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Stable short exact sequences and the maximal exact structure of an additive category

by

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Dedicated to B.V.M.

Abstract. It was recently proved that every additive category has a unique maximal exact structure, while it remained open whether the distinguished short exact sequences of this canonical exact structure coincide with the stable short exact sequences. The question is answered by a counterexample which shows that none of the steps to construct the maximal exact structure can be dropped.

1. Introduction. Exact categories, preconceived by Heller [6] (see [1, Appendix B]) and introduced by Quillen [13], provide a suitable framework for K-theory [13], relative homological algebra [7, 2, 5], and derived categories [12, 10]. For an additive category $\mathscr{A}$, a sequence of morphisms

$$A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C \tag{1}$$

with $a = \ker b$ and $b = \operatorname{cok} a$ is said to be a *short exact sequence* in $\mathscr{A}$. (In what follows, we depict kernels of morphisms by $\rightarrowtail$ and cokernels by $\twoheadrightarrow$.) For a desciption of the unique maximal exact structure of an additive category $\mathscr{A}$, one-sided exact structures had to be introduced [15]. By definition, a *left exact structure* on $\mathscr{A}$ is given by a class $\mathscr{D} \subset \mathscr{A}$ of cokernels, called *deflations*, satisfying:

- (C) $\mathscr{D}$ is a subcategory with $\operatorname{Ob}\mathscr{D} = \operatorname{Ob}\mathscr{A}$.
- (P) The pullback of any $c \in \mathscr{D}$ along an arbitrary morphism exists and belongs to $\mathscr{D}$.
- (Q) If $A \overset{a}{\to} B \overset{b}{\to} C$ belongs to $\mathscr{D}$ and b has a kernel, then $b \in \mathscr{D}$.

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In particular, the pullback of a deflation $b\colon B \twoheadrightarrow C$ along $0 \to C$ yields its kernel $a\colon A \rightarrowtail B$. Thus every deflation b gives rise to a short exact sequence (1). We refer to such sequences as *conflations*. Dually, a class $\mathscr{I}$ of kernels (called *inflations*) defines a *right exact structure* on $\mathscr{A}$ if $\mathscr{I}$ is a left exact structure on $\mathscr{A}^{\mathrm{op}}$. An *exact* category in the sense of Quillen [13] is an additive category with conflations coming simultaneously from a left and a right exact structure.

Every abelian category $\mathscr{A}$ has two extremal exact structures: the *maximal* one, which consists of all short exact sequences, and the *minimal* one, which consists of the split short exact sequences. More generally, the split short exact sequences of any additive category $\mathscr{A}$ determine the smallest exact structure, but it was discovered only recently that $\mathscr{A}$ also admits a unique maximal exact structure. For a triangulated category $\mathscr{A}$, the two extremal exact structures coincide. If $\mathscr{A}$ has kernels and cokernels, the existence of the maximal exact structure was first proved by Sieg and Wegner [16]. Crivei [4] extended the result to additive categories for which every split epimorphism has a kernel.

The general case was established in [15]. Using Quillen's axiom (Q), the "obscure axiom" [17] which is redundant for two-sided exact categories [9], we proved first that any left exact structure of an additive category $\mathscr{A}$ can be combined with any right exact structure to make $\mathscr{A}$ into an exact category [15, Theorem 1]. Of course, every exact category arises in this way. The existence of the maximal exact structure then follows from the more general result that every additive category $\mathscr{A}$ admits a unique maximal left exact structure [15, Corollary 1 to Theorem 2]. Its deflations are given explicitly in terms of two operators P and Q on $\mathscr{A}$ which are defined as follows [15].

