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Conformal measures and matings between Kleinian groups and quadratic polynomials

by

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Abstract. Following results of McMullen concerning rational maps, we show that the limit set of matings between a certain class of representations of $C_2 * C_3$ and quadratic polynomials carries δ-conformal measures, and that if the correspondence is geometrically finite then the real number δ is equal to the Hausdorff dimension of the limit set. Moreover, when f is the limit of a pinching deformation $\{f_t\}_{0\leq t<1}$ we give sufficient conditions for the dynamical convergence of $\{f_t\}$.

1. Introduction. An $m : n$ *holomorphic correspondence* is a multivalued map f on the Riemann sphere defined as $f : z \mapsto w$ if $p(z,w) = 0$ for a polynomial p of degree m in z and n in w. The theory of iterated holomorphic correspondences can be seen as a generalisation of both the theories of iterated rational maps and Kleinian groups: the grand orbits of a point under a degree d rational map $z \mapsto P(z)/Q(z)$ are the same as its grand orbits under the $d : 1$ correspondence $z \mapsto w$ if $wQ(z) - P(z) = 0$; and the orbit of a point under a finitely generated Kleinian group $G = \langle g_1, \ldots, g_k\rangle$ is the same as the grand orbit of the point under the $k : k$ correspondence $h_G : z \mapsto w$ if

$$(g_1(z) - w) \cdots (g_k(z) - w) = 0.$$

In [6] Bullett and Penrose introduced a 1-parameter family $\mathcal{F}$ of holomorphic 2 : 2 correspondences which are *matings* between the modular group $\mathrm{PSL}(2,\mathbb{Z})$ and degree two maps:

Definition 1. A 2 : 2 correspondence f is called a *mating* between $\mathrm{PSL}(2,\mathbb{Z})$ and a degree 2 map g_f if the action of f partitions the Riemann sphere into two completely invariant sets Ω and Λ such that:

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(1) Ω is open, simply connected and f restricted to Ω is a $2:2$ correspondence conformally conjugate to

$$h_{\mathrm{PSL}(2,\mathbb{Z})} : z \mapsto w \quad \text{if} \quad (\sigma\varrho(z) - w)(\sigma\varrho^2(z) - w) = 0$$

acting on the open upper half-plane, where $\sigma : z \mapsto -1/z$ and $\varrho : z \mapsto -1/(z+1)$ form a generating set for $\mathrm{PSL}(2,\mathbb{Z})$;

(2) there exists an involution J associated to f such that J restricted to Ω is conformally conjugate to σ;

(3) $\Lambda = \Lambda_+ \cup \Lambda_-$, where $\Lambda_+ \cap \Lambda_- = \{p\}$, and p is fixed by f;

(4) f restricted to Λ_- as domain and range is a holomorphic $2:1$ map denoted by g_f;

(5) $J(\Lambda_-) = \Lambda_+$ and J conjugates the action of f on Λ_- to that of f^{-1} on Λ_+;

(6) the remaining branch of f on Λ_- sends it homeomorphically to Λ_+.

Figures 1 and 2 show examples of matings in $\mathcal{F}$.

The correspondences in the family $\mathcal{F}$ introduced by Bullett and Penrose can be normalised to have the form $J \circ \mathrm{Cov}_0^Q$, where J is an involution and Q is the cubic polynomial $Q(z) = z^3 - 3z$ (this notation will be explained later). It then follows that p is a fixed point of both J and Cov_0^Q. Thus $\mathcal{F}$ is a one-complex-parameter family, the parameter being the "free" fixed point of J. We write f_a for the correspondence in $\mathcal{F}$ given by the parameter a.

CONJECTURE 1. *Let $f \in \mathcal{F}$ and let g_f be the $2:1$ restriction $f : \Lambda_- \to \Lambda_-$. Then g_f is conjugate to a quadratic polynomial $q_c : z \to z^2 + c$ acting on its (connected) filled Julia set. The conjugacy is conformal on interiors. Conversely, for any c in the Mandelbrot set, there exists a correspondence $f \in \mathcal{F}$ which mates* $\mathrm{PSL}(2,\mathbb{Z})$ *and q_c. The set $\mathcal{M} = \{a \in \mathbb{C} : f_a \in \mathcal{F}\}$ is homeomorphic to the Mandelbrot set.*

In Sections 3 and 4 of this paper we shall follow the work of McMullen in [10] to show that for any correspondence $f \in \mathcal{F}$ there is a unique normalised δ-conformal measure μ supported on $\partial\Lambda$. If f is *geometrically finite* then μ is supported on the *radial limit set* $L_{\mathrm{rad}}(f) \subset \partial\Lambda$, and the real number δ is equal to the Hausdorff dimension of $\partial\Lambda$.

Bullett and Haïssinsky [4] recently proved Conjecture 1 for a wide subclass of $\mathcal{F}$. As we shall see later, the obstruction to proving this result in general is the fact that the sets Λ_+ and Λ_- meet in the point p, which we shall refer to as the *pinch point*. This difficulty can be avoided if we do not insist that the group involved in the mating is the modular group, and allow its limit set to become totally disconnected. Then it is actually possible to construct a mating involving *any* quadratic polynomial q_c with c in the Mandelbrot set. For this purpose we consider the set of Kleinian groups with

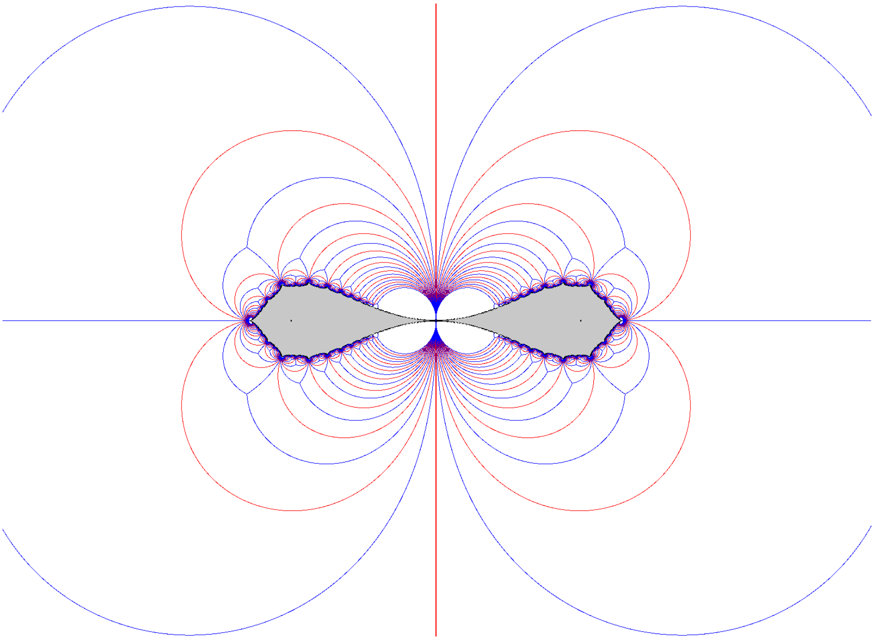

Fig. 1. This is a mating between $\mathrm{PSL}(2,\mathbb{Z})$ and $z \mapsto z^2$.

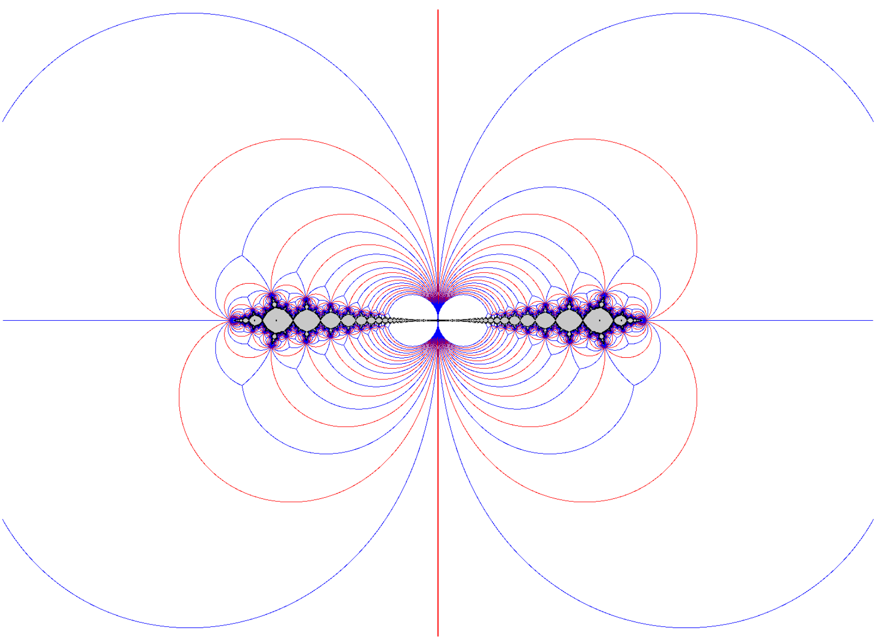

Fig. 2. This is a mating between $\mathrm{PSL}(2,\mathbb{Z})$ and $z \mapsto z^2 - 1$.

connected ordinary set which are faithful representations of the free product $C_2 * C_3$. There is a one-complex-parameter family of these groups, each having a Cantor limit set. In parameter space these groups define an open topological disc $\mathcal{U}$, and the modular group (itself being a representation of $C_2 * C_3$) corresponds to a cusp point on the boundary of $\mathcal{U}$. In fact, all the groups in $\mathcal{U}$ are quasi-Fuchsian, and the ordinary set of each group contains two completely invariant topological discs.

Definition 2. Let G be a group given by a parameter in the interior of $\mathcal{U}$ and let q_c be a quadratic polynomial with connected Julia set. A $2:2$ correspondence f *realises a mating* between G and q_c if the sphere is partitioned into an open simply connected set Ω, two disjoint closed and simply connected sets Λ_+ and Λ_- and a set $\mathcal{C}$ of curves such that:

(1) Ω is completely invariant and f restricted to Ω is a $2:2$ correspondence conformally conjugate to

$$h_G : z \to w \quad \text{if} \quad (\sigma\varrho(z) - w)(\sigma\varrho^2(z) - w) = 0,$$

restricted to a simply connected completely invariant subset of its ordinary set; here σ and ϱ are the order 2 and 3 generators of G respectively;

(2) there exists an involution J associated to f such that J restricted to Ω is conformally conjugate to σ;

(3) $\Lambda = \Lambda_+ \cup \Lambda_-$ is completely invariant and f restricted to Λ_- as domain and range is a holomorphic $2:1$ map conjugate to q_c restricted to its filled Julia set, the conjugacy being conformal on interiors with $\overline{\partial} = 0$ a.e. on Λ_-;

(4) $J(\Lambda_-) = \Lambda_+$ and J conjugates the action of f on Λ_- to that of f^{-1} on Λ_+;

(5) the remaining branch of f on Λ_- sends it homeomorphically to Λ_+;

(6) the set $\mathcal{C}$ of curves is completely invariant under f; it consists of the orbit under f of a curve γ connecting Λ_+ and Λ_- with end-points corresponding to the β-fixed point of q_c.

See Figure 3. Bullett and Harvey proved in [5]:

Theorem 1. *For any group G given by a parameter in $\mathcal{U}$ and any quadratic polynomial q_c with connected Julia set, there exists a $2:2$ correspondence f which realises a mating between the two. Up to Möbius conjugacy, f has the form $J \circ \mathrm{Cov}_0^Q$.*

A mating of this form is best understood as follows: if we remove from $\widehat{\mathbb{C}}$ the sets Λ_+ and Λ_- then we are left with a topological annulus. Cutting along the curves in $\mathcal{C}$ now turns this annulus into a topological disc. On this disc the correspondence is conjugate to the group acting on a simply

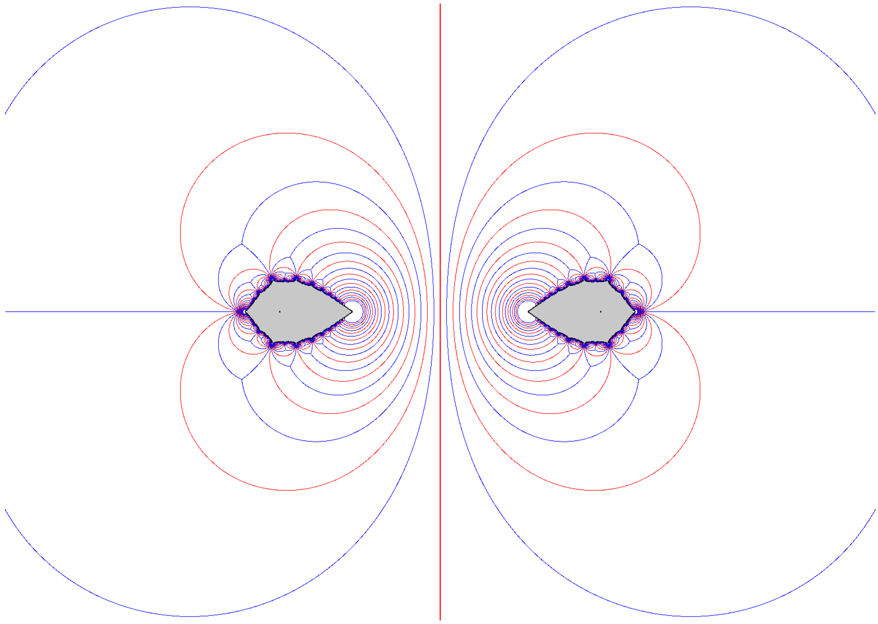

Fig. 3. This is an unpinched mating involving $z \mapsto z^2$. The line γ connects the cusps of the two grey regions Λ_+ and Λ_-.

connected subset of its regular set. The gaps in the Cantor limit set of the groups correspond to the curves in $\mathcal{C}$. Also notice that if we shrink or "pinch" the curve γ to a point then we expect to end up with a mating as described in Definition 1 which satisfies Conjecture 1. For this reason we refer to matings satisfying Definition 1 as *pinched matings* and to those satisfying Definition 2 as *unpinched matings*.

In [4] Bullett and Haïssinsky formalised this idea (for technical reasons they had to make an assumption about the nature of the quadratic polynomial involved):

Theorem 2. *Let q_c be a weakly hyperbolic quadratic polynomial with connected Julia set. Also assume that if the critical point 0 of q_c is recurrent then the β-fixed point of q_c is not in its ω-limit set. Let G_0 be a group corresponding to a parameter in $\mathcal{U}$ and let f_0 be a correspondence mating the two. Then there exists a path $\{f_t\}_{0\le t<1}$ of correspondences such that as $t \to 1$ the f_t converge uniformly to a correspondence $f \in \mathcal{F}$, which is a mating between q_c and the modular group in the sense of Definition 1.*

Such a path $\{f_t\}_{0\le t<1}$ is called a *pinching deformation* of f_0.

