The Covering Principle for Darboux Baire 1 functions

by

Piotr Szuca (Gdańsk)

Abstract. We show that the Covering Principle known for continuous maps of the real line also holds for functions whose graph is a connected $G_δ$ subset of the plane. As an application we find an example of an approximately continuous (hence Darboux Baire 1) function $f : [0,1] \to [0,1]$ such that any closed subset of $[0,1]$ can be translated so as to become an $ω$-limit set of $f$. This solves a problem posed by Bruckner, Ceder and Pearson [Real Anal. Exchange 15 (1989/90)].

1. Introduction. For $f \in \mathbb{R}^R$, $f^0$ is the identity function, and for any integer $n > 0$, the $n$th iterate of $f$ is defined by $f^n = f \circ f^{n-1}$.

We say that for a given function $f : \mathbb{R} \to \mathbb{R}$ a compact interval $I_1 f$-covers a compact interval $I_2$ if $f(I_1) \supset I_2$. We then write $I_1 \to_f I_2$ (or $I_1 \to I_2$ if $f$ is clear from the context).

It is easy to see that if $f$ is continuous and

$$I_1 \to I_2 \to I_3 \to \cdots$$

for a sequence $\{I_i\}_{i \in \mathbb{N}}$ of compact intervals then there is an $x \in I_1$ such that $f^i(x) \in I_{i+1}$ for each $i$. This fact, known as the Covering Principle (or Itinerary Lemma), is widely used in one-dimensional dynamics (see e.g. [6]). We generalize it to the class of real functions with connected $G_δ$ graph in the following

THEOREM 1.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a connected $G_δ$ function and there exists a sequence of compact intervals

$$I_1 \to I_2 \to I_3 \to \cdots$$

Then there exists an $x \in I_1$ such that $f^i(x) \in I_{i+1}$ for each $i \in \mathbb{N}$.

2000 Mathematics Subject Classification: Primary 26A18; Secondary 26A15, 26A21, 37E05, 54C30.

Key words and phrases: connectivity functions, Darboux functions, Baire 1 functions, Borel measurable functions, Covering Principle, Itinerary Lemma, sequences of intervals, $ω$-limit sets, attractors, $f$-cover.
Recall that in [10] we have proved Sharkovskii’s theorem for connected $G_\delta$ real functions using the following dual (“cycle”) version of Theorem 1.1.

**Theorem 1.2 ([10]).** Suppose that $f: \mathbb{R} \to \mathbb{R}$ is a connected $G_\delta$ function and there exists a cycle of compact intervals

$$I_1 \to I_2 \to \cdots \to I_N \to I_1.$$ 

Then there exists an $x \in I_1$ such that $f^N(x) = x$ and $f^i(x) \in I_{i+1}$ for each $i \in \{0, \ldots, N-1\}$.

We give a proof of Theorem 1.1 in Section 3. In Section 4 we show how it can be used to solve a problem from [3] regarding $\omega$-limit sets of Darboux Baire 1 functions (see definitions below).

2. Preliminaries. We denote by $[a, b]$ the compact interval (possibly degenerate) with endpoints $a$ and $b$. We do not assume $a < b$ unless explicitly stated.

A set $H$ is said to be $\tau_d$-open if $H$ has inner density 1 at every point $x \in H$. These sets form a completely regular (but not normal) topology $\tau_d$ (see e.g. [7]).

We identify every function with its graph. We consider the following classes of functions $f$ from $\mathbb{R}$ into $\mathbb{R}$ (or $[0, 1]$ into $[0, 1]$, after obvious modifications):

- $f$ is approximately continuous ($f \in \mathcal{A}$) if $f^{-1}(U) \in \tau_d$ for every open set $U \subset [0, 1]$.
- $f$ is connected or a connectivity function ($f \in \text{Conn}$) if $f$ is a connected subset of $\mathbb{R}^2$;
- $f$ is Darboux ($f \in \mathcal{D}$) if $f$ has the intermediate value property, i.e. $f(I)$ is an interval for every interval $I \subset \mathbb{R}$;
- $f$ is Baire class 1 ($f \in \mathcal{B}_1$) if $f$ is a pointwise limit of a sequence of continuous functions; this is equivalent to $f^{-1}(G)$ being an $F_\sigma$ subset of $\mathbb{R}$ for every open $G \subset \mathbb{R}$, and to $f|_K$ having a point of continuity for every non-empty closed set $K \subset \mathbb{R}$;
- $f$ is Darboux Baire 1 ($f \in \mathcal{DB}_1$) if $f$ is Darboux and Baire 1;
- $f \in G_\delta$ if $f$ is a $G_\delta$ subset of $\mathbb{R}^2$, i.e. $f = \bigcap_{n \in \mathbb{N}} G_n$, where all $G_n \subset \mathbb{R}^2$ are open.

