a member of \mathcal{D} , hence we may assume that

$$(3.5) \quad \mathcal{E} = \{D \in \mathcal{D} : D \subset V \cap Z \cap Z_0\} \text{ is a } \pi\text{-network in } V.$$

Now we distinguish two subcases.

SUBCASE 1: The collection

$$\mathcal{E}_0 = \{D \in \mathcal{E} : D^\circ \cap Z \cap Z_0 = \emptyset\}$$

is a π -network in V . In this case the family $\mathcal{F} = \{D^\circ : D \in \mathcal{E}_0\}$ is also a π -network in V because V is π -regular, hence by definition \mathcal{F} is a π -network in $V \setminus (Z \cap Z_0)$ as well. But every $D^\circ \in \mathcal{F}$ is n -resolvable by Claim 3.1.1 and (\circ_n) , hence so is $V \setminus (Z \cap Z_0)$ by Theorem 2.1. This, however, implies that V is $(n+1)$ -resolvable because $V \cap Z \cap Z_0$ is also dense in V .

SUBCASE 2: \mathcal{E}_0 is not a π -network in V , i.e. there is $U \in \text{RC}^+(V)$ such that if $D \in \mathcal{D}$ and $D \subset U \cap Z \cap Z_0$ then $D^\circ \cap U \cap Z \cap Z_0 \neq \emptyset$. Observe that

$$\mathcal{E}_1 = \{D \in \mathcal{D} : D \subset U \cap Z \cap Z_0\} = \{D \in \mathcal{E} : D \subset U\}$$

is a π -network in $U \cap Z \cap Z_0$. Therefore, the set

$$S = (U \cap Z \cap Z_0) \cap \bigcup \{D^\circ : D \in \mathcal{E}_1\}$$

is dense in $U \cap Z \cap Z_0$.

Now, it is easy to check that then the space $U \cap Z \cap Z_0$, the cardinal κ , the family \mathcal{E}_1 , and the dense subset S of $U \cap Z \cap Z_0$ satisfy all the assumptions of Lemma 2.4, hence $U \cap Z \cap Z_0$, and thus $U \cap Z_0$ as well, is 2-resolvable. But $U \setminus Z_0$ is clearly $(n-1)$ -resolvable, and so it follows that U is $(n+1)$ -resolvable. This completes the proof of (\circ'_{n+1}) in the case $\lambda = \kappa$.

CASE 2: $\lambda > \kappa$. Recall that our aim is to show that every space $X \in \mathcal{C}_\lambda$ has an $(n+1)$ -resolvable subspace. Assume first that there are $B \in \text{RC}^+(X)$ and a dense subset A of B with $|A| < \lambda$ such that $B \setminus A$ is κ -closed. Then $\Delta(B \setminus A) = \lambda$ because $|A| < \lambda$, and $\widehat{e}(B \setminus A) \leq \kappa$ because $B \setminus A$ is κ -closed, consequently $B \setminus A \in \mathcal{C}_\lambda$. So the (sub)inductive assumption (\circ_n) implies that $B \setminus A$ is n -resolvable and hence B is $(n+1)$ -resolvable.

Thus we may assume from here on that

$$(3.6) \quad \text{if } A \in [X]^{<\lambda} \text{ and } X \setminus A \text{ is } \kappa\text{-closed then } A \text{ is nowhere dense.}$$

Let us now apply Lemma 2.11 to the space X and the cardinals λ and κ^+ , which is possible because $\lambda \geq \kappa^+$. This way we obtain $Y \in \text{RC}^+(X)$ with a chain decomposition

$$\{Y_\alpha : \alpha < \text{cf}(\lambda)\} \subset [Y]^{<\lambda}$$

of length $\text{cf}(\lambda)$ such that for each $\alpha < \text{cf}(\lambda)$,

$$(3.7) \quad Y \setminus Y_\alpha \text{ is } \kappa\text{-closed for each } \alpha < \lambda.$$

Note that then each Y_α is nowhere dense by (3.6).

For any point $x \in Y$ we again define the ordinal $\alpha(x) < \text{cf}(\lambda)$ by the formula $x \in Y_{\alpha(x)+1} \setminus Y_{\alpha(x)}$ and call a set $E \subset Y$ *rare* if $|E \cap (Y_{\alpha+1} \setminus Y_\alpha)| \leq 1$ for all $\alpha < \text{cf}(\lambda)$.