For a class $\mathscr{D}$ of morphisms in $\mathscr{A}$, let $P\mathscr{D}$ denote the the class of morphisms $b\colon B \to C$ such that for every $f\colon C' \to C$ in $\mathscr{A}$, the pullback

$$(2) \qquad \begin{array}{ccc} B' & \xrightarrow{\;b'\;} & C' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\;b\;} & C \end{array}$$

exists and satisfies $b' \in \mathscr{D}$. The class $Q\mathscr{D}$ consists of the morphisms $b \in \mathscr{D}$ such that, for every pullback (2), the implication $e \in \mathscr{D} \Rightarrow f \in \mathscr{D}$ is satisfied. If $\mathscr{C}$ denotes the class of cokernels in $\mathscr{A}$, then $P\mathscr{C}$ consists of the *semistable* cokernels in the sense of Richman and Walker [14, 4]. Dually, a kernel in $\mathscr{A}$ is said to be *semistable* if it is a semistable cokernel in $\mathscr{A}^{\mathrm{op}}$. A short exact sequence (1) with a semistable kernel a and a semistable cokernel b is said to be *stable*. The main result of [15] states that the cokernels in $PQP\mathscr{C}$ define a unique maximal left exact structure on $\mathscr{A}$.

While every conflation of the maximal exact structure of $\mathscr{A}$ is stable, it remained open [15] whether every stable short exact sequence is a conflation. Phrased in a one-sided way, this amounts to the question whether there exist additive categories $\mathscr{A}$ for which the inclusion $QP\mathscr{C} \subset P\mathscr{C}$ is proper. A positive answer would imply that the "Quillen operator" Q figures in an essential way.

In this note, we show that this is indeed the case. Removing the generic module from the module category of a discrete valuation domain R, we obtain a category $\mathscr{A}$ with the proper inclusions

$$PQP\mathscr{C} \subsetneq QP\mathscr{C} \subsetneq P\mathscr{C} \subsetneq \mathscr{C}.$$

Our explicit description of the subcategories $PQP\mathscr{C} \subset QP\mathscr{C} \subset P\mathscr{C}$ shows that the case differentiation in $P\mathscr{C}$ (Proposition 1) and $QP\mathscr{C}$ (Proposition 3) disappears for the smaller subcategory $PQP\mathscr{C}$ which is given by a projective structure [5, 11]. Namely, there is an indecomposable injective R-module T such that a cokernel $b\colon B \twoheadrightarrow C$ belongs to $PQP\mathscr{C}$ if and only if every morphism $T \to C$ factors through b (Theorem 1). In other words, the injective R-module T becomes projective in the subcategory $\mathscr{A}$ and determines the left exact structure of $\mathscr{A}$. A similar result is proved for the subcategory of inflations (Theorem 2). As a consequence, we obtain a simple description of the maximal exact structure (Corollary 1) and exhibit a stable short exact sequence which is not a conflation (Corollary 2).

2. The counterexample. Let R be a discrete valuation domain with quotient field K. By $\mathbf{Mod}(R)$ we denote the category of all (left) R-modules. We consider the standard torsion theory where an R-module is *torsion* if it has no non-zero free submodules. The *torsion-free* R-modules are then the submodules of K-vector spaces. Up to isomorphism, K is the unique indecomposable torsion-free injective R-module. Equivalently, K belongs to the unique isomorphism class of *generic* R-modules, that is, non-finitely presented indecomposable R-modules of finite length over their endomorphism ring [3, 1.3]. Removing K from $\mathbf{Mod}(R)$, we obtain an additive category $\mathscr{A}$. There is a short exact sequence

$$R \hookrightarrow K \overset{c}{\twoheadrightarrow} K/R, \tag{3}$$

where R is a projective generator and K/R an injective cogenerator in $\mathbf{Mod}(R)$. So the embedding $\mathscr{A} \hookrightarrow \mathbf{Mod}(R)$ respects kernels and cokernels of morphisms. As a full embedding, $\mathscr{A} \hookrightarrow \mathbf{Mod}(R)$ also reflects kernels and cokernels. Therefore, a sequence of morphisms $A \to B \to C$ in $\mathscr{A}$ is short exact in $\mathscr{A}$ if and only if it is a short exact sequence in $\mathbf{Mod}(R)$. The simple R-module S relates R to K/R by homomorphisms

$$R \overset{r}{\twoheadrightarrow} S \overset{s}{\rightarrowtail} K/R,$$

which will be fixed in what follows.

