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The Covering Principle for Darboux Baire 1 functions

by

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Abstract. We show that the Covering Principle known for continuous maps of the real line also holds for functions whose graph is a connected G_δ subset of the plane. As an application we find an example of an approximately continuous (hence Darboux Baire 1) function $f\colon [0,1] \to [0,1]$ such that any closed subset of $[0,1]$ can be translated so as to become an ω-limit set of f. This solves a problem posed by Bruckner, Ceder and Pearson [Real Anal. Exchange 15 (1989/90)].

1. Introduction. For $f \in \mathbb{R}^{\mathbb{R}}$, f^0 is the identity function, and for any integer $n > 0$, the nth iterate of f is defined by $f^n = f \circ f^{n-1}$.

We say that for a given function $f\colon \mathbb{R} \to \mathbb{R}$ a compact interval I_1 *f-covers* a compact interval I_2 if $f(I_1) \supset I_2$. We then write $I_1 \to_f I_2$ (or $I_1 \to I_2$ if f is clear from the context).

It is easy to see that if f is continuous and

$$I_1 \to I_2 \to I_3 \to \cdots$$

for a sequence $\{I_i\}_{i\in\mathbb{N}}$ of compact intervals then there is an $x \in I_1$ such that $f^i(x) \in I_{i+1}$ for each i. This fact, known as the *Covering Principle* (or *Itinerary Lemma*), is widely used in one-dimensional dynamics (see e.g. [6]). We generalize it to the class of real functions with connected G_δ graph in the following

THEOREM 1.1. *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a connected G_δ function and there exists a sequence of compact intervals*

$$I_1 \to I_2 \to I_3 \to \cdots .$$

Then there exists an $x \in I_1$ such that $f^i(x) \in I_{i+1}$ for each $i \in \mathbb{N}$.

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Recall that in [10] we have proved Sharkovskiĭ's theorem for connected G_δ real functions using the following dual ("cycle") version of Theorem 1.1.

THEOREM 1.2 ([10]). *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a connected G_δ function and there exists a cycle of compact intervals*

$$I_1 \to I_2 \to \cdots \to I_N \to I_1.$$

Then there exists an $x \in I_1$ such that $f^N(x) = x$ and $f^i(x) \in I_{i+1}$ for each $i \in \{0, \ldots, N-1\}$.

We give a proof of Theorem 1.1 in Section 3. In Section 4 we show how it can be used to solve a problem from [3] regarding ω-limit sets of Darboux Baire 1 functions (see definitions below).

2. Preliminaries. We denote by $[a, b]$ the compact interval (possibly degenerate) with endpoints a and b. We do not assume $a < b$ unless explicitly stated.

A set H is said to be τ_{d}-*open* if H has inner density 1 at every point $x \in H$. These sets form a completely regular (but not normal) topology τ_{d} (see e.g. [7]).

We identify every function with its graph. We consider the following classes of functions f from $\mathbb{R}$ into $\mathbb{R}$ (or $[0, 1]$ into $[0, 1]$, after obvious modifications):

- f is *approximately continuous* ($f \in \mathcal{A}$) if $f^{-1}(U) \in \tau_{\mathrm{d}}$ for every open set $U \subset [0, 1]$.
- f is *connected* or a *connectivity function* ($f \in \mathrm{Conn}$) if f is a connected subset of $\mathbb{R}^2$;
- f is *Darboux* ($f \in \mathrm{D}$) if f has the intermediate value property, i.e. $f(I)$ is an interval for every interval $I \subset \mathbb{R}$;
- f is *Baire class 1* ($f \in \mathrm{B}_1$) if f is a pointwise limit of a sequence of continuous functions; this is equivalent to $f^{-1}(G)$ being an F_σ subset of $\mathbb{R}$ for every open $G \subset \mathbb{R}$, and to $f{\restriction}K$ having a point of continuity for every non-empty closed set $K \subset \mathbb{R}$;
- f is *Darboux Baire 1* ($f \in \mathrm{DB}_1$) if f is Darboux and Baire 1;
- $f \in G_\delta$ if f is a G_δ subset of $\mathbb{R}^2$, i.e. $f = \bigcap_{n\in\mathbb{N}} G_n$, where all $G_n \subset \mathbb{R}^2$ are open.