If Y is ω -resolvable then we are done. Otherwise we may apply Lemma 2.12 to obtain $T \in \text{RC}^+(Y)$ with a dense subset $D \subset T$ such that

$$(3.8) \quad \overline{Y_\alpha \cap D} \subset Y_\alpha \quad \text{for all } \alpha < \text{cf}(\lambda).$$

We claim that D has no rare subset of cardinality κ . Indeed, for any rare set $E \in [D]^\kappa$ we would have $E' = \emptyset$, contradicting $\widehat{e}(X) \leq \kappa$.

To see this, we pick any point $x \in Y$. Then $x \notin \overline{D \cap Y_{\alpha(x)}} \subset Y_{\alpha(x)}$ by (3.8), moreover $x \notin \overline{E \setminus Y_{\alpha(x)+1}}$ because $Y \setminus Y_{\alpha(x)+1}$ is κ -closed by (3.7). But $|E \cap (Y_{\alpha(x)+1} \setminus Y_{\alpha(x)})| \leq 1$, hence clearly $x \notin E'$.

Consequently, if we got this far, i.e. no $(n+1)$ -resolvable subspace has been found yet, we must have $\text{cf}(\lambda) < \kappa$. Indeed, since each Y_α is nowhere dense but D is not, there are cofinally many $\alpha < \text{cf}(\lambda)$ with $D \cap (Y_{\alpha+1} \setminus Y_\alpha) \neq \emptyset$, hence $\text{cf}(\lambda) \geq \kappa$ would imply the existence of a rare subset of D of size κ .

Let us now put $T_\alpha = T \cap Y_\alpha$ and $Z_\alpha = \overline{D \cap T_{\alpha+1}} \setminus T_\alpha$ for $\alpha < \text{cf}(\lambda)$. Then $Z_\alpha \subset T_{\alpha+1} \setminus T_\alpha$ by (3.8) and Z_α is κ -closed, being the intersection of a closed set and a κ -closed set. This clearly implies $\widehat{e}(Z_\alpha) \leq \kappa$.

Moreover,

$$Z = \bigcup_{\alpha < \text{cf}(\lambda)} Z_\alpha \subset T$$

is dense in T because $D \subset Z$. We also have $\widehat{e}(Z) \leq \kappa$ because $\widehat{e}(Z_\alpha) \leq \kappa$ for each $\alpha < \text{cf}(\lambda)$ and $\text{cf}(\lambda) < \kappa = \text{cf}(\kappa)$.

The following observation will be crucial in the rest of our proof.

CLAIM 3.1.3. *Every set $H \in [Z]^{\leq \kappa}$ is nowhere dense.*

Proof. Let us fix $H \in [Z]^{\leq \kappa}$ and pick $\alpha < \text{cf}(\lambda)$. Then we have $\overline{H \cap T_\alpha} \subset \overline{D \cap T_\alpha} \subset T_\alpha$ by (3.8) and $\overline{H \setminus T_{\alpha+1}} \subset T \setminus T_{\alpha+1}$ by (3.7). Moreover,

$$\overline{H \cap (T_{\alpha+1} \setminus T_\alpha)} \subset Z_\alpha \subset T_{\alpha+1} \setminus T_\alpha$$

because $Z_\alpha \subset T_{\alpha+1} \setminus T_\alpha$ is κ -closed, hence

$$\overline{H \cap (T_{\alpha+1} \setminus T_\alpha)} = \overline{H \cap (T_{\alpha+1} \setminus T_\alpha)}.$$

This then implies that $\{\overline{H \cap (T_{\alpha+1} \setminus T_\alpha)} : \alpha < \text{cf}(\lambda)\}$ is a partition of \overline{H} into relatively clopen subsets of size $< \lambda$. Consequently, for all $U \in \tau^+(\overline{H})$ we have $\Delta(U) < \lambda$, while for every $W \in \tau^+(X)$ we have $\Delta(W) = \lambda$. But this implies that $\text{Int}(\overline{H}) = \emptyset$, i.e. H (equivalently \overline{H}) is nowhere dense. ■

If there are an $\alpha < \text{cf}(\lambda)$ and an $R \in \text{RC}^+(Z_\alpha)$ with $\Delta(R) \geq \kappa$ ($\geq \widehat{e}(R)$) then, as $R \in \mathcal{C}$ and $|R| < \lambda$, our inductive hypothesis implies that R is ω -resolvable, hence we are done.