For properties of these and other Darboux-like classes of functions see e.g. the survey [4]. In particular, it is known that

$$\text{Conn} \subset \mathcal{D} \quad \text{and} \quad \mathcal{A} \subset \mathcal{DB}_1 \subset \text{Conn} \cap G_\delta,$$

and all these inclusions are proper. It follows that $\text{Conn} = \mathcal{D}$ within the class of Baire 1 functions. Moreover, every bounded approximately continuous
function is a derivative, and every derivative belongs to the class DB\(_1\) (see e.g. [2]).

A set \(W \subset \mathbb{R}\) is called an \(\omega\)-limit set for \(f\) if there is an \(x \in \mathbb{R}\) such that \(W\) is the cluster set of the sequence \(\{f^n(x)\} \ (n \in \mathbb{N})\), i.e.
\[
W = \bigcap_{i \in \mathbb{N}} \{f^j(x) : j > i\}.
\]
Denote this set by \(\omega_f(x)\) and let \(\Omega_f\) be the class of all \(\omega\)-limit sets of \(f\), i.e. \(\Omega_f = \{\omega_f(x) : x \in \mathbb{R}\}\). Clearly each element of \(\Omega_f\) is closed.

Agronsky, Bruckner, Ceder and Pearson have given in [1] the following characterization of \(\omega\)-limit sets: a non-empty closed set \(W \subset [0, 1]\) is an \(\omega\)-limit set of a continuous map \(f : [0, 1] \rightarrow [0, 1]\) if \(W\) is either a finite collection of nondegenerate closed intervals or is nowhere dense (the necessity of this condition was proved earlier by Sharkovskii in [9]).

The characterization of \(\omega\)-limit sets for a DB\(_1\) function was given by Bruckner, Ceder and Pearson in [3]: every non-empty closed subset of \([0, 1]\) is an \(\omega\)-limit set of a function \(f \in \text{DB}_1([0, 1])\). In Section 4 we show that there exists an approximately continuous (hence DB\(_1\)) function having a translation of every closed set as an \(\omega\)-limit set.

3. Proof of Theorem 1.1. Fix a Darboux function \(f : \mathbb{R} \rightarrow \mathbb{R}\). Suppose that there exist sequences \(\{a_i\}_{i \in \mathbb{N}}, \{b_i\}_{i \in \mathbb{N}}\) such that \(a_i < b_i\) and \([f(a_i), f(b_i)] \supset [a_{i+1}, b_{i+1}]\) for each \(i \in \mathbb{N}\). Clearly
\[
[a_1, b_1] \rightarrow [a_2, b_2] \rightarrow [a_3, b_3] \rightarrow \ldots .
\]
For every \(x \in \mathbb{R}\) and \(i \geq 1\) let
\[
\delta_i(x) = \begin{cases} 
  a_i & \text{if } x \leq a_i, \\
  b_i & \text{if } x \geq b_i, \\
  x & \text{if } x \in (a_i, b_i),
\end{cases}
\]
and \(\Delta_i = \delta_i \circ f \circ \delta_{i-1} \circ f \circ \delta_{i-2} \circ \cdots \circ \delta_2 \circ f \circ \delta_1\).

With every \(y \in \mathbb{R}\) we associate a sequence of symbols
\[
\alpha(y) = \alpha_1(y)\alpha_2(y)\alpha_3(y)\ldots
\]
with \(\alpha_i(y)\) being one of the symbols “\(L\)” , “\(U\)” or “\(C\)” given by the formula
\[
\alpha_i(x) = \begin{cases} 
  L & \text{if } \Delta_i(x) = a_i, \\
  U & \text{if } \Delta_i(x) = b_i, \\
  C & \text{otherwise.}
\end{cases}
\]
Clearly \(\alpha_i(a_1), \alpha_i(b_1) \in \{L, U\}\), and the sequence \(\alpha(a_1)\) differs from \(\alpha(b_1)\) at every position.

