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Extended Ramsey theory for words representing rationals

by

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Abstract. Ramsey theory for words over a finite alphabet was unified in the work of Carlson, who also presented a method to extend the theory to words over an infinite alphabet, but subject to a fixed dominating principle. In the present work we establish an extension of Carlson's approach to countable ordinals and Schreier-type families developing an extended Ramsey theory for dominated words over a doubly infinite alphabet (in fact for ω-$\mathbb{Z}^*$-located words), and we apply this theory, exploiting the Budak–Işik–Pym representation of rational numbers, to obtain an analogous partition theory for the set of rational numbers.

1. Introduction. We introduce (in Definition 2.1) the notion of ω-$\mathbb{Z}^*$-located words ($\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$) over an alphabet $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$ dominated by a two-sided sequence $\vec{k} = (k_n)_{n\in\mathbb{Z}^*}$ of natural numbers: a word $w = w_{n_1} \ldots w_{n_l}$ over Σ with domain $\{n_1 < \cdots < n_l\} \subseteq \mathbb{Z}^*$ is ω-$\mathbb{Z}^*$-located if for $1 \le i \le l$, $w_{n_i} \in \{\alpha_1, \ldots, \alpha_{k_{n_i}}\}$ if $n_i \in \mathbb{N}$ and $w_{n_i} \in \{\alpha_{-k_{n_i}}, \ldots, \alpha_{-1}\}$ if $-n_i \in \mathbb{N}$. The inspiration for this notion came from the representation of rational numbers introduced by T. Budak, N. Işik and J. Pym [BIP, Theorem 4.2], who proved that every rational number q has a unique representation as

$$q = \sum_{s=1}^{\infty} q_{-s} \frac{(-1)^s}{(s+1)!} + \sum_{r=1}^{\infty} q_r (-1)^{r+1} r!,$$

where $(q_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N} \cup \{0\}$ with $0 \le q_{-s} \le s$ for every $s > 0$, $0 \le q_r \le r$ for every $r > 0$ and $q_{-s} = q_r = 0$ for all but finitely many r, s. So, the set of non-zero rational numbers can be identified with the set of ω-$\mathbb{Z}^*$-located words over $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$, where $\alpha_{-n} = \alpha_n = n$ for $n \in \mathbb{N}$, dominated by the sequence $(k_n)_{n\in\mathbb{Z}^*}$, where $k_{-n} = k_n = n$ for $n \in \mathbb{N}$.

The entire infinitary Ramsey theory can be obtained for ω-$\mathbb{Z}^*$-located words; indeed, the classical Ramsey theory consisting of a partition the-

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orem for the family of m-tuples of ω-$\mathbb{Z}^*$-located words with m a natural number and a partition theorem for the family of infinite sequences of ω-$\mathbb{Z}^*$-located words considering suitable partition sets, as Borel sets with regard to the product topology, is presented in Section 2 (Theorems 2.5 and 2.6 respectively) and mainly follows from the fundamental work of Carlson [C] (Theorem 2.3 below).

In Section 3, we extend the classical Ramsey theory to partitions involving ξ-Schreier-type sequences of ω-$\mathbb{Z}^*$-located words for every countable ordinal ξ (Definition 3.2), which constitute the natural transfinite analogues of m-tuples of ω-$\mathbb{Z}^*$-located words (with m a natural number). The basic feature that distinguishes the families of ξ-Schreier-type sequences of ω-$\mathbb{Z}^*$-located words from each other is their complexity, as measured by a suitable Cantor–Bendixson type index introduced in Definition 3.15 (Proposition 3.18). Thus Theorem 3.5, a partition theorem for the family of ξ-Schreier sequences of ω-$\mathbb{Z}^*$-located words for each countable ordinal ξ, extends Theorem 2.6 (corresponding to ordinal level $\xi = m$, a natural number).

The main result of Section 3, and indeed of the paper, is Theorem 3.21, which on the one hand strengthens Theorem 3.5, in case the set of all finite sequences of variable ω-$\mathbb{Z}^*$-located words is partitioned by a family $\mathcal{F}$ which is a tree, and on the other hand implies a stronger countable ordinal form of Theorem 2.5 in case the partition sets are clopen in the product topology. More specifically, Theorem 3.5 provides no information on whether the ξ-homogeneous family, for a countable ordinal ξ, falls in $\mathcal{F}$ or in its complement, while Theorem 3.21 provides such a criterion in terms of a suitable Cantor–Bendixson type index of $\mathcal{F}$: if this index is greater than $\xi + 1$ then the ξ-homogeneous family falls in $\mathcal{F}$, and if less than ξ it falls in its complement.

The set of non-zero rational numbers with addition, using the representation given by Budak–Işik–Pym, can be identified with the set of ω-$\mathbb{Z}^*$-located words over $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$, where $\alpha_{-n} = \alpha_n = n$ for $n \in \mathbb{N}$, dominated by the sequence $(k_n)_{n\in\mathbb{Z}^*}$, where $k_{-n} = k_n = n$ for $n \in \mathbb{N}$, via the function $g : \widetilde{L}(\Sigma, \vec{k}) \to \mathbb{Q} \setminus \{0\}$ which sends a word $w = q_{t_1} \ldots q_{t_l} \in \widetilde{L}(\Sigma, \vec{k})$ to the rational number

$$q(w) = \sum_{t\in\mathrm{dom}^-(w)} q_t \frac{(-1)^{-t}}{(-t+1)!} + \sum_{t\in\mathrm{dom}^+(w)} q_t (-1)^{t+1} t!,$$

since g is one-to-one and onto and $g(w_1 \star w_2) = g(w_1) + g(w_2)$ for every $w_1 <_{R_1} w_2 \in \widetilde{L}(\Sigma \cup \{0\}, \vec{k})$. Applying the results of Sections 2 and 3, via the function g, to the rationals with addition, we obtain, in Section 4, an analogous Ramsey theory for the rational numbers, starting from the partition Theorem 4.1, a strengthened van der Waerden theorem for the set of rational numbers. Analogous partition theorems can be obtained for semigroups representable as ω-$\mathbb{Z}^*$-located words.

Notation. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z}^- = \{n \in \mathbb{Z} : n < 0\}$, $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ and $\mathbb{Q}$ the set of rational numbers. For a non-empty set X we denote by $[X]^{<\omega}$ the set of all finite subsets of X and by $[X]^{<\omega}_{>0}$ the set of non-empty finite subsets of X.

2. Classical Ramsey theory for ω-$\mathbb{Z}^*$-located words. The purpose of this section is to introduce ω-$\mathbb{Z}^*$-located words over an alphabet $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$ dominated by a two-sided sequence $\vec{k} = (k_n)_{n\in\mathbb{Z}^*}$ of natural numbers (Definition 2.1) and to develop the classical Ramsey theory for such words. These results mainly follow from the partition Theorem 2.3 for ω-located words (Definition 2.2) proved by Carlson [C]. Let us start with the necessary terminology and notation.

Definition 2.1. Let $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$ be an alphabet, $\vec{k} = (k_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N}$ a sequence such that $k_n \le k_{n+1}$ and $k_{-n} \le k_{-(n+1)}$ for every $n \in \mathbb{N}$, and let $\upsilon \notin \Sigma$ be an entity which is called a *variable*. We will call these assumptions "standard assumptions" throughout the paper.