Consequently, we may assume that

$$\mathcal{P}_\alpha = \{U \in \tau^+(Z_\alpha) : |U| < \kappa\}$$

is a π -base of Z_α for each $\alpha < \text{cf}(\lambda)$. For any $\alpha < \text{cf}(\lambda)$ let \mathcal{E}_α be a maximal disjoint subfamily of \mathcal{P}_α . It then follows that $E_\alpha = \bigcup \mathcal{E}_\alpha$ is a dense open subset of Z_α , consequently

$$E = \bigcup_{\alpha < \text{cf}(\lambda)} E_\alpha$$

is dense in Z and hence in T .

Let us now put $F_\alpha = Z_\alpha \setminus E_\alpha$ for all $\alpha < \text{cf}(\lambda)$, and

$$F = \bigcup_{\alpha < \text{cf}(\lambda)} F_\alpha.$$

Since F_α is closed in Z_α , we have $\widehat{e}(F_\alpha) \leq \widehat{e}(Z_\alpha) \leq \kappa$ and so, by $\text{cf}(\lambda) < \kappa$,

$$(3.9) \quad \widehat{e}(F) \leq \kappa$$

as well.

We claim that F is also dense in Z . Assume on the contrary that $F \cap V = \emptyset$ for some $V \in \text{RC}^+(Z)$, i.e. $V \subset E$. Then $V \cap Z_\alpha \subset E_\alpha$ for each $\alpha < \text{cf}(\lambda)$, hence $\mathcal{U}_\alpha = \{U \cap (V \cap Z_\alpha) : U \in \mathcal{E}_\alpha\}$ yields a partition of $V \cap Z_\alpha$ into (relatively) clopen subsets of $V \cap Z_\alpha$. But $V \cap Z_\alpha$ is closed in Z_α , and consequently $\widehat{e}(Z_\alpha) \leq \kappa$ implies $|\mathcal{U}_\alpha| < \kappa$. But then we also have $|V \cap Z_\alpha| < \kappa$, because $|U| < \kappa$ for each $U \in \mathcal{U}_\alpha$ and κ is a regular cardinal. This, in turn, implies $|V \cap Z| < \kappa$ because $\text{cf}(\lambda) < \kappa$. But $V \cap Z$ is somewhere dense, and this contradicts Claim 3.1.3. So F is indeed dense in Z .

As F is dense in Z , applying Claim 3.1.3 again we conclude that

$$(3.10) \quad \lambda \geq \Delta(F) > \kappa.$$

Putting (3.9) and (3.10) together, our inductive hypotheses, including (\circ_n) , imply that F is n -resolvable, hence Z is $(n+1)$ -resolvable because E is dense in Z and $E \cap F = \emptyset$. Thus (\circ'_{n+1}) is verified and the proof is complete. ■

Let us now make a few comments on the assumptions of our main Theorem 3.1. Although the uncountability of κ was used in our proof when we referred to [4, Theorem 2.2], Theorem 3.1 is valid for $\kappa = \omega$ as well. Indeed, to see this we note that $\widehat{e}(X) = \omega$ just means that X is countably compact, and Pytkeev [8] proved that crowded countably compact regular spaces are even ω_1 -resolvable.

The question whether the assumption of the regularity of κ is essential is more interesting and we do not know the answer. We only have the following partial positive result in the case when κ is a singular cardinal of countable cofinality.

THEOREM 3.2. *Let κ be a singular cardinal of countable cofinality. Then every regular space X satisfying*

$$\Delta(X) \geq \kappa \geq \widehat{e}(X)$$

is 2-resolvable.

Proof. Using Theorems 3.1 and 2.1 it clearly suffices to show that any regular space X with

$$|X| = \Delta(X) = \widehat{e}(X) = \kappa$$

has a 2-resolvable subspace.

