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On disjointness properties of some smooth flows

by

Krzysztof Frączek and **Mariusz Lemańczyk** (Toruń)

Abstract. Special flows over some locally rigid automorphisms and under L^2 ceiling functions satisfying a local L^2 Denjoy–Koksma type inequality are considered. Such flows are proved to be disjoint (in the sense of Furstenberg) from mixing flows and (under some stronger assumption) from weakly mixing flows for which the weak closure of the set of all instances consists of indecomposable Markov operators. As applications we prove that

- special flows built over ergodic interval exchange transformations and under functions of bounded variation are disjoint from mixing flows;
- ergodic components of flows coming from billiards on rational polygons are disjoint from mixing flows;
- smooth ergodic flows of compact orientable smooth surfaces having only non-degenerate saddles as isolated critical points (and having a "good" transversal) are disjoint from mixing and from Gaussian flows.

1. Introduction. One of the most important and still open problems in ergodic theory is whether each dynamics has a smooth model. To be more precise, we would like to know whether for any given ergodic flow $\{T_t\}_{t\in\mathbb{R}}$ one can find a smooth compact manifold M together with a smooth measure m on it and a smooth m-preserving flow $\{S_t\}_{t\in\mathbb{R}}$ so that the two flows are measure-theoretically isomorphic. Closely related to the smooth realization problem is A. Katok's program (see [8, Part III]) to describe (up to measure-theoretic isomorphism) all possible smooth dynamics on a given smooth compact manifold. Opening by the famous article by Anosov and Katok ([1]), there is a series of results toward realization of this program (see e.g. [8] and the references therein). An interesting related problem asked in [4] is to find a smooth realization of Gaussian flows.

In the present paper we will focus on the problem of smooth realization of Gaussian flows on smooth compact surfaces (i.e. on smooth compact manifolds of dimension 2). Suppose that M is a C^∞ compact orientable surface

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with negative Euler characteristic and m is a positive C^∞-measure on M. Consider the family of all m-invariant ergodic C^∞-flows on M. Typically such a flow has only non-degenerate saddles as isolated critical points and has a transversal. It follows that the flow is measure-theoretically isomorphic to a special flow built over an ergodic interval exchange transformation and under a function of the form

$$f(x) = g(x) - \sum_{i=1}^{k}(b_i \log(\{x-\beta_i\}) + c_i \log(\{\beta_i - x\})),$$

where $g : \mathbb{T} \to \mathbb{R}$ is a function of bounded variation, and b_i, c_i, $i = 1, \ldots, k$, are non-negative constants such that $\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} c_i > 0$ (see [11]). The family of all such functions will be denoted by LOGSYM$_+$. In this paper we deal with the special case where special flows are built over rotations (exchanges of two intervals) or exchanges of three intervals. We prove that for a typical such interval exchange transformation the dynamics of special flows built under a LOGSYM$_+$-function is totally different from those of "probability origin" (like Gaussian flows, Poisson suspension flows and dynamical systems coming from stationary symmetric α-stable processes). On the base of this result we conjecture that the above-mentioned flows have no smooth realization on smooth compact surfaces. Our approach is in fact via joinings. Historically, joinings were introduced by H. Furstenberg in his paper [7] on disjointness. Recall that two systems $\{T_t\}_{t\in\mathbb{R}}$ on $(X, \mathcal{B}, \mu)$ and $\{S_t\}_{t\in\mathbb{R}}$ on $(Y, \mathcal{C}, \nu)$ are *disjoint* if the only $\{T_t \times S_t\}_{t\in\mathbb{R}}$-invariant measure on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ which has marginal μ on X and ν on Y is the product measure. If a dynamical system $\{T_t\}_{t\in\mathbb{R}}$ is disjoint from some "known" system $\{S_t\}_{t\in\mathbb{R}}$ then the information we gain about $\{T_t\}_{t\in\mathbb{R}}$ is that its dynamics is completely different from that of $\{S_t\}_{t\in\mathbb{R}}$. In particular, the two systems have no common factors.

One of the features distinguishing dynamics of "probability origin" is the ELF property introduced in [5]. A flow $\{S_t\}_{t\in\mathbb{R}}$ has the ELF property if the weak closure of $\{S_t : t \in \mathbb{R}\}$ in the set of Markov operators on the underlying L^2-space consists of indecomposable Markov operators. The ELF property is satisfied by mixing flows, Gaussian flows ([12]), Poisson suspension flows and dynamical systems coming from stationary symmetric α-stable processes ([3]).

An approach which allows us to prove disjointness from ELF flows for special flows built over rigid automorphisms and under roof functions which satisfy a Denjoy–Koksma type inequality was developed in [5]. The main idea of this approach is the following. Suppose that $\{(T^f)_t\}_{t\in\mathbb{R}}$ is the special flow built from T and f. We then look at instances $(T^f)_t$, $t \in \mathbb{R}$, of $\{(T^f)_t\}_{t\in\mathbb{R}}$ as Markov operators on the underlying L^2-space and study the weak closure of the set of such operators. More precisely, it is proved that the integral operator $\int_{\mathbb{R}}(T^f)_{-t}\,dP(t)$ belongs to this weak closure, where P is a certain probability Borel measure which is determined by the roof function f. Finally, if P is not a Dirac measure, then T^f is disjoint from weakly mixing ELF flows.

In this paper we extend the above approach to special flows built over automorphisms having only some local rigidity property. Let (X,d) be a compact metric space and let $\mathcal{B}$ stand for the σ-algebra of Borel sets. Assume that $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is an ergodic automorphism, where μ is a probability measure. Assume that $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of towers for T such that $\mu(C_n)\to\alpha>0$, where C_n stands for the union of levels of ξ_n, and some iteration T^{q_n} of T transforms the bottom of ξ_n onto the set which is close (in the sense of d) to itself. Moreover, suppose that $f\in L^2(X,\mathcal{B},\mu)$ is a positive function and $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers such that the sequence $\{\int_{C_n}|f^{(q_n)}(x)-a_n|^2\,d\mu(x)\}_{n\in\mathbb{N}}$ is bounded. One of the main results of the present paper (see Theorem 6) states that under the above assumptions,

$$(T^f)_{a_n}\to\alpha\int_{\mathbb{R}}(T^f)_{-t}\,dP(t)+(1-\alpha)J, \tag{1}$$

where P is the weak limit of the sequence of distributions of $\{f^{(q_n)}-a_n:(C_n,\mu(\cdot\,|C_n))\to\mathbb{R}\}_{n\in\mathbb{N}}$ and J is a Markov operator. We show that once (1) is satisfied then T^f is disjoint from all mixing flows. Whenever in (1) P is not a Dirac measure then T^f is disjoint from weakly mixing ELF flows.

We then show that (1) holds for two classes of systems: special flows built over ergodic interval exchange transformations and under functions of bounded variation (see Section 5), and special flows built over some irrational rotations on the circle and under LOGSYM_+-functions. In the latter case P is not Dirac.

As an application we deduce that special flows built over ergodic interval exchange transformations and under functions of bounded variation, and consequently ergodic components of billiard flows on rational polygons, are disjoint from all mixing flows. This strengthens Katok's classical result saying that such flows are not mixing ([9]).

Furthermore, we prove that every special flow constructed over a circle rotation which admits a sufficiently fast approximation by rationals and under roof function belonging to LOGSYM_+ is disjoint from weakly mixing ELF flows. This essentially strengthens Kochergin's classical result saying that such special flows are not mixing ([11]).

2. Joinings and ELF property. Assume that $\mathcal{S}=\{S_t\}_{t\in\mathbb{R}}$ is a flow on $(X,\mathcal{B},\mu)$. Such a flow $\mathcal{S}=\{S_t\}_{t\in\mathbb{R}}$ determines a unitary action, still denoted by $\mathcal{S}$, of $\mathbb{R}$ on $L^2(X,\mathcal{B},\mu)$ by the formula

$$f \mapsto f \circ S_t.$$

By a flow we will always mean a *measurable flow*, i.e. we require that the above representation is continuous: the map $\mathbb{R} \ni t \mapsto \langle f \circ S_t, g\rangle \in \mathbb{C}$ is continuous for all $f, g \in L^2(X,\mathcal{B},\mu)$. Assume moreover that $\mathcal{S}$ is ergodic and let $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ be another ergodic flow defined on $(Y,\mathcal{C},\nu)$. By a *joining* between $\mathcal{S}$ and $\mathcal{T}$ we mean any $\{S_t \times T_t\}_{t\in\mathbb{R}}$-invariant probability measure on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ whose projections on X and Y are equal to μ and ν respectively. The set of joinings between $\mathcal{S}$ and $\mathcal{T}$ is denoted by $J(\mathcal{S},\mathcal{T})$. The subset of ergodic joinings is denoted by $J^{\mathrm{e}}(\mathcal{S},\mathcal{T})$ and we write $J(\mathcal{S})$ and $J^{\mathrm{e}}(\mathcal{S})$ instead of $J(\mathcal{S},\mathcal{S})$ and $J^{\mathrm{e}}(\mathcal{S},\mathcal{S})$ respectively. Ergodic joinings are exactly extremal points in the simplex $J(\mathcal{S},\mathcal{T})$. Given $\varrho \in J(\mathcal{S},\mathcal{T})$ define an operator $\Phi_\varrho : L^2(X,\mathcal{B},\mu) \to L^2(Y,\mathcal{C},\nu)$ by requiring that

$$\int_{X\times Y} f(x)g(y)\,d\varrho(x,y) = \int_Y \Phi_\varrho(f)(y)g(y)\,d\nu(y)$$

for each $f \in L^2(X,\mathcal{B},\mu)$ and $g \in L^2(Y,\mathcal{C},\nu)$. This operator has the following Markov property:

$$\Phi_\varrho 1 = \Phi_\varrho^* 1 = 1 \quad \text{and} \quad \Phi_\varrho f \geq 0 \quad \text{whenever } f \geq 0. \tag{2}$$

Moreover,

$$\Phi_\varrho \circ S_t = T_t \circ \Phi_\varrho \qquad \text{for each } t \in \mathbb{R}. \tag{3}$$

In fact, there is a one-to-one correspondence between the set of Markov operators $\Phi : L^2(X,\mathcal{B},\mu) \to L^2(Y,\mathcal{C},\nu)$ satisfying (3) and the set $J(\mathcal{S},\mathcal{T})$, where the joining ϱ given by Φ is determined by the formula

$$\varrho(A \times B) = \int_B \Phi(\chi_A)\,d\nu$$

for each $A \in \mathcal{B}$ and $B \in \mathcal{C}$ (see e.g. [15]). Markov operators corresponding to ergodic joinings will be called *indecomposable*. Notice that the product measure corresponds to the Markov operator denoted by $\int$, where $\int(f)$ equals the constant function $\int_X f\,d\mu$. On $J(\mathcal{S})$ we consider the weak operator topology. In this topology $J(\mathcal{S})$ becomes a metrizable compact semitopological semigroup in which $\varrho_n \to \varrho$ iff $\langle \Phi_{\varrho_n} f, g\rangle \to \langle \Phi_\varrho f, g\rangle$ for all $f, g \in L^2(X,\mathcal{B},\mu)$. For each $t \in \mathbb{R}$, S_t can be considered as a Markov operator on $L^2(X,\mathcal{B},\mu)$. The corresponding self-joining is denoted by μ_{S_t} and it is exactly the joining concentrated on the graph of S_t.