In Section 5 of this paper we will show that if q_c is geometrically finite and if $\{f_t\}_{0\le t<1}$ is a pinching deformation of a correspondence f_0 mating q_c with some group, then the limit sets $\Lambda_t = \Lambda_+^t \cup \Lambda_-^t$ converge to the limit set Λ of f in the Hausdorff topology. Under certain conditions we can also show that the Hausdorff dimensions of the limit sets $\partial\Lambda_t$ vary continuously with t and converge to the Hausdorff dimension of $\partial\Lambda_1$.

Most of the proofs in this paper derive from those given by McMullen for geometrically finite rational maps in [10].

2. Properties of matings. It turns out that correspondences which represent matings (pinched or unpinched) have a convenient description in terms of *covering correspondences*:

Definition 3. Let P be a polynomial of degree d. The *covering correspondence* Cov^P of P is the $d:d$ correspondence $\mathrm{Cov}^P : z \mapsto w$ if $P(z) - P(w) = 0$. It sends a point z to all the points which have the same image as z under P. The *deleted covering correspondence* Cov_0^P is the $d-1 : d-1$ correspondence $\mathrm{Cov}_0^P : z \mapsto w$ if $(P(z) - P(w))/(z-w) = 0$.

Note that each point has a finite grand orbit under Cov^P of size d. Critical points are fixed by both Cov^P and Cov_0^P and co-critical points (points that map to the same image as a critical point) have fewer than d (or $d-1$) images under Cov^P (or Cov_0^P). Away from critical and co-critical points of P, the action of Cov^P is reminiscent of the action of a cyclic group of order d.

Definition 4. By the *composition* $J \circ f$ of a correspondence f and a homeomorphism J we mean the correspondence $z \mapsto w$ if $z \mapsto J^{-1}(w)$ under f.

Definition 5. A *transversal* D_P for Cov_0^P is a maximal domain of injectivity of P. We have $\mathrm{Cov}_0^P(D_P) \cap D_P = \emptyset$ and $\mathrm{Cov}^P(\overline{D}_P) = \widehat{\mathbb{C}}$.

Consider the cubic $Q(z) = z^3 - 3z$. Then Cov_0^Q is the $2:2$ correspondence

$$z \mapsto w \quad \text{if} \quad z^2 + zw + w^2 = 3.$$

The finite critical points of Q are 1 and -1 with co-critical points -2 and 2.

The following two results are proved in [3] and [5] respectively:

Theorem 3. *A $2:2$ correspondence f represents an unpinched mating between a degree 2 holomorphic map and a group if and only if f is of the form $f = J \circ \mathrm{Cov}_0^Q$, where $Q(z) = z^3 - 3z$ and J is an involution with the following properties*:

(i) *there exists a fundamental domain D_J of J and a transversal D_Q of Q containing the point 2, such that $D_J^0 \cup D_Q^0 = \widehat{\mathbb{C}}$ (where D^0 denotes the interior of a set D)*;

(ii) *the point 2 is contained in the set $\Lambda_+ = \bigcap_{i=0}^{\infty} f^i(\widehat{\mathbb{C}} - D_J)$.*

THEOREM 4. *A* $2 : 2$ *correspondence* f *represents a pinched mating between a degree* 2 *holomorphic map and the modular group* $\mathrm{PSL}(2,\mathbb{Z})$ *if and only if* f *is of the form* $f = J \circ \mathrm{Cov}_0^Q$, *where* $Q(z) = z^3 - 3z$ *and* J *is an involution with the following properties*:

(i) *there exists a fundamental domain* D_J *of* J *and a transversal* D_Q *of* Q *containing the point* 2, *such that* $D_J^0 \cup D_Q^0 = \widehat{\mathbb{C}} - \{1\}$;

(ii) *the point* 1 *is a fixed point of* J;

(iii) *the point* 2 *is contained in the set* $\Lambda_+ = \bigcap_{i=0}^{\infty} f^i(\widehat{\mathbb{C}} - D_J)$.

Moreover, *the conjugacy* ϕ *from the upper half-plane to* Ω *extends to the points* 0 *and* ∞ *and sends both of them to the point* $1 = p = \Lambda_+ \cap \Lambda_-$.

Notice that in the case of an unpinched mating the set Λ_+ is the filled Julia set of a quadratic-like map. Let $D = \widehat{\mathbb{C}} - D_J$. The restriction of f to D is a $1 : 2$ map, and $f(D) \subset D$. Thus the inverse map g_f restricted to $f(D)$ is quadratic-like and Λ_+ is its filled Julia set. By Douady and Hubbard's straightening theorem [8] it follows immediately that on $f(D)$ the map g_f is quasi-conformally conjugate to a unique quadratic polynomial and that the conjugacy sends Λ_+ to the (connected) filled Julia set of the quadratic with $\overline{\partial} = 0$ a.e. on Λ_+. That is, the conjugacy is conformal if Λ_+ has interior.

In the case of a pinched mating we have a slightly different situation: let $D = \widehat{\mathbb{C}} - D_J$. The restriction of f to D is still $1 : 2$, but now we have $\overline{f(D)} \subset \overline{D}$ with $\partial D \cap \partial f(D) = \{1\}$. The inverse map g_f restricted to $f(D)$ is a degree 2 map, but not quite quadratic-like because the boundaries of D and $f(D)$ touch. This fact is the main obstacle to a complete proof of Conjecture 1, as here the straightening theorem cannot be applied.

We call such a map *pinched-quadratic-like* with pinch-point $1 = p = \partial D \cap \partial f(D)$. The set Λ_+ is the filled Julia set of g_f.

See Figure 4.

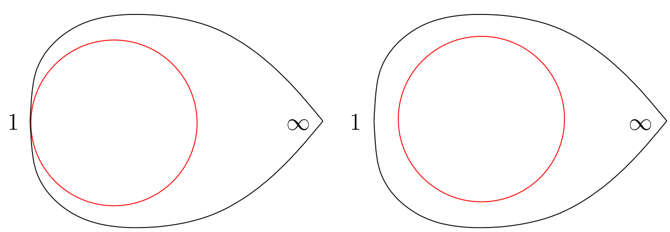

Fig. 4. The left-hand picture shows the regions D_Q and D_J for a pinched mating. D_Q is the inside of the outer curve, containing the point 1 on the left and ∞ on the right, and D_J is the outside of the inner circle. The right-hand figure shows these two regions for an unpinched mating. In both cases the inside of the inner circle maps $2 : 1$ under f^{-1} onto D_Q.

2.1. *Special points.* The point $p = 1$ and the "singular points" ± 2 play special roles in the dynamics of correspondences which represent matings. The following lemmas are easy to prove:

LEMMA 1. *Let $f = J \circ \mathrm{Cov}_0^Q$ be an unpinched mating and let g_f denote the inverse of f restricted to $f(D)$, where $D = \widehat{\mathbb{C}} - D_J$.*

(i) *The branch of f sending $p = 1$ to $J(-2)$ has critical point p. Thus in any neighbourhood of p this branch is a $2:1$ map.*
(ii) *The points 2 and -2 have unique images $J(-1)$ and $J(1)$ under f. Since $2 \in \Lambda_+$, it follows that 2 is the critical value of the map g_f, with critical point $J(-1)$.*

LEMMA 2. *Let $f \in \mathcal{F}$ be a pinched mating and let g_f denote the inverse of f restricted to Λ_+.*

(i) *The point $p = 1$ is fixed by one branch of f with derivative 1. For all but one correspondence in $\mathcal{F}$ (up to conjugacy), p has one petal. For the exceptional correspondence f, p has three petals and the pinched-quadratic-like map g_f has a unique fixed point. In this case g_f is conjugate to $z \mapsto z^2 + 1/4$.*
(ii) *The other branch of f sends 1 to $J(-2) \in \Lambda_+$ with derivative 0. The map g_f is $1:2$ on any neighbourhood of $J(-2)$. See Figure 5.*

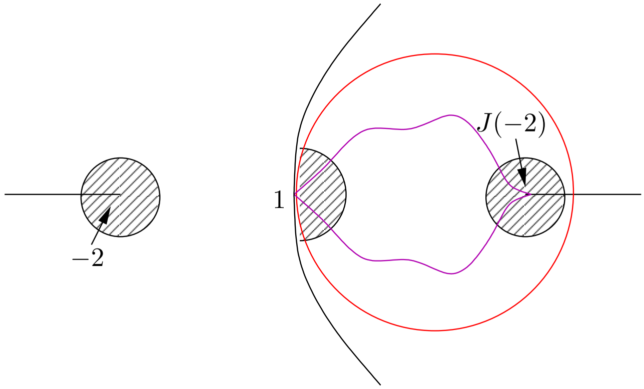

Fig. 5. This figure represents a pinched mating. The shaded region around the point 1 maps $2:1$ onto the cut disc at -2 under Cov_0^Q and then to the shaded disc at $J(-2)$ under J. The round circle represents ∂D_J, the closed curve within it represents $\partial \Lambda_+$, and the line tangent to the circle represents ∂D_Q.

(iii) *The point 2 has unique image $J(-1)$ under f and the point -2 has unique image $J(1)$ under f. The critical value of g_f is the point 2, with critical point $J(-1)$. Throughout, we denote this critical point $J(-1)$ by ω. See Figure 6.*

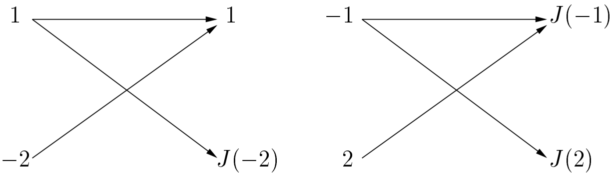

Fig. 6. These diagrams show how the singular points map to each other.

For an introduction to parabolic fixed points and petals see Chapter 6 of [1].

If a set E does not contain any forward (resp. backward) singular point of f, then we say that f (resp. f^{-1}) has two *single-valued branches* on E.

2.2. *Properties of* $\partial\Lambda_+$*.* In this section we list some useful results about $\partial\Lambda_+$. In the case of an unpinched mating f these results follow immediately from the fact that Λ_+ in this case has the properties of the filled Julia set of a quadratic map. For a pinched mating $f \in \mathcal{F}$ however, we need to give separate proofs.

Due to symmetry, corresponding results hold for $\partial\Lambda_-$.

PROPOSITION 1. *Let* $f \in \mathcal{F}$ *be a pinched mating and let* g_f *denote the branch of* f^{-1} *sending* $f(D)$ *to* D (*where* $D = \widehat{\mathbb{C}} - D_J$).

(a) *Let* $z \in D - \Lambda_+$*. The sets* $H_n = f^n(z)$ *converge to* $\partial\Lambda_+$ *in the Hausdorff topology.*
(b) *Given any open set* U *meeting* $\partial\Lambda_+$ *there exists a subset* S *of* $\partial\Lambda_+$ *contained in* U *and an integer* M *such that* $\partial\Lambda_+ \subset g_f^M(S)$.
(c) *For any* $z \in \partial\Lambda_+$ *the orbit* $f^n(z)$ *is dense in* $\partial\Lambda_+$.

Proof. (a) We recall the definition of convergence of compact sets in the Hausdorff topology. Let K_n be a sequence of compact subsets of $\widehat{\mathbb{C}}$. We define $\liminf K_n$ to be the set of points x such that every neighbourhood of x meets all but finitely many K_n, and we define $\limsup K_n$ to be the set of points x such that any neighbourhood of x meets infinitely many K_n. Then $K_n \to K$ if and only if

$$\liminf K_n = \limsup K_n = K.$$

It is obvious that any convergent sequence $\{y_n \in f^{k_n}(z)\}$ accumulates on $\partial\Lambda_+$, so $\limsup H_n \subset \partial\Lambda_+$. To show that $\partial\Lambda_+ \subset \liminf H_n$ we must show that for any $y \in \partial\Lambda_+$ and $\varepsilon > 0$ and for all n sufficiently large, H_n meets the ε-neighbourhood N of y. Consider a connected component U of $N \cap \Omega$. Let γ be a boundary component of U that lies in Ω. Let ϕ be the conjugacy from the upper half-plane to Ω which conjugates the generators $\sigma\varrho$ and $\sigma\varrho^2$ of $\mathrm{PSL}(2,\mathbb{Z})$ to the two branches of f. By Proposition 2.14 of [13], every boundary component of U which lies in Ω maps under ϕ^{-1} to a curve in the upper half-plane with distinct end-points in $\widehat{\mathbb{R}}_+$, so $\partial\phi^{-1}(U)$ contains real intervals.

Now $\phi^{-1}(z)$ is a point in the upper half-plane and since the limit set of the modular group is $\mathbb{R} \cup \infty$, it can be shown that for each integer n sufficiently large, there exists a finite sequence $i_1 \ldots, i_n$ of 1's and 2's such that

$$\sigma\varrho^{i_1} \cdots \sigma\varrho^{i_n}(\phi^{-1}(z)) \in \phi^{-1}(U).$$

The result now follows since each $\sigma\varrho^i$ is conjugate to a branch of f.

(b) Let U be an open set meeting $\partial\Lambda_+$ and let V denote a connected component of $U \cap \Omega$. As above, $\phi^{-1}(V)$ is an open set in the upper half-plane partially bounded by real intervals. It is a basic property of the modular group that any interval in the positive real line contains a subinterval of the form

$$[\sigma\varrho^{i_1} \cdots \sigma\varrho^{i_M}(0), \sigma\varrho^{i_1} \cdots \sigma\varrho^{i_M}(\infty)],$$

where the i_j are either 1 or 2, and M is a positive integer. Let $[a, b]$ be such a subinterval of $\partial\phi^{-1}(V)$. Let γ be a curve in the upper half-plane, contained in $\phi^{-1}(V)$ with end-points a and b, and let $\mathcal{U}'$ be the region bounded by γ and $[a, b]$.

By the last assertion of Theorem 4, $\phi(\gamma)$ is a curve in V with end-points on $\partial\Lambda_+$. The boundary of $U' = \phi(\mathcal{U}')$ consists of $\phi(\gamma)$ and a subset S of $\partial\Lambda_+$ which lies in U. Changing the curve γ if necessary (but not its end-points) we can ensure that the open set $g_f^i(U')$ does not contain $J(-2)$ or $J(\infty)$ for any $1 \le i \le M$. So g_f^M is an analytic homeomorphism on U'. Moreover, $g_f^M(\phi(\gamma))$ is a simple closed curve meeting $\partial\Lambda_+$ only in p. Since $g_f^M(U')$ is simply connected it follows that $g_f^M(S) = \partial\Lambda_+$.