If there is $R \in \text{RC}^+(X)$ with $e(R) < \kappa$ then R is ω -resolvable by Theorem 3.1, hence we may assume that $\widehat{e}(R) = \kappa$ for all $R \in \text{RC}^+(X)$. Also, if some $G \in \tau^+(X)$ has a dense subset Y of cardinality $< \kappa$ then $\Delta(G) = \Delta(X) = \kappa$ implies that $G \setminus Y$ is also dense in G , hence G is 2-resolvable. Thus we may also assume that every set $Y \in [X]^{<\kappa}$ is nowhere dense in X .

Since $\text{cf}(\kappa) = \omega$, we can fix a strictly increasing sequence of cardinals $\langle \kappa_n : n < \omega \rangle$ with $\kappa = \sum \{\kappa_n : n < \omega\}$. Then we may choose a sequence of sets $\{Y_n : n < \omega\} \subset [X]^{<\kappa}$ with $|Y_n| = \kappa_n$ and $\bigcup_{n \in \omega} Y_n = X$. Each

$\overline{Y_n}$ is nowhere dense, hence by a straightforward induction we may define a sequence $\{U_n : n < \omega\} \subset \tau^+(X)$ such that $\overline{U_{n+1}} \subsetneq U_n$ for all $n < \omega$, and moreover $\bigcap \{U_n : n < \omega\} = \emptyset$.

Then $R_n = \overline{U_n} \setminus \text{Int}(\overline{U_{n+1}}) \in \text{RC}^+(X)$, and $\{R_{2n} : n < \omega\}$ is clearly a discrete collection in X . But $\widehat{\varepsilon}(R_{2n}) = \kappa$ implies the existence of a set $D_n \in [R_{2n}]^{\kappa_n}$ that is closed discrete in R_{2n} and hence in X . Consequently,

$$D = \bigcup \{D_n : n < \omega\} \in [X]^\kappa$$

is also closed discrete in X , contradicting $\widehat{\varepsilon}(X) = \kappa$ and completing the proof. ■

4. Stepping-up resolvability. Let κ be an infinite cardinal and denote by \mathcal{L}_κ the class of all regular spaces X that are κ^+ -compact and satisfy $|X| = \Delta(X) = \kappa^+$. (We recall that the κ^+ -compactness of X means that $A^\circ \neq \emptyset$ for each $A \in [X]^{\kappa^+}$.) Our aim is then to prove the following stepping-up result.

THEOREM 4.1. *If every member of \mathcal{L}_κ is κ -resolvable then actually every member of \mathcal{L}_κ is κ^+ -resolvable.*

We need the following lemma.

LEMMA 4.2. *Assume that $X \in \mathcal{L}_\kappa$ has no κ^+ -approachable subset. Then*

- (i) *for any $A \in [X]^{\kappa^+}$ we have $A^\circ \in \mathcal{L}_\kappa$;*
- (ii) *there are κ^+ many pairwise disjoint sets of the form A° with $A \in [X]^{\kappa^+}$.*

Proof. (i) It is immediate from Lemma 2.6(2), applied to κ^+ instead of κ , that for any $A \in [X]^{\kappa^+}$ we have $|A^\circ| = \kappa^+$. Also, as A° is closed in X , it is κ^+ -compact. So, to prove $A^\circ \in \mathcal{L}_\kappa$ it only remains to show that $\Delta(A^\circ) = \kappa^+$.

To see this, assume that U is open with $U \cap A^\circ \neq \emptyset$, say $x \in U \cap A^\circ$. By the regularity of X the point x has an open neighbourhood V such that $\overline{V} \subset U$. Then $x \in A^\circ$ implies $|V \cap A| = \kappa^+$, hence $|(V \cap A)^\circ| = \kappa^+$. This, in turn, implies $|U \cap A^\circ| = \kappa^+$ because $(V \cap A)^\circ \subset \overline{V} \cap A^\circ \subset U \cap A^\circ$.

(ii) Let us note first of all that if we have $A, B \in [X]^{\kappa^+}$ with $A^\circ \setminus B^\circ \neq \emptyset$ then there is a set $C \in [A]^\kappa$ such that $C^\circ \cap B^\circ = \emptyset$. Indeed, if $x \in A^\circ \setminus B^\circ$ and V is an open neighbourhood of x with $|\overline{V} \cap B| \leq \kappa$ then the choice $C = V \cap A$ clearly works.