Following [7], $\mathcal{S}$ and $\mathcal{T}$ are called *disjoint* if $J(\mathcal{S},\mathcal{T}) = \{\mu \otimes \nu\}$. Equivalently, the operator $\int$ is the only Markov operator that intertwines S_t and T_t (for each $t \in \mathbb{R}$).

An ergodic flow $\mathcal{S} = \{S_t : t \in \mathbb{R}\}$ on a standard probability space $(X, \mathcal{B}, \mu)$ is said to have the *ELF property* if $\overline{\mathcal{S}} := \overline{\{S_t : t \in \mathbb{R}\}} \subset J^{\mathrm{e}}(\mathcal{S})$ (see [5]).

If $\mathcal{S}$ is mixing then $\overline{\mathcal{S}} = \{S_t\}_{t\in\mathbb{R}} \cup \{\int\}$, so it is an ELF flow. It is also easy to see that all ergodic flows with discrete spectrum have the ELF property. It was shown in [12] that Gaussian flows have the ELF property (see also [5] for a direct proof). Moreover, Poisson suspension flows and dynamical systems coming from stationary symmetric α-stable processes also enjoy the ELF property (see [3]).

Suppose that $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ is an ergodic flow on $(X, \mathcal{B}, \mu)$. Given a probability Borel measure P on $\mathbb{R}$ define the integral Markov operator $\int_{\mathbb{R}} T_s\, dP(s)$ on $L^2(X, \mathcal{B}, \mu)$ by

$$\Big\langle \Big(\int_{\mathbb{R}} T_s\, dP(s) \Big) f, g \Big\rangle = \int_{\mathbb{R}} \langle T_s f, g\rangle\, dP(s)$$

for all $f, g \in L^2(X, \mathcal{B}, \mu)$. Analysis similar to that in Proposition 3.2 of [5] gives the following result.

Proposition 1. *Suppose that $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ is an ergodic flow on $(Y, \mathcal{C}, \nu)$ for which there exist a sequence $\{t_n\} \subset \mathbb{R}$ and $0 < \alpha \le 1$ such that*

$$T_{t_n} \to \alpha \int_{\mathbb{R}} T_s\, dP(s) + (1-\alpha)J,$$

where P is a probability Borel measure on $\mathbb{R}$ and $J \in J(\mathcal{T})$. Then

(i) *$\mathcal{T}$ is disjoint from all mixing flows*;
(ii) *$\mathcal{T}$ is disjoint from all weakly mixing ELF flows whenever P is not a Dirac measure.*

The following two lemmas will be convenient in applications of Proposition 1.

Lemma 2. *Suppose that $T : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ is a bounded linear operator such that $\langle T\chi_A, \chi_B\rangle \ge 0$ for all $A, B \in \mathcal{B}$.*

(i) *If $T1 = T^*1 = 1$, then T is a Markov operator.*
(ii) *If $T1 = 0$, then $T \equiv 0$.*

Proof. (i) Since $\langle \chi_A, T^*\chi_B\rangle \ge 0$ for all $A, B \in \mathcal{B}$, we have $\langle f, T^*\chi_B\rangle \ge 0$ for each non-negative $f \in L^2(X, \mathcal{B}, \mu)$ and $B \in \mathcal{B}$. Hence $\langle Tf, \chi_B\rangle \ge 0$ for each $B \in \mathcal{B}$ and thus $Tf \ge 0$ for any $f \ge 0$.

(ii) By assumption, $T\chi_A \ge 0$ for every $A \in \mathcal{B}$. Consequently,

$$T\chi_A = T(1 - \chi_{A^{\mathrm{c}}}) = -T\chi_{A^{\mathrm{c}}} \le 0$$

and therefore $T\chi_A = 0$ for every $A \in \mathcal{B}$, which implies $T \equiv 0$. ∎

Lemma 3. *Let $\{T_n\}, J$ be Markov operators on $L^2(X,\mathcal{B},\mu)$. Suppose that the sequence $\{T_n\}$ converges in the weak operator topology and there exists $0<\alpha\le 1$ such that*

$$\lim_{n\to\infty}\langle T_n\chi_A,\chi_B\rangle \ge \alpha\langle J\chi_A,\chi_B\rangle \quad \textit{for all } A,B\in\mathcal{B}.$$

Then

$$T_n\to \alpha J+(1-\alpha)J' \quad \textit{weakly},$$

where J' is also a Markov operator.

Proof. Denote by $T: L^2(X,\mathcal{B},\mu)\to L^2(X,\mathcal{B},\mu)$ the weak limit of the sequence $\{T_n\}$. By assumption, we have

$$\langle T\chi_A,\chi_B\rangle \ge \alpha\langle J\chi_A,\chi_B\rangle \quad \text{for all } A,B\in\mathcal{B}. \tag{4}$$

Case 1: $\alpha<1$. Let $J': L^2(X,\mathcal{B},\mu)\to L^2(X,\mathcal{B},\mu)$ be defined by $J'=\frac{1}{1-\alpha}(T-\alpha J)$. From (4), we have $\langle J'\chi_A,\chi_B\rangle\ge 0$ for all $A,B\in\mathcal{B}$. Since $J'1=J'^*1=1$, J' is a Markov operator as well, by Lemma 2. Consequently, $T_n\to\alpha J+(1-\alpha)J'$.

Case 2: $\alpha=1$. From (4), we have $\langle (T-J)\chi_A,\chi_B\rangle\ge 0$ for all $A,B\in\mathcal{B}$. Since $(T-J)1=0$, $T\equiv J$, by Lemma 2. Consequently, $T_n\to J$. ∎

3. Special flows. Let T be an ergodic automorphism of a standard probability space $(X,\mathcal{B},\mu)$. Denote by λ Lebesgue measure on $\mathbb{R}$. Assume that $f\in L^1(X,\mathcal{B},\mu)$ is a positive function. The *special flow* $T^f=\{(T^f)_t\}_{t\in\mathbb{R}}$ built from T and f is defined on the space $X^f=\{(x,t)\in X\times\mathbb{R}: 0\le t<f(x)\}$ (considered with $\mathcal{B}^f$, the restriction of the product σ-algebra, and μ^f, the restriction of the product measure $\mu\otimes\lambda$ to $X\times\mathbb{R}$). Under the action of the special flow each point in X^f moves vertically at unit speed, and we identify the point $(x,f(x))$ with $(Tx,0)$ (see e.g. [2, Chapter 11]). Given $m\in\mathbb{Z}$ we put

$$f^{(m)}(x)=\begin{cases} f(x)+f(Tx)+\cdots+f(T^{m-1}x) & \text{if } m>0,\\ 0 & \text{if } m=0,\\ -(f(T^mx)+\cdots+f(T^{-1}x)) & \text{if } m<0.\end{cases}$$

The action of T^f can be well understood when we consider the following actions on the space $(X\times\mathbb{R},\mu\otimes\lambda)$. First, let $S_{-f}:(X\times\mathbb{R},\mu\otimes\lambda)\to(X\times\mathbb{R},\mu\otimes\lambda)$ denote the skew product given by

$$S_{-f}(x,r)=(Tx,r-f(x)).$$

Notice that $(S_{-f})^k(x,r)=(T^kx,r-f^{(k)}(x))$ for each $k\in\mathbb{Z}$. Consider the quotient space $\Gamma^f=X\times\mathbb{R}/\sim$, where the relation $\sim$ is defined by $(x,r)\sim(x',r')$ iff $(x,r)=(S_{-f})^k(x',r')$ for an integer k. Since $f^{(k)}(x)\to+\infty$ μ-a.e., with no loss of generality we can assume that the set

$$\{(x,r)\in X\times\mathbb{R}: 0\le r<f(x)\}$$

intersects each equivalence class of $\sim$ in exactly one point (and hence can be identified with Γ^f). Let $\sigma = \{\sigma_t\}_{t\in\mathbb{R}}$ stand for the flow on $(X \times \mathbb{R}, \mu \otimes \lambda)$ given by

$$\sigma_t(x, r) = (x, r + t).$$

Notice that σ_t commutes with S_{-f}. Then the special flow T^f can be seen as the quotient flow of the action σ by the relation $\sim$. It follows that given $(x, r) \in X^f$ and $t \in \mathbb{R}$ there exists a unique $k \in \mathbb{Z}$ such that

$$(T^f)_t(x, r) = (S_{-f})^k \circ \sigma_t(x, r).$$

REMARK 1. For all measurable $A, B \subset X^f$ we have

$$\mu^f((T^f)_t A \cap B) = \sum_{k\in\mathbb{Z}} \mu \otimes \lambda((S_{-f})^k \sigma_t A \cap B).$$

REMARK 2. Suppose that $A, B \subset X \times \mathbb{R}$ are measurable rectangles of the form $A = A_1 \times A_2$, $B = B_1 \times B_2$. Then

$$\mu \otimes \lambda((S_{-f})^k A \cap B) = \int\limits_{T^k A_1 \cap B_1} \lambda((A_2 + f^{(-k)}(x)) \cap B_2)\, d\mu(x).$$

The proofs of the remarks are straightforward and can be found in [5].

Given $A \in \mathcal{B}$ of positive measure consider the induced automorphism $T_A : A \to A$ and the first return time map $\tau_A : A \to \mathbb{N}$ to A. Given a positive integrable function $f : X \to \mathbb{R}$ let $f_A : A \to \mathbb{R}$ be defined by $f_A(x) = f^{(\tau_A(x))}(x)$. Then the special flows T^f and $(T_A)^{f_A}$ are metrically isomorphic.