LEMMA 1. *Every non-split short exact sequence $R \rightarrowtail X \twoheadrightarrow K/R$ of R-modules is isomorphic to the total complex of a double complex*

$$(4)\qquad \begin{array}{ccc} R & \xhookrightarrow{\;i\;} & K \\ {\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ E & \xhookrightarrow{\;j\;} & K/R \end{array}$$

Proof. Since K is injective, we have a commutative diagram

$$\begin{array}{ccccc} R & \rightarrowtail & X & \twoheadrightarrow & K/R \\ \| & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ R & \xhookrightarrow{\;i\;} & K & \xrightarrow{\;c\;}\!\!\!\!\!\!\twoheadrightarrow & K/R \end{array}$$

As the upper row does not split, it follows that $q \neq 0$. Hence q is epic. So the right-hand square is a pullback and pushout, which implies that p is epic, too. Thus $E := \operatorname{Ker} p \cong \operatorname{Ker} q$ is isomorphic to a proper factor module of R. By [8, Theorem 23], the short exact sequence $E \hookrightarrow X \overset{p}{\twoheadrightarrow} K$ splits. So the short exact sequence $R \rightarrowtail X \twoheadrightarrow K/R$ is the total complex of a double complex (4) which is a pullback and pushout. Note that f is epic since $f \neq 0$. ∎

The semistable kernels and cokernels of $\mathscr{A}$ can be characterized as follows.

PROPOSITION 1. *Let $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ be a short exact sequence in $\mathscr{A}$.*

(a) *The kernel a is semistable if and only if $C \cong K/R$ implies that, for any R-linear map $f\colon A \to R$, the map $rf\colon A \to S$ factors through a.*
(b) *The cokernel b is semistable if and only if $A \cong R$ implies that, for any R-linear map $g\colon K/R \to C$, the map $gs\colon S \to C$ factors through b.*

Proof. (a) The embedding $\mathscr{A} \hookrightarrow \mathbf{Mod}(R)$ respects and reflects kernels and cokernels. Hence a is not semistable if and only if there is a pushout

$$\begin{array}{ccccc} A & \overset{a}{\rightarrowtail} & B & \overset{b}{\twoheadrightarrow} & C \\ {\scriptstyle f}\big\downarrow & & \big\downarrow & & \| \\ D & \rightarrowtail & K & \twoheadrightarrow & C \end{array}$$

in $\mathbf{Mod}(R)$ with $D \in \operatorname{Ob}\mathscr{A}$. Then $C, D \neq 0$, which implies that $C \cong K/R$ and $D \cong R$. So we can replace D by R, and then rf does not factor through a. Conversely, assume that $C \cong K/R$ and there is an R-linear map $f\colon A \to R$ such that rf does not factor through a. By Lemma 1, the pushout of a along f gives a non-split short exact sequence $R \overset{\binom{i}{e}}{\rightarrowtail} K \oplus E \twoheadrightarrow C$ with $i\colon R \hookrightarrow K$ and a non-invertible epimorphism $e\colon R \twoheadrightarrow E$. If $E \neq 0$, then $r = pe$ for some $p\colon E \twoheadrightarrow S$. Hence $r = (0\ p)\binom{i}{e}$, and thus rf factors through a, contrary to the assumption. Thus $E = 0$, which shows that a is not semistable.

(b) The cokernel b is not semistable if and only if there is a pullback

$$\begin{array}{ccccc} A & \rightarrowtail & K & \twoheadrightarrow & D \\ \| & & \downarrow & & \downarrow{\scriptstyle g} \\ A & \overset{a}{\rightarrowtail} & B & \overset{b}{\twoheadrightarrow} & C \end{array}$$

in $\mathbf{Mod}(R)$ with $D \in \operatorname{Ob} \mathscr{A}$. Then $A \cong R$ and $D \cong K/R$. So we can assume that $D = K/R$, and then gs does not factor through b. Conversely, assume that $A \cong R$ and there is an R-linear map $g\colon K/R \to C$ such that gs does not factor through b. By Lemma 1, the pullback of b along g gives a non-split short exact sequence $A \rightarrowtail K \oplus E \twoheadrightarrow K/R$, and a similar argument to the above shows that $E = 0$. ∎

Lemma 2. *Every cokernel $b\colon B \twoheadrightarrow K/R$ in $\mathscr{A}$ is semistable.*