Now we distinguish two cases.

CASE 1: There is a disjoint family $\{X_\xi : \xi < \kappa^+\} \subset [X]^{\kappa^+}$ such that the increasing κ^+ -sequence $\{Y_\xi^\circ : 0 < \xi < \kappa^+\}$, where $Y_\xi = \bigcup_{\eta < \xi} X_\eta$, does not stabilize. This means that the set $I = \{\xi < \kappa^+ : Y_{\xi+1}^\circ \setminus Y_\xi^\circ \neq \emptyset\}$

has cardinality κ^+ . By our remark above, for each $\xi \in I$ there is a set $C_\xi \in [Y_{\xi+1}]^{\kappa^+}$ such that $C_\xi^\circ \cap Y_\xi^\circ = \emptyset$. But this means that the members of the family $\{C_\xi^\circ : \xi \in I\}$ are pairwise disjoint, and we are done.

CASE 2: For every disjoint family $\{X_\xi : \xi < \kappa^+\} \subset [X]^{\kappa^+}$ the sequence $\{Y_\xi^\circ : \xi < \kappa^+\}$, as defined above, does stabilize. Let us then fix a disjoint family $\{X_\xi : \xi < \kappa^+\} \subset [X]^{\kappa^+}$, and for any $\xi < \eta < \kappa^+$ let us put $Y_{\xi,\eta} = \bigcup \{X_i : \xi \leq i < \eta\}$. But now for each fixed $\xi < \kappa^+$ the sequence

$$\{Y_{\xi,\eta}^\circ : \xi < \eta < \kappa^+\}$$

stabilizes, i.e. there is an $\eta(\xi) < \kappa^+$ such that

$$Y_{\xi,\zeta}^\circ = Y_{\xi,\eta(\xi)}^\circ = F_\xi \quad \text{whenever } \eta(\xi) \leq \zeta < \kappa^+.$$

The sequence $\{F_\xi : \xi < \kappa^+\}$ is clearly decreasing and we claim that it cannot stabilize. Indeed, assume on the contrary that there is some $\xi_0 < \kappa^+$ such that $F_\xi = F_{\xi_0}$ for all $\xi_0 < \xi < \kappa^+$. We may then select a set $I \subset [\kappa^+ \setminus \xi_0]^{\kappa^+}$ such that $\eta(\xi) < \zeta$ whenever $\{\xi, \zeta\} \in [I]^2$ and $\xi < \zeta$. But then the disjoint collection $\{Y_{\xi,\eta(\xi)} : \xi \in I\} \subset [X]^{\kappa^+}$ would witness that the set F_{ξ_0} is κ^+ -approachable in X , contradicting our assumption.

Consequently, the set $J = \{\xi < \kappa^+ : F_\xi \setminus F_{\xi+1} \neq \emptyset\}$ has cardinality κ^+ and by our introductory remark for each $\xi \in J$ there is a set $C_\xi \in [Y_{\xi,\eta(\xi)}]^{\kappa^+}$ such that $C_\xi^\circ \cap F_{\xi+1} = \emptyset$. But we also have $C_\xi^\circ \subset F_\xi$ for each $\xi \in J$, so the sets $\{C_\xi^\circ : \xi \in J\}$ are pairwise disjoint, completing the proof of (ii). ■

Proof of Theorem 4.1. Let us assume, to begin with, that every member of \mathcal{L}_κ is κ -resolvable. Our aim is to show that then every member of \mathcal{L}_κ is κ^+ -resolvable. Since \mathcal{L}_κ is regular closed hereditary, by Corollary 2.2 it suffices to prove that every member of \mathcal{L}_κ has a κ^+ -resolvable subspace.

Now, if $X \in \mathcal{L}_\kappa$ is such that its κ^+ -approachable subsets form a π -network in X then Theorem 2.7 implies that X is κ^+ -resolvable. Therefore, it will suffice to show that any space $X \in \mathcal{L}_\kappa$ that has no κ^+ -approachable subset contains a κ^+ -resolvable subspace.