4. Special flows over automorphisms having a local rigidity property. Let (X, d) be a compact metric space. Let $\mathcal{B}$ stand for the σ-algebra of all Borel sets and let μ be a probability Borel measure on X. Suppose that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic measure-preserving automorphism and there exist an increasing sequence $\{q_n\}$ of natural numbers and a sequence $\{C_n\}$ of Borel sets such that

$$\mu(C_n) \to \alpha > 0, \quad \mu(C_n \mathbin{\triangle} T^{-1}C_n) \to 0, \quad \sup_{x\in C_n} d(x, T^{q_n}x) \to 0.$$

Let $f \in L^2(X, \mu)$ be a positive Borel function. Suppose that there exists a sequence $\{a_n\}$ of real numbers such that the sequence $\{\int_{C_n} |f_n(x)|^2\, d\mu(x)\}$ is bounded, where $f_n := f^{(q_n)} - a_n$ for $n \in \mathbb{N}$. As the distributions

$$\left\{\frac{1}{\mu(C_n)}\,(f_n|_{C_n})_*(\mu|_{C_n}) : n \in \mathbb{N}\right\}$$

are uniformly tight, by passing to a further subsequence if necessary we can assume that

$$\frac{1}{\mu(C_n)}\,(f_n|_{C_n})_*(\mu|_{C_n}) \to P$$

weakly in $\mathcal{P}(\mathbb{R})$, the set of probability Borel measures on $\mathbb{R}$. Let $C(\overline{\mathbb{R}})$ denote the set of all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to-\infty} \varphi(x) = \lim_{x\to+\infty} \varphi(x)$. Then

$$\int_{C_n} \varphi(f_n(x))\, d\mu(x) \to \alpha \int_{\mathbb{R}} \varphi(t)\, dP(t) \tag{5}$$

for every $\varphi \in C(\overline{\mathbb{R}})$.

LEMMA 4. *For every $\varphi \in C(\overline{\mathbb{R}})$, $g \in L^1(X, \mathcal{B}, \mu)$ and any measurable function $h : X \to \mathbb{R}$ we have*

$$\int_{C_n} \varphi(f_n(x) + h(x))g(x)\, d\mu(x) \to \alpha \int_X \int_{\mathbb{R}} \varphi(t + h(x))g(x)\, dP(t)\, d\mu(x).$$

Proof. We first prove our claim in the case where $h \equiv 0$, i.e.

$$\int_{C_n} (\varphi \circ f_n) \cdot g\, d\mu \to \alpha \int_{\mathbb{R}} \varphi\, dP \int_X g\, d\mu \tag{6}$$

whenever $\varphi \in C(\overline{\mathbb{R}})$ and $g \in L^1(X, \mathcal{B}, \mu)$. It follows from (5) that (6) holds for constant functions g and thus it is enough to prove that the limit is 0 when $\int_X g\, d\mu = 0$. Since the coboundaries are dense in the subspace of functions with zero mean, we can restrict ourselves to the case $g = \xi - \xi \circ T$, $\xi \in L^1(X, \mathcal{B}, \mu)$. Then

$$\begin{aligned}\Big| \int_{C_n} (\varphi \circ f_n) \cdot g\, d\mu\Big| &= \Big| \int_{C_n} (\varphi \circ f_n) \cdot \xi\, d\mu - \int_{C_n} (\varphi \circ f_n) \cdot (\xi \circ T)\, d\mu\Big| \\ &= \Big| \int_{T^{-1}C_n} \varphi(f_n(Tx))\xi(Tx)\, d\mu(x) - \int_{C_n} \varphi(f_n(x))\xi(Tx)\, d\mu(x)\Big| \\ &\le \Big| \int_{C_n} (\varphi(f_n(Tx)) - \varphi(f_n(x)))\xi(Tx)\, d\mu(x)| \\ &\quad + \int_{T^{-1}C_n \triangle C_n} \|\varphi\|_\infty |\xi(Tx)|\, d\mu(x).\end{aligned}$$

We will now prove that

$$\chi_{C_n} \cdot (f \circ T^{q_n} - f) = \chi_{C_n} \cdot (f_n \circ T - f_n)$$

converges to 0 in measure, i.e. for every $a > 0$,

$$\mu(\{x \in C_n : |f(x) - f(T^{q_n}x)| \ge a\}) \to 0.$$

Fix $\varepsilon > 0$ and $a > 0$. Then there exists a compact set $B_\varepsilon \subset X$ such that $\mu(B_\varepsilon^{\mathrm{c}}) < \varepsilon/2$ and $f : B_\varepsilon \to \mathbb{R}$ is uniformly continuous. Then there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < a$ for all $x, y \in B_\varepsilon$. By assumption, there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ and $x \in C_n$ we have $d(x, T^{q_n}x) < \delta$. It follows that if $n \ge n_0$ and $x \in C_n \cap B_\varepsilon \cap T^{-q_n}B_\varepsilon$, then

$|f(x) - f(T^{q_n}x)| < a$. Consequently,

$$\mu(\{x \in C_n : |f(x) - f(T^{q_n}x)| \geq a\}) \leq \mu(C_n \cap (B_\varepsilon^{\mathrm{c}} \cup T^{-q_n} B_\varepsilon^{\mathrm{c}})) < \varepsilon$$

for $n \geq n_0$, and therefore $\chi_{C_n} \cdot (f_n \circ T - f_n)$ converges to 0 in measure. As φ is uniformly continuous, $\chi_{C_n} \cdot (\varphi \circ f_n \circ T - \varphi \circ f_n)$ still converges to zero in measure and thus

$$\int_{C_n} (\varphi(f_n(Tx)) - \varphi(f_n(x)))\xi(Tx)\, d\mu \to 0,$$

because φ is also bounded. Since $\mu(T^{-1}C_n \triangle C_n) \to 0$, (6) follows.

Let us return to the proof of the assertion of the lemma. Since every measurable function can be approximated in measure by functions taking only finitely many values and φ is uniformly continuous and bounded, we can restrict ourselves to the case $h = \sum_{j=1}^{k} h_j \cdot \chi_{A_j}$, where $\{A_j : j = 1, \ldots, k\}$ are pairwise disjoint and $h_j \in \mathbb{R}$, $j = 1, \ldots, k$. Then from (6) we obtain

$$\begin{aligned}\int_{C_n} \varphi(f_n(x) + h(x))g(x)\, d\mu(x) &= \sum_{j=1}^{k} \int_{C_n} \varphi(f_n(x) + h_j)(g \cdot \chi_{A_j})(x)\, d\mu(x) \\ &\to \alpha \sum_{j=1}^{k} \int_{\mathbb{R}} \varphi(t + h_j)\, dP(t) \int_X (g \cdot \chi_{A_j})(x)\, d\mu(x) \\ &= \alpha \int_X \int_{\mathbb{R}} \varphi(t + h(x))g(x)\, dP(t)\, d\mu(x),\end{aligned}$$

which completes the proof. ∎

LEMMA 5. *For every $\varphi \in C(\overline{\mathbb{R}})$, $g, \xi \in L^\infty(X, \mathcal{B}, \mu)$ and any measurable function $h : X \to \mathbb{R}$ we have*

$$\int_{C_n} \varphi(f_n(x) + h(x))g(x)\xi(T^{q_n}x)\, d\mu(x) \to \alpha \int_X \int_{\mathbb{R}} \varphi(t + h(x))g(x)\xi(x)\, dP(t)\, d\mu(x).$$

Proof. By the proof of Lemma 4, $\chi_{C_n} \cdot (\xi - \xi \circ T^{q_n})$ converges to zero in measure and since φ, g, ξ are bounded,

$$\begin{aligned}\Big| \int_{C_n} \varphi(f_n(x) + h(x))g(x)\xi(T^{q_n}x)\, d\mu(x) - \int_{C_n} \varphi\,(f_n(x) + h(x))\, g(x)\xi(x)\, d\mu(x)\Big| \\ \leq \|\varphi\|_\infty \|g\|_\infty \int_{C_n} |\xi(x) - \xi(T^{q_n}x)| d\mu(x) \to 0.\end{aligned}$$

The result follows directly from Lemma 4. ∎

THEOREM 6. *Let* $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ *be an ergodic automorphism and* $f \in L^2(X, \mu)$ *a positive function for which there exists* $c > 0$ *such that* $0 < c \leq f(x)$ *for a.a.* $x \in X$. *Suppose that* $\{C_n\}$ *is a sequence of Borel subsets of* X, $\{q_n\}$ *is an increasing sequence of natural numbers, and* $\{a_n\}$ *is a sequence of real numbers such that*

- $\mu(C_n) \to \alpha > 0$ *as* $n \to \infty$,
- $\mu(C_n \triangle T^{-1}C_n) \to 0$ *as* $n \to \infty$,
- $\sup_{x \in C_n} d(x, T^{q_n}x) \to 0$ *as* $n \to \infty$,
- *the sequence* $\{\int_{C_n} |f_n(x)|^2 \, d\mu(x)\}$ *is bounded, where* $f_n := f^{(q_n)} - a_n$,
- $\frac{1}{\mu(C_n)}(f_n|_{C_n})_*(\mu|_{C_n}) \to P$ *weakly in* $\mathcal{P}(\mathbb{R})$,
- *the sequence* $\{(T^f)_{a_n}\}$ *converges in the weak operator topology.*

Then $\{(T^f)_{a_n}\}$ *converges weakly to the operator*

$$\alpha \int_{\mathbb{R}} (T^f)_{-t} \, dP(t) + (1 - \alpha)J,$$

where $J \in J(T^f)$.