(c) This follows immediately from (b). ∎

LEMMA 3. *Let $f \in \mathcal{F}$ be a pinched mating and let $q \in \partial\Lambda_+$ be a parabolic periodic point of g_f of period k. Then there exists a neighbourhood N of q such that each component of $N \cap \partial\Lambda_+ - \{q\}$ is contained in a repelling petal for g_f^k at q.*

Proof. We can analytically continue the branch h of f^{-k} which fixes q to a neighbourhood N of q. Its dynamics gives rise to attracting petals. If a component C of $\partial\Lambda_+ \cap N - \{q\}$ is contained in an attracting petal, then so is a point $x \in \Omega \cap N$, since petals are open. If $q \neq p$, this contradicts the fact that all iterates of x under $g_f = f^{-1}$ accumulate at $\partial\Lambda_-$. If $q = p$ then, by the last assertion of Theorem 4, the dynamics around p can be transferred via the map ϕ to dynamics around the points 0 and ∞ in the boundary of the upper half-plane. It is then easy to check that C being contained in an attracting petal contradicts the action of $\sigma\varrho$ and $\sigma\varrho^2$ near 0 and ∞. ∎

LEMMA 4. *Let $f \in \mathcal{F}$ be a pinched mating. Suppose the critical point ω of g_f is contained in $\partial \Lambda_+$ and its orbit under g_f eventually lands on a periodic cycle, but not on p. Then this cycle is repelling.*

Proof. Suppose that the cycle has period k, that it is non-repelling and that q is a point of the cycle. Suppose that there exists a neighbourhood U of q such that for all n the branch h_n of f^{nk} which fixes q is an analytic homeomorphism on U. Since $h_n(x) \to \Omega$ as $n \to \infty$ for any $x \in \Omega$ and since Λ_+ maps into itself under each h_n, we see that the images of U under h_n miss out more than three points of the sphere and hence the family $\{h_n\}$ is a normal family. Let ϕ be the limit of a converging subsequence $\{h_{n_j}\}$. Then ϕ is injective or constant and $\phi(q) = q$. Moreover, for all $x \in \Omega$ there exists N such that $x \notin h_n(U)$ for all $n > N$ and hence $\phi(U)$ does not meet Ω. But if ϕ is not constant then q lies in the interior of $\phi(U)$, so ϕ must be constant with value q.

But this implies that q is a repelling periodic point of g_f, contradicting our assumption. Therefore, either $q = p$ or the orbit of a critical point of g_f other than ω accumulates at q. But ω is the only critical point of g_f, a contradiction. ■

LEMMA 5. *Let $f \in \mathcal{F}$ be a pinched mating. Now suppose that for some k we have $\omega \in f^k(p)$ and let S denote the branch of f^k sending $p=1$ to ω. Then we can extend S to an analytic homeomorphism in a neighbourhood of p.*

Proof. Let N' be a neighbourhood of the critical value $g_f(\omega) = 2$ and denote by N the open neighbourhood arising from N' by removing a curve γ connecting $\partial N'$ and 2, together with its end-points. Let T_1 and T_2 denote the two branches of g_f^{k-1} on N such that p lies on the boundary of each $T_i(N)$. Similarly, let S_1 and S_2 denote the two branches of f defined on N. Define the maps $S_i T_i^{-1} : T_i(N) \to S_i(N)$. These can be continuously extended to the images of the curve γ under T_1 and T_2 to give a continuous map S from a neighbourhood of p to a neighbourhood of ω. At any point apart from p, S is the composition of two holomorphic maps and hence is also holomorphic. The fact that it is holomorphic at p itself follows since S is bounded. ■

3. The radial limit set. Many of the results of this paper concern matings for which the map g_f is "geometrically finite".

DEFINITION 6. We say that a correspondence f which represents a mating, pinched or unpinched, is *geometrically finite* if

(1) the intersection of the "post-critical set" $P(f) = \overline{\{g_f^n(\omega) : n \in \mathbb{N}\}}$ (recall that ω is the critical point of g_f) with $\partial \Lambda_+$ is finite;
(2) there are no irrationally indifferent periodic points of g_f in $\partial \Lambda_+$.

This definition may be somewhat surprising, since for rational maps the definition of geometrically finite only requires a weaker version of point (1), namely that the orbits of any critical points in the Julia set be finite. However, for rational maps, this immediately implies points (1) and (2) in the definition above. For pinched matings though, the corresponding result is not immediately obvious, since we cannot assume that $\partial\Lambda_+$ is a genuine quadratic filled Julia set (unless the mating is a limit of a pinching deformation) and therefore do not have the extensive list of results that describe the structure of quadratic filled Julia sets. Rather than attempting to prove these results here, we have simply taken points (1) and (2) as the definition of geometrically finite.

Definition 7. Let f be a mating, pinched or unpinched, let r be a positive real number and let $x \in \partial\Lambda_+$. We say that $x \in L_{\mathrm{rad}}(f,r)$ if for all $\varepsilon > 0$ there exists a neighbourhood U of x with $\mathrm{diam}(U) < \varepsilon$ and a positive integer n such that g_f^n is an analytic homeomorphism on U and $g_f^n(U) = B(g_f^n(x), r)$, the ball of radius r and centre $g_f^n(x)$.

Define the radial limit set $L_{\mathrm{rad}}(f) = \bigcup_r L_{\mathrm{rad}}(f,r)$.

Lemma 6. *The radial limit set does not contain any parabolic periodic points or the critical point of g_f, or any of their pre-images* (*this includes the pinch-point p if f is a pinched mating*).

Proof. If $x\in L_{\mathrm{rad}}(f,r)$ for some r then clearly $\limsup_{n\to\infty} |(g_f^n)'(x)|=\infty$, and this is not the case for any of the points mentioned. ∎

We will show

Theorem 5. *For f a pinched or unpinched geometrically finite mating*, *$\partial\Lambda_+ - L_{\mathrm{rad}}(f)$ consists of the parabolic periodic points* (*including p if f is pinched*) *and the critical point of g_f* (*if this is contained in $\partial\Lambda_+$*), *together with their inverse images under g_f.*

The proof of this is essentially the same as that of Theorem 6.5 in [10]. The main steps are the following two lemmas:

Lemma 7. *Let f be a geometrically finite mating*, *pinched or unpinched. Suppose that $x \in \partial\Lambda_+$ is not equal to the pinch-point p*, *a point in the post-critical set $P(f)$*, *or any of their pre-images under g_f. Then there exists $s > 0$ such that for all $N \in \mathbb{N}$ there exists $n > N$ such that on $B(g_f^n(x), s)$ the inverse branch of g_f^n sending $g_f^n(x)$ to x is an analytic homeomorphism.*

Proof. We assume here that f is pinched. If it is not, we can use the same argument, treating the set $\{p\}$ consisting of the pinch-point as the empty set. Let $q_1, \ldots, q_m$ be the points of $P(f) \cap \partial\Lambda_+$ and let $x_n = g_f^n(x)$. Now for some $l \geq 1$ we see that $\{q_l, \ldots, q_m\}$ forms a repelling or parabolic periodic orbit. Since the orbit of x gets repelled from this cycle and from the point p, by Lemma 3, we deduce that whenever it gets close to a point in the cycle or p it must first have come close to an inverse image y of some q_j or p which is distinct from all the q_i, $l \leq i \leq m$, and p. Hence if for some subsequence $\{n_k\}$ we have $\lim_{k\to\infty} x_{n_k} = q_j$ for some j, or $\lim_{k\to\infty} x_{n_k} = p$, then there exists a subsequence $\{n_i\}$ such that $\lim_{i\to\infty} x_{n_i} = y$, where y is not one of the q_i or p. Hence there exists $s > 0$ such that $\liminf d(p, x_{n_i}) > s$ and $\liminf d(q_j, x_{n_i}) > s$ for all $1 \leq j \leq m$, where d denotes distance in the spherical metric. Thus for all sufficiently large i the ball with centre x_{n_i} and radius s does not meet the post-critical set or p and hence the result holds. ■

LEMMA 8. *Let f be a geometrically finite pinched or unpinched mating. For every point $x \in \partial\Lambda_+$ whose orbit under g_f does not land on the pinch-point p (if f is a pinched mating) or on the post-critical set $P = P(f)$ we have*

$$\|(g_f^n)'(x)\| \to \infty$$

in the Poincaré metric on $D - \{P\}$, where $D = \widehat{\mathbb{C}} - D_J$.

Proof. Let $P_n = g_f^{-n}(P)$, a sequence of compact sets increasing in size. Now $g_f^n : g_f^{-n}(D) - P_n \to D - P$ is a proper local homeomorphism and hence a covering map. Therefore g_f^n is a local isometry from the Poincaré metric on $g_f^{-n}(D) - P_n$ to the Poincaré metric on $D - P$. Let $\iota_n : g_f^{-n}(D) - P_n \to D - P$ be the inclusion map. Then by Theorem 2.25 of [11] we have $\|\iota_n'(x)\| = o(|s\log(s)|)$, where s is the distance from x to $(D - P) - (g_f^{-n}(D) - P_n)$ in the Poincaré metric on $D - P$. By Proposition 1 we have $f^m(y) \to \partial\Lambda_+$ for all $y \in \partial D$, so the distance between x and $(D - P) - (g_f^{-n}(D) - P_n)$ tends to zero in the spherical metric and hence in the Poincaré metric on $D - P$. Therefore $\|\iota_n'(x)\| \to 0$. It now follows that the map $g_f^n \circ \iota_n^{-1}$ expands the Poincaré metric on $D - P$ and the expansion factor tends to infinity as n tends to infinity. ■

Proof of Theorem 5. By Lemma 6 no point whose orbit under g_f eventually lands on p, on a parabolic periodic point or on the critical point ω of g_f can lie in $L_{\mathrm{rad}}(f)$. Suppose that $x \in \partial\Lambda_+$ is not such a point. If the orbit of x meets the post-critical set, then it lands on a parabolic or repelling periodic point because f is geometrically finite. The former case is ruled out by our assumption and Lemma 4, hence the orbit lands on a repelling periodic point and therefore is in $L_{\mathrm{rad}}(f)$. If this orbit does not meet the post-critical set then by Lemma 7 there exists a sequence $\{n_j\}$ of integers and a real $s > 0$ such that the inverse branch h_j of $f_f^{n_j}$ sending $g_f^{n_j}(x)$ to x is an analytic homeomorphism on $B(g_f^{n_j}(x), s)$. By the Koebe distortion theorem, the image U_j of $B(g_f^{n_j}(x), s)$ under h_j satisfies $\mathrm{diam}(U_j) \asymp |(g_f^{n_j})'(x))|^{-1}$. By Lemma 8, $\|(g_f^{n_j})'(x))\|^{-1} \to 0$ as $j \to \infty$ in the Poincaré metric on

$D - P$. Hence the same is true for the spherical metric and $\operatorname{diam}(U_j) \to 0$ as $j \to \infty$. ∎

We pause for a moment to consider the relationship between $L_{\mathrm{rad}}(f)$, the radial Julia set of q_c and the radial (or conical) limit set of the group G. Recall that the *radial Julia set* of a geometrically finite quadratic polynomial q_c consists of the Julia set minus the inverse orbit of any parabolic point and the inverse orbit of the critical point if it lies in the Julia set. The *radial limit set* L_G of a finitely generated Kleinian group G consists of all limit points which are not parabolic fixed points. Therefore, if f is an un-pinched geometrically finite mating between q_c and G, then the radial limit set of f corresponds exactly to that of q_c. The group G in this case has no parabolic fixed points, so assuming that the conjugacy $\phi : \mathcal{D}_G \to \Omega$ extends to $\partial\mathcal{D}_G \to \partial\Lambda \cup \mathcal{C}$ we see that $\phi^{-1}(L_{\mathrm{rad}}(f)) \subseteq L_G$ with equality if and only if q_c has no parabolic periodic point and its critical point 0 does not lie in the Julia set.

If f is a pinched mating between q_c and $G = \mathrm{PSL}(2,\mathbb{Z})$, then the β-fixed point of q_c corresponds to the pinch-point p. So, assuming that we have a homeomorphism $\psi : \Lambda_+ \to K_c$ conjugating g_f to q_c, we see that $\psi(L_{\mathrm{rad}}(f)) \subseteq L_{\mathrm{rad}}(q_c)$ with equality if and only if the β-fixed point of q_c is parabolic. This is satisfied if and only if $c = 1/4$.

The radial limit set of the modular group consists of those points which are not in the orbit of 0 or ∞. The conjugacy $\phi : \mathbb{H} \to \Omega$ extends to 0 and ∞ and sends both to the pinch-point p. Thus, provided that ϕ extends to $\widehat{\mathbb{R}}$, we have $\phi^{-1}(L_{\mathrm{rad}}(f)) \subseteq L_{\mathrm{PSL}(2,\mathbb{Z})}$ with equality if and only if q_c has no parabolic periodic point other than possibly the β-fixed point, and its critical point either does not lie in the Julia set or lands on the β-fixed point.

4. Conformal measures

DEFINITION 8. Let f be a pinched or unpinched mating. An *α-conformal f-invariant measure* is a positive Borel regular probability measure μ supported on the Riemann sphere such that for any Borel set E and for any branch h of f or f^{-1} which is injective and single-valued on E we have

$$\mu(h(E)) = \int_E |h'(z)|^\alpha \, d\mu(z). \tag{1}$$

We also assume that the support of μ does not consist solely of the point $p = 1$ (the pinch-point) if f is pinched. The *critical dimension* $\alpha(f)$ is defined as

$$\alpha(f) = \inf\{\alpha \geq 0 : \exists \text{ an } \alpha\text{-conformal } f\text{-invariant measure supported on } \Lambda_+\}.$$

In this section we construct α-conformal measures on the limit sets $\partial\Lambda$ of pinched or unpinched matings f. If f is unpinched, then the results in this section can be proved directly by application of proofs in [10], since in this case the map g_f is quadratic-like and Λ_+ is its filled Julia set. If f is pinched we have to be a little more careful in dealing with the existence of the pinch-point p. In order to avoid having to switch from the pinched to the unpinched case all the time, we assume throughout this section that f is pinched, keeping in mind that the same results hold, and are easier to prove, for unpinched matings.