An *ω-$\mathbb{Z}^*$-located word* over Σ dominated by $\vec{k}$ is a function w from a non-empty, finite subset F of $\mathbb{Z}^*$ into Σ such that $w(n) = w_n \in \{\alpha_1, \ldots, \alpha_{k_n}\}$ for every $n \in F \cap \mathbb{N}$ and $w_n \in \{\alpha_{-k_n}, \ldots, \alpha_{-1}\}$ for every $n \in F \cap \mathbb{Z}^-$.

A *variable ω-$\mathbb{Z}^*$-located word* over Σ dominated by $\vec{k}$ is a function w from a non-empty, finite subset F of $\mathbb{Z}^*$ into $\Sigma \cup \{\upsilon\}$ such that $w(n) = w_n \in \{\upsilon, \alpha_1, \ldots, \alpha_{k_n}\}$ for every $n \in F \cap \mathbb{N}$ and $w_n \in \{\upsilon, \alpha_{-k_n}, \ldots, \alpha_{-1}\}$ for every $n \in F \cap \mathbb{Z}^-$, and there exist $n_1 \in F \cap \mathbb{N}$ and $n_2 \in F \cap \mathbb{Z}^-$ such that $w_{n_1} = w_{n_2} = \upsilon$.

So, the set $\widetilde{L}(\Sigma, \vec{k})$ of all (constant) ω-$\mathbb{Z}^*$-located words over Σ dominated by $\vec{k}$ is

$$\widetilde{L}(\Sigma, \vec{k}) = \{w = w_{n_1} \ldots w_{n_l} : l \in \mathbb{N}, n_1 < \cdots < n_l \in \mathbb{Z}^* \text{ and } w_{n_i} \in \{\alpha_1, \ldots, \alpha_{k_{n_i}}\} \text{ if } n_i > 0,\ w_{n_i} \in \{\alpha_{-k_{n_i}}, \ldots, \alpha_{-1}\} \text{ if } n_i < 0 \text{ for every } 1 \le i \le l\},$$

and the set of *variable ω-$\mathbb{Z}^*$-located words* over Σ dominated by $\vec{k}$ is

$$\widetilde{L}(\Sigma, \vec{k}; \upsilon) = \{w = w_{n_1} \ldots w_{n_l} : l \in \mathbb{N},\ n_1 < \cdots < n_l \in \mathbb{Z}^*,\ w_{n_i} \in \{\upsilon, \alpha_1, \ldots, \alpha_{k_{n_i}}\} \text{ if } n_i > 0,\ w_{n_i} \in \{\upsilon, \alpha_{-k_{n_i}}, \ldots, \alpha_{-1}\} \text{ if } n_i < 0 \text{ for all } 1 \le i \le l \text{ and there exist } n_1 < 0 < n_2 \text{ with } w_{n_1} = w_{n_2} = \upsilon\}.$$

We set $\widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k}) = \widetilde{L}(\Sigma, \vec{k}) \cup \widetilde{L}(\Sigma, \vec{k}; \upsilon)$.

For $w = w_{n_1} \dots w_{n_l} \in \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})$ the set $\mathrm{dom}(w) = \{n_1 < \dots < n_l\}$ is the *domain* of w. Let $\mathrm{dom}^-(w) = \{n \in \mathrm{dom}(w) : n < 0\}$ and $\mathrm{dom}^+(w) = \{n \in \mathrm{dom}(w) : n > 0\}$.

For $w = w_{n_1} \dots w_{n_r}, u = u_{m_1} \dots u_{m_l} \in \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})$ with $\mathrm{dom}(w) \cap \mathrm{dom}(u) = \emptyset$ we define their concatenation located on the union of the domains of w, u as

$$w \star u = z_{q_1} \dots z_{q_{r+l}} \in \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k}),$$

where $\{q_1 < \dots < q_{r+l}\} = \mathrm{dom}(w) \cup \mathrm{dom}(u)$, $z_i = w_i$ if $i \in \mathrm{dom}(w)$ and $z_i = u_i$ if $i \in \mathrm{dom}(u)$.

We endow the set $\widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})$ with a relation $<_{R_1}$ by defining for $w, u \in \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})$,

$$w <_{R_1} u \;\Leftrightarrow\; \begin{array}{l} \mathrm{dom}(u) = A_1 \cup A_2 \text{ with } A_1, A_2 \neq \emptyset \text{ such that} \\ \max A_1 < \min \mathrm{dom}(w) \leq \max \mathrm{dom}(w) < \min A_2. \end{array}$$

We define

$$\widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon) = \{(w_n)_{n\in\mathbb{N}} \subseteq \widetilde{L}(\Sigma, \vec{k}; \upsilon) : w_n <_{R_1} w_{n+1} \text{ for every } n \in \mathbb{N}\}.$$

For $m \in \mathbb{N}$ we set

$$\widetilde{L}^m(\Sigma, \vec{k}; \upsilon) = \{(w_1, \dots, w_m) : w_1 <_{R_1} \dots <_{R_1} w_m \in \widetilde{L}(\Sigma, \vec{k}; \upsilon)\}.$$

For every $(p, q) \in \mathbb{N} \times \mathbb{N} \cup \{(0, 0)\}$ we define the functions

$$T_{(p,q)} : \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k}) \to \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})$$

setting, for $w = w_{n_1} \dots w_{n_l} \in \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})$, $T_{(0,0)}(w) = w$ and, for $(p, q) \in \mathbb{N} \times \mathbb{N}$, $T_{(p,q)}(w) = u_{n_1} \dots u_{n_l}$, where, for $1 \leq i \leq l$,

$$u_{n_i} = \begin{cases} w_{n_i} & \text{if } w_{n_i} \in \Sigma, \\ \alpha_p & \text{if } w_{n_i} = \upsilon,\ n_i > 0 \text{ and } p \leq k_{n_i}, \\ \alpha_{k_{n_i}} & \text{if } w_{n_i} = \upsilon,\ n_i > 0 \text{ and } p > k_{n_i}, \\ \alpha_{-q} & \text{if } w_{n_i} = \upsilon,\ n_i < 0 \text{ and } q \leq k_{n_i}, \\ \alpha_{-k_{n_i}} & \text{if } w_{n_i} = \upsilon,\ n_i < 0 \text{ and } q > k_{n_i}. \end{cases}$$

We remark that for every $(p, q) \in \mathbb{N} \times \mathbb{N} \cup \{(0, 0)\}$ we have $\mathrm{dom}(T_{(p,q)}(w)) = \mathrm{dom}(w)$ for $w \in \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})$, $T_{(p,q)}(w) = w$ for $w \in \widetilde{L}(\Sigma, \vec{k})$ and $T_{(p,q)}(w \star u) = T_{(p,q)}(w) \star T_{(p,q)}(u)$ for every $w, u \in \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})$ with $\mathrm{dom}(w) \cap \mathrm{dom}(u) = \emptyset$. Also, $T_{(p,q)}(\widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})) \subseteq \widetilde{L}(\Sigma, \vec{k})$ for every $(p, q) \in \mathbb{N} \times \mathbb{N}$.

Extracted ω-$\mathbb{Z}^*$-located words, extractions. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. We fix a sequence $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$.

An *extracted variable ω-$\mathbb{Z}^*$-located word* of $\vec{w}$ is a variable ω-$\mathbb{Z}^*$-located word $u \in \widetilde{L}(\Sigma, \vec{k}; \upsilon)$ such that

$$u = T_{(p_1,q_1)}(w_{n_1}) \star \dots \star T_{(p_\lambda,q_\lambda)}(w_{n_\lambda}),$$

where $\lambda \in \mathbb{N}$, $n_1 < \cdots < n_\lambda \in \mathbb{N}$, $(p_i, q_i) \in \mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ with $0 \leq p_i \leq k_{n_i}$, $0 \leq q_i \leq k_{-n_i}$ for every $1 \leq i \leq \lambda$ and $(0,0) \in \{(p_1, q_1), \ldots, (p_\lambda, q_\lambda)\}$. The set of extracted variable ω-$\mathbb{Z}^*$-located words of $\vec{w}$ is denoted by $\widetilde{\mathrm{EV}}(\vec{w})$.