To see this, we may apply Lemma 4.2 to obtain a family

$$\{C_\alpha : \alpha < \kappa^+\} \subset [X]^{\kappa^+}$$

such that the sets $A_\alpha = C_\alpha^\circ$ are pairwise disjoint. Since each $A_\alpha \in \mathcal{L}_\kappa$ is κ -resolvable by our assumption, it follows that the closed set

$$F = \bigcap_{\alpha < \kappa^+} \overline{\bigcup \{A_\beta : \beta \in \kappa^+ \setminus \alpha\}}$$

is κ -nice in the sense of Definition 2.14.

We claim that $F \in \mathcal{L}_\kappa$ also holds, and this will follow if we can show that $\Delta(F) = \kappa^+$. So let U be open with $U \cap F \neq \emptyset$, and pick $x \in U \cap F$.

By the regularity of X the point x has an open neighbourhood V such that $\overline{V} \subset U$. Clearly,

$$I = \{\alpha : V \cap A_\alpha \neq \emptyset\} \in [\kappa^+]^{\kappa^+}.$$

Let us pick for each $\alpha \in I$ a point $x_\alpha \in V \cap A_\alpha$ and consider the set $B = \{x_\alpha : \alpha \in I\}$. Then $B \in [X]^{\kappa^+}$ and $B^\circ \subset \overline{V} \cap F \subset U \cap F$, hence $|U \cap F| = \kappa^+$. So we indeed have $F \in \mathcal{L}_\kappa$, and therefore F is both κ -nice and κ -resolvable.

This argument can also be applied to any regular closed subset R in $\text{RC}^+(X)$ to obtain a subset of R that is both κ -nice and κ -resolvable. Thus we have concluded that the sets that are both κ -nice and κ -resolvable form a π -network in X . So if \mathcal{F} is any maximal disjoint collection of such subsets of X then Corollary 2.15 can be applied to X and \mathcal{F} to conclude that X has a κ^+ -resolvable (open) subspace, and thus the proof is complete. ■

Since Lindelöf spaces are trivially ω_1 -compact, from Theorem 4.1 and the case $\kappa = \omega_1$ of Theorem 3.1 we immediately obtain the following result.

COROLLARY 4.3. *Every ω_1 -compact (hence every Lindelöf) regular space X with $|X| = \Delta(X) = \omega_1$ is ω_1 -resolvable.*

We have been unable to answer the natural question whether, in Corollary 4.3, ω_1 -compactness can be relaxed to having countable extent.

Acknowledgments. This research was supported by OTKA grant no. K 83726.

References

- [1] A. G. El'kin, *On the maximal resolvability of products of topological spaces*, Soviet Math. Dokl. 10 (1969), 659–662.
- [2] M. A. Filatova, *Resolvability of Lindelöf spaces*, Fundam. Prikl. Mat. 11 (2005), no. 5, 225–231 (in Russian); English transl.: J. Math. Sci. (N.Y.) 146 (2007), 5603–5607.
- [3] A. Illanes, *Finite and ω -resolvability*, Proc. Amer. Math. Soc. 124 (1996), 1243–1246.
- [4] I. Juhász, L. Soukup and Z. Szentmiklóssy, *Resolvability of spaces having small spread or extent*, Topology Appl. 154 (2007), 144–154.
- [5] I. Juhász and Z. Szentmiklóssy, *Interpolation of κ -compactness and PCF*, Comment. Math. Univ. Carolin. 50 (2009), 315–320.
- [6] V. I. Malykhin, *Borel resolvability of compact spaces and their subspaces*, Math. Notes 64 (1998), 607–615.
- [7] O. Pavlov, *On resolvability of topological spaces*, Topology Appl. 126 (2002), 37–47.
- [8] E. G. Pytkeev, *Resolvability of countably compact regular spaces*, Proc. Steklov Inst. Math. 2002, Suppl. 2, S152–S154.

István Juhász, Lajos Soukup
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
13–15 Reáltanoda u.
1053 Budapest, Hungary
E-mail: juhasz@renyi.hu
soukup@renyi.hu
<http://www.renyi.hu/~soukup>

Zoltán Szentmiklóssy
Institute of Mathematics
Faculty of Science
Eötvös Loránd University
Pázmány Péter sétány 1/C
1117 Budapest, Hungary
E-mail: zoli@renyi.hu

*Received 11 November 2013;
in revised form 18 June 2014*