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The equivariant universality and couniversality of the Cantor cube

by

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Abstract. Let $\langle G, X, \alpha\rangle$ be a G-space, where G is a non-Archimedean (having a local base at the identity consisting of open subgroups) and second countable topological group, and X is a zero-dimensional compact metrizable space. Let $\langle H(\{0,1\}^{\aleph_0}), \{0,1\}^{\aleph_0}, \tau\rangle$ be the natural (evaluation) action of the full group of autohomeomorphisms of the Cantor cube. Then

(1) there exists a topological group embedding $\varphi : G \hookrightarrow H(\{0,1\}^{\aleph_0})$;

(2) there exists an embedding $\psi : X \hookrightarrow \{0,1\}^{\aleph_0}$, equivariant with respect to φ, such that $\psi(X)$ is an equivariant retract of $\{0,1\}^{\aleph_0}$ with respect to φ and ψ.

1. Introduction. The Cantor cube $\mathcal{C} = \{0,1\}^{\aleph_0}$ is a universal space in the class of zero-dimensional, separable, metrizable spaces, that is, every such space can be topologically embedded into $\mathcal{C}$. In particular, every *compact*, zero-dimensional, metrizable space is homeomorphic to a *closed* subset of $\mathcal{C}$. Sierpiński [15] showed that every non-empty closed subset of $\mathcal{C}$ is a retract of $\mathcal{C}$. This gives us the following well-known fact.

FACT 1.1. *Every non-empty, compact, zero-dimensional, metrizable space is homeomorphic to a retract of* $\mathcal{C}$.

Our Main Theorem 3.5 is an equivariant generalization of Fact 1.1 for *non-Archimedean* acting groups. A topological group is *non-Archimedean* if it has a local base at the identity consisting of open subgroups. The class of non-Archimedean groups includes:

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• the prodiscrete (in particular, the profinite) groups;

• the groups arising in *non-Archimedean functional analysis* [14] (for example, the additive groups of the fields of p-adic numbers);

• the group $\mathrm{Is}(X,d)$ of all isometries of an ultrametric space (X,d), with the topology of pointwise convergence;

• the locally compact, totally disconnected groups [3];

• the *symmetric group* S_∞ on a countably infinite set, with the topology of pointwise convergence;

• the full group $H(X)$ of autohomeomorphisms of X, with the compact-open topology, where X is a compact Hausdorff zero-dimensional space (see Lemma 3.2 below).

In fact, a topological group G is non-Archimedean iff G is a topological subgroup of $H(X)$ for some appropriate compact Hausdorff zero-dimensional space X. This complete characterization of the non-Archimedean groups is a part of Theorem 3.3 below. It is easy to show that the class of all non-Archimedean groups is a *variety* in the sense of [12]. That is, this class is closed under the formation of topological subgroups, products and quotient groups.

Note that the transformation groups having zero-dimensional (in particular, ultrametric) phase spaces have many applications in descriptive set theory [1, 6, 7].

2. Preliminaries and conventions. All topological spaces in this paper are assumed to be Hausdorff. The neutral element of a group G is denoted by e_G. The *weight* $w(X)$ of a topological space X is defined to be $\tau(X)\cdot\aleph_0$, where $\tau(X)$ denotes the minimal cardinality of a base for X.

For information on *uniform spaces*, we refer the reader to [4]. If μ is a uniformity for X, then the collection of elements of μ which are *finite* coverings of X forms a base for a uniformity for X, which we denote by μ_{fin}. If (X,μ) is a uniform space, the uniform completion $(\widehat{X},\widehat{\mu}_{\mathrm{fin}})$ of (X,μ_{fin}) is a compact uniform space known as the *Samuel compactification* of (X,μ). A *partition* of a set X is a covering of X consisting of pairwise disjoint subsets of X. Following [14], we say that a uniform space (X,μ) is *non-Archimedean* if it has a base consisting of partitions of X. Equivalently, μ is generated by a system $\{d_i\}$ of *ultrapseudometrics*, that is, pseudometrics, each of which satisfies the *strong triangle inequality* $d_i(x,z)\le\max\{d_i(x,y),d_i(y,z)\}$. Clearly, a non-Archimedean uniform space is zero-dimensional in the uniform topology. A topological group is non-Archimedean iff its right uniformity is non-Archimedean.