An *extracted* ω*-*$\mathbb{Z}^*$*-located word* of $\vec{w}$ is an ω-$\mathbb{Z}^*$-located word $z \in \widetilde{L}(\Sigma, \vec{k})$ with

$$z = T_{(p_1,q_1)}(w_{n_1}) \star \ldots \star T_{(p_\lambda,q_\lambda)}(w_{n_\lambda}),$$

where $\lambda \in \mathbb{N}$, $n_1 < \cdots < n_\lambda \in \mathbb{N}$ and $(p_i, q_i) \in \mathbb{N}\times\mathbb{N}$ with $1 \leq p_i \leq k_{n_i}$, $1 \leq q_i \leq k_{-n_i}$ for every $1 \leq i \leq \lambda$. The set of extracted ω-$\mathbb{Z}^*$-located words of $\vec{w}$ is denoted by $\widetilde{E}(\vec{w})$. Let

$$\widetilde{\mathrm{EV}}^{\infty}(\vec{w}) = \{\vec{u} = (u_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon) : u_n \in \widetilde{\mathrm{EV}}(\vec{w}) \text{ for every } n \in \mathbb{N}\}.$$

For $m \in \mathbb{N}$ we set

$$\widetilde{\mathrm{EV}}^{m}(\vec{w}) = \{(u_1, \ldots, u_m) : u_1 <_{R_1} \cdots <_{R_1} u_m \in \widetilde{\mathrm{EV}}(\vec{w})\}.$$

If $\vec{u} \in \widetilde{\mathrm{EV}}^{\infty}(\vec{w})$, then we say that $\vec{u}$ is an *extraction* of $\vec{w}$ and we write $\vec{u} \prec \vec{w}$. Notice that for $\vec{u}, \vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ we have $\vec{u} \prec \vec{w}$ if and only if $\widetilde{\mathrm{EV}}(\vec{u}) \subseteq \widetilde{\mathrm{EV}}(\vec{w})$.

DEFINITION 2.2. Let $\Sigma = \{\alpha_n : n \in \mathbb{N}\}$ be an alphabet, $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ a sequence such that $k_n \leq k_{n+1}$ for every $n \in \mathbb{N}$ and let $\upsilon \notin \Sigma$ be a *variable.*

An ω*-located word* over the alphabet Σ dominated by $\vec{k}$ is a function w from a non-empty, finite subset F of $\mathbb{N}$ into Σ such that $w(n) = w_n \in \{\alpha_1, \ldots, \alpha_{k_n}\}$ for every $n \in F$.

A *variable* ω*-located word* over Σ dominated by $\vec{k}$ is a function w from a non-empty, finite subset F of $\mathbb{N}$ into $\Sigma \cup \{\upsilon\}$ such that $w(n) = w_n \in \{\upsilon, \alpha_1, \ldots, \alpha_{k_n}\}$ for every $n \in F$ and there exists $n_1 \in F$ such that $w_{n_1} = \upsilon$.

So, the set of ω-located words over Σ dominated by $\vec{k}$ is

$$\begin{aligned} L(\Sigma, \vec{k}) = \{w = w_{n_1} \ldots w_{n_l} : l \in \mathbb{N},\ & n_1 < \cdots < n_l \in \mathbb{N}, \\ & w_{n_i} \in \{\alpha_1, \ldots, \alpha_{k_{n_i}}\} \text{ for every } 1 \leq i \leq l\}, \end{aligned}$$

and the set of variable ω-located words over Σ dominated by $\vec{k}$ respectively, is

$$\begin{aligned} L(\Sigma, \vec{k}; \upsilon) = \{w = w_{n_1} \ldots w_{n_l} :\ & l \in \mathbb{N},\ n_1 < \cdots < n_l \in \mathbb{N},\ w_{n_i} \in \\ & \{\upsilon, \alpha_1, \ldots, \alpha_{k_{n_i}}\} \text{ for every } 1 \leq i \leq l \text{ and there exists} \\ & 1 \leq i \leq l \text{ with } w_{n_i} = \upsilon\}. \end{aligned}$$

Let $L(\Sigma \cup \{\upsilon\}, \vec{k}) = L(\Sigma, \vec{k}) \cup L(\Sigma, \vec{k}; \upsilon)$.

For $w, u \in L(\Sigma \cup \{\upsilon\}, \vec{k})$ we write

$$w <_{R_2} u \ \Leftrightarrow\ \max \mathrm{dom}(w) < \min \mathrm{dom}(u).$$

We define

$$L^\infty(\Sigma,\vec{k};\upsilon)=\{(w_n)_{n\in\mathbb{N}}\subseteq L(\Sigma,\vec{k};\upsilon): w_n<_{R_2} w_{n+1} \text{ for every } n\in\mathbb{N}\}.$$

For every $p\in\mathbb{N}\cup\{0\}$ we define the functions

$$T_p: L(\Sigma\cup\{\upsilon\},\vec{k})\to L(\Sigma\cup\{\upsilon\},\vec{k})$$

setting for $w=w_{n_1}\dots w_{n_l}\in L(\Sigma\cup\{\upsilon\},\vec{k})$: $T_0(w)=w$ and, for $p\in\mathbb{N}$, $T_p(w)=u_{n_1}\dots u_{n_l}$, where, for $1\le i\le l$, $u_{n_i}=w_{n_i}$ if $w_{n_i}\in\Sigma$, $u_{n_i}=\alpha_p$ if $w_{n_i}=\upsilon$ and $p\le k_{n_i}$ and finally, $u_{n_i}=\alpha_{k_{n_i}}$ if $w_{n_i}=\upsilon$ and $p>k_{n_i}$.

Let $\vec{w}=(w_n)_{n\in\mathbb{N}}\in L^\infty(\Sigma,\vec{k};\upsilon)$. The set $\mathrm{EV}(\vec{w})$ of *extracted variable ω-located words* of $\vec{w}$ consists of all words of the form

$$T_{p_1}(w_{n_1})\star\dots\star T_{p_\lambda}(w_{n_\lambda}),$$

where $\lambda\in\mathbb{N}$, $n_1<\dots<n_\lambda\in\mathbb{N}$ and $p_1,\dots,p_\lambda\in\mathbb{N}\cup\{0\}$ such that $0\le p_i\le k_{n_i}$ for every $1\le i\le\lambda$ and $0\in\{p_1,\dots,p_\lambda\}$, and the set $E(\vec{w})$ of *extracted ω-located words* of $\vec{w}$ consists of all words of the form $T_{p_1}(w_{n_1})\star\dots\star T_{p_\lambda}(w_{n_\lambda})$, where $\lambda\in\mathbb{N}$, $n_1<\dots<n_\lambda\in\mathbb{N}$ and $p_1,\dots,p_\lambda\in\mathbb{N}$ such that $1\le p_i\le k_{n_i}$ for every $1\le i\le\lambda$. Let

$$\mathrm{EV}^\infty(\vec{w})=\{\vec{u}=(u_n)_{n\in\mathbb{N}}\in L^\infty(\Sigma,\vec{k};\upsilon): u_n\in\mathrm{EV}(\vec{w}) \text{ for every } n\in\mathbb{N}\}.$$

If $\vec{u}\in\mathrm{EV}^\infty(\vec{w})$, then we say that $\vec{u}$ is an *extraction* of $\vec{w}$ and we write $\vec{u}\prec\vec{w}$. Notice that $\vec{u}\prec\vec{w}$ if and only if $\mathrm{EV}(\vec{u})\subseteq\mathrm{EV}(\vec{w})$.