Notice that by definition, for any interval \(I\) and for every \(i \in \mathbb{N}\), either \(\delta_i(I) = \{a_i\}\), or \(\delta_i(I) = \{b_i\}\), or \(\delta_i(I) \subset I\). Consequently, for any \(x_1, x_2 \in \mathbb{R}\)
either the sequences $\alpha(x_1)$ and $\alpha(x_2)$ agree for sufficiently large indices, or

$$[\Delta_n(x_1), \Delta_n(x_2)] \rightarrow [\Delta_{n+1}(x_1), \Delta_{n+1}(x_2)]$$

for each $n$. So, we have the following

**Remark 3.1.** If $x_1, x_2 \in \mathbb{R}$ and the sequence $\alpha(x_1)$ differs from $\alpha(x_2)$ at infinitely many positions then

$$[\Delta_1(x_1), \Delta_1(x_2)] \rightarrow [\Delta_2(x_1), \Delta_2(x_2)] \rightarrow [\Delta_3(x_1), \Delta_3(x_2)] \rightarrow \cdots$$

**Lemma 3.2.** Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is connected and there exists a sequence of compact intervals

$$I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow \cdots$$

Then there exists an $x \in I_1$ such that for every open neighbourhood $G$ of $\langle x, f(x) \rangle$ there exists a sequence of compact intervals

$$I'_1 \rightarrow I'_2 \rightarrow I'_3 \rightarrow \cdots$$

such that $I'_1 \times I'_2 \subset G$ and $I'_i \subset I_i$ for each $i \geq 1$.

**Proof.** Note that if $I_n$ is degenerate for some $n \in \mathbb{N}$, then there exists an $x \in I_1$ such that $f^i(x) \in I_{i+1}$ for $i = 1, \ldots, n-1$ and the family $\{I'_i\}_{i \in \mathbb{N}}$ of degenerate intervals $I'_i = \{f^{i-1}(x)\}$ is as desired. So, we assume that all intervals $I_n$ are non-degenerate.

First we claim that there exist sequences $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$ such that for each $i \in \mathbb{N}$, $a_i < b_i$, $[a_i, b_i] \subset I_i$ and

$$[a_{i+1}, b_{i+1}] \subset [f(a_i), f(b_i)].$$

Indeed, by the intermediate value property of $f$, for each $i$ there exist $p_i, q_i \in I_i$ such that $f(p_i) = \inf I_{i+1}$ and $f(q_i) = \sup I_{i+1}$. Clearly the sequences of points

$$a_i = \min\{p_i, q_i\} \quad \text{and} \quad b_i = \max\{p_i, q_i\}$$

are as desired.

For any $x, y \in [a_1, b_1]$ let

$$\text{diff}(x, y) = \{n \in \mathbb{N} : \alpha_n(x) \neq \alpha_n(y)\}.$$

Let

$$A = \{\langle x, f(x) \rangle \in f|[a_1, b_1] : \text{diff}(a_1, x) \text{ is finite}\},$$

$$B = \{\langle x, f(x) \rangle \in f|[a_1, b_1] : \text{diff}(a_1, x) \text{ is infinite}\}.$$

Since $\langle a_1, f(a_1) \rangle \in A$, and $\alpha(a_1)$ differs from $\alpha(b_1)$ at every position, both sets are non-empty. Clearly $f|[a_1, b_1] = A \cup B$ and $A \cap B = \emptyset$. Since $f|[a_1, b_1]$ is connected, there is an $\langle x_0, f(x_0) \rangle$ in $(A \cap B) \cup (\overline{A} \cap B)$.

If $G$ is an open neighbourhood of $\langle x_0, f(x_0) \rangle$ then there exist $x_1, x_2 \in [a_1, b_1]$ with $\alpha(x_1)$ and $\alpha(x_2)$ differing at infinitely many positions and

$$[x_1, x_2] \times [f(x_1), f(x_2)] \subset G.$$
If we set \( I'_i = [\Delta_i(x_1), \Delta_i(x_2)] \) then \( I'_1 \times I'_2 \subset G \), \( I'_i \subset [a_i, b_i] \subset I_i \) for each \( i \), and by Remark 3.1,
\[
I'_1 \to I'_2 \to I'_3 \to \cdots. \quad \blacksquare
\]

**Lemma 3.3.** Suppose that \( f: \mathbb{R} \to \mathbb{R} \) is connected, \( f \subset G \) for an open set \( G \subset \mathbb{R} \times \mathbb{R} \) and there exists a sequence of compact intervals
\[
I_1 \to I_2 \to I_3 \to \cdots.
\]
Then there exists a sequence of compact intervals
\[
J_1 \to J_2 \to J_3 \to \cdots
\]
such that \( J_i \subset I_i \) and \( J_i \times J_{i+1} \subset G \) for each \( i \in \mathbb{N} \).