Proof. By Lemma 3, it is easy to see that all we need to show is that

$$\mu^f((T^f)_{a_n}(A \cap (C_n \times \mathbb{R})) \cap B) \to \alpha \int_{\mathbb{R}} \mu^f((T^f)_{-t}A \cap B) \, dP(t)$$

for any pair of measurable rectangles $A, B \subset X^f$ of the form $A = A_1 \times A_2$, $B = B_1 \times B_2$ such that $A_2, B_2 \subset \mathbb{R}$ are bounded. By Remarks 1 and 2,

$$\begin{aligned}
&\mu^f((T^f)_{a_n}((A_1 \cap C_n) \times A_2) \cap B_1 \times B_2) \\
&\qquad = \sum_{k \in \mathbb{Z}} \mu \otimes \lambda((S_{-f})^k (S_{-f})^{q_n} \sigma_{a_n}((A_1 \cap C_n) \times A_2) \cap B_1 \times B_2)
\end{aligned}$$

and

$$\begin{aligned}
&\mu \otimes \lambda((S_{-f})^k (S_{-f})^{q_n} \sigma_{a_n}((A_1 \cap C_n) \times A_2) \cap B_1 \times B_2) \\
&\quad = \int_{T^{q_n+k}(A_1 \cap C_n) \cap B_1} \lambda((A_2 + a_n + f^{(-q_n-k)}(x)) \cap B_2) \, d\mu(x) \\
&\quad = \int_{T^{q_n+k}(A_1 \cap C_n) \cap B_1} \lambda((A_2 + a_n - f^{(q_n)}(T^{-q_n-k}x) + f^{(-k)}(x)) \cap B_2) \, d\mu(x) \\
&\quad = \int_{A_1 \cap C_n \cap T^{-q_n-k}B_1} \lambda((A_2 - f_n(x) - f^{(k)}(T^{q_n}x)) \cap B_2) \, d\mu(x).
\end{aligned}$$

Set $s := \mathrm{diam}(A_2 \cup B_2)$. Let k be an integer such that $|k| > s/c$. Then

$$\begin{aligned}
&\mu\otimes\lambda((S_{-f})^k(S_{-f})^{q_n}\sigma_{a_n}((A_1\cap C_n)\times A_2)\cap B_1\times B_2)\\
&\qquad\le\int_{C_n}\lambda((A_2-f_n(x)-f^{(k)}(T^{q_n}x))\cap B_2)\,d\mu(x)\\
&\qquad=\int_{\{x\in C_n\,:\,|f_n(x)+f^{(k)}(T^{q_n}x)|\le s\}}\lambda((A_2-f_n(x)-f^{(k)}(T^{q_n}x))\cap B_2)\,d\mu(x)\\
&\qquad\le s\mu(\{x\in C_n:|f_n(x)+f^{(k)}(T^{q_n}x)|\le s\})\\
&\qquad\le s\mu(\{x\in C_n:|f_n(x)|\ge c|k|-s\})\le sC/(c|k|-s)^2
\end{aligned}$$

by Chebyshev's inequality, where $C=\sup_n\int_{C_n}|f_n(x)|^2\,d\mu(x)$. Putting $b_k:=sC/(c|k|-s)^2$ whenever $|k|>s/c$ and $b_k:=s$ otherwise we obtain

$$\mu\otimes\lambda((S_{-f})^k(S_{-f})^{q_n}\sigma_{a_n}((A_1\cap C_n)\times A_2)\cap B_1\times B_2)\le b_k \tag{7}$$

for each natural n and moreover $\sum_{k\in\mathbb{Z}}b_k<\infty$.

On the other hand, given an integer k, for any natural n,

$$\begin{aligned}
&\mu\otimes\lambda((S_{-f})^k(S_{-f})^{q_n}\sigma_{a_n}((A_1\cap C_n)\times A_2)\cap B_1\times B_2)\\
&\quad=\int_{T^{q_n+k}(A_1\cap C_n)\cap B_1}\lambda((A_2+a_n+f^{(-q_n-k)}(x))\cap B_2)\,d\mu(x)\\
&\quad=\int_{T^{q_n+k}(A_1\cap C_n)\cap B_1}\lambda((A_2+a_n-f^{(q_n)}(T^{-q_n}x)+f^{(-k)}(T^{-q_n}x))\cap B_2)\,d\mu(x)\\
&\quad=\int_{T^k(A_1\cap C_n)\cap T^{-q_n}B_1}\lambda((A_2+a_n-f^{(q_n)}(x)+f^{(-k)}(x))\cap B_2)\,d\mu(x).
\end{aligned}$$

Since $\mu(T^kC_n\triangle C_n)\to 0$ as $n\to\infty$,

$$\begin{aligned}
&\Big|\mu\otimes\lambda((S_{-f})^k(S_{-f})^{q_n}\sigma_{a_n}((A_1\cap C_n)\times A_2)\cap B_1\times B_2)\\
&\qquad\qquad-\int_{T^kA_1\cap C_n\cap T^{-q_n}B_1}\lambda((A_2-f_n(x)+f^{(-k)}(x))\cap B_2)\,d\mu(x)\Big|\\
&\qquad\le\mu(T^kC_n\triangle C_n)\lambda(B_2)\to 0
\end{aligned}$$

as $n\to\infty$. Furthermore, by Lemma 5, given $k\in\mathbb{Z}$,

$$\begin{aligned}
&\int_{T^kA_1\cap C_n\cap T^{-q_n}B_1}\lambda((A_2-f_n(x)+f^{(-k)}(x))\cap B_2)\,d\mu(x)\\
&\qquad=\int_{C_n}\lambda((A_2-f_n(x)+f^{(-k)}(x))\cap B_2)\chi_{T^kA_1}(x)\chi_{B_1}(T^{q_n}x)\,d\mu(x)\\
&\qquad\to\alpha\int_{T^kA_1\cap B_1}\int_{\mathbb{R}}\lambda((A_2-t+f^{(-k)}(x))\cap B_2)\,dP(t)\,d\mu(x)
\end{aligned}$$

as $n\to\infty$. Consequently, given $k\in\mathbb{Z}$,

$$\mu\otimes\lambda((S_{-f})^k(S_{-f})^{q_n}\sigma_{a_n}((A_1\cap C_n)\times A_2)\cap B_1\times B_2)$$
$$\to\alpha\int_{T^kA_1\cap B_1}\int_{\mathbb{R}}\lambda((A_2-t+f^{(-k)}(x))\cap B_2)\,dP(t)\,d\mu(x)$$

as $n\to\infty$. By (7), it follows that

$$\mu^f((T^f)_{a_n}((A_1\cap C_n)\times A_2)\cap B_1\times B_2)$$
$$\to\alpha\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\int_{T^kA_1\cap B_1}\lambda((A_2-t+f^{(-k)}(x))\cap B_2)\,d\mu(x)\,dP(t).$$

On the other hand,

$$\int_{\mathbb{R}}\mu^f((T^f)_{-t}(A_1\times A_2)\cap B_1\times B_2)\,dP(t)$$
$$=\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}\mu\otimes\lambda((S_{-f})^k\sigma_{-t}(A_1\times A_2)\cap B_1\times B_2)\,dP(t)$$
$$=\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}\int_{T^kA_1\cap B_1}\lambda((A_2-t+f^{(-k)}(x))\cap B_2)\,d\mu(x)\,dP(t)$$
$$=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\int_{T^kA_1\cap B_1}\lambda((A_2-t+f^{(-k)}(x))\cap B_2)\,d\mu(x)\,dP(t),$$

which completes the proof. ■

5. Special flows over interval exchange transformations. Consider a permutation π of $\{1,\dots,m\}$, a vector $\lambda=(\lambda_1,\dots,\lambda_m)$ in the interior of the unit simplex, i.e. $\lambda_i>0$ for $i=1,\dots,m$ and $\sum_{i=1}^m\lambda_i=1$, and a vector $\varepsilon=(\varepsilon_1,\dots,\varepsilon_m)$ whose coordinates are either 1 or -1. Let $\beta_0=0$ and $\beta_i=\lambda_1+\dots+\lambda_i$ for $i=1,\dots,m$. The *interval exchange transformation* $T_{\lambda,\pi,\varepsilon}:[0,1]\to[0,1]$ is the map that is linear and Lebesgue measure preserving on every interval (β_{i-1},β_i), rearranges those intervals according to the permutation π, and preserves or reverses orientation on (β_{i-1},β_i) according to the sign of ε_i $(i=1,\dots,m)$. Denote by Leb the Lebesgue measure on $[0,1]$.

THEOREM 7. *Let $T_{\lambda,\pi,\varepsilon}:([0,1],\mathrm{Leb})\to([0,1],\mathrm{Leb})$ be an ergodic interval exchange transformation and let $f:[0,1]\to\mathbb{R}$ be a positive function of bounded variation for which there exists a positive constant c such that $f\ge c>0$. Then the special flow $T^f_{\lambda,\pi,\varepsilon}$ is disjoint from all mixing flows.*

Proof. To shorten notation, we will write T instead of $T_{\lambda,\pi,\varepsilon}$. A collection $\Xi=\{T^iI\}_{0\le i<q}$ of pairwise disjoint subintervals is called a *tower of intervals* for T. The measure of Ξ, i.e. the number $\sum_{0\le i<q}|T^iI|$, will be denoted by $|\Xi|$. As shown by Katok in [9], there exist two sequences

$\Xi_n = \{T^i I_n\}_{0\le i<p_n}$, $\Xi'_n = \{T^i J_n\}_{0\le i<q_n}$ of towers of intervals for T such that

$$|I_n| \to 0 \quad \text{as } n \to \infty, \tag{8}$$

$$|\Xi_n| \ge \frac{1}{m+1}, \tag{9}$$

$$J_n \subset I_n \quad \text{and} \quad T^{q_n} J_n \subset I_n, \tag{10}$$

$$\bigcup_{0\le i<p_n} T^i I_n \cap \bigcup_{p_n \le i<q_n} T^i J_n = \emptyset, \tag{11}$$

$$|J_n| \ge \frac{1}{m+1}\,|I_n|. \tag{12}$$

Put $C_n := \bigcup_{0\le i<p_n} T^i J_n$. From (9) and (12) we obtain $|C_n| \ge 1/(m+1)^2$. Moreover, from (8) we have $|C_n \triangle T^{-1} C_n| \le 2|I_n| \to 0$ as $n \to \infty$. Suppose that $x \in C_n$, say $x \in T^k J_n$, where $0 \le k < p_n$. Then

$$T^i x \in \begin{cases} T^{i+k} I_n & \text{if } 0 \le i < p_n - k, \\ T^{i+k} J_n & \text{if } p_n - k \le i < q_n - k, \\ T^{i+k-q_n} I_n & \text{if } q_n - k \le i \le q_n, \end{cases}$$

by (10). In particular $x, T^{q_n} x \in T^k I_n$, and hence $\sup_{x\in C_n} |x - T^{q_n} x| \le |I_n| \to 0$ as $n \to \infty$. Set

$$a_n := \frac{1}{|I_n|} \int\limits_{\bigcup_{0\le i<p_n} T^i I_n} f(t)\,dt + \frac{1}{|J_n|} \int\limits_{\bigcup_{p_n\le i<q_n} T^i J_n} f(t)\,dt.$$