The following is an important property of conformal measures:

THEOREM 6. *Let $f \in \mathcal{F}$ be a pinched mating, and let μ be a β-conformal f-invariant measure supported on $\partial\Lambda$. Then for any $r > 0$ and any $x \in L_{\mathrm{rad}}(f,r) \subset \Lambda_+$ there exist arbitrarily small balls $B(x,s)$ such that $\mu(B(x,s)) \asymp s^\beta$, where the constants involved in the "$\asymp$" are independent of x and s.*

Proof. This is the same as Proposition 2.3 in [10]. Since it is short, we will outline the proof here. Note that for any $r > 0$ there exists a non-zero lower bound $a(r)$ for $\mu(B(x,r))$, where $x \in L_{\mathrm{rad}}(f,r)$. Let $x \in L_{\mathrm{rad}}(f,r)$. Since x is in the radial limit set and by the Koebe distortion theorem, given any $s' > 0$ there exists $0 < s < s'$ and an integer n such that $g_f^n(B(x,s))$ contains the ball $B(g_f^n(x), r/32)$. Then

$$1 \geq \mu(g_f^n(B(x,s))) \geq \mu(B(g_f^n(x), r/32)) > a(r/32).$$

Moreover, there exist constants $0 < b(r) < B(r) < \infty$ depending only on r such that for all $z \in B(x,s)$ we have $b(r)/s < |(g_f^n)'(z)| < B(r)/s$. Let h denote the branch of g_f^{-n} sending $g_f^n(x)$ to x. Then $\mu(B(x,s)) = \int_{g_f^n(B(x,s))} |h'(z)|^\beta \, d\mu(z)$. Hence

$$\mu(g_f^n(B(x,s)))B(r)^{-\beta}s^\beta < \mu(B(x,s)) < \mu(g_f^n(B(x,s)))b(r)^{-\beta}s^\beta,$$

so

$$a(r/32)B(r)^{-\beta}s^\beta < \mu(B(x,s)) < b(r)^{-\beta}s^\beta. \blacksquare$$

4.1. *Poincaré series.* Let $x \in \Omega$ be a point whose orbit under f does not land on the singular point ∞. Then for each integer n let $S_{1,n}, S_{2,n}, \ldots, S_{2^n,n}$ denote the branches of f^n at x.

DEFINITION 9. We define the *Poincaré series*

$$P_s(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2^n} |S'_{j,n}(x)|^s.$$

We also define

$$\delta(x) = \inf\{s > 0 : P_s(x) < \infty\} \quad \text{and} \quad \delta(f) = \inf\{\delta(x) : x \in \Omega\}.$$

We will see later (Theorems 7 and 11) that for f geometrically finite, $\delta(x)$ is independent of x for all $x \in \Omega$.

PROPOSITION 2. *Let* $D = \Omega \cap (\widehat{\mathbb{C}} - D_J)$ *and* $x \in D$. *Then*

(1) $\delta(x) \leq 2$;
(2) $P_2(x) < \infty$;
(3) $P_\beta(x) < \infty$ *if* x *meets the support of a* β*-conformal* f*-invariant measure.*

Proof. This follows the proof of Proposition 4.3 in [10]. Since $x \in D$ we have $f^n(x) \in D$ for all n. In particular $f^n(x) \neq \infty$ for all n. Hence there exists a ball B centred at x such that all branches of f^n are analytic homeomorphisms on B. Moreover, we can choose B so that the images of B under branches of iterates f^n are disjoint. The total spherical area of these images is finite, as they all are contained in D. But the area of each image U is proportional to the square of the derivative of the branch of f^n sending B to U at x (by the Koebe distortion theorem) and hence $P_2(x) < \infty$ and $\delta(x) \leq 2$.

Now suppose that x meets the support of a β-conformal measure μ. Then

$$\infty > \mu\Big(\bigcup_n f^n(B)\Big) = \sum_n \sum_{j=1}^{2^n} \mu(S_{j,n}(B)) \asymp P_\beta(x). \ \blacksquare$$

4.2. *Constructing conformal measures.* We now use the approach of Patterson and Sullivan (as used in Theorem 4.1 in [10] for rational maps) to construct a δ-conformal measure supported on the boundary of the set Λ for $f \in \mathcal{F}$.

THEOREM 7. *Let* $f \in \mathcal{F}$ *and let* $x \in \Omega \cap (\widehat{\mathbb{C}} - D_J)$. *Then* $\partial\Lambda$ *carries a* $\delta(x)$*-conformal* f*-invariant measure* μ *with no atoms on repelling or parabolic periodic points of* g_f *or any of their inverse images under* g_f.

Proof. By Proposition 2 we know that $\delta = \delta(x) < \infty$. We first construct a measure on $\partial\Lambda_+$. Let $s > \delta$ and for any Borel set E define

$$\mu_s(E) = \frac{1}{P_s(x)} \sum_{n=0}^{\infty} \sum_{j=1}^{2^n} |S'_{j,n}(x)|^s \delta_{S_{j,n}(x)}(E),$$

where $\delta_{S_{j,n}(x)}(E) = 1$ if $S_{j,n}(x) \in E$ and 0 otherwise.

Let E be a Borel set in $\overline{D}$ such that a branch h of f^{-1} is injective and single-valued on E and such that $h(E) \subset \overline{D}$. Then

$$\begin{aligned}\mu_s(h(E)) &= \frac{1}{P_s(x)} \sum_{n=0}^{\infty} \sum_{j=1}^{2^n} |S'_{j,n}(x)|^s \delta_{S_{j,n}(x)}(h(E)) \\ &= \frac{1}{P_s(x)} \sum_{n=0}^{\infty} \sum_{j=1}^{2^n} \frac{|(h^{-1}S_{j,n}(x))'(x)|^s}{|(h^{-1})'(S_{j,n}(x))|^s} \delta_{h^{-1}S_{j,n}(x)}(E)\end{aligned}$$

$$= \frac{1}{P_s(x)} \sum_{n=0}^{\infty} \sum_{j=1}^{2^n} |(h^{-1}S_{j,n})'(x)|^s |(h'(h^{-1}S_{j,n}(x)))|^s \delta_{h^{-1}S_{j,n}(x)}(E)$$

$$= \int_E |h'(z)|^s \, d\mu_s(z) - \begin{cases} |h'(x)|^s / P_s(x) & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, one gets the corresponding result for h a branch of f injective and single-valued on a Borel set $E \subset \overline{D}$.

If $P_s(x)$ diverges at δ, let μ be a weak accumulation point of the μ_s as $s \to \delta$. If $P_s(x)$ does not diverge at $s = \delta$, we use a standard trick to force $P_s(x) \to \infty$. As $s \to \delta$, change a large but finite number of terms in $P_s(x)$ and the definition of μ_s from $|S'_{j,n}(x)|^s$ to $|S'_{j,n}(x)|^t$, where $t = 2\delta - s$. Using the same notation as above, this gives measures μ_s satisfying

$$\int_E \min\{|h'(z)|^s, |h'(z)|^t\} \, d\mu_s(z) \le \mu_s(h(E))$$

$$\le \int_E \max\{|h'(z)|^s, |h'(z)|^t\} \, d\mu_s(z),$$

for $x \notin E$ and with $t \to \delta$ as $s \to \delta$. See [10, Chapter 4] for details. Again, let μ be a weak accumulation point of the μ_s.

In both cases, μ is a probability measure with support on $\partial\Lambda_+$. Since it is constructed using weak limits, it follows from the Riesz representation theorem that it is Borel regular. We will show that μ is a δ-conformal measure: Let A be a subset of $\partial\Lambda_+$ such that a branch h of $g_f = f^{-1}$ is injective and single-valued on A. For the moment, assume that A does not contain the critical point ω of g_f.

We cover A by open neighbourhoods $U(z)$, $z \in A$, such that

- h restricted to $U(z)$ is injective and single-valued,
- $\mu(\partial U(z)) = \mu(\partial h(U(z))) = 0$,
- $\overline{U(z)} \cap \{J(-2)\} = 0$,
- $\int_{(\bigcup U(z)) - A} |h'(z)|^\delta \, d\mu(z) < \varepsilon$ for some given $\varepsilon > 0$.

We choose a countable subcover $\{U_n\}$ and define sets $A_1 = U_1$ and $A_n = U_n - \bigcup_{k<n} U_k$. A standard result from measure theory states that if measures μ_s converge weakly to a measure μ, and if A is a Borel set with $\mu(\partial A) = 0$, then $\mu(h(A_k)) = \lim_{s\to\delta} \mu_s(h(A_k))$. Hence,

$$\mu(h(A_k)) = \lim_{s\to\delta} \int_{A_k} |h'(z)|^s d\mu_s(z).$$

But on each A_k the functions $|h'(z)|^s$ are uniformly bounded above and converge uniformly to $|h'(z)|^\delta$, so we have $\mu(h(A_k)) = \int_{A_k} |h'(z)|^\delta \, d\mu(z)$.

Now

$$\mu(h(A)) = \mu\Big(\bigcup_k h(A \cap A_k)\Big)$$
$$\leq \sum_k \mu(h(A_k)) = \sum_k \int_{A_k} |h'(z)|^\delta \, d\mu(z)$$
$$= \int_A |h'(z)|^\delta \, d\mu(z) + \sum_k \int_{A_k - A} |h'(z)|^\delta \, d\mu(z) \leq \int_A |h'(z)|^\delta \, d\mu(z) + \varepsilon.$$

On the other hand, we have

$$\infty > \mu(h(A)) = \mu\Big(\bigcup_k h(A \cap A_k)\Big)$$
$$= \sum_k \mu(h(A \cap A_k)) = \sum_k (\mu(h(A_k)) - \mu(h(A_k - A)))$$
$$\geq \sum_k \Big(\int_{A_k} |h'(z)|^\delta \, d\mu(z) - \int_{A_k - A} |h'(z)|^\delta \, d\mu(z)\Big)$$
$$= \int_{\bigcup A_k} |h'(z)|^\delta \, d\mu(z) - \int_{\bigcup A_k - A} |h'(z)|^\delta \, d\mu(z) \geq \int_A |h'(z)|^\delta \, d\mu(z) - \varepsilon.$$

Since ε was arbitrary we have $\mu(h(A)) = \int_A |h'(z)|^\delta \, d\mu(z)$ as required.

Now suppose that the critical point ω of g_f lies in A and that g_f is injective and single-valued on A. Then since $g_f'(\omega) = 0$ we have

$$\int_A |g_f'(z)|^\delta \, d\mu(z) = \int_{A-\omega} |g_f'(z)|^\delta \, d\mu(z).$$

Thus, if there is no atom at $g_f(\omega)$, then

$$\mu(g_f(A)) = \mu(g_f(A - \omega)) = \int_{A-\omega} |g_f'(z)|^\delta \, d\mu(z) = \int_A |g_f'(z)|^\delta \, d\mu(z).$$

To show that there is indeed no atom at the critical value $g_f(\omega)$, note that on a *punctured* neighbourhood of ω the map g_f is locally injective and single-valued and hence transforms the measures μ_s by the rule that gives rise to δ-conformality of μ. Since ω is a critical point, we can find neighbourhoods of ω on which the derivative of g_f is arbitrarily small. This means that for any $\varepsilon > 0$ there is a punctured neighbourhood N of $g_f(\omega)$ with $\mu_s(N) < \varepsilon$ for all s sufficiently close to δ. Moreover, we have $\mu_s(g_f(\omega)) = 0$ for all s because the μ_s do not assign any mass to points in $\partial\Lambda_+$. Hence

$$\limsup_{s\to\delta} \mu_s(N \cup g_f(\omega)) < \varepsilon,$$

so μ has no atom at $g_f(\omega)$.

Similar arguments work for h a branch of f, proving that μ is a δ-conformal measure.

Next, we show that there are no atoms at parabolic or repelling periodic points of g_f which lie in $\partial\Lambda_+$, or any of their images under f. The proof for repelling or parabolic periodic points of g_f follows from the local dynamics of the correspondence f and is the same as that for rational maps in Theorem 4.1 of [10]. The key idea here is to cover a neighbourhood of the periodic point q of g_f by *fundamental regions* for the linearised dynamics of g_f: if q is repelling, these regions are annuli A_n, nesting down to q with $g_f^n : A_n \to A_0$ satisfying $|(g_f^n)'| \asymp \lambda^n$ for some $\lambda > 1$. Using the fact that the μ_s behave like s-conformal measures we deduce that $\mu_s(U - q) = O(\lambda^{-Ns})$, where $U = \{q\} \cup \bigcup_{n=N}^{\infty} A_n$. Thus for N large enough and s sufficiently close to δ we get $\mu_s(U - q) < \varepsilon$. Since each μ_s does not have an atom at q we get $\mu(U) \leq \limsup \mu_s(U) < \varepsilon$ and hence there is no atom at q.

If q is parabolic with one petal we recall that locally g_f acts like the Möbius transformation $T : z \mapsto z/(1 - z)$ around its parabolic fixed point 0 (Chapter 2 of [7]). The boundary of an attracting petal of g_f corresponds to a curve in the T-plane which at its cusp is asymptotic to the positive real line. By Lemma 3 there exists a neighbourhood N of q such that every component of the intersection of N and $\partial\Lambda_+$ lies in a repelling petal, so we can deduce that $\partial\Lambda_+$ corresponds to a region asymptotic to the positive real line in the T-plane. We can now use the dynamics of T in a neighbourhood of the positive real line to find fundamental regions for the action of g_f near q and estimate the $|(g_f^n)'|$ on these regions, thus obtaining the result as above. If q has more petals the result can be proved similarly; for more details on the methods for the parabolic case see Theorem 4.1 of [10].

We can now deduce that there are no atoms at pre-periodic points q of g_f (which do not land on p) in the same way as in [10]. Suppose that $g_f^i(q) = g_j^{i+j}(q)$ for some $i, j > 0$. If q is not a critical point of g_f^i, in other words if its orbit does not land on the critical point ω of g_f, then an atom at q would give rise to an atom at the periodic point $g_f^i(q)$, a contradiction.

If q is a pre-critical point then, as explained in the proof of Theorem 4.1 of [10], we consider the homeomorphic branches of $g_f^{-i} \circ g_f^j \circ g_f^i(q)$ on a *punctured* neighbourhood of q to obtain the result.

Now suppose that $q \neq \omega$ is a pre-periodic point whose orbit lands on p. The pinch-point p is a parabolic fixed point of g_f (see Lemma 2), and the above arguments apply. Hence there is no atom at p, and we can find a neighbourhood U of p, not containing x, such that

$$\limsup_s \mu_s(U - \{p\}) < 2\varepsilon,$$

for any given $\varepsilon > 0$. By the construction of the μ_s, any part of U which carries positive measure lies in D_Q. Let $U_0 = U \cap D_Q$. Then $\mu(U) = \mu(U_0)$.

Let h denote the branch of f^n (for the appropriate n) sending p to q.

Since U_0 is contained in D_Q we see that h restricted to U_0 is an analytic homeomorphism. Its image is a topological disc B with a cut $\mathcal{C}$ from the point q to the boundary of B. See Figure 5. The cut $\mathcal{C}$ is the image of $\partial D_Q \cap \partial U_0$. The measures μ_s are constructed using the orbit under f of a point $x \in \Omega \cap (\widehat{\mathbb{C}} - D_J)$. The points in this orbit never land on ∂D_Q or any of its images and hence the cut $\mathcal{C}$ carries no positive measure for any of the μ_s. Hence,

$$\mu_s(B - \{q\}) = \mu_s(B - \mathcal{C}) = \int_{U_0} |h'(z)|^s \, d\mu_s(z)$$

for all s. Now p is a critical point of the branch h, so for small enough U we have $|h'(z)| < 1/2$ for $z \in U$. Then $\mu_s(B - \{q\}) < \mu_s(U_0)/2 \le \varepsilon$. This works for all s sufficiently close to δ and any given ε, so μ has no atom at q.