With the previous terminology we can state the following fundamental partition theorem of Carlson for infinite sequences of variable ω-located words, which is a corollary of the much stronger Theorem 15 of [C]. According to the latter, the partition families $A_1,\dots,A_r$, referred to in the statement of Theorem 2.3 below, can be members of a wider class of sets, but we restrict to the class of Borel sets, since it is a sufficiently large, universally understood class.

THEOREM 2.3 (Carlson, [C]). *Let $\Sigma=\{\alpha_n:n\in\mathbb{N}\}$ be an infinite countable alphabet, $\upsilon\notin\Sigma$ a variable, $\vec{k}=(k_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}$ an increasing sequence and $r\in\mathbb{N}$. If $L^\infty(\Sigma,\vec{k};\upsilon)=A_1\cup\dots\cup A_r$ where A_i is a Borel set (with regard to the product topology on sequences of elements of $L(\Sigma,\vec{k};\upsilon)$, where $L(\Sigma,\vec{k};\upsilon)$ has the discrete topology) for all $i=1,\dots,r$, then there exists a sequence $\vec{w}=(w_n)_{n\in\mathbb{N}}\in L^\infty(\Sigma,\vec{k};\upsilon)$ and $1\le i_0\le r$ such that*

$$\mathrm{EV}^\infty(\vec{w})\in A_{i_0}.$$

For the sake of completeness, we will state the following partition theorem for variable ω-located words. It follows from the stronger Theorem 2.3, but it can also be proved independently either similarly to Lemma 5.9 in [C], as indicated in [C], or as Theorem 1.4 in [F4].

THEOREM 2.4 (Carlson, [C]). *Let $\Sigma = \{\alpha_n : n \in \mathbb{N}\}$ be an infinite countable alphabet, $\upsilon \notin \Sigma$ a variable, $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence and $r, s \in \mathbb{N}$. If $L(\Sigma, \vec{k}; \upsilon) = A_1 \cup \cdots \cup A_r$ and $L(\Sigma, \vec{k}) = C_1 \cup \cdots \cup C_s$, then there exists a sequence $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)$ and $1 \le i_0 \le r$, $1 \le j_0 \le s$ such that*

$$\mathrm{EV}(\vec{w}) \in A_{i_0} \quad \textit{and} \quad E(\vec{w}) \in C_{j_0}.$$

Now we will prove a partition theorem for infinite sequences of variable ω-$\mathbb{Z}^*$-located words using Theorem 2.3.

THEOREM 2.5. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and let $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$, $r \in \mathbb{N}$. If $\widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon) = A_1 \cup \cdots \cup A_r$, where A_i is a Borel set (with regard to the product topology on sequences of elements of $\widetilde{L}(\Sigma, \vec{k}; \upsilon)$, where $\widetilde{L}(\Sigma, \vec{k}; \upsilon)$ has the discrete topology) for all $i = 1, \ldots, r$, then there exists an extraction $\vec{u} = (u_n)_{n\in\mathbb{N}}$ of $\vec{w}$ and $1 \le i_0 \le r$ such that*

$$\widetilde{\mathrm{EV}}^\infty(\vec{u}) \subseteq A_{i_0}.$$

Proof. We will order the set $\mathbb{N} \times \mathbb{N}$. For $(p, q) \in \mathbb{N} \times \mathbb{N}$ we set $i(p, q)$ equal to the least $n \in \mathbb{N}$ such that $p \le k_n$ and $q \le k_{-n}$; then we define the order $<_*$ of $\mathbb{N} \times \mathbb{N}$ so that $(p_1, q_1) <_* (p_2, q_2)$ if and only if either $i(p_1, q_1) < i(p_2, q_2)$ or both $i(p_1, q_1) = i(p_2, q_2)$ and (p_1, q_1) is less than (p_2, q_2) in the lexicographical ordering (i.e. either $p_1 < p_2$ or both $p_1 = p_2$ and $q_1 < q_2$).

Let $\mathbb{N} \times \mathbb{N} = \{\beta_1 <_* \beta_2 <_* \cdots\}$ and let $\vec{l} = (l_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be the increasing sequence such that $\beta_{l_n} = (k_n, k_{-n})$. We set $\widetilde{\Sigma} = \{\beta_n : n \in \mathbb{N}\}$ and we consider the function $h : L(\widetilde{\Sigma}, \vec{l}; \upsilon) \to \widetilde{\mathrm{EV}}(\vec{w})$ which sends $t_{n_1} \ldots t_{n_\lambda} \in L(\widetilde{\Sigma}, \vec{l}; \upsilon)$ to

$$h(t_{n_1} \ldots t_{n_\lambda}) = T_{(p_1,q_1)}(w_{n_1}) \star \cdots \star T_{(p_\lambda,q_\lambda)}(w_{n_\lambda}),$$

where for $1 \le i \le \lambda$, $(p_i, q_i) = (0, 0)$ if $t_{n_i} = \upsilon$ and $(p_i, q_i) = (\mu_1, \mu_2)$ if $t_{n_i} = \beta_\mu = (\mu_1, \mu_2)$. The function h is one-to-one and onto $\widetilde{\mathrm{EV}}(\vec{w})$.

Now, we define an extension $\overline{h}$ of h to $L^\infty(\widetilde{\Sigma}, \vec{l}; \upsilon)$ setting

$$\overline{h} : L^\infty(\widetilde{\Sigma}, \vec{l}; \upsilon) \to \widetilde{\mathrm{EV}}^\infty(\vec{w}) \quad \text{with} \quad \overline{h}((t_n)_{n\in\mathbb{N}}) = (h(t_n))_{n\in\mathbb{N}}$$

for every $(t_n)_{n\in\mathbb{N}} \in L^\infty(\widetilde{\Sigma}, \vec{l}; \upsilon)$. Note that $\overline{h}$ is a homeomorphism with respect to the product topologies.

By Theorem 2.3, there exist a sequence $\vec{s} = (s_n)_{n\in\mathbb{N}} \in L^\infty(\widetilde{\Sigma}, \vec{l}; \upsilon)$ and $1 \le i_0 \le r$ such that $\mathrm{EV}^\infty(\vec{s}) \subseteq (\overline{h})^{-1}(A_{i_0})$. Set $u_n = h(s_n) \in \widetilde{\mathrm{EV}}(\vec{w})$ for every $n \in \mathbb{N}$ and $\vec{u} = (u_n)_{n\in\mathbb{N}}$. Then $\vec{u} = (u_n)_{n\in\mathbb{N}} \prec \vec{w}$ and $\widetilde{\mathrm{EV}}^\infty(\vec{u}) \subseteq \overline{h}(\mathrm{EV}^\infty(\vec{s})) \subseteq A_{i_0}$. ■

Theorem 2.5 implies the following partition theorem for ordered m-tuples of variable ω-$\mathbb{Z}^*$-located words for every natural number m.