For properties of these and other Darboux-like classes of functions see e.g. the survey [4]. In particular, it is known that

$$\mathrm{Conn} \subset \mathrm{D} \quad \text{and} \quad \mathcal{A} \subset \mathrm{DB}_1 \subset \mathrm{Conn} \cap G_\delta,$$

and all these inclusions are proper. It follows that Conn = D within the class of Baire 1 functions. Moreover, every bounded approximately continuous function is a derivative, and every derivative belongs to the class DB_1 (see e.g. [2]).

A set $W \subset \mathbb{R}$ is called an *ω-limit set* for f if there is an $x \in \mathbb{R}$ such that W is the cluster set of the sequence $\{f^n(x)\}$ ($n \in \mathbb{N}$), i.e.

$$W = \bigcap_{i \in \mathbb{N}} \overline{\{f^j(x) : j > i\}}.$$

Denote this set by $\omega_f(x)$ and let Ω_f be the class of all ω-limit sets of f, i.e. $\Omega_f = \{\omega_f(x) : x \in \mathbb{R}\}$. Clearly each element of Ω_f is closed.

Agronsky, Bruckner, Ceder and Pearson have given in [1] the following characterization of ω-limit sets: a non-empty closed set $W \subset [0,1]$ is an ω-limit set of a continuous map $f\colon [0,1] \to [0,1]$ if W is either a finite collection of nondegenerate closed intervals or is nowhere dense (the necessity of this condition was proved earlier by Sharkovskiĭ in [9]).

The characterization of ω-limit sets for a DB_1 function was given by Bruckner, Ceder and Pearson in [3]: every non-empty closed subset of $[0,1]$ is an ω-limit set of a function $f \in DB_1([0,1])$. In Section 4 we show that there exists an approximately continuous (hence DB_1) function having a translation of every closed set as an ω-limit set.

3. Proof of Theorem 1.1. Fix a Darboux function $f\colon \mathbb{R} \to \mathbb{R}$. Suppose that there exist sequences $\{a_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ such that $a_i < b_i$ and $[f(a_i), f(b_i)] \supset [a_{i+1}, b_{i+1}]$ for each $i \in \mathbb{N}$. Clearly

$$[a_1, b_1] \to [a_2, b_2] \to [a_3, b_3] \to \cdots .$$

For every $x \in \mathbb{R}$ and $i \geq 1$ let

$$\delta_i(x) = \begin{cases} a_i & \text{if } x \leq a_i, \\ b_i & \text{if } x \geq b_i, \\ x & \text{if } x \in (a_i, b_i), \end{cases}$$

and $\Delta_i = \delta_i \circ f \circ \delta_{i-1} \circ f \circ \delta_{i-2} \circ \cdots \circ \delta_2 \circ f \circ \delta_1$.

With every $y \in \mathbb{R}$ we associate a sequence of symbols

$$\alpha(y) = \alpha_1(y)\alpha_2(y)\alpha_3(y)\ldots$$

with $\alpha_i(y)$ being one of the symbols "L", "U" or "C" given by the formula

$$\alpha_i(x) = \begin{cases} L & \text{if } \Delta_i(x) = a_i, \\ U & \text{if } \Delta_i(x) = b_i, \\ C & \text{otherwise.} \end{cases}$$

Clearly $\alpha_i(a_1), \alpha_i(b_1) \in \{L, U\}$, and the sequence $\alpha(a_1)$ differs from $\alpha(b_1)$ at every position.