Then for $x \in T^k J_n$ we have

$$\begin{aligned} |f^{(q_n)}(x) - a_n| &\le \sum_{k\le i<p_n} \frac{1}{|I_n|} \int\limits_{T^i I_n} |f(T^{i-k}x) - f(t)|\,dt \\ &\quad + \sum_{p_n\le i<q_n} \frac{1}{|J_n|} \int\limits_{T^i J_n} |f(T^{i-k}x) - f(t)|\,dt \\ &\quad + \sum_{0\le i<k} \frac{1}{|I_n|} \int\limits_{T^i I_n} |f(T^{q_n-k+i}x) - f(t)|\,dt \\ &\le \sum_{0\le i<p_n} \operatorname{Var}_{T^i I_n} f + \sum_{p_n\le i<q_n} \operatorname{Var}_{T^i J_n} f \le \operatorname{Var}_{[0,1]} f, \end{aligned}$$

by (11). Consequently, $|f^{(q_n)}(x) - a_n| \le \operatorname{Var}_{[0,1]} f$ for all $x \in C_n$. Hence the sequences $\{C_n\}$, $\{q_n\}$ and $\{a_n\}$ satisfy the assumptions of Theorem 6. Therefore by passing to a further subsequence of $\{a_n\}$ if necessary, we have

$$T^f_{a_n} \to \alpha \int\limits_{\mathbb{R}} T^f_t \,dP(t) + (1-\alpha)J,$$

where $0 < \alpha \le 1$, $P \in \mathcal{P}(\mathbb{R})$ and $J \in J(T^f)$. An application of Proposition 1 completes the proof. ∎

5.1. *Billiards in rational polygons.* Let P be a connected polygon in $\mathbb{R}^2$. The billiard flow $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ in P is the flow on the space $P\times\mathbb{S}^1$ of all unit tangent vectors to $\mathbb{R}^2$ with footpoints in P described as follows. A vector v with footpoint $p \in P$ moves with unit speed along the straight line $p + vt$ until it reaches the boundary of P, then it changes its direction according to the rule of reflection and continues its movement. Assume that P is a rational k-gon, i.e. all angles of P have the form $\pi m_j/n_j$, $j = 1,\dots,k$, where all m_j, n_j are natural numbers. Then the phase space $P \times \mathbb{S}^1$ splits into invariant sets M_c, $0 \le c \le \pi/N$, where N is the least common multiple of $\{n_j : j = 1,\dots,k\}$ (see [10]). Moreover, for every $0 \le c \le \pi/N$ the flow $\mathcal{T}$ restricted to M_c has a natural representation as the special flow built over an interval exchange transformation $I_c : [0,1] \to [0,1]$ and under a piecewise linear function $f_c : [0,1] \to \mathbb{R}$. Suppose that μ is a Borel non-atomic probability measure on $[0,1]$ invariant and ergodic with respect to I_c. By Lemma 1 in [9], $I_c : ([0,1],\mu) \to ([0,1],\mu)$ is metrically isomorphic to an interval exchange transformation $I' : ([0,1],\mathrm{Leb}) \to ([0,1],\mathrm{Leb})$ via a monotone function $R : [0,1] \to [0,1]$. Consequently, the special flow over $I_c : ([0,1],\mu) \to ([0,1],\mu)$ and under f_c is metrically isomorphic to the special flow $I'^{f_c\circ R}$. Moreover, $f_c\circ R : [0,1] \to \mathbb{R}$ is of bounded variation because it is a difference of two monotone functions. Thus, if ν is an ergodic $\mathcal{T}$-invariant measure then $(P\times\mathbb{S}^1,\nu,\mathcal{T})$ is metrically isomorphic to either a periodic orbit or the special flow built over an ergodic interval exchange transformation and under a function of bounded variation. Now by Theorem 7, we obtain the following.

THEOREM 8. *Let P be a rational connected polygon. Assume that ν is a probability measure on $P\times\mathbb{S}^1$ invariant and ergodic with respect to the billiard flow $\mathcal{T}$ in P. Then the flow $(P \times \mathbb{S}^1,\nu,\mathcal{T})$ is disjoint from all mixing flows.*

6. Flows on surfaces which are disjoint from ELF flows. Let M be a smooth orientable surface (two-dimensional manifold) with negative Euler characteristic. Assume that $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ is a C^3-flow for which the only isolated critical points are non-degenerate saddles and which carries a positive C^1-measure invariant and ergodic with respect to ϕ. Suppose that ϕ has a smooth closed transversal τ whose first return map is monotone (that is, the Poincaré map preserves orientation on τ). Then this map is isomorphic to the rotation on the circle by an irrational number denoted by $\alpha(\phi)$. As shown by Kochergin in [11], the flow ϕ is metrically isomorphic to the special flow built over the rotation by $\alpha(\phi)$ and under a positive function $F : \mathbb{T} \to \mathbb{R}$ of the form

$$F(x) = G(x) - \sum_{i=1}^{k}(b_i \log(\{x-\beta_i\}) + c_i \log(\{\beta_i - x\})),$$

where $G:\mathbb{T}\to\mathbb{R}$ is a function of bounded variation, and b_i, c_i, $i=1,\dots,k$, are nonnegative constants such that $\sum_{i=1}^k b_i=\sum_{i=1}^k c_i>0$. The family of all such functions will be denoted by LOGSYM_+. Moreover, he proved that if $\alpha=\alpha(\phi)$ satisfies

$$\liminf_n\,(\log q_n)q_n\|q_n\alpha\|<\infty, \tag{13}$$

where $\{q_n\}$ is the sequence of denominators of α, then the flow ϕ is not mixing. In this section we give more information about this flow. More precisely, we show that under the assumption (13) the flow ϕ is disjoint from all weakly mixing ELF flows. We should mention that if $\sum_{i=1}^k b_i\neq\sum_{i=1}^k c_i$, $G\in C^2(\mathbb{T}\setminus\bigcup_{i=1}^k\{\beta_i\})$, G'' is bounded and α satisfies a Diophantine condition, then the special flow built over the rotation by α and under F is mixing (see [16]).

Let $\alpha\in\mathbb{T}$ be an irrational number and let $T=T_\alpha$ stand for the rotation by α on the circle. Assume that α satisfies (13). By passing to a subsequence of $\{q_n\}$ if necessary we can assume that $\{(\log q_n)q_n\|q_n\alpha\|\}$ is bounded. For every $\beta\in\mathbb{T}$ denote by $f_\beta:\mathbb{T}\to\mathbb{R}$ the function $f_\beta(x)=l(x)+l(\beta-x)$, where $l:\mathbb{T}\to\mathbb{R}$ is given by $l(x)=-\log\{x\}$. Put $q:=q_n$. Consider the function $l_q:\mathbb{T}\to\mathbb{R}$ given by

$$l_q(x)=\sum_{j=0}^{q-1}l(x+j/q).$$

We will denote by $z_q:\mathbb{T}\to\mathbb{T}$ the function $z_q(x)=\{qx\}$. Define

$$A_q:=z_q^{-1}([2q\|q\alpha\|,1-2q\|q\alpha\|]).$$

Then $|A_q|=1-4q\|q\alpha\|$.

LEMMA 9 (see Lemma 9 in [6]). *There exists a positive constant c_1 such that*

$$|l^{(q)}(x)-l_q(x)|\le c_1\quad \textit{for all } x\in A_q \textit{ and all } q.$$

Set $B_{q,\beta}=z_q^{-1}([0,\{q\beta\}])$ and $B'_{q,\beta}=z_q^{-1}([\{q\beta\},1])$. For every $a\in[0,1)$ let $\delta_a:[0,1)\to\mathbb{R}$ be defined by

$$\delta_a(x)=\begin{cases}x & \text{for } x\in[0,a/2)\cup[a,(1+a)/2),\\ a-x & \text{for } x\in[a/2,a),\\ 1+a-x & \text{for } x\in[(1+a)/2,1),\end{cases}$$

and put

$$\gamma_a:=\max_{a/2\le x\le 2}|\log\Gamma(x)|.$$

Finally, denote by $r_q:\mathbb{T}\to\mathbb{R}$ the function

$$r_q(x):=2q-\{q\beta\}\log q-\log\delta_{\{q\beta\}}(\{qx\}).$$

Lemma 10. *There exists a positive constant* c_2 *such that*

$$|l_q(x)+l_q(\beta-x)-r_q(x)-\log q|\le c_2+2\gamma_{\{q\beta\}} \tag{14}$$

for every $x\in B_{q,\beta}$, *and*

$$|l_q(x)+l_q(\beta-x)-r_q(x)|\le c_2+2\gamma_1 \tag{15}$$

for every $x\in B'_{q,\beta}$.

Proof. First recall that

$$0\le -\sum_{k=0}^{n}\log(x+k)-(-\log n!-x\log(n+1)+\log\Gamma(x))\le x\log 2 \tag{16}$$

for all $x>0$ and $n\in\mathbb{N}$ (see the proof of Theorem 8.19 in [14]). Moreover, by the Stirling formula, there exists $c'>0$ such that

$$\left|\log n^n+\frac{1}{2}\log 2\pi n-\log n!-n\right|\le c' \tag{17}$$

for every natural n.