Now suppose that the orbit of the critical point ω of g_f eventually lands on p. Let h denote the branch of g_f^n which sends ω to p. By Lemma 5 we can extend h to an analytic homeomorphism on a neighbourhood V of ω. Hence there exists $a > 0$ such that $|h'(z)| \ge a$ for all $z \in V$. If there is an atom at the point ω then there exists $\varepsilon > 0$ such that for any neighbourhood U of the pinch-point p we have $\mu_s(h^{-1}(U)) > \varepsilon$ for all s sufficiently close to δ. Since h is injective and single-valued on $h^{-1}(U)$, we have

$$\mu_s(U) = \int_{h^{-1}(U)} |h'(z)|^s \, d\mu_s(z) \ge a^s \mu_s(h^{-1}(U)) > a^\delta \varepsilon.$$

Hence there is an atom at p, a contradiction.

So far we have constructed μ with support on $\partial\Lambda_+$. For E a Borel set meeting $\partial\Lambda_-$ we define $\mu(E) = \int_{J(E)} |J'(z)|^\delta \, d\mu(z)$. We then normalise so that $\mu(\partial\Lambda) = 1$. A simple calculation now shows that μ is a δ-conformal measure. ∎

Corollary 1. *If f is geometrically finite, then the measure μ we have constructed is supported on the radial limit set.* ∎

Theorem 8. *The Hausdorff dimension* $\mathrm{HD}(L_{\mathrm{rad}}(f))$ *of the radial limit set is equal to $\alpha(f)$.*

Proof. The fact that the Hausdorff dimension of the radial limit set is at most $\alpha = \alpha(f)$ can be proved as for rational maps in Corollary 2.4 of [10]. Suppose that μ is an α-conformal measure. Let $n \in \mathbb{N}$. For any $\varepsilon > 0$ find a point $x \in L_{\mathrm{rad}}(f, 1/n)$ and $0 < s < \varepsilon$ satisfying $\mu(B(x,s)) \asymp s^\alpha$. Inductively, define more balls $B(x_i, s_i)$ with the same property, each disjoint from the ones before. Now if for some $x \in L_{\mathrm{rad}}(f, 1/n)$ the ball $B(x,s)$ was not chosen, then it must be contained in a ball previously chosen, so $L_{\mathrm{rad}}(f, 1/n) \subset \bigcup_i B(x_i, 3s_i)$. Moreover, we have $\mu(B(x_i, s_i)) \asymp s_i^\alpha$, so

$$\sum (\mathrm{diam}(B(x_i, 3s_i)))^\alpha \asymp \sum \mu(B(x_i, s_i)) \le \mu(\partial\Lambda_+).$$

Thus the α-dimensional Hausdorff measure of $L_{\mathrm{rad}}(f, 1/n)$ is finite and the Hausdorff dimension of $L_{\mathrm{rad}}(f, 1/n)$ is at most α. This works for all n, and the result now follows because $L_{\mathrm{rad}}(f) = \bigcup_n L_{\mathrm{rad}}(f, 1/n)$.

For the proof in the other direction we recall some definitions: We say that a compact set $X \subset \partial\Lambda_+$ is *hyperbolic* if there exists an m such that for all $x \in X$ we have $\|(g_f^m)'(x)\| > 1$ (in the spherical metric) and $p \neq g_f^n(x)$ for all n. We define $\mathrm{hypdim}(f) = \sup\{\mathrm{HD}(X) : X \text{ is hyperbolic}\}$. Moreover we define $L_{\mathrm{hyp}}(f)$ to be the union of the hyperbolic sets for g_f. Then

$$\mathrm{HD}(\mathrm{hypdim}(f)) \leq \mathrm{HD}(L_{\mathrm{hyp}}(f)) \leq \mathrm{HD}(L_{\mathrm{rad}}(f))$$

since the expansion property on L_{hyp} ensures that $L_{\mathrm{hyp}}(f) \subset L_{\mathrm{rad}}(f)$. Now for a rational map R one knows that $\alpha(R) \leq \mathrm{hypdim}(R) \leq \mathrm{HD}(J_{\mathrm{rad}}(R))$, where $J_{\mathrm{rad}}(R)$ is the radial Julia set for R. We will prove the same result for our correspondence f in a very similar way, using results from [14].

The general idea is to construct measures m_n on compact subsets K_n of $\partial\Lambda_+$, which behave very much like conformal measures. The K_n tend to $\partial\Lambda_+$ as $n \to \infty$, but each K_n does not contain the inverse orbits under g_f of the pinch-point p and the critical point ω. This fact enables us to show that the dimensions of the measures m_n are at most $\mathrm{hypdim}(K_n) \leq \mathrm{hypdim}(\partial\Lambda_+)$ for each n. As $n \to \infty$ they weakly converge to a conformal measure m on $\partial\Lambda_+$ of dimension at most $\mathrm{hypdim}(\partial\Lambda_+) \leq \mathrm{HD}(L_{\mathrm{rad}}(f))$, which proves the result.

For each n we define an open set V_n as follows: if $p \in \overline{\bigcup_{n=0}^{\infty} g_f^n(\omega)} = P$ or if $\omega \notin \partial\Lambda_+$ we define V_n to be the disc of radius $1/n$ and centre p. Otherwise, define V_n to consist of two open discs $\mathcal{A}_n$ centred at p and $\mathcal{B}_n$ centred at $v_\omega = g_f(\omega) = 2$, both of radius $1/n$. We will see in the proposition following this proof that either $p \in P$, or $\limsup |(g_f^n)'(v_\omega)| > 1$. Define K_n to be the set of points in $\partial\Lambda_+$ whose orbit under g_f never enters V_n. Then K_n is compact. Clearly we have $g_f(K_n) \subset K_n$.

Choose n large enough so that g_f is injective on V_n. Then every point in K_n has at least one inverse image outside of V_n, which implies that every point in K_n also lies in $g_f(K_n)$, hence $g_f(K_n) = K_n$. The function $|g_f'|$ is bounded on each K_n and g_f can be extended analytically to a neighbourhood of K_n. This enables us to use a construction presented in Chapter 10 of [14] to obtain measures m_n supported on K_n which, regarded as measures on all of $\partial\Lambda_+$, satisfy

$$m_n(g_f(E)) = \int_E |(g_f)'(z)|^{s_n}\, dm_n(z)$$

for all Borel sets E on which g_f is injective and single-valued and which satisfy $E \cap \overline{V}_n = \emptyset$, and

$$m_n(g_f(E)) \geq \int_E |(g_f)'(z)|^{s_n}\, dm_n(z)$$

for all Borel sets E on which g_f is injective and single-valued and which satisfy $E \cap \overline{V}_n \neq \emptyset$. The real numbers s_n involved here are non-decreasing with n. Moreover, they satisfy

$$s_n \leq \operatorname{hypdim}(K_n) \leq \operatorname{hypdim}(\partial \Lambda_+) \leq \operatorname{HD}(L_{\mathrm{rad}}(f)).$$

As n tends to infinity, the measures m_n converge weakly to a measure m supported on $\partial \Lambda_+$. It is easy to show that for any Borel set E on which g_f is injective and single-valued, and which does not contain p or v_ω, we have

$$m(g_f(E)) = \int_E |g_f'(z)|^s \, dm(z),$$

where $s = \lim_{n\to\infty} s_n$. An argument similar to that used in the proof of Theorem 7 shows that in fact m has no atoms at p or its inverse orbit under g_f. Using the properties of the m_n, one can also show that $m(g_f(v_\omega)) \geq |g_f'(v_\omega)|^s \mu(v_\omega)$. However, since

$$\limsup_{n\to\infty} |(g_f^n)'(v_\omega)| = \limsup_{n\to\infty} |(g_f^n)'(g_f(v_\omega))| > 1,$$

the measure m cannot possibly ascribe any mass to $g_f(v_\omega)$ as otherwise the point masses along its orbit would add up to infinity. It follows that m is an s-conformal measure supported on $\partial \Lambda_+$ and therefore $\alpha(f) \leq \operatorname{HD}(L_{\mathrm{rad}}(f))$. ■

PROPOSITION 3. *Suppose that $\omega \in \partial \Lambda_+$. Then either $p \in P$ or*

$$\limsup_{n\to\infty} |(g_f^n)'(v_\omega)| \geq 1.$$

Proof. Suppose that $p \notin P$, so there exists $\varepsilon > 0$ such that the distance between any point in P and $J(-2)$ is greater than ε. Assume that

$$\limsup |(g_f^n)'(v_\omega)| < 1.$$

For every integer n, let r_n be the maximal real number such that the iterate g_f^n is single-valued on the disc $B_n = B(\omega, r_n)$. In other words we have $J(-2) \notin g_f^i(B_n)$ for all $0 \leq i < n$. Then $r_n \to 0$ as $n \to \infty$, as by Proposition 1 any open set meeting $\partial \Lambda_+$ maps to all of $\partial \Lambda_+$ under a finite number of iterations of g_f.

Let $\{r_{n_k}\}$ be a strictly decreasing subsequence. Then for all k there exists $n_k \leq j_k < n_{k+1}$ such that $J(-2) \in g_f^{j_k}(B_{n_k})$, so the diameter of $g_f^{j_k}(B_{n_k})$ is greater than ε for all k. Now if infinitely many of the $g_f^{j_k - 1}$ are univalent on $g_f(B_{n_k})$, then by the fact that $r_{n_k} \to 0$ and the Koebe distortion theorem, we get $\lim_{k\to\infty} |(g_f^{j_k-1})'(v_\omega)| = \infty$, contradicting our assumption.

Thus for any k, there exists $n_k \leq i_k < j_k$ such that $\omega \in g_f^{i_k}(B_{n_k})$. For k large enough, g_f is a strong contraction of B_{n_k}, since ω is a critical point of g_f. Moreover, the growth of $|(g_f^n)'(v_\omega)|$ is bounded by 1, so for k large enough, the diameter of $g_f^{i_k}(B_{n_k})$ is less than half the diameter of B_{n_k}. Since $\omega \in g_f^{i_k}(B_{n_k})$ we have $g_f^{i_k}(B_{n_k}) \subset B_{n_k}$, contradicting Proposition 1. ∎

4.3. *Dynamics on the radial limit set.* So far we have shown that the Hausdorff dimension of the radial limit set of any $f \in \mathcal{F}$ is equal to $\alpha(f)$ and that for any $f \in \mathcal{F}$ there exists a δ-conformal measure μ supported on $\partial\Lambda$ for some finite real number δ. Moreover, we know that if f is geometrically finite, then the Hausdorff dimension of $\partial\Lambda_+$ equals that of $L_{\mathrm{rad}}(f)$ (because $\partial\Lambda_+ - L_{\mathrm{rad}}(f)$ is countable) and that the measure μ is supported on the radial limit set. In this section we will prove:

THEOREM 9. *For any $f \in \mathcal{F}$ there exists at most one normalised conformal measure supported on the radial limit set. The measure is $\alpha(f)$-conformal and ergodic with respect to the action of g_f.*

THEOREM 10. *If the canonical $\alpha(f)$-conformal measure exists then*

- *$P_s(x)$ diverges at $s = \alpha(f)$ for all $x \in \overline{\Omega \cap (\widehat{\mathbb{C}} - D_J)}$;*
- *if A is a Borel set with $g_f(A) \subset A$ then A has either zero or full measure.*

THEOREM 11. *If $f \in \mathcal{F}$ is geometrically finite then*

$$\delta(f) = \mathrm{HD}(L_{\mathrm{rad}}(f)) = \mathrm{HD}(\partial\Lambda_+) = \alpha(f).$$

Moreover, the measure μ constructed in Theorem 7 is the unique normalised $\delta(f)$-conformal measure with support in $\overline{\Omega} - \{p\}$.

COROLLARY 2. *If $f \in \mathcal{F}$ is geometrically finite and μ is a conformal measure supported on $\partial\Lambda_+$ then either it is the canonical measure μ constructed in Theorem 7, or it is an atomic measure of dimension greater than $\alpha(f)$ supported on the orbit under f of parabolic periodic points and the critical point of g_f.*

COROLLARY 3. *If $f \in \mathcal{F}$ is geometrically finite then* $\mathrm{HD}(\partial\Lambda_+) < 2$.

Proof of Theorem 9. This is Theorem 5.1 of [10]. Let ν be a β-conformal measure and let μ be an $\alpha(f)$-conformal measure, both with support on the radial limit set. Let $r > 0$. By Theorem 6, for all $x \in L_{\mathrm{rad}}(f, r)$ there exist arbitrarily small balls satisfying

$$\frac{\nu(B(x,s))}{\mu(B(x,s))} \asymp \frac{s^{\beta}}{s^{\alpha(f)}}.$$

If $\beta > \alpha(f)$ then $s^{\beta-\alpha(f)} \to 0$ as $s \to 0$ and

$$\lim_{s\to 0} \frac{\nu(B(x,s))}{\mu(B(x,s))} = 0,$$

and hence

$$\nu(L_{\mathrm{rad}}(f,r)) = 0.$$

This contradicts the fact that ν is supported on the radial limit set, so $\beta = \alpha(f)$.

Now suppose that ν_1 and ν_2 are two $\alpha(f)$-conformal measures. Let E be a set such that $\nu_1(E) = 0$. Let s_n be a sequence of positive real numbers tending to 0. Then for each n we can find a cover $\mathcal{V}_n$ of E such that:

- for all $x \in E$ there exists r with $0 < r < s_n$ such that $B(x, r) \in \mathcal{V}_n$,
- $\nu_i(B(x, r)) \asymp r^\alpha$ for $i = 1, 2$, $\alpha = \alpha(f)$.

By the Besicovitch covering lemma, for each n there exists a countable subcover $\mathcal{U}_n$ of $\mathcal{V}_n$ consisting of balls which we label $B(x_{j,n}, r_{j,n})$, and such that the balls $B(x_{j,n}, r_{j,n}/a)$ are disjoint, where $a > 0$ is some constant.