The following result is well known (see, for example, [4] and [5]).

LEMMA 2.1. *Let (X,μ) be a non-Archimedean uniform space. Then both (X,μ_{fin}) and the uniform completion $(\widehat{X},\widehat{\mu})$ of (X,μ) are non-Archimedean uniform spaces.*

A *topological transformation group*, or *G-space*, is a triple $\langle G,X,\alpha\rangle$, where G is a topological group (called the *acting group*), X is a topological space (called the *phase space*), and $\alpha : G\times X\to X$ is a continuous action. For each $g\in G$, the *g-transition map* is the function $\alpha^g : X\to X$, $\alpha^g(x)=gx$.

DEFINITION 2.2. Let $\langle G_1,X_1,\alpha_1\rangle$ be a G_1-space, and let $\langle G_2,X_2,\alpha_2\rangle$ be a G_2-space. Suppose that $\varphi : G_1\hookrightarrow G_2$ is a topological group embedding.

(1) A continuous function $\psi : X_1\to X_2$ is *equivariant with respect to* φ (or, simply, *equivariant*, if φ is clear from the context) if, for all $g\in G_1$ and $x\in X_1$, $\psi(gx)=\varphi(g)\psi(x)$.

(2) Let $\psi : X_1\to X_2$ be an equivariant embedding with respect to φ. We say that $\psi(X_1)$ is an *equivariant retract* of X_2 (with respect to φ and ψ) if there is a continuous retraction $r : X_2\to\psi(X_1)$ which is equivariant with respect to $\varphi^{-1}:\varphi(G_1)\to G_1$.

Let $\langle G,X,\alpha\rangle$ be a G-space. If $\langle G,Y,\gamma\rangle$ is a compact Hausdorff G-space and $\psi : X\to Y$ is equivariant, then Y is called a *G-compactification* of X. If, in addition, ψ is a topological embedding, then Y is a *proper G-compactification* of X. A G-space $\langle G,X,\alpha\rangle$ is *G-Tikhonov* if it has a proper G-compactification. Not every Tikhonov G-space is G-Tikhonov [8]. De Vries [19] proved that if G is locally compact, then *every* Tikhonov G-space is G-Tikhonov. For every G-space X there exists a (possibly improper) *maximal* G-compactification $\beta_G X$ (see [18]). For more information on G-compactifications, as well as for a general method of constructing Tikhonov G-spaces which are *not* G-Tikhonov, see [11].

Let G be a topological group. Recall [2] that the collection of coverings $\{Ux : x\in G\}$, where U is a neighborhood of e_G, forms a base for the *right uniformity* μ_{R} for G. In 1957, Teleman [16] proved that for arbitrary Hausdorff G, the Samuel compactification $\widehat{G}$ of G with respect to its right uniformity is a proper G-compactification of the G-space $\langle G,G,\alpha_{\mathrm{L}}\rangle$, where α_{L} is the usual *left action* of G on itself. In fact, $\widehat{G}$ is isomorphic to $\beta_G G$ and is called the *greatest ambit* (see, for example, [20]). $\beta_G G$ is the maximal *proper* G-compactification of $\langle G,G,\alpha_{\mathrm{L}}\rangle$.

To the best of our knowledge, very little is known about the dimension of $\beta_G X$. Some special results can be found in [8, 9]. The dimension of the greatest ambit $\beta_G G$ may be greater than the topological dimension of G (simply take a cyclic dense subgroup G of the circle group; then $\dim G=0$ and $\dim\beta_G G=1$). However, in the case of the Euclidean group $G=\mathbb{R}^n$, we have $\dim \beta_G G = \dim G$. This follows from Theorem 5.12 of [4]. By a result of Pestov [13], one has $\dim \beta_G G = 0$ iff G is non-Archimedean. An alternative proof of this will be given in Theorem 3.3 below.