The functions $l_q(\,\cdot\,)+l_q(\beta-\,\cdot\,)$ and $\log\delta_{\{q\beta\}}(\{q\,\cdot\})$ are invariant under the rotation by $1/q$. Clearly the former function is invariant under the symmetry

$$x\mapsto\beta-x. \tag{18}$$

By considering x so that consecutively

$$\{qx\}\in[0,\{q\beta\}/2),\qquad \{qx\}\in[\{q\beta\}/2,\{q\beta\}),$$
$$\{qx\}\in[\{q\beta\},(1+\{q\beta\})/2),\qquad \{qx\}\in[\{q\beta\},1),$$

also the latter function is invariant under (18). Therefore in order to show (14) we only need to consider $0\le x\le\{q\beta\}/2q$. Notice that for such an x we have

$$\{x+j/q\}=x+j/q,\qquad j=0,\dots,q-1.$$

Moreover,

$$\{\{\beta-x+j/q\}:j=0,\dots,q-1\}$$
$$=\{\{\beta-[q\beta]/q-x+j/q\}:j=0,\dots,q-1\}$$

and

$$-x+(j+\{q\beta\})/q\in[0,1)\qquad\text{for } j=0,\dots,q-1.$$

Consequently, by (16),

$$\begin{aligned}
&|l_q(x)+l_q(\beta-x)-r_q(x)-\log q| \\
&\qquad = \bigg|-\sum_{j=0}^{q-1}\log\Big(x+\frac{j}{q}\Big)-\sum_{j=0}^{q-1}\log\Big(-x+\frac{j+\{q\beta\}}{q}\Big) \\
&\qquad\quad -r_q(x)-\log q\pm\log(x+1)\bigg| \\
&\qquad \le \bigg|-\sum_{j=1}^{q}\log(qx+j)-\sum_{j=0}^{q-1}\log(-qx+j+\{q\beta\}) \\
&\qquad\quad -\log x+2\log q^q-r_q(x)-\log q\bigg|+\log 2 \\
&\qquad \le |-2\log(q-1)!-(qx+1)\log q-(-qx+\{q\beta\})\log q \\
&\qquad\quad +\log\Gamma(qx+1)+\log\Gamma(-qx+\{q\beta\}) \\
&\qquad\quad -\log x-r_q(x)-\log q+2\log q^q|+\frac{5}{2}\log 2.
\end{aligned}$$

Now

$$1\le qx+1\le 1+\{q\beta\}/2\le 2 \quad\text{and}\quad \{q\beta\}/2\le -qx+\{q\beta\}\le 1,$$

so by (17),

$$\begin{aligned}
&|l_q(x)+l_q(\beta-x)-r_q(x)-\log q| \\
&\le |-2\log q!-\{q\beta\}\log q-\log x-r_q(x)+2\log q^q|+\frac{5}{2}\log 2+2\gamma_{\{q\beta\}} \\
&\le |-2\log q!-2q+\log q+2\log q^q|+\frac{5}{2}\log 2+2\gamma_{\{q\beta\}} \\
&\le 2c'+\log 2\pi+\frac{5}{2}\log 2+2\gamma_{\{q\beta\}}.
\end{aligned}$$

The proof of (14) is now complete.

Similarly, we only need to show (15) for $\{q\beta\}/q\le x\le(1+\{q\beta\})/2q$. Applying again (16) and (17) we see that under this condition,

$$\begin{aligned}
&|l_q(x)+l_q(\beta-x)-r_q(x)| \\
&= \bigg|-\sum_{j=0}^{q-1}\log\Big(x+\frac{j}{q}\Big)-\sum_{j=1}^{q}\log\Big(-x+\frac{j+\{q\beta\}}{q}\Big)-r_q(x)\pm\log(x+1)\bigg| \\
&\le \bigg|-\sum_{j=1}^{q}\log(qx+j)-\sum_{j=1}^{q}\log(-qx+j+\{q\beta\})-\log x+2\log q^q-r_q(x)\bigg|+\log 2
\end{aligned}$$

$$\begin{aligned}
&\le |-2\log(q-1)! - (qx+1)\log q - (-qx+1+\{q\beta\})\log q \\
&\quad + \log\Gamma(qx+1) + \log\Gamma(-qx+1+\{q\beta\}) \\
&\quad - \log x - r_q(x) + 2\log q^q| + 4\log 2 \\
&\le |-2\log q! - \{q\beta\}\log q - \log x - r_q(x) + 2\log q^q| + 4\log 2 + 2\gamma_1 \\
&\le |-2\log q! - 2q + \log q + 2\log q^q| + 4\log 2 + 2\gamma_1 \\
&\le 2c' + \log 2\pi + 4\log 2 + 2\gamma_1. \ \blacksquare
\end{aligned}$$

Suppose that $\{(\log q_n)q_n\|q_n\alpha\|\}$ is bounded. Given $\beta\in\mathbb{T}$ put

$$\begin{aligned}
C_n^0(\beta) = C_n^0 &:= B_{q_n,\beta}\cap A_{q_n}\cap(\beta - A_{q_n}) \\
&= z_{q_n}^{-1}([2q_n\|q_n\alpha\|, \{q_n\beta\} - 2q_n\|q_n\alpha\|]), \\
C_n^1(\beta) = C_n^1 &:= B'_{q_n,\beta}\cap A_{q_n}\cap(\beta - A_{q_n}) \\
&= z_{q_n}^{-1}([\{q_n\beta\} + 2q_n\|q_n\alpha\|, 1 - 2q_n\|q_n\alpha\|]), \\
a_n^0(\beta) = a_n^0 &:= 2q_n + (1-\{q_n\beta\})\log q_n - (2c_1 + c_2 + 2\gamma_{\{q_n\beta\}}), \\
a_n^1(\beta) = a_n^1 &:= 2q_n - \{q_n\beta\}\log q_n - (2c_1 + c_2 + 2\gamma_1).
\end{aligned}$$

LEMMA 11. *Under the above notation, we have*

$$C_n^0\cap C_n^1 = \emptyset, \qquad |C_n^0| = \{q_n\beta\} - 4q_n\|q_n\alpha\|, \qquad |C_n^1| = 1 - \{q_n\beta\} - 4q_n\|q_n\alpha\|; \tag{19}$$

$$f_\beta^{(q_n)}(x) - a_n^i \ge -\log\delta_{\{q_n\beta\}}(\{q_n x\}) \ge 0 \quad \textit{for } x\in C_n^i \textit{ and } i = 0,1; \tag{20}$$

$$\int_{C_n^0} |f_\beta^{(q_n)}(x) - a_n^0|^2\,dx \le 8((2c_1 + c_2 + 2\gamma_{\{q_n\beta\}})^2 + 1); \tag{21}$$

$$\int_{C_n^1} |f_\beta^{(q_n)}(x) - a_n^1|^2 dx \le 8((2c_1 + c_2 + 2\gamma_1)^2 + 1); \tag{22}$$

for every $2q_n\|q_n\alpha\|\le t$ and $s>0$ we have

$$\begin{aligned}
&|\{x\in C_n^0\cap z_{q_n}^{-1}([2q_n\|q_n\alpha\|, t]) : f_\beta^{(q_n)}(x) - a_n^0 > s\}| \\
&\qquad \ge \min(t, e^{-s}, \{q_n\beta\}/2) - 2q_n\|q_n\alpha\|;
\end{aligned} \tag{23}$$

and for every $2q_n\|q_n\alpha\|\le t$ and $s>0$ we have

$$\begin{aligned}
&|\{x\in C_n^1\cap z_{q_n}^{-1}([2q_n\|q_n\alpha\|, t] + \{q_n\beta\}) : f_\beta^{(q_n)}(x) - a_n^1 > s\}| \\
&\qquad \ge \min(t, e^{-s} - \{q_n\beta\}, (1-\{q_n\beta\})/2) - 2q_n\|q_n\alpha\|.
\end{aligned} \tag{24}$$

Proof. (19) follows immediately. In order to get (20) we apply consecutively Lemmas 9 and 10 (estimating from below), and the definitions of r_{q_n} and a_n^i. Applying Lemmas 9 and 10 (estimating from above) for $x\in C_n^0$ we obtain

$$f_\beta^{(q_n)}(x)-a_n^0\le -\log\delta_{\{q_n\beta\}}(\{q_nx\})+2(2c_1+c_2+2\gamma_{\{q_n\beta\}}).$$

Hence

$$\int\limits_{C_n^0}|f_\beta^{(q_n)}(x)-a_n^0|^2\,dx\le 2\Big(4(2c_1+c_2+2\gamma_{\{q_n\beta\}})^2+\int\limits_{C_n^0}\log^2\delta_{\{q_n\beta\}}(\{q_nx\})\,dx\Big).$$

Noticing once more that $\delta_{\{q_n\beta\}}(\{q_n\cdot\})$ is invariant under $x\mapsto x+1/q_n$ and $x\mapsto\beta-x$ we obtain

$$\int\limits_{C_n^0}\log^2\delta_{\{q_n\beta\}}(\{q_nx\})\,dx\le 2q_n\int\limits_0^{\{q_n\beta\}/2q_n}\log^2(q_nx)\,dx$$

$$\le 2q_n\int\limits_0^{1/q_n}\log^2(q_nx)\,dx=2\int\limits_0^1\log^2x\,dx=4,$$

and (21) is proved. In the same manner one can prove (22). To prove (23) suppose that

$$x\in D:=z_{q_n}^{-1}([2q_n\|q_n\alpha\|,\min(t,e^{-s},\{q_n\beta\}/2)).$$

Then

$$x\in C_n^0\cap z_{q_n}^{-1}([2q_n\|q_n\alpha\|,t])\quad\text{and}\quad\{q_nx\}<\min(\{q_n\beta\}/2,e^{-s}).$$

By (20),

$$f_\beta^{(q_n)}(x)-a_n^0\ge-\log\delta_{\{q_n\beta\}}(\{q_nx\})=-\log\{q_nx\}>s.$$

Now (23) follows from the fact that

$$|D|=\max(0,\min(t,e^{-s},\{q_n\beta\})-2q_n\|q_n\alpha\|).$$

In the same manner one can prove (24); this time we define D as

$$z_{q_n}^{-1}([2q_n\|q_n\alpha\|,\min(t,e^{-s}-\{q_n\beta\},(1-\{q_n\beta\})/2))+\{q_n\beta\}).\ ■$$

THEOREM 12. *Assume that $\alpha\in\mathbb{T}$ is an irrational number satisfying* (13). *Let $F:\mathbb{T}\to\mathbb{R}$ be a positive function of the form*

$$F(x)=G(x)+\sum_{j=1}^k\kappa_jf_{\beta_j}(x-\omega_j),$$

where $G:\mathbb{T}\to\mathbb{R}$ is of bounded variation, κ_j is a positive number and $\beta_j,\omega_j\in\mathbb{T}$ for $j=1,\ldots,k$. Then the special flow T_α^F is disjoint from all weakly mixing ELF flows.

Proof. Given $0\le a\le 1$ set

$$a^\varepsilon=\begin{cases}a & \text{if }\varepsilon=0,\\ 1-a & \text{if }\varepsilon=1.\end{cases}$$

By passing to a subsequence of $\{q_n\}_{n\in\mathbb{N}}$ if necessary, we can assume that $\{(\log q_n)q_n\|q_n\alpha\|\}_{n\in\mathbb{N}}$ is bounded and $\{q_n\beta_j\} \to \overline{\beta}_j$ as $n \to \infty$ for all $j = 1,\ldots,k$. In the first part of the proof we give a general recipe for a construction of sequences $\{C_n\}_{n\in\mathbb{N}}$ and $\{a_n\}_{n\in\mathbb{N}}$ which are necessary if we want to apply Theorem 6. The main ingredients are: a finite sequence $\{\varepsilon_j\}_{j=1}^k$ of elements of $\{0,1\}$ and a sequence $\{J_n\}_{n\in\mathbb{N}}$ of subintervals of $\mathbb{T}$. In the second part of the proof we will give concrete ingredients which will depend on $\overline{\beta}_j$, $j=1,\ldots,k$.