Notice that by definition, for any interval I and for every $i \in \mathbb{N}$, either $\delta_i(I) = \{a_i\}$, or $\delta_i(I) = \{b_i\}$, or $\delta_i(I) \subset I$. Consequently, for any $x_1, x_2 \in \mathbb{R}$

either the sequences $\alpha(x_1)$ and $\alpha(x_2)$ agree for sufficiently large indices, or

$$[\Delta_n(x_1), \Delta_n(x_2)] \to [\Delta_{n+1}(x_1), \Delta_{n+1}(x_2)]$$

for each n. So, we have the following

REMARK 3.1. If $x_1, x_2 \in \mathbb{R}$ and the sequence $\alpha(x_1)$ differs from $\alpha(x_2)$ at infinitely many positions then

$$[\Delta_1(x_1), \Delta_1(x_2)] \to [\Delta_2(x_1), \Delta_2(x_2)] \to [\Delta_3(x_1), \Delta_3(x_2)] \to \cdots .$$

LEMMA 3.2. *Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is connected and there exists a sequence of compact intervals*

$$I_1 \to I_2 \to I_3 \to \cdots .$$

Then there exists an $x \in I_1$ such that for every open neighbourhood G of $\langle x, f(x)\rangle$ there exists a sequence of compact intervals

$$I'_1 \to I'_2 \to I'_3 \to \cdots$$

such that $I'_1 \times I'_2 \subset G$ and $I'_i \subset I_i$ for each $i \ge 1$.

Proof. Note that if I_n is degenerate for some $n \in \mathbb{N}$, then there exists an $x \in I_1$ such that $f^i(x) \in I_{i+1}$ for $i = 1, \ldots, n-1$ and the family $\{I'_i\}_{i\in\mathbb{N}}$ of degenerate intervals $I'_i = \{f^{i-1}(x)\}$ is as desired. So, we assume that all intervals I_n are non-degenerate.

First we claim that there exist sequences $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ such that for each $i \in \mathbb{N}$, $a_i < b_i$, $[a_i, b_i] \subset I_i$ and

$$[a_{i+1}, b_{i+1}] \subset [f(a_i), f(b_i)].$$

Indeed, by the intermediate value property of f, for each i there exist $p_i, q_i \in I_i$ such that $f(p_i) = \inf I_{i+1}$ and $f(q_i) = \sup I_{i+1}$. Clearly the sequences of points

$$a_i = \min\{p_i, q_i\} \quad \text{and} \quad b_i = \max\{p_i, q_i\}$$

are as desired.

For any $x, y \in [a_1, b_1]$ let

$$\operatorname{diff}(x, y) = \{n \in \mathbb{N} : \alpha_n(x) \ne \alpha_n(y)\}.$$

Let

$$A = \{\langle x, f(x)\rangle \in f\restriction[a_1, b_1] : \operatorname{diff}(a_1, x) \text{ is finite}\},$$
$$B = \{\langle x, f(x)\rangle \in f\restriction[a_1, b_1] : \operatorname{diff}(a_1, x) \text{ is infinite}\}.$$

Since $\langle a_1, f(a_1)\rangle \in A$, and $\alpha(a_1)$ differs from $\alpha(b_1)$ at every position, both sets are non-empty. Clearly $f\restriction[a_1, b_1] = A \cup B$ and $A \cap B = \emptyset$. Since $f\restriction[a_1, b_1]$ is connected, there is an $\langle x_0, f(x_0)\rangle$ in $(A \cap \overline{B}) \cup (\overline{A} \cap B)$.

If G is an open neighbourhood of $\langle x_0, f(x_0)\rangle$ then there exist $x_1, x_2 \in [a_1, b_1]$ with $\alpha(x_1)$ and $\alpha(x_2)$ differing at infinitely many positions and

$$[x_1, x_2] \times [f(x_1), f(x_2)] \subset G.$$

If we set $I_i' = [\Delta_i(x_1), \Delta_i(x_2)]$ then $I_1' \times I_2' \subset G$, $I_i' \subset [a_i, b_i] \subset I_i$ for each i, and by Remark 3.1,

$$I_1' \to I_2' \to I_3' \to \cdots . \blacksquare$$

LEMMA 3.3. *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is connected, $f \subset G$ for an open set $G \subset \mathbb{R} \times \mathbb{R}$ and there exists a sequence of compact intervals*

$$I_1 \to I_2 \to I_3 \to \cdots .$$

Then there exists a sequence of compact intervals

$$J_1 \to J_2 \to J_3 \to \cdots$$

such that $J_i \subset I_i$ and $J_i \times J_{i+1} \subset G$ for each $i \in \mathbb{N}$.