As $n \to \infty$, the coverings $\mathcal{U}_n$ tend to E, hence

$$\lim_{n\to\infty} \nu_1\Big(\bigcup_{U_j \in \mathcal{U}_n} U_j\Big) = \nu_1(E) = 0.$$

Moreover, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \nu_1(B(x_{j,n}, r_{j,n}/a)) &= \nu_1\Big(\bigcup_j B(x_{j,n}, r_{j,n}/a)\Big) \\ &\le \nu_1\Big(\bigcup_j B(x_{j,n}, r_{j,n})\Big) = \nu_1\Big(\bigcup_{U_j \in \mathcal{U}_n} U_j\Big),\end{aligned}$$

so

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \nu_1(B(x_{j,n}, r_{j,n}/a)) = 0.$$

Since each $\nu_1(B(x_{j,n}, r_{j,n}/a))$ is proportional to $r_{j,n}^\alpha$, this implies that

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} r_{j,n}^\alpha = 0.$$

Now

$$\nu_2\Big(\bigcup_{U_j \in \mathcal{U}_n} U_j\Big) \le \sum_{j=1}^{\infty} \nu_2(B(x_{j,n}, r_{j,n})) \asymp \sum_{j=1}^{\infty} r_{j,n}^\alpha,$$

so

$$\nu_2(E) = \lim_{n\to\infty} \nu_2\Big(\bigcup_{U_j \in \mathcal{U}_n} U_j\Big) = 0,$$

hence ν_1 and ν_2 are absolutely continuous with respect to each other.

Next we show ergodicity of any $\alpha(f)$-conformal measure ν: Suppose that $g_f(E) = E$ and that $\nu(E) > 0$. Then ν restricted to E is also an $\alpha(f)$-conformal measure and hence absolutely continuous with respect to ν. Hence $\nu(\partial\Lambda_+ - E) = \nu|_E(\partial\Lambda_+ - E) = 0$ and E has full measure.

Now if ν_1 and ν_2 are two $\alpha(f)$-conformal measures, then their Radon–Nikodym derivative is a g_f-invariant Borel function. It is well known that this, together with ergodicity, implies that the Radon–Nikodym derivative is constant (see for example Lemma 9.1 in [12]). ∎

Proof of Theorem 10. This is Theorem 5.2 of [10]. Let μ be the canonical measure which is supported on the radial limit set. We say that a ball B' is a *descendant* of a ball B which meets $\partial\Lambda_+$ if for some $n > 0$ the map g_f^n restricted to B' is univalent and analytic with bounded distortion and image B. Let $r > 0$ be such that $\mu(J_{\mathrm{rad}}(f,r)) > 0$. One can show that there exists a finite set of balls $\{B_1, \ldots, B_n\}$ such that every $x \in J_{\mathrm{rad}}(f,r)$ is contained in infinitely many descendants of balls in that set. Now let A_i be the set of points in $J_{\mathrm{rad}}(f,r)$ which are contained in infinitely many descendants of B_i. Then for some i we have $\mu(A_i) > 0$. By the Borel–Cantelli lemma we have $\sum \mu(B') = \infty$, where the sum is taken over all descendants of B_i.

Now for any $x \in B_i$ we see that any descendant B' contains a point y which maps to x under an iterate of g_f. Let h denote the branch of f^n sending x to y. Then

$$\mu(B') = \mu(h(B_i)) = \int_{B_i} |h'(z)|^{\alpha(f)}\, d\mu(z).$$

Hence by the Koebe distortion theorem we have $\mu(B') \asymp |h'(x)|^{\alpha(f)}$. But $\sum \mu(B') = \infty$ and each branch of f^n contributes to the Poincaré series, so $P_s(x) = \infty$ at $s = \alpha(f)$ for all $x \in B_i$.

If $x \in \overline{\Omega \cap (\widehat{\mathbb{C}} - D_J)}$ then we can find a point in the orbit of x under f which lies in one of the B_i, by Proposition 1.

For the second point let A be such that $g_f(A) \subset A$ and such that $\mu(A) > 0$. One can modify the proof of the classical Lebesgue density theorem so that it works in the more general setting of finite Borel measures, and so we know that there exists a Lebesgue density point x with

$$\lim_{s\to 0} \frac{\mu(B(x,s)\cap A)}{\mu(B(x,s))} = 1$$

and $x \in L_{\mathrm{rad}}(f,r)$ for some $r > 0$. We can find sequences $\{s_n\}$ and $\{k_n\}$ such that $g_f^{k_n} : B(x,s_n) \to D_n$ is univalent and analytic with bounded distortion and $B(g_f^{k_n}(x), r/16) \subset D_n$. Moreover, $g_f^{k_n}(x) \in A$, so

$$\frac{\mu(A\cap D_n)}{\mu(D_n)} = \frac{\mu(g_f^{k_n}(A\cap B_n))}{\mu(g_f^{k_n}(B_n))} = \frac{\int_{A\cap B_n} |(g_f^{k_n})'(z)|^{\alpha}\, d\mu(z)}{\int_{B_n} |(g_f^{k_n})'(z)|^{\alpha}\, d\mu(z)},$$

where $B_n = B(x,s_n)$ and $\alpha = \alpha(f)$. Since x is a density point, the expression tends to 1 as $n \to \infty$. Now choose a subsequence such that $D_n \to D_\infty$ in the Hausdorff topology, so $\mu(D_\infty) = \mu(A\cap D_\infty)$. But D_∞ contains an open set meeting $\partial\Lambda_+$ and so by Proposition 1 there exists an integer n and a subset U of $\partial\Lambda_+$ contained in D_∞ such that $\partial\Lambda_+ \subset g_f^n(U)$. This gives

$$\begin{aligned}\mu(\partial\Lambda_+) &= \mu(g_f^n(D_\infty)) = \mu(g_f^n(A \cap D_\infty)) \\ &= \mu(g_f^n(A) \cap g_f^n(D_\infty)) = \mu(g_f^n(A) \cap \partial\Lambda_+) = \mu(g_f^n(A)) \le \mu(A),\end{aligned}$$

so A has full measure. ∎

Proof of Theorem 11. We know that if f is geometrically finite then the measure μ constructed in Theorem 7 is supported on the radial limit set. It follows from Theorem 9 that the set has dimension $\alpha(f)$. Hence $\delta(x) = \delta(f) = \alpha(f)$ for all $x \in D$. Moreover, by Theorem 8, $\alpha(f)$ is equal to the Hausdorff dimension of the radial limit set. But this is equal to the Hausdorff dimension of $\partial\Lambda_+$ by Theorem 5.

Now consider a normalised $\delta(f)$-invariant measure ν which has support in $\overline{\Omega} - \{p\}$. Then by the dynamics of f, we see that ν has support in $\overline{D} = \overline{\Omega \cap (\widehat{\mathbb{C}} - D_J)}$. If ν has support the interior of D, then by Proposition 2 we have $P_\alpha(x) < \infty$, contradicting Theorem 10. Hence Ω does not meet the support of μ. We will show that ν is non-atomic: Suppose that ν has an atom at a point x which lies in the orbit under g_f of the critical point ω of g_f. Then $\omega \in \partial\Lambda_+$ and the orbit of ω lands on a periodic cycle and since ν is $\delta(f)$-conformal this cycle has multiplier of modulus one, contradicting Lemma 4. So suppose that ν has an atom at a point x which does not lie in the orbit under g_f of the point ω. We may assume that $x \ne p$. Suppose that $g_f(y) = x$. Since $y \notin \{c, J(-2)\}$ we have $\nu(x) = |g_f'(y)|^\delta \nu(y)$ and hence $\nu(x)/\nu(y) = |g_f'(y)|^\delta$. But then

$$P_\delta(x) = \sum_{g_f^n(y)=x} |(g_f^n)'(y)|^{-\delta} = \sum_{g_f^n(y)=x} \nu(y)/\nu(x) \le \nu(D)/\nu(x) < \infty,$$

a contradiction. It follows that ν has no atoms. Since $\partial\Lambda_+ - L_{\mathrm{rad}}(f)$ is countable, ν is supported on the radial limit set, and hence ν is equal to the canonical measure μ from Theorem 7. ∎

Proof of Corollary 2. Suppose that ν is a conformal measure supported on $\partial\Lambda$. If it is $\delta(f)$-conformal then by Theorem 11 it is equal to μ. If it is β-conformal for $\beta > \delta(f)$ then it must be supported on $\partial\Lambda_+ - L_{\mathrm{rad}}(f)$ by Theorem 9. ∎

Proof of Corollary 3. By Theorem 11 we have $\delta(f) = \alpha(f) = \mathrm{HD}(\partial\Lambda_+) \le 2$. If $\mathrm{HD}(\partial\Lambda_+) = 2$ then both μ and 2-dimensional Lebesgue measure are 2-conformal measures. But μ is not equal to Lebesgue measure as it is supported only on $\partial\Lambda_+$. This contradicts Theorem 11. ∎

5. Pinching deformations. In this section we investigate how the Hausdorff dimension of the limit set varies along a pinching deformation

$\{f_t\}_{0\le t<1}$ with limit a correspondence $f \in \mathcal{F}$. The most complete result occurs in the case where all the correspondences f and f_t, $t < 1$, are given by real parameters.

DEFINITION 10 ([4]). Let f_0 be an unpinched mating between a quadratic polynomial q_c and a representation of $C_2 * C_3$ with connected regular set, let p_0 denote the fixed point of f_0^{-1} which corresponds to the landing point of the external ray of argument 0 of q_c (known as the *β-fixed point* of q_c), and let γ be an arc in Ω with end-points $p_0 \in \Lambda_+^0$ and $J(p_0) \in \Lambda_-^0$. A *pinching deformation* $\{f_t\}_{0\le t<1}$ of f_0 is given by a family of quasi-conformal maps h_t, $0 \le t < 1$, such that each correspondence $f_t = h_t \circ f_0 \circ h_t^{-1}$ is holomorphic and such that

- the pairs (f_t, h_t) converge uniformly to a pair (f, h) as $t \to 1$, and
- the non-trivial fibers of h are exactly the closure of the connected components of the orbit of γ.

Since a geometrically finite quadratic polynomial satisfies the two conditions in Theorem 2 we have:

THEOREM 12. *Let f_0 be an unpinched mating between a representation G of $C_2 * C_3$ with connected regular set and a geometrically finite quadratic polynomial q_c. Then there exists a pinching deformation of f_0 such that the f_t converge uniformly to a mating $f \in \mathcal{F}$ between the modular group and q_c.*

We pause briefly to recall the construction of a pinching deformation given in [4]: let f_0 be the initial unpinched mating, which gives rise to sets Ω_0 and $\Lambda_0 = \Lambda_+^0 \cup \Lambda_-^0$. Let p_0 be the point corresponding to the β-fixed point of the quadratic-like map g_{f_0}.

We choose a curve γ in $\widehat{\mathbb{C}} - \Lambda_0$ which connects p_0 and $J(p_0)$. We now construct a *collar neighbourhood* $\mathcal{N}$ of γ, that is, a neighbourhood bounded by two curves, both with end-points p_0 and $J(p_0)$, which lie on either side of γ. Thus γ divides $\mathcal{N}$ into two parts, B_+ and B_-. For each t we now define an almost complex structure σ_t on $B_- \cup B_+$ by first defining it on a model strip L in the complex plane and then transferring it onto B_+ and B_- by means of conformal homeomorphisms $\psi_- : L \to B_-$ and $\psi_+ : L \to B_+$. We spread σ_t to the images of $\mathcal{N}$ using the dynamics of f_0. By the measurable Riemann mapping theorem there exists a quasi-conformal homeomorphism h_t which integrates σ_t. The family $\{f_t = h_t f_0 h_t^{-1}\}_{t<1}$ can be proved to be a pinching deformation with limit f. Moreover, f is a mating between the modular group and the quadratic involved.

Before stating the main results of this section we recall the following definitions of [10]:

DEFINITION 11. Suppose that a sequence $\{\lambda_n\} \subset \mathbb{C} - \{0\}$ tends to 1 and let $\lambda_n = \exp(L_n + i\theta_n)$. We say that λ_n tends to 1 *radially* if

$$\theta_n = O(L_n).$$

We say that λ_n tends to 1 *horocyclically* if

$$\theta_n^2/L_n \to 0.$$

For a sequence $\{\lambda_n\} \to \lambda$ we say that the convergence is *radial* (or *horocyclical*) if $\lambda_n/\lambda \to 1$ radially (or horocyclically).

Given a pinching deformation $f_t \to f$, let q be a parabolic periodic point of g_f with multiplier λ and let q_n be the corresponding periodic points of g_{f_t} with multipliers λ_t. Then we say that $q_n \to q$ *radially* or *horocyclically* if $\lambda_n \to \lambda$ radially or horocyclically.

THEOREM 13. *If f_t is a pinching deformation with limit f then the limit sets $\partial\Lambda_+^t$ of f_t converge to the limit set $\partial\Lambda_+$ of f in the Hausdorff topology.*

THEOREM 14. *Let $q_c : z \mapsto z^2 + c$ be a geometrically finite quadratic polynomial with $c \neq 1/4$, with connected Julia set and such that the critical point of q_c does not land on the β-fixed point. Let f_0 be a mating between q_c and a representation of $C_2 * C_3$, and let $\{f_t\}_{0\le t<1}$ be a pinching deformation with limit f. Let p_t denote the β-fixed points of g_{f_t}, so $p_t \to p$. If either*

(i) *$p_t \to p$ radially, or*
(ii) *$p_t \to p$ horocyclically and* $\liminf(\mathrm{HD}(\partial\Lambda_+^t)) > 1$,

then $\mathrm{HD}(\partial\Lambda_+^t) \to \mathrm{HD}(\partial\Lambda_+)$.

Proof of Theorem 13. Since $\partial\Lambda_+^0$ is compact and since the h_t converge uniformly to h, we see that the images $h_t(\partial\Lambda_+^0)$ converge to $h(\partial\Lambda_+^0)$ in the Hausdorff topology. Moreover, since each h_t is a conjugacy for $t < 1$ and since the h_t converge uniformly to h, it follows that $h_t(\partial\Lambda_+^0) = \partial\Lambda_+^t$ and that $h(\partial\Lambda_+^0) = \partial\Lambda_+$. The result follows. ■

LEMMA 9. *Let $\{f_t\}_{0\le t<1}$ be a converging pinching deformation with limit f. Then each f_t is an unpinched mating between a group G_t and the quadratic q_c. Moreover, we can assume that each f_t for $0 \le t < 1$ is of the form $J_t \circ \mathrm{Cov}_0^Q$, where $Q(z) = z^3 - 3z$.*

Proof. By definition the initial correspondence is an unpinched mating which partitions the sphere into sets Ω_0, Λ_0 and $\mathcal{C}_0$. These give rise to sets Ω_t, Λ_t and $\mathcal{C}_t$ for each $0 < t < 1$.