3. Proof of the main results

FACT 3.1 ([9]). *Every G-Tikhonov G-space X has a proper G-compactification Y such that $w(Y) \leq w(X) \cdot w(G)$ and $\dim Y \leq \dim \beta_G X$.*

LEMMA 3.2. *If X is a compact Hausdorff zero-dimensional space, then $H(X)$ is a non-Archimedean group.*

Proof. For each two-element compact clopen partition $\{K_1, K_2\}$ of X, define

$$B(K_1, K_2) = \{\varphi \in H(X) : \varphi(K_1) = K_1,\ \varphi(K_2) = K_2\}.$$

Let $\mathcal{B} = \{B(K_1, K_2) : \{K_1, K_2\}$ is a compact clopen partition of $X\}$. Then $\mathcal{B}$ is a local base at $e_{H(X)}$ consisting of clopen subgroups, and, hence, $H(X)$ is non-Archimedean. ■

The following theorem provides a useful characterization of non-Archimedean groups. (As noted before, the equivalence of (i) and (ii) was established by Pestov [13].)

THEOREM 3.3. *The following assertions are equivalent*:

(i) *G is a non-Archimedean topological group*;

(ii) $\dim \beta_G G = 0$;

(iii) *G is a topological subgroup of $H(X)$ for some compact Hausdorff zero-dimensional space X such that $w(X) = w(G)$.*

Proof. (i)⇒(ii). Suppose G is non-Archimedean. Then the right uniformity μ_{R} for G is a non-Archimedean uniformity. By Lemma 2.1, the precompact uniformity $(\mu_{\mathrm{R}})_{\mathrm{fin}}$ for G is also a non-Archimedean uniformity. Let $(\widehat{G}, \widehat{\mu})$ be the uniform completion of $(G, (\mu_{\mathrm{R}})_{\mathrm{fin}})$. Then, again by Lemma 2.1, $\widehat{\mu}$ is a non-Archimedean uniformity, and, hence, $\widehat{G}$ is zero-dimensional. But $(\widehat{G}, \widehat{\mu})$ is exactly $\beta_G G$.

(ii)⇒(iii). By Fact 3.1, there exists a zero-dimensional proper G-compactification $\langle G, X, \alpha_{\mathrm{L}}^* \rangle$ of $\langle G, G, \alpha_{\mathrm{L}} \rangle$ such that $w(X) = w(G)$. Let $\psi : G \to X$ be the corresponding equivariant embedding.

We will show that the map $\varphi : G \to H(X)$ defined by $\varphi(g) = (\alpha_{\mathrm{L}}^*)^g$ is a topological group embedding. Observe that φ is one-to-one because α_{L}^* extends the action α_{L}. To prove the continuity of φ, suppose $\alpha^g \in O = \{f \in H(X) : f(K) \subseteq U\}$, where $K \subseteq X$ is compact and $U \subseteq X$ is open. Using the compactness of K and the continuity of α_{L}^*, we can find a neighborhood V of g such that $\varphi(V) \subseteq O$. Hence, φ is continuous.

It remains to show that if $O \subseteq G$ is open, then $\varphi(O)$ is open in $\varphi(G)$. Let $O \subseteq G$ be open. Then $\psi(O)$ is open in $\psi(G)$. Let $W \subseteq X$ be open such that $\psi(O) = W \cap \psi(G)$. Define $B = \{f \in H(X) : f(\psi(e_G)) \in W\}$. Then B is open in $H(X)$ and $\varphi(O) = B \cap \varphi(G)$. Hence, $\varphi(O)$ is open in $\varphi(G)$.

(iii)$\Rightarrow$(i) follows directly by Lemma 3.2. ■

FACT 3.4 (Brouwer). *The Cantor cube* $\{0,1\}^{\aleph_0}$ *is the unique* (*up to homeomorphism*) *non-empty*, *compact*, *metrizable*, *zero-dimensional*, *perfect space.*

Now we are ready to prove our main result.