**Proof.** By Lemma 3.2 we can find a sequence \( \{I'_i\}_{i \geq 1} \) of compact intervals such that:

- \( I'_i \subset I_i \) for each \( i \geq 1 \);
- \( I'_1 \times I'_2 \subset G \);
- \( I'_1 \to I'_2 \to I'_3 \to \cdots \).

Continuing inductively (using Lemma 3.2 for the sequence \( \{I'^{n+1}_i\}_{i \geq n+1} \)) we can build a sequence \( \{I'^{n+1}_i\}_{i \geq n+1} \) of compact intervals such that for every \( n > 1 \):

- \( I'^{n+1}_i \subset I'_i \) for each \( i \geq n+1 \);
- \( I'^{n+1}_i \times I'^{n+1}_{i+1} \subset G \);
- \( I'^{n+1}_i \to I'^{n+1}_{i+2} \to I'^{n+1}_{i+3} \to \cdots \).

The assertion of the lemma follows from a diagonal argument, since:

- \( I'^n_i \subset I_i \) for each \( n \geq 1 \);
- \( I'^n_i \times I'^{n+1}_{i+1} \subset G \) for each \( n \geq 1 \);
- \( I'^1_1 \to I'^2_1 \to I'^3_1 \to \cdots \).

**Proof of Theorem 1.1.** Since \( f \) is \( G_\delta \) there exists a sequence \( \{G_n\}_{n \in \mathbb{N}} \) of open subsets of \( \mathbb{R}^2 \) such that \( f = \bigcap_{n \in \mathbb{N}} G_n \). Using Lemma 3.3 we can find a sequence \( \{J^1_i\}_{i \geq 1} \) of compact intervals such that:

- \( J^1_i \subset I_i \) for each \( i \);
- \( J^1_i \times J^1_{i+1} \subset G_1 \) for each \( i \);
- \( J^1_1 \to J^1_2 \to J^1_3 \to \cdots \).

Continuing inductively for every \( n \geq 2 \) we can define a sequence \( \{J^n_i\}_{i \in \mathbb{N}} \) of compact intervals such that:

1. \( J^n_i \subset J^{n-1}_i \subset I_i \) for each \( i \geq 1 \);
2. \( J^n_i \times J^{n+1}_{i+1} \subset G_n \) for each \( i \geq 1 \);
3. \( J^n_1 \to J^n_2 \to J^n_3 \to \cdots \).
It follows that for every $i \in \mathbb{N}$ there exists an $x_i \in \bigcap_{n \in \mathbb{N}} J^n_i$. By (1), $x_i \in I_i$ for each $i$. By (2),

$$\langle x_i, x_{i+1} \rangle \in \bigcap_{n \in \mathbb{N}} J^n_i \times J^n_{i+1} \subset \bigcap_{n \in \mathbb{N}} G_n = f,$$

so $x_{i+1} = f(x_i)$ for each $i$. Therefore $f^i(x_1) \in I_{i+1}$ for every $i \in \mathbb{N}$. 

Remark 3.4. Using a technique similar to that in [11] we can prove a generalization of Theorem 1.1 to the class of all finite compositions of connected $G_\delta$ functions (in [10] we have constructed an example of a connected $G_\delta$ function $f$ such that $f^2 \notin G_\delta$.)

4. A “universal” dynamical system generated by a Darboux Baire 1 map of the interval. The authors of [1] formulated the problem of existence of a “universal” continuous function, i.e. they asked if there exists a continuous function $f : [0,1] \to [0,1]$ such that $\Omega_f$ contains a homeomorphic copy of every $\omega$-limit set possible for continuous functions. An affirmative answer was given by Pokluda and Šmítal in [8].

The related problem for Darboux Baire 1 functions was formulated in [3], where Bruckner, Ceder and Pearson constructed a DB$_1$ function $f$ with $\Omega_f$ containing a homeomorphic copy of every $\omega$-limit set possible for continuous functions, and asked if there exists a DB$_1$ function $g$ such that $\Omega_g$ contains a homeomorphic copy of every non-empty closed set. (Recall that in [5] Keller gives a simple example of a function $f : [0,1] \to [0,1]$ continuous everywhere except for a single point such that any nowhere dense compact set $W \subset [0,1]$ has a homeomorphic copy in $\Omega_f$.) In Corollary 1 we answer this question in the affirmative. Moreover, we show that a “universal” Darboux Baire 1 function can be approximately continuous and bounded (hence a derivative).