Part I. Suppose that $\{\varepsilon_j\}_{j=1}^k$ is a sequence of elements of $\{0,1\}$ for which there is $\varrho>0$ such that

$$\Big|\bigcap_{j=1}^{k}(C_n^{\varepsilon_j}(\beta_j)+\omega_j)\Big|\ge\varrho$$

for all $n\in\mathbb{N}$ large enough. Since $C_n^{\varepsilon_j}(\beta_j)+\omega_j=z_{q_n}^{-1}(I_j^n)$, where $I_j^n\subset\mathbb{T}$ is an interval for any $j=1,\ldots,k$, we have

$$\bigcap_{j=1}^{k}(C_n^{\varepsilon_j}(\beta_j)+\omega_j)=z_{q_n}^{-1}\Big(\bigcap_{j=1}^{k}I_j^n\Big)$$

and $|\bigcap_{j=1}^k I_j^n|\ge\varrho$. Moreover, $\bigcap_{j=1}^k I_j^n$ is the union of at most k intervals. Denote by $\mathcal{A}_n$ the family of such intervals. Next suppose that $\{J_n\}_{n\in\mathbb{N}}$ is a sequence of intervals such that $J_n\in\mathcal{A}_n$ and $|J_n|\ge\tau>0$ for all n large enough. Put

$$C_n:=z_{q_n}^{-1}(J_n)\quad\text{and}\quad a_n:=q_n\int_{\mathbb{T}}G(x)\,dx+\sum_{j=1}^{k}\kappa_j a_n^{\varepsilon_j}(\beta_j).$$

Then $|C_n|\ge\tau$ for all n large enough and $|C_n\triangle T^{-1}C_n|\to0$ as $n\to\infty$. We also have (see (19))

$$\varrho\le|C_n^{\varepsilon_j}(\beta_j)|=|I_j^n|=\{q_n\beta_j\}^{\varepsilon_j}-4q_n\|q_n\alpha\|,$$

so

$$\min\{\{q_n\beta_j\}^{\varepsilon_j}:j=1,\ldots,k\}\ge\varrho/2 \tag{25}$$

for all n large enough. By the Denjoy–Koksma inequality (applied to G) we have

$$\begin{aligned}|F^{(q_n)}(x)-a_n|&\le\Big|G^{(q_n)}(x)-q_n\int_{\mathbb{T}}G(t)\,dt\Big|+\sum_{j=1}^{k}\kappa_j|f_{\beta_j}^{(q_n)}(x-\omega_j)-a_n^{\varepsilon_j}(\beta_j)|\\&\le\operatorname{Var}(G)+\sum_{j=1}^{k}\kappa_j|f_{\beta_j}^{(q_n)}(x-\omega_j)-a_n^{\varepsilon_j}(\beta_j)|.\end{aligned}$$

Hence by the Cauchy–Bunyakovskiĭ–Schwarz inequality,

$$|F^{(q_n)}(x) - a_n|^2 \le (k+1)\Big(\operatorname{Var}^2(G) + \sum_{j=1}^{k} \kappa_j^2 |f_{\beta_j}^{(q_n)}(x-\omega_j) - a_n^{\varepsilon_j}(\beta_j)|^2\Big).$$

It follows that (notice that $C_n \subset \bigcap_{j=1}^k (C_n^{\varepsilon_j}(\beta_j)+\omega_j)$)

$$\begin{aligned}
&\int_{C_n} |F^{(q_n)}(x) - a_n|^2\,dx \\
&\le (k+1)\Big(\operatorname{Var}^2(G) + \sum_{j=1}^{k} \kappa_j^2 \int_{C_n^{\varepsilon_j}(\beta_j)+\omega_j} |f_{\beta_j}^{(q_n)}(x-\omega_j) - a_n^{\varepsilon_j}(\beta_j)|^2\,dx\Big) \\
&\le (k+1)\Big(\operatorname{Var}^2(G) + \sum_{j=1}^{k} \kappa_j^2 \int_{C_n^{\varepsilon_j}(\beta_j)} |f_{\beta_j}^{(q_n)}(x) - a_n^{\varepsilon_j}(\beta_j)|^2\,dx\Big) \\
&\le (k+1)\Big(\operatorname{Var}^2(G) + 8((2c_1+c_2+\gamma_{\varrho/2})^2+1)\sum_{j=1}^{k} \kappa_j^2\Big)
\end{aligned}$$

for all n large enough, by (21), (22) and (25).

PART II. By passing to a further subsequence of $\{q_n\}_{n\in\mathbb{N}}$ if necessary, we can assume that

$$\frac{1}{|C_n|}(F_n|_{C_n})_*(\mathrm{Leb}|_{C_n}) \to P$$

weakly in $\mathcal{P}(\mathbb{R})$, where $F_n = F^{(q_n)} - a_n$. We will show that for a careful choice of $\{\varepsilon_j\}$ and $\{J_n\}$ the topological support of P will be unbounded, in particular, P will not be a Dirac measure.

CASE 1. Suppose that $\overline{\beta}_j = 0$ for all $j = 1,\dots,k$. Put $\varepsilon_j = 1$ for all $j = 1,\dots,k$. Then

$$\Big|\bigcap_{j=1}^{k} (C_n^{\varepsilon_j}(\beta_j)+\omega_j)\Big| \ge 1 - \sum_{j=1}^{k}\{q_n\beta_j\} - 4kq_n\|q_n\alpha\| \ge 1/2$$

for all n large enough. Next let J_n be the longest interval from $\mathcal{A}_n$. Then $|J_n| \ge 1/2k$. Fix $s>0$ which is not an atom of P. There exists $1\le j_0\le k$ such that the left endpoint of J_n coincides with the left endpoint of $I_{j_0}^n$. Then

$$C_n - \omega_{j_0} = C_n^1(\beta_{j_0}) \cap z_{q_n}^{-1}([2q_n\|q_n\alpha\|, t_n] + \{q_n\beta_{j_0}\}),$$

where $t_n = |J_n| + 2q_n\|q_n\alpha\|$. Next notice that if

$$x \in C_n^1(\beta_{j_0}) \cap z_{q_n}^{-1}([2q_n\|q_n\alpha\|, t_n] + \{q_n\beta_{j_0}\}) + \omega_{j_0}$$

satisfies

$$f_{\beta_{j_0}}^{(q_n)}(x-\omega_{j_0}) - a_n^1(\beta_{j_0}) > \frac{s+\mathrm{Var}(G)}{\kappa_{j_0}}$$

then $x \in C_n$, so $x - \omega_j \in C_n^1(\beta_j)$ for $j = 1, \ldots, k$, and by (20),

$$\begin{aligned} F^{(q_n)}(x) - a_n &= G^{(q_n)}(x) - q_n \int_{\mathbb{T}} G(t)\,dt + \sum_{j=1}^{k} \kappa_j (f_{\beta_j}^{(q_n)}(x-\omega_j) - a_n^{\varepsilon_j}(\beta_j)) \\ &\geq -\mathrm{Var}(G) + \kappa_{j_0}(f_{\beta_{j_0}}^{(q_n)}(x-\omega_{j_0}) - a_n^{\varepsilon_{j_0}}(\beta_{j_0})) > s. \end{aligned}$$

Therefore by (24),

$$\begin{aligned} &P(\{t \in \mathbb{R} : t > s\}) \\ &= \lim_{n\to\infty} \frac{1}{|C_n|} |\{x \in C_n : F^{(q_n)}(x) - a_n > s\}| \\ &\geq \liminf_n |\{x \in C_n^1(\beta_{j_0}) \cap z_{q_n}^{-1}([2q_n\|q_n\alpha\|, t_n] + \{q_n\beta_{j_0}\}) : \\ &\qquad\qquad f_{\beta_{j_0}}^{(q_n)}(x) - a_n^1(\beta_{j_0}) > (s + \mathrm{Var}(G))/\kappa_{j_0}\}| \\ &\geq \liminf_n [\min(t_n, e^{-(s+\mathrm{Var}(G))/\kappa_{j_0}} - \{q_n\beta_{j_0}\}, (1 - \{q_n\beta_{j_0}\})/2) - 2q_n\|q_n\alpha\|] \\ &\geq \min(1/2k, e^{-(s+\mathrm{Var}(G))/\kappa_{j_0}}) > 0. \end{aligned}$$

Consequently, the support of P is unbounded.

Case 2. Suppose that $\overline{\beta}_1 > 0$. Put $\varepsilon_1 = 0$. For $\varepsilon = 0, 1$ and $j = 1, \ldots, k$ let $I_{j,\varepsilon}^n \subset \mathbb{T}$ be an interval such that $C_n^\varepsilon(\beta_j) + \omega_j = z_{q_n}^{-1}(I_{j,\varepsilon}^n)$. Then

$$\Big| I_{1,0}^n \cap \bigcap_{j=2}^{k} \bigcup_{\varepsilon=0,1} I_{j,\varepsilon}^n \Big| \geq \{q_n\beta_1\} - 8kq_n\|q_n\alpha\|, \tag{26}$$

in particular

$$\Big| I_{1,0}^n \cap \bigcap_{j=2}^{k} \bigcup_{\varepsilon=0,1} I_{j,\varepsilon}^n \Big| \geq \overline{\beta}_1/2$$

for all n large enough and $I_{1,0}^n \cap \bigcap_{j=2}^k \bigcup_{\varepsilon=0,1} I_{j,\varepsilon}^n$ is the union of at most $2k$ intervals. Denote by $\overline{\mathcal{A}}_n$ the family of those intervals. Since $\#\overline{\mathcal{A}}_n \leq 2k$ and (26) holds, it is easy to see that we can choose a sequence $\{J_n\}$ of intervals such that $J_n \in \overline{\mathcal{A}}_n$, $|J_n| \geq \varrho > 0$ for all n large enough and the distance, denoted by d_n, between the left endpoint of J_n and the left endpoint of $I_{1,0}^n$ tends to zero as $n \to \infty$. By passing to a subsequence of $\{J_n\}$ if necessary, we can assume that there exist $\varepsilon_2, \ldots, \varepsilon_k \in \{0, 1\}$ such that