THEOREM 2.6. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and let $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ and $r, m \in \mathbb{N}$. If $\widetilde{L}^m(\Sigma, \vec{k}; \upsilon) = C_1 \cup \cdots \cup C_r$, then there exists an extraction $\vec{u} = (u_n)_{n\in\mathbb{N}}$ of $\vec{w}$ and $1 \le i_0 \le r$ such that*

$$\widetilde{\mathrm{EV}}^{m}(\vec{u}) \subseteq C_{i_0}.$$

Proof. Set $A_i = \{(x_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon) : (x_1, \ldots, x_m) \in C_i\}$ for $i = 1, \ldots, r$. Then $\widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon) = A_1 \cup \cdots \cup A_r$, and $A_1, \ldots, A_r$ are Borel subsets of $\widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$. The conclusion follows from Theorem 2.5. ∎

REMARK 2.7. 1) The initial case ($m = 1$) of Theorem 2.6 has a proof independent of Theorem 2.5 applying Theorem 2.4 via the function $h : L(\widetilde{\Sigma}, \vec{l}; \upsilon) \to \widetilde{\mathrm{EV}}(\vec{w})$ defined in the proof of Theorem 2.5.

2) Theorem 2.6 can be proved by induction from its initial case $m = 1$, using Lemma 3.6 as in the proof of Theorem 3.5.

3) Theorems 2.5 and 2.6 imply the analogous partition theorems for constant ω-$\mathbb{Z}^*$-located words.

3. Ramsey-theoretic results involving Schreier systems for ω-$\mathbb{Z}^*$-located words. The starting point of this section is Theorem 3.5, an extended Ramsey-type partition theorem ([R]) for variable ω-$\mathbb{Z}^*$-located words over an alphabet $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$, dominated by a sequence $\vec{k} = (k_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N}$. It is an extension to every countable order ξ of Theorem 2.5 corresponding to the case $\xi = m$, a natural number. As a consequence of Theorem 3.5 we can get an extended (to every countable order) Ramsey-type partition theorem for ω-located words (see Corollary 3.8).

The main result of this section is Theorem 3.21. It is a strengthening of Theorem 3.5 in case the partition family $\mathcal{F}$ is a tree, providing a criterion, in terms of a Cantor–Bendixson type index for $\mathcal{F}$, to decide whether the ξ-homogeneous family falls into $\mathcal{F}$ or into its complement. We note, in Theorem 3.24 and Remark 3.25, that Theorem 3.21 can be considered as a strengthening of the particular case of Theorem 2.5 in which the partition families of $\widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ are clopen in the product topology.

The vehicle for proving these extended Ramsey type partition Theorems 3.5 and 3.21 is the Schreier systems $(\widetilde{L}^{\xi}(\Sigma, \vec{k}))_{\xi<\omega_1}$ and $(\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))_{\xi<\omega_1}$, consisting of all families of finite orderly sequences of (constant and variable respectively) ω-$\mathbb{Z}^*$-located words over the alphabet Σ, dominated by the sequence $\vec{k}$ (given in Definition 3.2). Instrumental for this definition are the Schreier sets $\mathcal{A}_{\xi}$, consisting of finite subsets of $\mathbb{N}$, which are defined below in Definition 3.1 (employing in case (3iii) the Cantor normal form of ordinals; cf. [KM], [L]). Schreier sets were systematically studied in [F1] and [F3].

NOTATION. For $s_1, s_2 \in [\mathbb{N}]^{<\omega}_{>0}$ we write $s_1 < s_2 \Leftrightarrow \max s_1 < \min s_2$.

Definition 3.1 (The Schreier systems, [F1, Def. 7], [F2, Def. 1.5], [F3, Def. 1.4]). For every non-zero, countable, limit ordinal λ choose a strictly increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of successor ordinals smaller than λ with $\sup_n \lambda_n = \lambda$ (i.e. λ is the least ordinal such that $\lambda_n \leq \lambda$ for all $n \in \mathbb{N}$). The system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is defined recursively as follows:

(1) $\mathcal{A}_0 = \{\emptyset\}$ and $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$;

(2) $\mathcal{A}_{\zeta+1} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \{n\} \cup s_1$, where $n \in \mathbb{N}$, $\{n\} < s_1$ and $s_1 \in \mathcal{A}_\zeta\}$;

(3i) $\mathcal{A}_{\omega^{\beta+1}} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \bigcup_{i=1}^n s_i$, where $n = \min s_1$, $s_1 < \cdots < s_n$ and $s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\}$;

(3ii) for a non-zero, countable limit ordinal λ, $\mathcal{A}_{\omega^\lambda} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s \in \mathcal{A}_{\omega^{\lambda_n}}$ with $n = \min s\}$;

(3iii) for a limit ordinal ξ such that $\omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$, if $\xi = \omega^\alpha p + \sum_{i=1}^m \omega^{a_i} p_i$, where $m \in \mathbb{N}$ with $m \geq 0$, $p, p_1, \ldots, p_m$ are natural numbers with $p, p_1, \ldots, p_m \geq 1$ (so that either $p > 1$, or $p = 1$ and $m \geq 1$) and $a, a_1, \ldots, a_m$ are ordinals with $a > a_1 > \cdots > a_m > 0$, then

$$\mathcal{A}_\xi = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = s_0 \cup \textstyle\bigcup_{i=1}^m s_i \text{ with } s_m < \cdots < s_1 < s_0, s_0 = s^0_1 \cup \cdots \cup s^0_p \text{ with } s^0_1 < \cdots < s^0_p \in \mathcal{A}_{\omega^a}, \text{ and } s_i = s^i_1 \cup \cdots \cup s^i_{p_i} \text{ with } s^i_1 < \cdots < s^i_{p_i} \in \mathcal{A}_{\omega^{a_i}} \text{ for every } 1 \leq i \leq m\}.$$

The Schreier systems are special systems of Ramsey families defined in [F3].

A *system of Ramsey families* is a collection $\mathcal{A} = (\mathcal{A}_\xi)_{\xi<\omega_1}$ of finite subsets of $\mathbb{N}$ defined recursively, by fixing for every non-zero, limit countable ordinal ξ a strictly increasing sequence of successor ordinals smaller than ξ with $\sup_n \xi_n = \xi$, as follows:

$$\mathcal{A}_0 = \{\emptyset\} \quad \text{and} \quad \mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\};$$

for every countable ordinal ζ,

$$\mathcal{A}_{\zeta+1} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \{n\} \cup s_1, \text{ where } n \in \mathbb{N}, \{n\} < s_1 \text{ and } s_1 \in \mathcal{A}_\zeta\};$$

and for every non-zero, limit countable ordinal ξ,

$$\mathcal{A}_\xi = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s \in \mathcal{A}_{\xi_n} \text{ with } n = \min s\}.$$

With suitable choices of sequences $(\xi_n)_{n\in\mathbb{N}}$ for each countable, non-zero limit ordinal ξ one can define interesting systems of Ramsey families. The Schreier systems are the simplest systems of Ramsey families defined by employing the Cantor normal form of countable ordinals. We note that although a Schreier system is a purely combinatorial entity, it nevertheless arose gradually in connection with the theory of Banach spaces (more details can be found in the introduction of [FN2]).

We point out that the results in this section can be stated for systems of Ramsey families instead of Schreier systems and also that they do not depend on the particular choice of the converging sequences, as the complexity of the family $\mathcal{A}_\xi$, as measured by its Cantor–Bendixson index, is independent of the particular choice of the converging sequences.

We will now define the Schreier systems of ω-$\mathbb{Z}^*$-located words.