In this section we prove the following

**Theorem 4.1.** There exists an approximately continuous function $f : [0,1] \to [0,1]$ such that every closed set $F \subset [0,1]$ with $0 \in F$ is an $\omega$-limit set for $f$.

The above theorem has a somewhat surprising

**Corollary 1.** There exists an approximately continuous function $f : [0,1] \to [0,1]$ such that any non-empty closed set $F \subset [0,1]$ can be translated so as to become an $\omega$-limit set for $f$.

To prove Theorem 4.1 we need the following lemma. Recall that a set $H$ is of type $M_5$ if $H$ is $F_\sigma$ and $\tau_4$-open.
Lemma 4.2 ([12]). If $E \subset H \subset [0, 1]$, $E$ is closed and $H$ is of type $M_5$, then there exists an approximately continuous function $f: [0, 1] \to [0, 1]$ such that
\[
\begin{cases}
  f(x) = 0 & \text{if } x \notin H, \\
  f(x) = 1 & \text{if } x \in E, \\
  0 < f(x) < 1 & \text{otherwise}.
\end{cases}
\]

Proof of Theorem 4.1. Let $E = \{0\}$ and $H$ be any $\tau_d$-open subset of $[0, 1]$ of type $F_\sigma$ containing zero such that $H$ and $[0, 1] \setminus H$ are dense in $[0, 1]$. By Lemma 4.2 there exists an approximately continuous function $f: [0, 1] \to [0, 1]$ with $f(0) = 1$ and with both sets
\[
\{x : f(x) = 0\} \text{ and } \{x : f(x) > 0\}
\]
being dense in $[0, 1]$. Observe that for such an $f$ and for every $x_1 \neq x_2$ there exists an arbitrarily small $\tau > 0$ such that
\[
(\star) \quad [x_1, x_2] \to [0, \tau] \to [0, 1].
\]
Let $F$ be a closed subset of $[0, 1]$ with $0 \in F$. Let $D = \{d_i : i \in \mathbb{N}\}$ be such that $F$ is the cluster set of $D$, and $\{D_i\}_{i \in \mathbb{N}}$ be a sequence of non-degenerate compact intervals such that $d_i \in D_i \subset [0, 1]$ and the diameter of $D_i$ is less than $i^{-1}$ for each $i > 0$. By $(\star)$, for every $i > 0$ there exists a $\tau_i \in (0, i^{-1})$ such that
\[
D_i \to_f [0, \tau_i] \to_f D_{i+1}.
\]
By Theorem 1.1 there exists an $x_0 \in [0, 1]$ such that $f^{2i}(x_0) \in D_i$ and $f^{2i+1}(x_0) \in [0, \tau_i]$. Thus
\[
\omega_f(x_0) = \bigcap_{i \in \mathbb{N}} \{f^{2j}(x_0) : j > i\} \cup \bigcap_{i \in \mathbb{N}} \{f^{2j+1}(x_0) : j > i\}
= \bigcap_{i \in \mathbb{N}} D_i \cup \bigcap_{i \in \mathbb{N}} [0, \tau_i] = F \cup \{0\} = F.
\]

The next argument (borrowed from [5]) shows that the “up to translation” part cannot be omitted in Corollary 1. Indeed, suppose that
\[
\{\{0, p\}, \{p, 1\} : p \in \mathbb{Q}\} \subset \Omega_f,
\]
where $\mathbb{Q}$ denotes the set of all rationals in $[0, 1]$. Then for every $p \in \mathbb{Q}$ and open $U_0 \ni 0, \ U_1 \ni 1, \ V \ni p, \ f(V) \cap U_0 \neq \emptyset \ \text{and} \ f(V) \cap U_1 \neq \emptyset,$
so the oscillation of $f$ is 1 at each point of $\mathbb{Q}$. Thus $f$ is nowhere continuous, hence not Baire 1.

Acknowledgments. We would like to thank the referee for valuable remarks, especially for simplified arguments for Remark 3.1.
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Department of Mathematics
Gdańsk University
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: pszuca@radix.com.pl

Received 20 June 2005;
in revised form 28 September 2006