THEOREM 3.5. *Let G be a non-Archimedean and second countable group*, *and let X be a compact*, *metrizable*, *zero-dimensional G-space. Then*

(1) *there exists a topological group embedding* $\varphi : G \hookrightarrow H(\mathcal{C})$;

(2) *there exists an embedding* $\psi : X \hookrightarrow \mathcal{C}$, *equivariant with respect to* φ, *such that* $\psi(X)$ *is an equivariant retract of* $\mathcal{C}$ *with respect to* φ *and* ψ.

Proof. By Theorem 3.3, there exists a compact, second countable (and thus metrizable) zero-dimensional space Y such that $H(Y)$ contains G as a topological subgroup. We may as well assume that all homeomorphisms of Y corresponding to elements of G transform a certain base point $y_0 \in Y$ onto itself (if not, replace Y with a disjoint union $Y \cup \{y_0\}$ and redefine those homeomorphisms in an obvious way).

Let us identify the action of G on X with a homomorphism $w : G \to H(X)$, and let $\mathcal{D}$ be a copy of the Cantor set. By Brouwer's theorem, the space $\mathcal{C} = X \times Y \times \mathcal{D}$ is homeomorphic to the Cantor set, and, clearly, the map $\varphi : G \to H(\mathcal{C})$,

$$g \mapsto (w(g), g, \mathrm{id}_D) \in H(X) \times H(Y) \times H(\mathcal{D}) \subseteq H(\mathcal{C}),$$

is a continuous homomorphism, thus turning $\mathcal{C}$ into a G-space. This homomorphism is also an embedding, for its composition with the projection onto $H(Y)$ is the identity mapping, so it is one-to-one and the inverse is continuous.

We define $\psi : X \to \mathcal{C}$ by $x \mapsto (x, y_0, d_0)$, where $d_0 \in \mathcal{D}$ is any base point, and the retraction $r : \mathcal{C} \to \psi(X)$ by $r(x, y, d) = (x, y_0, d_0)$. Then ψ and r are equivariant, and the proof is complete. ■

THEOREM 3.6. $H(\mathcal{C})$ *is universal in the class of all non-Archimedean*, *second countable groups*, *that is*, *every such group is topologically isomorphic to a subgroup of* $H(\mathcal{C})$.

FINAL REMARKS. (1) By Theorem 1.5.1 of [1], the group S_∞ is also universal in this class.

(2) The group $H(I^{\aleph_0})$ is universal in the class of all second countable topological groups, where I is the closed interval $[0, 1]$ (see [17]). Moreover,

by [10], the topological transformation group $\langle H(I^{\aleph_0}), I^{\aleph_0}\rangle$ is universal in the class of all compact, metrizable G-spaces with second countable acting group G.

(3) The action on $\mathcal{C}$ which we defined in the proof of Theorem 3.5 intrinsically depends on the original action of G on X, as the following example shows.

EXAMPLE 3.7. Let $\alpha : S_\infty \times \mathcal{C} \to \mathcal{C}$ be the natural "permutation of coordinates" action

$$\alpha(g,(x_n)) = (x_{g(n)}).$$

Let $\overline{0}$ and $\overline{1}$ denote the two constant sequences of $\mathcal{C}$. Let $H = \{\overline{0}, \overline{1}\} \subseteq \mathcal{C}$. Consider H as an S_∞-subspace of $\mathcal{C}$.

CLAIM. *H is* not *an equivariant retract of $\mathcal{C}$ with respect to $\varphi = \mathrm{id}_{S_\infty}$ and $\psi = \mathrm{id}_H$.*

Proof. The Cantor cube $\mathcal{C}$ is an S_∞-*ambit* under the action α, that is, it contains a point whose orbit is dense in $\mathcal{C}$. In fact, all points which contain infinitely many 0's and infinitely many 1's have dense orbits. Hence, every image of $\mathcal{C}$ under an equivariant map is also an S_∞-ambit. However, H is *not* an S_∞-ambit. ∎

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