NOTATION. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions (see Definition 2.1). We define the *finite orderly sequences of ω-$\mathbb{Z}^*$-located words* over Σ dominated by $\vec{k}$ as follows:

$$\widetilde{L}^{<\infty}(\Sigma,\vec{k}) = \{\mathbf{w} = (w_1,\ldots,w_l) : l \in \mathbb{N},\ w_1 <_{R_1} \cdots <_{R_1} w_l \in \widetilde{L}(\Sigma,\vec{k})\} \cup \{\emptyset\},$$

$$\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) = \{\mathbf{w} = (w_1,\ldots,w_l) : l \in \mathbb{N},\ w_1 <_{R_1} \cdots <_{R_1} w_l \in \widetilde{L}(\Sigma,\vec{k};\upsilon)\} \cup \{\emptyset\},$$

$$\widetilde{L}^{<\infty}(\Sigma\cup\{\upsilon\},\vec{k}) = \widetilde{L}^{<\infty}(\Sigma,\vec{k}) \cup \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon).$$

DEFINITION 3.2 (Schreier systems $(\widetilde{L}^\xi(\Sigma,\vec{k}))_{\xi<\omega_1}$ and $(\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon))_{\xi<\omega_1}$). We define

$$\widetilde{L}^0(\Sigma,\vec{k}) = \{\emptyset\} = \widetilde{L}^0(\Sigma,\vec{k};\upsilon), \qquad \xi \geq 1,$$

and for every countable ordinal

$$\widetilde{L}^\xi(\Sigma,\vec{k}) = \{(w_1,\ldots,w_l) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k}) : \{\min \mathrm{dom}^+(w_1),\ldots,\min \mathrm{dom}^+(w_l)\} \in \mathcal{A}_\xi\},$$

$$\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon) = \{(w_1,\ldots,w_l) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) : \{\min \mathrm{dom}^+(w_1),\ldots,\min \mathrm{dom}^+(w_l)\} \in \mathcal{A}_\xi\}.$$

REMARK 3.3.

(i) $\emptyset \notin \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$ for every $\xi \geq 1$.

(ii) $\widetilde{L}^m(\Sigma,\vec{k};\upsilon) = \{(w_1,\ldots,w_m) : w_1 <_{R_1} \cdots <_{R_1} w_m \in \widetilde{L}(\Sigma,\vec{k};\upsilon)\}$ for $m \in \mathbb{N}$.

(iii) $\widetilde{L}^\omega(\Sigma,\vec{k};\upsilon) = \{(w_1,\ldots,w_n) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) : n \in \mathbb{N}$, and $\min \mathrm{dom}^+(w_1) = n\}$.

(iv) Alternatively we could define the sets $\widetilde{L}^\xi(\Sigma,\vec{k})$, $\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$ via the negative part of the domain of words as follows:

$$\widetilde{L}^\xi(\Sigma,\vec{k}) = \{(w_1,\ldots,w_l) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k}) : \{|\max \mathrm{dom}^-(w_1)|,\ldots,|\max \mathrm{dom}^-(w_l)|\} \in \mathcal{A}_\xi\},$$

$$\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon) = \{(w_1,\ldots,w_l) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) : \{|\max \mathrm{dom}^-(w_1)|,\ldots,|\max \mathrm{dom}^-(w_l)|\} \in \mathcal{A}_\xi\}.$$

The following proposition justifies the recursiveness of $(\widetilde{L}^\xi(\Sigma,\vec{k}))_{\xi<\omega_1}$ and $(\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon))_{\xi<\omega_1}$.

For a family $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma \cup \{\upsilon\}, \vec{k})$ and $t \in \widetilde{L}(\Sigma \cup \{\upsilon\}, \vec{k})$, we set

$$\mathcal{F}(t) = \{\mathbf{w} \in \widetilde{L}^{<\infty}(\Sigma \cup \{\upsilon\}, \vec{k}) : \text{either } \mathbf{w} = (w_1, \ldots, w_l) \neq \emptyset \text{ and } (t, w_1, \ldots, w_l) \in \mathcal{F}, \text{ or } \mathbf{w} = \emptyset \text{ and } (t) \in \mathcal{F}\},$$

$$\mathcal{F} - t = \{\mathbf{w} \in \mathcal{F} : \text{either } \mathbf{w} = (w_1, \ldots, w_l) \neq \emptyset \text{ and } t <_{R_1} w_1, \text{ or } \mathbf{w} = \emptyset\}.$$

PROPOSITION 3.4. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. For each countable ordinal $\xi \geq 1$, there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of countable ordinals with $\xi_n < \xi$ such that for $s \in \widetilde{L}(\Sigma, \vec{k})$ and $t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon)$, with* $\min \mathrm{dom}^+(s) = \min \mathrm{dom}^+(t) = n$, *we have*

$$\widetilde{L}^{\xi}(\Sigma, \vec{k})(s) = \widetilde{L}^{\xi_n}(\Sigma, \vec{k}) \cap (\widetilde{L}^{<\infty}(\Sigma, \vec{k}) - s),$$
$$\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) = \widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap (\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) - t).$$

Moreover, $\xi_n = \zeta$ for every $n \in \mathbb{N}$ if $\xi = \zeta + 1$, and $(\xi_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $\sup_n \xi_n = \xi$ *if ξ is a limit ordinal.*

Proof. This follows from Theorem 1.6 in [F3], according to which for each countable ordinal $\xi > 0$ there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of countable ordinals with $\xi_n < \xi$ such that $\mathcal{A}_\xi(n) = \mathcal{A}_{\xi_n} \cap [\{n+1, n+2, \ldots\}]^{<\omega}$ for every $n \in \mathbb{N}$, where $\mathcal{A}_\xi(n) = \{s \in [\mathbb{N}]^{<\omega} : s \in [\mathbb{N}]^{<\omega}_{>0}, n < \min s \text{ and } \{n\} \cup s \in \mathcal{A}_\xi$ or $s = \emptyset$ and $\{n\} \in \mathcal{A}_\xi\}$. Moreover, $\xi_n = \zeta$ for every $n \in \mathbb{N}$ if $\xi = \zeta + 1$, and $(\xi_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $\sup_n \xi_n = \xi$ if ξ is a limit ordinal. ∎

To state the following Ramsey type partition theorem for Schreier families of variable ω-$\mathbb{Z}^*$-located words, we need the following notation:

Notation. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. For $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$, $\mathbf{w} = (w_1, \ldots, w_l) \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ and $t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon)$, we set:

$$\widetilde{\mathrm{EV}}^{<\infty}(\vec{w}) = \{\mathbf{u} = (u_1, \ldots, u_l) \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) : l \in \mathbb{N},\ u_1, \ldots, u_l \in \widetilde{\mathrm{EV}}(\vec{w})\} \cup \{\emptyset\},$$

$$\widetilde{\mathrm{EV}}(\mathbf{w}) = \{T_{(p_1,q_1)}(w_{n_1}) \star \cdots \star T_{(p_\lambda,q_\lambda)}(w_{n_\lambda}) \in \widetilde{L}(\Sigma, \vec{k}; \upsilon) : 1 \leq n_1 < \cdots < n_\lambda \leq l \text{ and } (p_i, q_i) \in (\mathbb{N} \times \mathbb{N}) \cup \{(0,0)\} \text{ with } 0 \leq p_i \leq k_{n_i},\ 0 \leq q_i \leq k_{-n_i} \text{ for every } 1 \leq i \leq \lambda \text{ and } (0,0) \in \{(p_1,q_1), \ldots, (p_\lambda,q_\lambda)\}\},$$ and

$$\widetilde{\mathrm{EV}}^{<\infty}(\mathbf{w}) = \{\mathbf{u} = (u_1, \ldots, u_l) \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) : l \in \mathbb{N}, u_1, \ldots, u_l \in \widetilde{\mathrm{EV}}(\mathbf{w})\} \cup \{\emptyset\}.$$