Proof. If b is a split epimorphism or $\operatorname{Ker} b \ncong R$, the cokernel b is semistable by Proposition 1. Otherwise, Lemma 1 yields

$$b\colon B \xrightarrow{\sim} K \oplus E \overset{(f\ j)}{\twoheadrightarrow} K/R$$

with an inclusion $j\colon E \hookrightarrow K/R$ and a finitely generated R-module $E \neq 0$. Hence $s\colon S \rightarrowtail K/R$ factors through $(f\ j)$. By Proposition 1, this implies that b is semistable. ∎

Proposition 1(b) allows a more succinct description. Let $c\colon K \twoheadrightarrow K/R$ denote the natural map in (3).

Proposition 2. *Let $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ be a short exact sequence in $\mathscr{A}$. Then b is not semistable if and only if $b\colon B \cong K \oplus D \overset{\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\twoheadrightarrow} K/R \oplus D \cong C$ with $D \neq 0$.*

Proof. Assume that b is not semistable. As in the proof of Proposition 1, this implies that there is a pullback

$$\begin{array}{ccccc} R & \hookrightarrow & K & \overset{c}{\twoheadrightarrow} & K/R \\ \downarrow{\scriptstyle \wr} & & \downarrow & & \downarrow{\scriptstyle g} \\ A & \overset{a}{\rightarrowtail} & B & \overset{b}{\twoheadrightarrow} & C \end{array}$$

If gs factors through b, then s factors through c, which is impossible. Thus $gs \neq 0$, which shows that g is monic. So we obtain a commutative diagram

$$\begin{array}{ccccc} K & \rightarrowtail & B & \overset{p}{\twoheadrightarrow} & D \\ {\scriptstyle c}\downarrow & & {\scriptstyle b}\downarrow & & \| \\ K/R & \overset{g}{\rightarrowtail} & C & \overset{q}{\twoheadrightarrow} & D \end{array}$$

with exact rows. By Lemma 2, $D \neq 0$. As the upper row splits, there is a morphism $h\colon D \to B$ with $ph = 1$. Thus $q \cdot bh = 1$, which yields the desired decomposition of b. The converse follows by Proposition 1. ∎

Remark. There is no dual version of Proposition 2. Necessary conditions for a to be non-semistable are $C \cong K/R$, and that b has a factorization $b\colon B \twoheadrightarrow K \twoheadrightarrow C$.

In what follows, let $\mathscr{C}$ denote the class of cokernels in $\mathscr{A}$.

Proposition 3. *Let $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ be a short exact sequence in $\mathscr{A}$. The cokernel b belongs to $QP\mathscr{C}$ if and only if one of the following is satisfied:*

(a) *$C \cong K/R$.*
(b) *For any split epimorphism $p\colon C \twoheadrightarrow K/R$ the map pb is a split epimorphism.*

Proof. If $C \cong K/R$, Lemma 2 implies that $b \in QP\mathscr{C}$. Assume that (b) holds, and suppose that b is not semistable. By Proposition 2, we can assume that b is of the form $b\colon K \oplus D \xrightarrow{\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)} K/R \oplus D$. Condition (b) then yields a map $\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)\colon K/R \to K \oplus D$ with $ci = (1\,0)\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right) = 1$, a contradiction. Thus $b \in P\mathscr{C}$.

Now let (2) be a pullback in $\mathscr{A}$ with $e \in P\mathscr{C}$. Suppose that $f \notin P\mathscr{C}$. By Proposition 2, we can assume that $f\colon K \oplus D \xrightarrow{\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)} K/R \oplus D$. Condition (b) implies that the first component of $b = \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)\colon B \twoheadrightarrow K/R \oplus D$ is a split epimorphism. Hence there is an R-linear map $g\colon K/R \to B$ with $b_1 g = 1$. By Proposition 1, $gs = eh$ for some $h\colon S \to B'$. So we get a commutative diagram

$$\begin{array}{ccccc} S & \overset{h}{\longrightarrow} & B' & \overset{b'}{\twoheadrightarrow} & K \oplus D \\ {\scriptstyle s}\downarrow & & {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)} \\ K/R & \overset{g}{\longrightarrow} & B & \overset{b}{\twoheadrightarrow} & K/R \oplus D \end{array}$$

The first component of $b'h$ is zero. Hence $s = b_1 gs = 0$, a contradiction. Thus $b \in QP\mathscr{C}$.