Since f_0 is an unpinched mating, there exists a conformal homeomorphism $\phi : \Omega_0 \to \mathcal{D}$, where $\mathcal{D}$ is a completely invariant subset of the regular set of the group G_0, conjugating f_0 to the action of the group G_0. Composing each h_t with ϕ, we get quasi-conformal maps $\phi_t : \Omega_t \to \mathcal{D}$. Transferring the standard complex structure on Ω_t to $\mathcal{D}$ using ϕ_t, and then spreading it to all of $\widehat{\mathbb{C}}$ using the involution χ, we get an almost complex structure on the sphere which can be integrated by the measurable Riemann mapping theorem. The conjugate of G_0 under the integrating map now gives the required group G_t. We know that any mating can be conjugated to one of the form $J_t \circ \mathrm{Cov}^Q$, and the rest follows. ∎

Proof of Theorem 14. By Lemma 9 for each $t < 1$ the correspondence f_t is an unpinched mating between q_c and a group. The results of the previous section hold for these unpinched matings as well, and since q_c is geometrically finite, each $\partial\Lambda_+^t$ carries a unique normalised conformal measure μ_t of dimension δ_t, where δ_t is equal to the Hausdorff dimension of $\partial\Lambda_+^t$. Choose a subsequence $\{t_n\}$ such that the $\mu_{t_n} = \mu_n$ tend to a measure ν in the weak topology as $t_n \to 1$ and such that $\delta_{t_n} = \delta_n \to \delta$ as $n \to \infty$. Then the measure ν is supported on $\partial\Lambda_+$ by Theorem 13. In fact, similar arguments to those used in the proof of Theorem 7 show that ν is δ-conformal on any Borel set A not containing the singular points of f and f^{-1} in its interior. Now if ν has no atoms on (pre-) periodic points of g_f then by Corollary 2 we have $\delta = \mathrm{HD}(\partial\Lambda_+)$, and this proves the theorem.

To show that there are indeed no atoms at these points we follow the proof of Theorem 11.2 of [10].

For a periodic point q of g_f which lies in $\partial\Lambda_+$, let q_n denote the corresponding periodic points of g_{f_n}, so $q_n \to q$. Then there exists a neighbourhood of q on which all g_{f_n} as well as g_f are analytic homeomorphisms. Let h and h_n denote the local inverses of g_f and g_{f_n} which fix q and q_n. Now if q is repelling for g_f (and therefore attracting for h), we can find a fundamental annulus A_0 around q such that $\{q\} \cup \bigcup_{i=0}^{\infty} h^i(A_0)$ covers a neighbourhood V of q. Enlarging A_0 slightly, we can also assume that

$$V \subset \{q_n\} \cup \bigcup_{i=0}^{\infty} h_n^i(A_0).$$

Since q and the q_n are attracting for h and h_n, we have $|h_n'| < \lambda < 1$ for some λ and all n sufficiently large in a neighbourhood of q. Then, for V sufficiently small, we see that for any $x \in A_0$, any $\varepsilon > 0$ and all n sufficiently large,

$$\sum_{h_n^i(x) \in V} |(h_n^i)'(x)|^{\delta_n} < \varepsilon.$$

Since the μ_n have no atoms, we find that

$$\mu_n(V) \le \sum_{i=0}^{\infty} \mu_n(h_n^i(A_0) \cap V) = \int_{A_0} \sum_{h_n^i(x) \in V} |(h_n^i)'(x)|^{\delta_n}\, d\mu_n(x) < \varepsilon \mu_n(A_0) < \varepsilon.$$

Since ε was arbitrary, we conclude that there is no atom at q. See Theorem 11.2 in [10] for details.

If q is parabolic and not equal to p then, since each g_{f_n} is conjugate to q_c on $\Lambda_+^{t_n}$, each g_{f_n} has a parabolic periodic point q_n of the same period as q and with the same petal number. It follows that the derivatives of g_{f_n} at q_n

are equal for all n and hence that $q_n \to q$ radially. If $q = p$ then, since the quadratic involved is not $z \mapsto z^2 + 1/4$, we see that all the q_n are repelling, and we have radial convergence as one of the assumptions in our theorem. Thus, in any case, we are able to apply Theorem 10.2 of [10], which implies that for any $\varepsilon > 0$ and any compact set A_0, there exists a neighbourhood V of q such that

$$\sum_{h_n^i(x) \in V} |(h_n^i)'(x)|_n^\delta < \varepsilon$$

for all n sufficiently large. We now proceed as in the repelling case, by finding a fundamental region for the action of h near q. Again see Theorem 11.2 in [10] for details.

It remains to prove that there are no atoms on pre-periodic points. For a pre-periodic point whose orbit under g_f does not land on p this can be shown as in Theorem 11.2 in [10]. Suppose that q is a point whose orbit under g_f eventually lands on p. Then by our assumption it is not the critical point of g_f. We will show that given any $\varepsilon > 0$ there exists a neighbourhood N of q such that $\mu_n(N) < \varepsilon$ for all n sufficiently large. Let h denote the branch of an iterate of f which sends p to q. Then h is a $2:1$ analytic map on some neighbourhood of p with p as a critical point. Since there is no atom at p, we can choose a neighbourhood U of p such that $\mu_n(U) < \varepsilon$ for all n sufficiently large. Let h_n be the branch corresponding to h, sending $h_n(p_n)$ to q_n for some q_n. Then each h_n is a $2:1$ analytic map on U with critical point p. Also observe that $h_n \to h$ uniformly on some neighbourhood of p containing U, so $h_n' \to h'$ uniformly on U. Thus, shrinking U if necessary, we can assume that for all n sufficiently large we have $|h_n'(z)| < 1$.

Let D_Q be the transversal for Q defined earlier. Let $U_0 = U \cap D_Q$, so that each h_n is injective on U_0 and

$$\mu_n(h_n(U_0)) = \int_{U_0} |h_n'(z)|^{\delta_n} \, d\mu_n(z) \leq \varepsilon.$$

The set $h_n(U_0)$ forms a cut neighbourhood of q_n for each n, and, by construction of the measures μ_n, the cut $\mathcal{C}_n$ carries no mass under μ_n, so we can assume that for n sufficiently large and $N_n = h_n(U_0) \cup \mathcal{C}_n$ we have $\mu_n(N) < \varepsilon$. Moreover, the N_n converge to a neighbourhood of q, and we deduce that there exists a neighbourhood N of q with $\mu_n(N) < \varepsilon$ for all n sufficiently large. ∎

REMARK 1. We had to exclude the case where the quadratic in the mating is $z \mapsto z^2 + 1/4$ because in this case the fixed points of the correspondences in the pinching deformation which tend to p are parabolic with one petal. The limit however will be the unique correspondence for which $p = 1$ has three petals. In this situation Theorem 10.2 in [10] is not applicable.

6. Real pinching deformations. In this section we will see that if we assume that the quadratic q_c involved in a mating has real parameter c then the assumption of radial convergence in Theorem 14 is automatically satisfied. Moreover, if c is real, then the result of Theorem 14 also holds when the orbit of the critical point does land on the β-fixed point of q_c. We say that a mating $f = J \circ \mathrm{Cov}_0^Q$ is *real* if both the fixed points of J are real.

THEOREM 15. *Let $q_c : z \mapsto z^2 + c$ with $c \in \mathbb{R} - \{1/4\}$ be geometrically finite with connected Julia set. Then there exists a pinched mating $f \in \mathcal{F}$ between q_c and $\mathrm{PSL}(2, \mathbb{Z})$, and a pinching deformation $\{f_t\}_{0 \le t < 1}$ with limit f such that $\mathrm{HD}(\partial \Lambda_+^t) \to \mathrm{HD}(\partial \Lambda_+)$.*

In order to prove this result we will need:

THEOREM 16. *Let G be a faithful discrete representation of $C_2 * C_3$ with connected regular set and let $q_c : z \mapsto z^2 + c$ be a quadratic polynomial with connected Julia set. Then there exists an unpinched real mating $f = J \circ \mathrm{Cov}^Q$, where $Q(z) = z^3 - 3z$, between G and q_c if and only if G is a Fuchsian group and c is real.*

Before proving Theorem 16 we prove the following lemma:

LEMMA 10. *Let $g : U \to V$ be a quadratic-like map with connected Julia set K_g. If g commutes with complex conjugation then it is hybrid-equivalent to a quadratic $q_c : z \mapsto z^2 + c$ with c real.*

Proof. Since g commutes with complex conjugation, we see that U, and hence V, are both symmetric with respect to the real axis.

We will sketch the proof of the straightening theorem, keeping track of what happens to the symmetry arising from complex conjugation. Let U' and V' be two round open discs, centred at the origin, such that $q_0(U') = V'$ and $U' \subset V'$ (where $q_0 : z \mapsto z^2$). Clearly, q_0 commutes with complex conjugation as well. Let c_1 denote complex conjugation in the g-plane and c_2 complex conjugation in the q_0-plane.

Let ζ be a point on the real line contained in the interior of the disc $\widehat{\mathbb{C}} - V$ in the g-plane. Then there exists a unique Riemann map R sending $\widehat{\mathbb{C}} - V$ to $\widehat{\mathbb{C}} - V'$ with $R(\zeta) = \infty$ and $R'(\zeta) > 0$.

Now the map $c_2 R c_1$ also sends $\widehat{\mathbb{C}} - V$ to $\widehat{\mathbb{C}} - V'$ with $R(\zeta) = \infty$ and $R'(\zeta) > 0$, so by uniqueness we have $c_2 R c_1 = R$.

Let A be the annulus $V - U$ and A' the annulus $V' - U'$. The Riemann map R extends to the outer boundary ∂V of A, which we assume to be smooth. We can extend R to a map $R : \partial U \to \partial U'$ between the inner boundaries of the two annuli by the rule $g(z) = R^{-1} q_0 R(z)$ for all $z \in \partial U$. Finally, we extend R quasi-conformally to the interior of A so that $R(A) = A'$. Since the regions involved are symmetric with respect to the real axis and since the maps involved all commute with complex conjugation, we can define R on A so that $c_2 R c_1 = R$ everywhere (this is done by first defining R on the intersection of A and the upper half-plane and then extending this to the lower half-plane accordingly).

Next, we define a degree two map F on $\widehat{\mathbb{C}}$ by $F(z) = g(z)$ on U and $F(z) = R^{-1} q_0 R(z)$ elsewhere. We also have an orientation-reversing involution J on $\widehat{\mathbb{C}}$ given by $J(z) = c_1$ on V and $J(z) = R^{-1} c_2 R(z)$ elsewhere. By definition, J commutes with F.

We now define an almost complex structure σ that is preserved by F in the usual way. Note that σ is also preserved by J.

By the measurable Riemann mapping theorem, there exists a quasi-conformal homeomorphism ϕ which carries σ to the standard complex structure. The conjugate of F by ϕ is a degree two holomorphic map of the complex plane and therefore a quadratic polynomial q. The conjugate $\widetilde{J}$ of J by ϕ is an orientation-reversing involution of the plane that fixes a curve pointwise and preserves the standard complex structure. Any such map is conformally conjugate to complex conjugation. Moreover, q commutes with $\widetilde{J}$, so it follows that g is hybrid-equivalent to a quadratic which commutes with complex conjugation. The result follows. ∎

Proof of Theorem 16. We recall briefly the construction given in [5] of the mating f_0 using quasi-conformal surgery: Firstly, the group G is a representation of the free product $C_2 * C_3$ and has connected regular set and hence a Cantor limit set. Let σ and ϱ denote the order 2 and 3 generators of G respectively. There exists a unique involution χ of the sphere which conjugates each of σ and ϱ to its inverse. Moreover, for the group $\langle \sigma, \varrho, \chi \rangle$ there exists a fundamental domain $\varDelta$ as shown in Figure 7. Here P and P' denote the fixed points of ϱ, Q and Q' the fixed points of σ, and W and T the fixed points of $\varrho\chi$ and $\sigma\chi$ respectively.

The quotient of $\varDelta \cup \varrho(\varDelta) \cup \varrho^2(\varDelta)$ by χ is an annulus A which carries a $2:2$ correspondence arising from $\varrho \cup \varrho^2$, and whose inner boundary maps $2:1$ onto the outer boundary under the projections of $\sigma\varrho$ and $\sigma\varrho^2$. The projection of σ gives an involution on the outer boundary. See Figures 8 and 9.

For the quadratic q_c one can choose a topological disc V bounded by an equipotential such that $q_c^{-1}(V)$ is a topological disc as well. The inner boundary of the annulus $B = V - q_c^{-1}(V)$ maps $2:1$ onto the outer boundary under q_c; and the outer boundary carries an involution j coming from sending an external angle t to $1-t$.

The surgery construction in [5] now matches the annuli A and B to give a mating $f = J \circ \mathrm{Cov}_0^Q$, where $Q(z) = z^3 - 3z$. The annulus A corresponds to the intersection of a transversal D_Q of Cov_0^Q and a fundamental domain D_J of J.

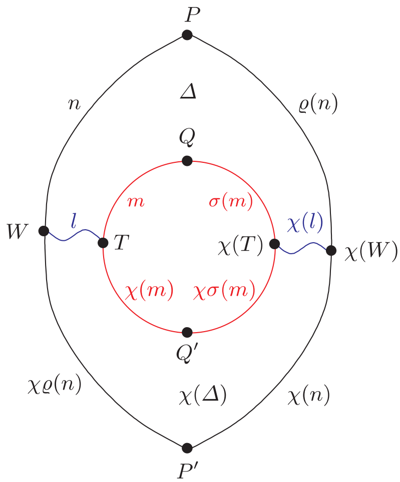

Fig. 7. A fundamental domain Δ of the group, made up of the images of a line n connecting P and W, a line m connecting Q and T and a line l connecting W and T, under the group elements ϱ, σ and χ.

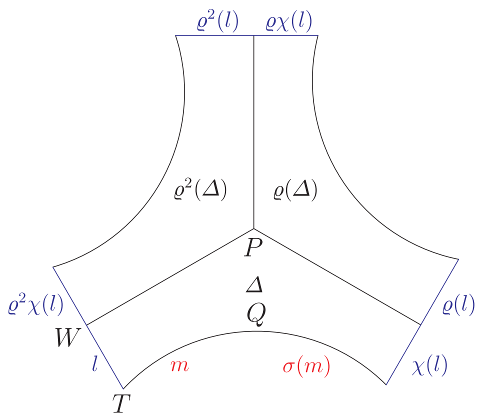

Fig. 8. Three copies of the fundamental domain Δ.

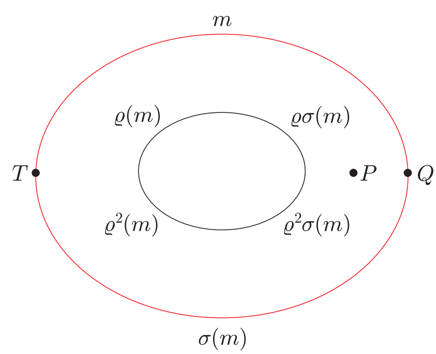

Fig. 9. The quotient annulus A.