Observe that the set $\widetilde{\mathrm{EV}}(\mathbf{w})$ is finite. Also, we set

$$\vec{w} - t = (w_n)_{n\geq l} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon), \text{ where } l = \min\{n \in \mathbb{N} : t <_{R_1} w_n\},$$
$$\vec{w} - \mathbf{w} = \vec{w} - w_l,$$

$$\mathbf{w} - t = \begin{cases} (w_n, \dots, w_l) & \text{for } n = \min\{1 \le i \le l : t <_{R_1} w_i\}, \\ & \text{if } \{1 \le i \le l : t <_{R_1} w_i\} \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

THEOREM 3.5 (Ramsey type partition theorem for Schreier families of variable ω-$\mathbb{Z}^*$-located words). *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. For every countable ordinal $\xi \ge 1$, every family $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ and every infinite orderly sequence $\vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ of variable ω-$\mathbb{Z}^*$-located words there exists a variable extraction $\vec{u} \prec \vec{w}$ of $\vec{w}$ such that: either*

- $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, *or*
- $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.

This theorem can be deduced from the stronger Theorem 2.5, as the partition family $\mathcal{F}$ of $\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ can be extended to a partition family A_1 of $\widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ which is clopen (and consequently Borel) in the product topology. But, in view of Proposition 3.4 on the recursiveness of a Schreier system, we provide for completeness a proof by induction, starting from the initial case ($m = 1$) of Theorem 2.6, which, as we have mentioned (in Remark 2.7), has a proof independent of Theorem 2.5.

In the proof of this partition theorem we will make use of a diagonal argument, contained in the following lemma.

LEMMA 3.6. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$, and*

$$\varPi = \{(t, \vec{s}) : t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon),\ \vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon) \text{ with } \vec{s} \prec \vec{w} \text{ and } t <_{R_1} s_1\}.$$

If a subset $\mathcal{R}$ of $\varPi$ satisfies

(i) *for every $(t, \vec{s}) \in \varPi$, there exists $(t, \vec{s}_1) \in \mathcal{R}$ with $\vec{s}_1 \prec \vec{s}$;*
(ii) *for every $(t, \vec{s}) \in \mathcal{R}$ and $\vec{s}_1 \prec \vec{s}$, we have $(t, \vec{s}_1) \in \mathcal{R}$,*

then there exists $\vec{u} \prec \vec{w}$ such that $(t, \vec{s}) \in \mathcal{R}$ for all $t \in \widetilde{\mathrm{EV}}(\vec{u})$ and $\vec{s} \prec \vec{u} - t$.

Proof. Let $u_0 = w_1$. By (i), there exists $\vec{s}_1 = (s_n^1)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ with $\vec{s}_1 \prec \vec{w} - u_0$ such that $(u_0, \vec{s}_1) \in \mathcal{R}$. Let $u_1 = s_1^1$. Then $u_0 <_{R_1} u_1$ and $u_0, u_1 \in \widetilde{\mathrm{EV}}(\vec{w})$. We now assume that $\vec{s}_1, \dots, \vec{s}_n \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ and $u_0, u_1, \dots, u_n \in \widetilde{\mathrm{EV}}(\vec{w})$ have been constructed with $\vec{s}_n \prec \dots \prec \vec{s}_1 \prec \vec{w}$, $u_0 <_{R_1} u_1 <_{R_1} \dots <_{R_1} u_n$ and $(t, \vec{s}_i) \in \mathcal{R}$ for all $t \in \widetilde{\mathrm{EV}}((u_0, \dots, u_{i-1}))$ and $1 \le i \le n$.

We will construct $\vec{s}_{n+1}$ and u_{n+1}. Let $\{t_1, \dots, t_l\} = \widetilde{\mathrm{EV}}((u_0, \dots, u_n))$. By (i), there exist $\vec{s}_{n+1}^{\,1}, \dots, \vec{s}_{n+1}^{\,l} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ such that $\vec{s}_{n+1}^{\,l} \prec \dots \prec \vec{s}_{n+1}^{\,1} \prec \vec{s}_n - u_n$ and $(t_i, \vec{s}_{n+1}^{\,i}) \in \mathcal{R}$ for every $1 \le i \le l$. Set $\vec{s}_{n+1} = \vec{s}_{n+1}^{\,l}$. If

$\vec{s}_{n+1} = (s_m^{n+1})_{m\in\mathbb{N}}$, set $u_{n+1} = s_1^{n+1}$. Of course $u_n <_{R_1} u_{n+1}$, $u_{n+1} \in \widetilde{\mathrm{EV}}(\vec{w})$ and, by (ii), $(t_i, \vec{s}_{n+1}) \in \mathcal{R}$ for all $1 \le i \le l$.

Set $\vec{u} = (u_0, u_1, u_2, \ldots) \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. Then $\vec{u} \prec \vec{w}$, since $u_0 <_{R_1} u_1 <_{R_1} \cdots \in \widetilde{\mathrm{EV}}(\vec{w})$. Let $t \in \widetilde{\mathrm{EV}}(\vec{u})$ and $\vec{s} \prec \vec{u} - t$. Set $n_0 = \min\{n \in \mathbb{N} : t \in \widetilde{\mathrm{EV}}((u_0, u_1, \ldots, u_n))\}$. Since $t \in \widetilde{\mathrm{EV}}((u_0, u_1, \ldots, u_{n_0}))$, it follows that $(t, \vec{s}_{n_0+1}) \in \mathcal{R}$. Then, by (ii), we have $(t, \vec{u} - u_{n_0}) \in \mathcal{R}$, since $\vec{u} - u_{n_0} \prec \vec{s}_{n_0+1}$, and also $(t, \vec{s}) \in \mathcal{R}$, since $\vec{s} \prec \vec{u} - u_{n_0} = \vec{u} - t$. ∎

Proof of Theorem 3.5. Let $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ and $\vec{w} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. For $\xi = 1$ the conclusion is valid, according to Theorem 2.5. Let $\xi > 1$. Assume that the conclusion holds for every $\zeta < \xi$. Let $t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon)$ with $\min \mathrm{dom}^+(t) = n$ and $\vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ with $\vec{s} \prec \vec{w}$ and $t <_{R_1} s_1$. According to Proposition 3.4, there exists $\xi_n < \xi$ such that

$$\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon)(t) = \widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap (\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) - t).$$

Using the induction hypothesis, there exists $\vec{s}_1 \prec \vec{s}$ such that either

- $\widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}_1) \subseteq \mathcal{F}(t)$, or
- $\widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}_1) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}(t)$.

Then $\vec{s}_1 \prec \vec{s} \prec \vec{w}$, and either

- $\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}_1) \subseteq \mathcal{F}(t)$, or
- $\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}_1) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}(t)$.

Let

$$\begin{aligned}\mathcal{R}_1 = \{(t, \vec{s}) : {} & t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon),\ \vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon), \vec{s} \prec \vec{w}, t <_{R_1} s_1, \\ & \text{and either } \widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}) \subseteq \mathcal{F}(t), \text{ or } \widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon)(t) \cap \\ & \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}(t)\}.\end{aligned}$$

The family $\mathcal{R}_1$ satisfies the conditions (i) (by the above arguments) and (ii) (obviously) of Lemma 3.6. Hence, there exists $\vec{w}_1 = (w_n^1)_{n\in\mathbb{N}} \prec \vec{w}$ such that $(t, \vec{s}) \in \mathcal{R}_1$ for all $t \in \widetilde{\mathrm{EV}}(\vec{w}_1)$ and $\vec{s} \prec \vec{w}_1 - t$.