Conversely, assume that $b \in QP\mathscr{C}$ and $C \not\cong K/R$. Suppose that (b) does not hold. Then $C = T \oplus C'$ with $T \cong K/R$ such that the component b_1 of $b = \binom{b_1}{b_2} \colon B \twoheadrightarrow T \oplus C'$ is not a split epimorphism. Consider the pullback

$$\begin{array}{ccc} D & \overset{d}{\twoheadrightarrow} & K \oplus C' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)} \\ B & \overset{b}{\twoheadrightarrow} & T \oplus C' \end{array}$$

with $p \colon K \overset{c}{\twoheadrightarrow} K/R \xrightarrow{\sim} T$. For any $g \colon K/R \to B$, the composition $b_1 g$ is not invertible. Hence $b_1 gs = 0$. Therefore, bgs factors through $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, by the pullback property, gs factors through e. By Proposition 1, this implies that $e \in P\mathscr{C}$. So we obtain $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in P\mathscr{C}$, contrary to Proposition 1. ∎

Now we can prove our first main result.

THEOREM 1. *Let* $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ *be a short exact sequence in* $\mathscr{A}$. *Then* b *is a deflation with respect to the maximal left exact structure of* $\mathscr{A}$ *if and only if every* R*-linear map* $f \colon K/R \to C$ *factors through* b.

Proof. By [15, Theorem 2], the cokernel b is a deflation if and only if $b \in PQP\mathscr{C}$. Assume that $b \in PQP\mathscr{C}$. For a given $f \colon K/R \to C$, consider the pullback

$$\begin{array}{ccc} D & \overset{d}{\twoheadrightarrow} & K/R \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \overset{b}{\twoheadrightarrow} & C \end{array}$$

We add a second pullback

$$\begin{array}{ccc} D \oplus R & \overset{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\twoheadrightarrow} & K/R \oplus R \\ {\scriptstyle (1\ 0)}\downarrow & & \downarrow{\scriptstyle (1\ 0)} \\ D & \overset{d}{\twoheadrightarrow} & K/R \end{array}$$

Thus $\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right) \colon D \oplus R \to K/R \oplus R$ belongs to $QP\mathscr{C}$. By Proposition 3, the map $(d\ 0) = (1\ 0)\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right) \colon D \oplus R \to K/R$ is a split epimorphism. Hence d is a split epimorphism, and thus f factors through b.

Conversely, assume that every morphism $K/R \to C$ factors through b. By Proposition 1, this implies that $b \in P\mathscr{C}$. Let $p \colon C \twoheadrightarrow K/R$ be a split epimorphism. Then $pi = 1$ for some $i \colon K/R \to C$. Hence there exists an R-linear map $j \colon K/R \to B$ with $i = bj$. Thus $pbj = pi = 1$. By Proposition 3, it follows that $b \in QP\mathscr{C}$. Finally, the condition of the theorem is preserved under pullback. Hence $b \in PQP\mathscr{C}$. ∎

Remark. Proposition 3 and Theorem 1 show that none of the three operators in $PQP\mathscr{C}$ can be dropped. In other words, there are proper inclusions

$$PQP\mathscr{C} \subsetneq QP\mathscr{C} \subsetneq P\mathscr{C} \subsetneq \mathscr{C}.$$

For example, the cokernel $\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)\colon K \oplus R \twoheadrightarrow K/R \oplus R$ is not semistable, the semistable cokernel $\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)\colon K \oplus K \twoheadrightarrow K/R \oplus K/R$ does not belong to $QP\mathscr{C}$, and the cokernel $(c\ 0)\colon K \oplus R \twoheadrightarrow K/R$ in $QP\mathscr{C}$ is not a deflation.

Since $\mathbf{Mod}(R)$ is not self-dual, the proof of Theorem 1 cannot simply be dualized. Nevertheless, the dual of Theorem 1 is true.