Proof. By Lemma 3.2 we can find a sequence $\{I_i^1\}_{i\geq 1}$ of compact intervals such that:

- $I_i^1 \subset I_i$ for each $i \geq 1$;
- $I_1^1 \times I_2^1 \subset G$;
- $I_1^1 \to I_2^1 \to I_3^1 \to \cdots$.

Continuing inductively (using Lemma 3.2 for the sequence $\{I_i^n\}_{i\geq n+1}$) we can build a sequence $\{I_i^{n+1}\}_{i\geq n+1}$ of compact intervals such that for every $n > 1$:

- $I_i^{n+1} \subset I_i^n$ for each $i \geq n+1$;
- $I_{n+1}^{n+1} \times I_{n+2}^{n+1} \subset G$;
- $I_{n+1}^{n+1} \to I_{n+2}^{n+1} \to I_{n+3}^{n+1} \to \cdots$.

The assertion of the lemma follows from a diagonal argument, since:

- $I_n^n \subset I_n$ for each $n \geq 1$;
- $I_n^n \times I_{n+1}^{n+1} \subset G$ for each $n \geq 1$;
- $I_1^1 \to I_2^2 \to I_3^3 \to \cdots$. $\blacksquare$

Proof of Theorem 1.1. Since f is G_δ there exists a sequence $\{G_n\}_{n\in\mathbb{N}}$ of open subsets of $\mathbb{R}^2$ such that $f = \bigcap_{n\in\mathbb{N}} G_n$. Using Lemma 3.3 we can find a sequence $\{J_i^1\}_{i\geq 1}$ of compact intervals such that:

- $J_i^1 \subset I_i$ for each i;
- $J_i^1 \times J_{i+1}^1 \subset G_1$ for each i;
- $J_1^1 \to J_2^1 \to J_3^1 \to \cdots$.

Continuing inductively for every $n \geq 2$ we can define a sequence $\{J_i^n\}_{i\in\mathbb{N}}$ of compact intervals such that:

(1) $J_i^n \subset J_i^{n-1} \subset I_i$ for each $i \geq 1$;
(2) $J_i^n \times J_{i+1}^n \subset G_n$ for each $i \geq 1$;
(3) $J_1^n \to J_2^n \to J_3^n \to \cdots$.

It follows that for every $i \in \mathbb{N}$ there exists an $x_i \in \bigcap_{n\in\mathbb{N}} J_i^n$. By (1), $x_i \in I_i$ for each i. By (2),

$$\langle x_i, x_{i+1}\rangle \in \bigcap_{n\in\mathbb{N}} J_i^n \times J_{i+1}^n \subset \bigcap_{n\in\mathbb{N}} G_n = f,$$

so $x_{i+1} = f(x_i)$ for each i. Therefore $f^i(x_1) \in I_{i+1}$ for every $i \in \mathbb{N}$. ■

REMARK 3.4. Using a technique similar to that in [11] we can prove a generalization of Theorem 1.1 to the class of all finite compositions of connected G_δ functions (in [10] we have constructed an example of a connected G_δ function f such that $f^2 \notin G_\delta$.)

4. A "universal" dynamical system generated by a Darboux Baire 1 map of the interval. The authors of [1] formulated the problem of existence of a "universal" continuous function, i.e. they asked if there exists a continuous function $f: [0,1] \to [0,1]$ such that Ω_f contains a homeomorphic copy of every ω-limit set possible for continuous functions. An affirmative answer was given by Pokluda and Smítal in [8].

The related problem for Darboux Baire 1 functions was formulated in [3], where Bruckner, Ceder and Pearson constructed a DB_1 function f with Ω_f containing a homeomorphic copy of every ω-limit set possible for continuous functions, and asked if there exists a DB_1 function g such that Ω_g contains a homeomorphic copy of every non-empty closed set. (Recall that in [5] Keller gives a simple example of a function $f: [0,1] \to [0,1]$ continuous everywhere except for a single point such that any nowhere dense compact set $W \subset [0,1]$ has a homeomorphic copy in Ω_f.) In Corollary 1 we answer this question in the affirmative. Moreover, we show that a "universal" Darboux Baire 1 function can be approximately continuous and bounded (hence a derivative).