If G is Fuchsian, the orientation-reversing involution $\chi\mathcal{C}$, where $\mathcal{C}$ denotes complex conjugation, descends to a reflectional involution on A. Similarly, if c is real, complex conjugation $\mathcal{C}$ gives a reflectional involution of B. These involutions are matched by the surgery construction and the resulting mating f then commutes with an orientation-reversing involution I of all of $\widehat{\mathbb{C}}$ which preserves the standard complex structure. Such an involution is conformally conjugate to complex conjugation.

The fixed points of J correspond to the projections onto the annulus A of the points Q and T, which lie on the line of symmetry of the reflectional involution on A. Since this involution passes to complex conjugation, we deduce that J has two real fixed points.

It remains to prove the other direction of the theorem: Suppose that $f = J \circ \mathrm{Cov}_0^Q$ and that the involution J has real fixed points $x_1 < y_1$. We note that the disc D_Q bounded by the line $\{x \pm i\sqrt{3x^2 - 3} : x \geq 1\}$ and containing the point 2 is a transversal for Q with the property that for $z \in \partial D_Q$ we have $\overline{z} = \mathcal{C}(z) = \mathrm{Cov}_0^Q(z) \cap \partial D_Q$.

Now, since f is a mating, there exists a fundamental domain D_J of J such that $D_Q^0 \cup D_J^0 = \widehat{\mathbb{C}}$. Images of the annulus $D_Q \cap D_J$ tile the regular set Ω of the correspondence, and it is this annulus, cut along an appropriate line, that corresponds to the regular set of the group.

Let D be the bounded component of the complement of the circle that passes through the two real fixed points of J and is centred on the real line. This clearly is a fundamental domain of J with the property that if $z \in \partial D$, then $J(z) = \overline{z}$. Moreover, D is properly contained in D_Q: Suppose that it is not and that ∂D_Q and ∂D meet in a point z. Then, clearly, z and $\overline{z}$ are fixed points of f and hence lie in Λ. Since $\overline{z} = J(z)$ and since J sends Λ_+ to Λ_- we deduce that one of z and $\overline{z}$ lies in Λ_+ while the other lies in Λ_-. However, since f commutes with complex conjugation we see that $z \in \Lambda_+$ if and only if $\overline{z} \in \Lambda_+$, a contradiction. Hence the boundaries of D_Q and D do not meet. It follows that the interior of $\widehat{\mathbb{C}} - D$ and the interior of D_Q together cover the sphere, and so we can take $\widehat{\mathbb{C}} - D$ to be the fundamental domain D_J of J mentioned above.

Now the annulus $D_Q \cap D_J = D_Q - D$ is invariant under complex conjugation. If we quotient it by the action of the branches of Cov_0^Q together with the involution J, we get a sphere S with four cone-points: $\widetilde{P}$ of order $2\pi/3$ corresponding to ∞, $\widetilde{Q}$ of order π corresponding to the (real) fixed point x_1 of J (see Figure 10), $\widetilde{T}$ of order π corresponding to the (real) fixed point x_2 of J, and $\widetilde{W}$ of order π corresponding to the fixed point 1 of Cov_0^Q. Since f is a mating, the sphere S is precisely the orbifold of the group $\langle \sigma, \varrho, \chi \rangle$. The cone-point $\widetilde{P}$ corresponds to the fixed point P of ϱ, $\widetilde{Q}$ corresponds to the fixed point Q of σ, $\widetilde{T}$ corresponds to T and $\chi(T)$ and $\widetilde{W}$ corresponds to W and $\chi(W)$ (see Figure 10). Complex conjugation in the correspondence plane now descends to an orientation-reversing involution on S which fixes pointwise a closed curve through the four cone-points.

Now consider the component of this fixed curve that connects the cone-point $\widetilde{P}$ to the cone-point $\widetilde{Q}$ and contains the other two cone-points. Cutting the sphere along this component gives a fundamental domain Δ of the group as in Figure 7. The involution on S now gives an orientation-reversing involution I, sending Δ to itself and fixing a curve that runs from the point P to the point Q and interchanging the points T and $\chi(T)$ and

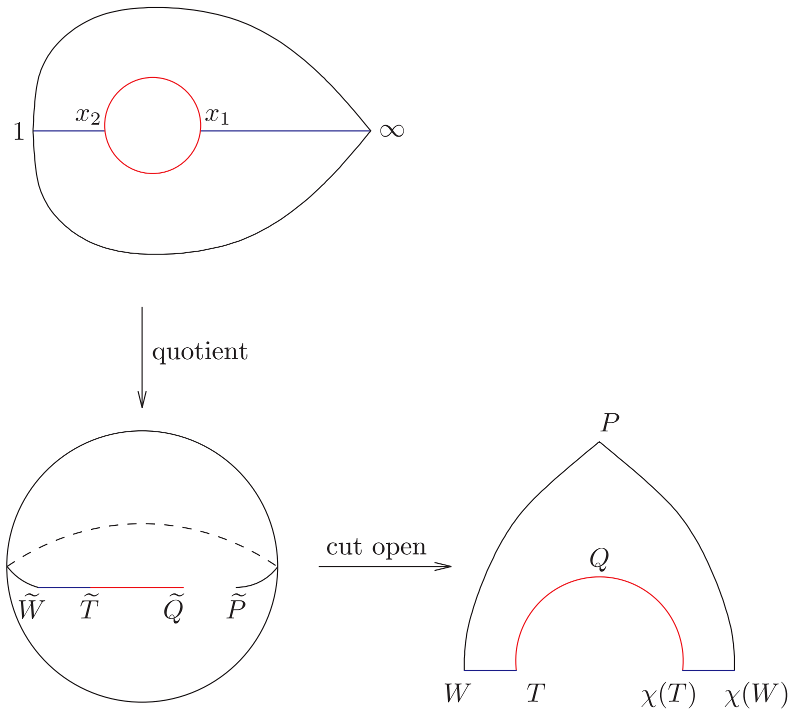

Fig. 10. The top figure shows the annulus $D_Q - D$. The line segments connecting 1 and x_2 and x_1 and ∞ mark its intersection with the real line. The bottom left figure shows the orbifold sphere and the curve along which it is cut. Cutting it gives the fundamental domain in the bottom right figure.

the points W and $\chi(W)$. Since we have cut S along a curve fixed by the involution on S, and since in the correspondence plane complex conjugation coincides with J on ∂D_J and with a branch of Cov_0^Q on ∂D_Q, we also have

- $\varrho(x) = I(x)$ for x lying in the boundary component of Δ that connects W to P,
- $\varrho^{-1}(x) = I(x)$ for x lying in the boundary component of Δ that connects $\chi(W)$ to P,
- $\sigma(x) = I(x)$ for x lying in the boundary component of Δ that connects T to $\chi(T)$,
- $\chi(x) = I(x)$ for x lying in the boundary component of Δ that connects W to T.

Let $J = \chi \circ I$, so that J sends Δ to $\chi(\Delta)$. Using the group elements σ and ϱ we can extend J to an orientation-reversing involution defined on all the copies of the fundamental domain Δ, which together make up the regular set of the group. We do this as follows: If w is a word in σ, ϱ and ϱ^2, then we define

$$J(w(\mathit{\Delta} \cup \chi(\mathit{\Delta}))) = w(J(\mathit{\Delta} \cup \chi(\mathit{\Delta}))).$$

By definition, J commutes with both σ and ϱ. It also fixes pointwise the images under the group of the boundary components l and $\chi(l)$ of $\mathit{\Delta}$.

Now, since every point in the limit set of our group is an accumulation point of copies of the fundamental domain $\mathit{\Delta}$, one might hope that the definition of J can be extended to the limit set. As the following argument shows, this is indeed the case. Whether or not our group is Fuchsian, it certainly is quasi-conformally conjugate to a Fuchsian group, since all the groups involved in our matings come from one quasi-conformal conjugacy class. Hence, all the copies of our fundamental domain $\mathit{\Delta}$ have quasi-conformal images which are copies of a fundamental domain of a Fuchsian group. Moreover, the combinatorics of our involution J, in other words the way in which it permutes copies of the fundamental domain, is exactly the same as that of complex conjugation in the Fuchsian case. Since complex conjugation in the Fuchsian case extends to the limit set, we deduce that J can be extended to the limit set analogously.

Since it comes from complex conjugation in the correspondence plane, the involution J preserves angles everywhere, except possibly on the limit set of the group. But as mentioned before, our group is quasi-conformally conjugate to a Fuchsian group by a quasi-conformal homeomorphism ϕ. The limit set of a Fuchsian group is contained in $\widehat{\mathbb{R}}$, which maps to a quasi-circle under ϕ. Now post-composing J with a map of the form

$$z \mapsto \frac{a\overline{z} + b}{c\overline{z} + d}$$

gives a homeomorphism that is conformal everywhere, except possibly on a set contained within a quasi-circle. But quasi-circles are removable for conformal homeomorphisms; in other words, a map that is defined on $\widehat{\mathbb{C}}$ and conformal everywhere off a quasi-circle is also conformal *on* the quasi-circle (this is a standard result, see for example Proposition 2 in [2]). Therefore our composition is conformal on the whole sphere and it follows that J is conformally conjugate to complex conjugation.

Thus, σ and ϱ commute with a conformal conjugate of complex conjugation and this implies that the group they generate is Fuchsian.

Lastly, we need to show that the quadratic q_c has real parameter c. Since f commutes with complex conjugation the quadratic-like map $g = f^{-1} : f(D_J) \to D_J$ commutes with it as well. The result now follows from Lemma 10. ∎

Proof of Theorem 15. Let f_0 be a mating between q_c and a Fuchsian representation G of $C_2 * C_3$. By Theorem 16, f_0 is real. We will show that there exists a pinching deformation $f_t \to f$ such that all f_t are real. The fixed points p_0 and $J(p_0)$ of f_0 are real and one can check ([4]) that we can choose for the curve γ the real interval $[J(p_0), p_0]$. Moreover, we can choose the collar neighbourhood $\mathcal{N}$ to be symmetric with respect to complex conjugation. Hence we can choose $\psi_- = \overline{\psi}_+$. Since f_0 commutes with complex conjugation $\mathcal{C}$, we see that f_t commutes with $h_t \circ \mathcal{C} \circ h_t^{-1}$ and that this map preserves the standard complex structure. It follows that $h_t \circ \mathcal{C} \circ h_t^{-1}$ is conformally conjugate to complex conjugation. Hence, conjugating suitably, we ensure that f_t commutes with complex conjugation with the real line in the f_0-plane corresponding to the real line in the f_t-plane, and therefore J_t has two real fixed points.

It follows that the fixed points $p_t \to p$ of the correspondences f_t are all real. The first derivative of the branch of f_t that fixes p_t commutes with complex conjugation and therefore it is real at p_t. Hence $p_t \to p$ radially.

The only case not covered by this argument is when the critical point of q_c lands on the β-fixed point. Since c is real, the only quadratic with this property is $q_{-2} : z \mapsto z^2 - 2$. This has Julia set the real interval $[-2, 2]$. By similar considerations to those in the proof of Theorem 16, the image of this interval under the hybrid-equivalence to g_{f_t} is a real interval, so Λ_+^t is a real interval for all $t < 1$. Now there is a unique mating f such that g_f satisfies the same critical relation as q_{-2}, namely $f = J \circ \mathrm{Cov}_0^Q$, where

$$J(z) = \frac{5z - 8}{2z - 5},$$

the involution fixing the points 1 and 4. This f is the limit of the f_t. It is easy to check that Λ_+ is a real interval as well, so $\mathrm{HD}(\Lambda_+^t) = 1$ for all $0 \le t \le 1$. ∎

7. Generalisations. In [3] we presented families of $(n-1) : (n-1)$ correspondences representing matings between the nth Hecke group H_n and Chebyshev-like maps of degree $n-1$, for each integer $n \ge 3$. The *nth Hecke group* is a Fuchsian group isomorphic to $C_2 * C_n$ with limit set the real line union infinity. It is generated by the matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varrho = \begin{pmatrix} 0 & 1 \\ -1 & -2\cos(\pi/n) \end{pmatrix}.$$

The group H_3 is the modular group. *Chebyshev-like maps* are maps with just two critical values, one being fixed and one being free.

Using analogous methods to those used in this paper for $n = 3$, it is possible to prove the results of Sections 1–4 of this paper for these higher "degree" matings.

In [9] it was shown that for each $n > 3$ there exist unpinched matings between quasi-Fuchsian groups with Cantor limit sets and certain polynomials with disconnected Julia sets, however the existence of converging pinching deformations in this context has not yet been proved. Assuming that these pinching deformations can indeed be constructed by methods similar to those in [4], we conjecture that the results of Section 5 of this paper will hold true for any of the matings presented in [3].

References

[1] A. Beardon, *Iteration of Rational Functions*, Grad. Texts in Math. 132, Springer, New York, 1991.

[2] C. Bishop, *Some homeomorphisms of the sphere conformal off a curve*, Ann. Acad. Sci. Fenn. Ser. A I Math. 19 (1994), 323–338.

[3] S. Bullett and M. Freiberger, *Holomorphic correspondences mating Chebyshev-like maps with Hecke groups*, Ergodic Theory Dynam. Systems 25 (2005), 1057–1090.

[4] S. Bullett and P. Haïssinsky, *Pinching holomorphic correspondences*, preprint, 2004.

[5] S. Bullett and W. J. Harvey, *Mating quadratic maps with Kleinian groups via quasiconformal surgery*, Electron. Res. Announc. Amer. Math. Soc. 6 (2000), 21–30.

[6] S. Bullett and C. Penrose, *Mating quadratic maps with the modular group*, Invent. Math. 115 (1994), 483–511.

[7] L. Carleson and T. Gamelin, *Complex Dynamics*, Springer, New York, 1993.

[8] A. Douady and J. H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. 18 (1985), 287–343.

[9] M. Freiberger, *Mating Kleinian groups isomorphic to* $C_2 * C_5$ *with quadratic polynomials*, Conform. Geom. Dyn. 7 (2003), 11–33.

[10] C. T. McMullen, *Hausdorff dimension and conformal dynamics II*: *Geometrically finite rational maps*, Comment. Math. Helv. 75 (2000), 535–593.

[11] —, *Complex Dynamics and Renormalization*, Ann. of Math. Stud. 135, Princeton Univ. Press, Princeton, NJ, 1994.

[12] M. Pollicott and M. Yuri, *Dynamical Systems and Ergodic Theory*, London Math. Soc. Student Texts 40, Cambridge Univ. Press, Cambridge, 1998.

[13] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Grundlehren Math. Wiss. 299, Springer, Berlin, 1992.

[14] F. Przytycki and M. Urbański, *Fractals in the Plane—the Ergodic Theory Methods*, Cambridge Univ. Press, Cambridge, to appear.

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