Theorem 2. *Let $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ be a short exact sequence in $\mathscr{A}$. Then a is an inflation with respect to the maximal left exact structure of $\mathscr{A}$ if and only if every R-linear map $f\colon A \to R$ factors through a.*

Proof. Let $\mathscr{K}$ be the class of kernels in $\mathscr{A}$. Applying P, Q to $\mathscr{A}^{\mathrm{op}}$, we get operators P', Q' on $\mathscr{A}$. The dual of [15, Theorem 2] then implies that $P'Q'P'\mathscr{K}$ is the class of inflations. Assume that $a \in P'Q'P'\mathscr{K}$. For a given $f\colon A \to R$, the pushout

$$\begin{array}{ccccc}
A & \overset{a}{\rightarrowtail} & B & \overset{b}{\twoheadrightarrow} & C \\
\downarrow f & \text{PO} & \downarrow g & & \| \\
R & \overset{d}{\rightarrowtail} & D & \twoheadrightarrow & C \\
\downarrow \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) & & \downarrow \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) & & \| \\
R \oplus K/R & \overset{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\rightarrowtail} & D \oplus K/R & \twoheadrightarrow & C
\end{array}$$

of a along $\left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right)$ yields a kernel $\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right) \in Q'P'\mathscr{K}$. Consider the pushout

$$\begin{array}{ccc}
R \oplus K/R & \overset{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\rightarrowtail} & D \oplus K/R \\
\downarrow & & \downarrow \\
K \oplus K/R & \overset{\left(\begin{smallmatrix} e & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\rightarrowtail} & E \oplus K/R
\end{array}$$

By Proposition 1, the left-hand inclusion is not semistable. So there is a morphism $(p\ q)\colon D \oplus K/R \to R$ such that $r(p\ q)$ does not factor through $D \oplus K/R \hookrightarrow E \oplus K/R$. Hence $r(p\ q)\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is non-zero on R, that is, $rpd \neq 0$. Thus pd is an isomorphism, which implies that d is a split monomorphism. Hence f factors through a.

Conversely, assume that every morphism $f\colon A \to R$ factors through a. Then Proposition 1 gives $a \in P'\mathscr{K}$. Consider a pushout

$$\begin{array}{ccc} A & \overset{a}{\rightarrowtail} & B \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ A' & \overset{a'}{\rightarrowtail} & B' \end{array}$$

with $v \in P'\mathscr{K}$. To show that $u \in P'\mathscr{K}$, we apply Proposition 1. So we can assume that $\operatorname{Cok} u = \operatorname{Cok} v = K/R$. Let $f\colon A \to R$ be given. Then $f = ga$ for some $g\colon B \to R$. Hence $rg = hv$ for some $h\colon B' \to S$, which yields $rf = rga = hva = ha'u$. This proves that $a \in Q'P'\mathscr{K}$. Finally, let

$$\begin{array}{ccc} A & \overset{a}{\rightarrowtail} & B \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A' & \overset{a'}{\rightarrowtail} & B' \end{array}$$

be any pushout in $\mathscr{A}$. Then the pushout property implies that a' satisfies the condition of the theorem. Hence $a \in P'Q'P'\mathscr{K}$. ∎

Combining Theorems 1 and 2, we get the following explicit description of the maximal exact structure on $\mathscr{A}$.

Corollary 1. *A short exact sequence* $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ *in* $\mathscr{A}$ *is a conflation with respect to the maximal exact structure of* $\mathscr{A}$ *if and only if any* R*-linear map* $A \to R$ *factors through* a *and every* $K/R \to C$ *factors through* b.

Proof. This follows immediately by Theorems 1 and 2, and [15, Theorem 1]. ∎

Corollary 2. *Not every stable short exact sequence in* $\mathscr{A}$ *is a conflation with respect to the maximal exact structure.*

Proof. Consider a projective resolution

$$R^{(J)} \hookrightarrow R^{(I)} \twoheadrightarrow K \oplus K/R.$$

This short exact sequence is stable by Proposition 1. By Corollary 1, it fails to be a conflation. ∎

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