In this section we prove the following

THEOREM 4.1. *There exists an approximately continuous function f: $[0,1] \to [0,1]$ such that every closed set $F \subset [0,1]$ with $0 \in F$ is an ω-limit set for f.*

The above theorem has a somewhat surprising

COROLLARY 1. *There exists an approximately continuous function f: $[0,1] \to [0,1]$ such that any non-empty closed set $F \subset [0,1]$ can be translated so as to become an ω-limit set for f.*

To prove Theorem 4.1 we need the following lemma. Recall that a set H is of *type* M_5 if H is F_σ and τ_{d}-open.

LEMMA 4.2 ([12]). *If $E \subset H \subset [0,1]$, E is closed and H is of type M_5, then there exists an approximately continuous function $f\colon [0,1] \to [0,1]$ such that*

$$\begin{cases} f(x) = 0 & \text{if } x \notin H,\\ f(x) = 1 & \text{if } x \in E,\\ 0 < f(x) < 1 & \text{otherwise.}\end{cases}$$

Proof of Theorem 4.1. Let $E = \{0\}$ and H be any τ_{d}-open subset of $[0,1]$ of type F_σ containing zero such that H and $[0,1] \setminus H$ are dense in $[0,1]$. By Lemma 4.2 there exists an approximately continuous function $f\colon [0,1] \to [0,1]$ with $f(0) = 1$ and with both sets

$$\{x : f(x) = 0\} \quad \text{and} \quad \{x : f(x) > 0\}$$

being dense in $[0,1]$. Observe that for such an f and for every $x_1 \neq x_2$ there exists an arbitrarily small $\tau > 0$ such that

$$(\star) \qquad [x_1, x_2] \to [0,\tau] \to [0,1].$$

Let F be a closed subset of $[0,1]$ with $0 \in F$. Let $D = \{d_i : i \in \mathbb{N}\}$ be such that F is the cluster set of D, and $\{D_i\}_{i\in\mathbb{N}}$ be a sequence of non-degenerate compact intervals such that $d_i \in D_i \subset [0,1]$ and the diameter of D_i is less than i^{-1} for each $i > 0$. By $(\star)$, for every $i > 0$ there exists a $\tau_i \in (0, i^{-1})$ such that

$$D_i \to_f [0,\tau_i] \to_f D_{i+1}.$$

By Theorem 1.1 there exists an $x_0 \in [0,1]$ such that $f^{2i}(x_0) \in D_i$ and $f^{2i+1}(x_0) \in [0,\tau_i]$. Thus

$$\begin{aligned}\omega_f(x_0) &= \bigcap_{i\in\mathbb{N}} \overline{\{f^{2j}(x_0) : j > i\}} \cup \bigcap_{i\in\mathbb{N}} \overline{\{f^{2j+1}(x_0) : j > i\}}\\ &= \bigcap_{i\in\mathbb{N}} \overline{\bigcup_{j>i} D_j} \cup \bigcap_{i\in\mathbb{N}} \overline{\bigcup_{j>i} [0,\tau_j]} = F \cup \{0\} = F. \ \blacksquare\end{aligned}$$

The next argument (borrowed from [5]) shows that the "up to translation" part cannot be omitted in Corollary 1. Indeed, suppose that

$$\{\{0,p\},\{p,1\} : p \in \mathbb{Q}\} \subset \Omega_f,$$

where $\mathbb{Q}$ denotes the set of all rationals in $[0,1]$. Then for every $p \in \mathbb{Q}$ and open $U_0 \ni 0$, $U_1 \ni 1$, $V \ni p$,

$$f(V) \cap U_0 \neq \emptyset \quad \text{and} \quad f(V) \cap U_1 \neq \emptyset,$$

so the oscillation of f is 1 at each point of $\mathbb{Q}$. Thus f is nowhere continuous, hence not Baire 1.

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