$$J_n \subset \bigcap_{j=1}^{k} I_{j,\varepsilon_j}^n.$$

Then $C_n - \omega_1 = C_n^0(\beta_1) \cap z_{q_n}^{-1}([2q_n\|q_n\alpha\|, t_n] + d_n)$, where $t_n = |J_n| + 2q_n\|q_n\alpha\|$. Repeating the arguments from Case 1, using (23) instead of (24) and taking into account that $d_n \to 0$ we obtain

$$\begin{aligned} P(\{t \in \mathbb{R} : t > s\}) &= \lim_{n\to\infty} \frac{1}{|C_n|} |\{x \in C_n : F^{(q_n)}(x) - a_n > s\}| \\ &\geq \liminf_n |\{x \in C_n^0(\beta_1) \cap z_{q_n}^{-1}([2q_n\|q_n\alpha\|, t_n] + d_n) : \\ &\qquad\qquad f_{\beta_1}^{(q_n)}(x) - a_n^0(\beta_1) > (s + \mathrm{Var}(G))/\kappa_1\}| \\ &\geq \liminf_n |\{x \in C_n^0(\beta_1) \cap z_{q_n}^{-1}([2q_n\|q_n\alpha\|, t_n]) : \\ &\qquad\qquad f_{\beta_1}^{(q_n)}(x) - a_n^0(\beta_1) > (s + \mathrm{Var}(G))/\kappa_1\}| \\ &\geq \liminf_n [\min(t_n, e^{-(s+\mathrm{Var}(G))/\kappa_1}, \{q_n\beta_1\}/2) - 2q_n\|q_n\alpha\|] \\ &\geq \min(\varrho, e^{-(s+\mathrm{Var}(G))/\kappa_1}, \overline{\beta}_1/2) > 0, \end{aligned}$$

which proves that the support of P is unbounded. The reasoning is unchanged if $\overline{\beta}_j > 0$ for some $2 \leq j \leq k$. An application of Theorem 6 and Proposition 1 completes the proof. ∎

We will now argue that the family of functions which appear in Theorem 12 coincides with LOGSYM$_+$. Let $F : \mathbb{T} \to \mathbb{R}$ be a function of the form

$$F(x) = -\sum_{i=1}^{k} (b_i \log(\{x - \beta_i\}) + c_i \log(\{\beta_i - x\})),$$

where b_i, c_i, $i = 1, \ldots, k$, are non-negative constants such that $\sum_{i=1}^k b_i = \sum_{i=1}^k c_i$ and $\beta_j \in \mathbb{T}$, $j = 1, \ldots, k$, are pairwise distinct. Define $N(F) = \sharp\{1 \leq j \leq k : b_j > 0\} + \sharp\{1 \leq j \leq k : c_j > 0\}$. Suppose that $N(F) > 0$. Let b_{j_0} and c_{j_1} be positive numbers. Assume that $b_{j_0} \geq c_{j_1}$ (in the opposite case the reasoning below will be the same). Then

$$F(x) = \widetilde{F}(x) + c_{j_1} f_{\beta_{j_1} - \beta_{j_0}}(x - \beta_{j_0}),$$

where

$$\widetilde{F}(x) = -\sum_{i=1}^{k} (\widetilde{b_i} \log(\{x - \beta_i\}) + \widetilde{c_i} \log(\{\beta_i - x\}))$$

and

$$\widetilde{b_j} = \begin{cases} b_j - c_{j_1} & \text{if } j = j_0, \\ b_j & \text{if } j \neq j_0, \end{cases} \qquad \widetilde{c_j} = \begin{cases} 0 & \text{if } j = j_1, \\ c_j & \text{if } j \neq j_1. \end{cases}$$

Then $N(\widetilde{F}) < N(F)$ and $\sum_{i=1}^k \widetilde{b_i} = \sum_{i=1}^k \widetilde{c_i}$. We then apply the same reasoning to $\widetilde{F}$ and it follows that in a finite number of steps we will find a natural number $\widehat{k}$, elements of the circle $\widehat{\beta}_j, \omega_j$, $j = 1, \ldots, \widehat{k}$, and positive numbers κ_j, $j = 1, \ldots, \widehat{k}$, such that

$$F(x) = \sum_{j=1}^{\widehat{k}} \kappa_j f_{\widehat{\beta}_j}(x - \omega_j).$$

By Theorem 12, we have the following.

THEOREM 13. *Assume that* $\alpha \in \mathbb{T}$ *is an irrational number satisfying* (13). *Let* $F : \mathbb{T} \to \mathbb{R}$ *be a positive function of the form*

$$F(x) = G(x) - \sum_{i=1}^{k} (b_i \log(\{x - \beta_i\}) + c_i \log(\{\beta_i - x\})),$$

where $G : \mathbb{T} \to \mathbb{R}$ *is a function of bounded variation*, b_i, c_i, $i = 1, \ldots, k$, *are non-negative constants such that* $\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} c_i > 0$ *and* β_j, $j = 1, \ldots, k$, *are pairwise distinct. Then the special flow* T_α^F *is disjoint from all weakly mixing ELF flows.*

Thus, following Kochergin [11], we obtain the following.

COROLLARY 14. *Let* M *be a smooth orientable surface with negative Euler characteristic. Assume that* $\phi = \{\phi_t\}_{t \in \mathbb{R}}$ *is a* C^3*-flow for which the only isolated critical points are non-degenerate saddles and which has a positive* C^1*-measure* ν *invariant and ergodic with respect to* ϕ. *Suppose that* ϕ *has a smooth closed transversal whose first return map is monotone. If* $\alpha(\phi)$ *satisfies* (13) *then the flow* (M, ν, ϕ) *is disjoint from all weakly mixing ELF flows.*

7. Special flows over exchange of three intervals. In this section we consider special flows built over three-interval exchange transformations and under LOGSYM$_+$-functions. Applying Rauzy induction (see [13]) we will represent every three-interval exchange transformation as a certain integral transformation over a rotation on the circle. This will allow us to see every special flow over a three-interval exchange transformation and under a LOGSYM$_+$-function as the special flow over a rotation on the circle and under another LOGSYM$_+$-function.

Let $\varDelta := \{(x, y) \in (0, 1) \times (0, 1) : x + y < 1\}$. Given any $(\alpha, \beta) \in \varDelta$ we denote by $T_{\alpha,\beta} : [0, 1) \to [0, 1)$ the symmetric exchange of three intervals $[0, \alpha)$, $[\alpha, \alpha + \beta)$ and $[\alpha + \beta, 1)$, i.e.

$$T_{\alpha,\beta} = T_{(\alpha,\beta,1-\alpha-\beta),\pi,(1,1,1)},$$

where $\pi(1) = 3$, $\pi(2) = 2$ and $\pi(3) = 1$. Next define

$$a(\alpha,\beta) = \begin{cases} 1-\alpha & \text{if } \alpha \le 1-\alpha-\beta, \\ \alpha+\beta & \text{if } \alpha > 1-\alpha-\beta, \end{cases}$$

$$A(\alpha,\beta) = [0, a(\alpha,\beta)),$$

$$B(\alpha,\beta) = \begin{cases} [0,\alpha) & \text{if } \alpha \le 1-\alpha-\beta, \\ [2\alpha+\beta-1,\alpha) & \text{if } \alpha > 1-\alpha-\beta, \end{cases}$$

and the function $R : \varDelta \to [0,1)$ by

$$R(\alpha,\beta) = \begin{cases} \dfrac{1-2\alpha-\beta}{1-\alpha} & \text{if } \alpha \le 1-\alpha-\beta, \\ \dfrac{1-\alpha}{\alpha+\beta} & \text{if } \alpha > 1-\alpha-\beta. \end{cases}$$

Given any $0 \le a < 1$ let $M_a : [0,1) \to [0,a)$ stand for the linear scaling $M_a x = ax$. It is easy to check that the induced transformation $(T_{\alpha,\beta})_{A(\alpha,\beta)}$ is isomorphic via the linear scaling $M_{a(\alpha,\beta)}$ to the rotation of the circle by $R(\alpha,\beta)$. Moreover, the first return time map $\tau_{A(\alpha,\beta)}$ equals $1+\chi_{B(\alpha,\beta)}$. Consequently, $T^F_{\alpha,\beta}$ is isomorphic to $T^{F_{\alpha,\beta}}_{R(\alpha,\beta)}$ (see the last sentence of Section 3), where

$$F_{\alpha,\beta}(x) = \begin{cases} F(M_{a(\alpha,\beta)}x) + F(M_{a(\alpha,\beta)}x + 1 - \alpha) & \text{if } x \in M^{-1}_{a(\alpha,\beta)}B(\alpha,\beta), \\ F(M_{a(\alpha,\beta)}x) & \text{otherwise.} \end{cases}$$

It is clear that if $(\alpha,\beta) \in \varDelta$ and $F \in \mathrm{LOGSYM}_+$, then $F_{\alpha,\beta} \in \mathrm{LOGSYM}_+$. In view of Theorem 13, we have the following.

THEOREM 15. *Let $F : \mathbb{T} \to \mathbb{R}$ be a positive function of the form*

$$F(x) = G(x) - \sum_{i=1}^{k} (b_i \log(\{x-\beta_i\}) + c_i \log(\{\beta_i - x\})),$$

where $G : \mathbb{T} \to \mathbb{R}$ is a function of bounded variation, b_i, c_i, $i = 1,\ldots,k$, *are non-negative constants such that* $\sum_{i=1}^k b_i = \sum_{i=1}^k c_i > 0$, *and* $\beta_j \in \mathbb{T}$, $j = 1,\ldots,k$, *are pairwise distinct. If* $R(\alpha,\beta)$ *satisfies* (13), *then the special flow* $T^F_{\alpha,\beta}$ *is disjoint from all weakly mixing ELF flows.*

REMARK 3. Since the set $\mathcal{D} \subset \mathbb{T}$ of all irrational numbers satisfying (13) has full Lebesgue measure and is G_δ and dense, it is easy to check that the set $R^{-1}\mathcal{D} \subset \varDelta$ has full Lebesgue measure and contains a G_δ and dense subset as well. Consequently, for a typical $(\alpha,\beta) \in \varDelta$ and for every $F \in \mathrm{LOGSYM}_+$ the special flow $T^F_{\alpha,\beta}$ is disjoint from all weakly mixing ELF flows.

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Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: fraczek@mat.uni.torun.pl
mlem@mat.uni.torun.pl

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