Let

$$\mathcal{F}_1 = \{t \in \widetilde{\mathrm{EV}}(\vec{w}_1) : \widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{w}_1 - t) \subseteq \mathcal{F}(t)\}.$$

We use the induction hypothesis for $\xi = 1$ (Theorem 2.5). There exists a variable extraction $\vec{u} \prec \vec{w}_1$ of $\vec{w}_1$ such that

$$\text{either } \widetilde{\mathrm{EV}}(\vec{u}) \subseteq \mathcal{F}_1, \text{ or } \widetilde{\mathrm{EV}}(\vec{u}) \subseteq \widetilde{L}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}_1.$$

Since $\vec{u} \prec \vec{w}_1$ we have $\widetilde{\mathrm{EV}}(\vec{u}) \subseteq \widetilde{\mathrm{EV}}(\vec{w}_1)$. Thus either

- $\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u} - t) \subseteq \mathcal{F}(t)$ for all $t \in \widetilde{\mathrm{EV}}(\vec{u})$, or
- $\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u} - t) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}(t)$ for all $t \in \widetilde{\mathrm{EV}}(\vec{u})$.

Hence, either

- $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, or
- $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$. ∎

REMARK 3.7. (1) The particular case $\xi = m \in \mathbb{N}$ of Theorem 3.5 coincides with Theorem 2.6.

(2) In the case $\xi = \omega$, Theorem 3.5 takes the form: if $\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) = A_1 \cup \cdots \cup A_r$, $r \in \mathbb{N}$ and $\vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$, then there exists an extraction $\vec{u} \prec \vec{w}$ of $\vec{w}$ and $1 \leq i_0 \leq r$ such that the set $\{(z_1, \ldots, z_n) \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)) : n \in \mathbb{N}, \min \mathrm{dom}^+(z_1) = n$ and $z_1, \ldots, z_n \in \widetilde{\mathrm{EV}}(\vec{u})\}$ is contained in A_{i_0}.

(3) In analogy to Theorem 3.5 one can prove a Ramsey type partition theorem for Schreier families of (constant) ω-$\mathbb{Z}^*$-located words.

As a consequence of Theorem 3.5 we will prove a Ramsey type partition theorem for Schreier families of variable ω-located words.

NOTATION. Let $\Sigma = \{\alpha_1, \alpha_2, \ldots\}$ be an infinite countable alphabet, $\upsilon \notin \Sigma$ a variable and $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence. We define the *finite orderly sequences of variable ω-located words* over Σ dominated by $\vec{k}$ as follows:

$$L^{<\infty}(\Sigma, \vec{k}; \upsilon) = \{\mathbf{w} = (w_1, \ldots, w_l) : \\ l \in \mathbb{N},\ w_1 <_{R_2} \cdots <_{R_2} w_l \in L(\Sigma, \vec{k}; \upsilon)\} \cup \{\emptyset\}.$$

For every countable ordinal $\xi \geq 1$, we set

$$L^{\xi}(\Sigma, \vec{k}; \upsilon) = \{(w_1, \ldots, w_l) \in L^{<\infty}(\Sigma, \vec{k}; \upsilon) : \\ \{\min \mathrm{dom}(w_1), \ldots, \min \mathrm{dom}(w_l)\} \in \mathcal{A}_{\xi}\}.$$

For $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^{\infty}(\Sigma, \vec{k}; \upsilon)$ we set

$$\mathrm{EV}^{<\infty}(\vec{w}) = \{\mathbf{u} = (u_1, \ldots, u_l) \in L^{<\infty}(\Sigma, \vec{k}; \upsilon) : \\ l \in \mathbb{N},\ u_1, \ldots, u_l \in \mathrm{EV}(\vec{w})\} \cup \{\emptyset\}.$$

COROLLARY 3.8 (Ramsey type partition theorem for Schreier families of variable ω-located words). *Let $\Sigma = \{\alpha_1, \alpha_2, \ldots\}$ be an infinite countable alphabet, $\upsilon \notin \Sigma$ a variable, $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence and $\xi \geq 1$ a countable ordinal. For a partition family $\mathcal{F} \subseteq L^{<\infty}(\Sigma, \vec{k}; \upsilon)$ and $\vec{w} \in L^{\infty}(\Sigma, \vec{k}; \upsilon)$, there exists an extraction $\vec{u} \prec \vec{w}$ of $\vec{w}$ such that either*

- $L^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \mathrm{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, *or*
- $L^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \mathrm{EV}^{<\infty}(\vec{u}) \subseteq L^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.

Proof. We set $\widetilde{\Sigma} = \{\alpha_n : n \in \mathbb{Z}^*\}$, $\vec{k}_* = (\widetilde{k}_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N}$ and $\vec{w}_* = (\widetilde{w}_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\widetilde{\Sigma}, \vec{k}_*; \upsilon)$, where $\alpha_{-n} = \alpha_n$, $\widetilde{k}_{-n} = \widetilde{k}_n = k_n$ and $\widetilde{w}_n = \upsilon_{-n} \star w_n$

for every $n \in \mathbb{N}$. Let $\varphi : \widetilde{L}(\widetilde{\Sigma}, \vec{k}_*; \upsilon) \to L(\Sigma, \vec{k}; \upsilon)$ with $\varphi(w_{n_1} \ldots w_{n_l}) = w_{n_{i_0}} \ldots w_{n_l}$, where $n_{i_0} = \min \mathrm{dom}^+(w_{n_1} \ldots w_{n_l})$ and $\widetilde{\varphi} : \widetilde{L}^{<\infty}(\widetilde{\Sigma}, \vec{k}_*; \upsilon) \to L^{<\infty}(\Sigma, \vec{k}; \upsilon)$ with $\widetilde{\varphi}(u_1, \ldots, u_l) = (\varphi(u_1), \ldots, \varphi(u_l))$. Then we apply Theorem 3.5 for the family $\widetilde{\varphi}^{-1}(\mathcal{F})$ and the sequence $\vec{w}_*$. ∎

In order to prove Theorem 3.21, a strengthening of Theorem 3.5 in case the partition family $\mathcal{F}$ is a tree, we will prove three basic properties of the Schreier families of variable ω-$\mathbb{Z}^*$-located words (Propositions 3.10, 3.11 and 3.18 below).

Let us start with the necessary notations and definitions.

Definition 3.9. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$.

(i) $\mathcal{F}$ is *thin* if there are no $\mathbf{w}, \mathbf{u} \in \mathcal{F}$ with $\mathbf{w} \propto \mathbf{u}$ and $\mathbf{w} \neq \mathbf{u}$.
(ii) $\mathcal{F}^* = \{\mathbf{w} \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) : \mathbf{w} \propto \mathbf{u}$ for some $\mathbf{u} \in \mathcal{F}\} \cup \{\emptyset\}$.
(iii) $\mathcal{F}$ is a *tree* if $\mathcal{F}^* = \mathcal{F}$.
(iv) $\mathcal{F}_* = \{\mathbf{w} \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) : \mathbf{w} \in \widetilde{\mathrm{EV}}^{<\infty}(\mathbf{u})$ for some $\mathbf{u} \in \mathcal{F}\} \cup \{\emptyset\}$.
(v) $\mathcal{F}$ is *hereditary* if $\mathcal{F}_* = \mathcal{F}$.

Proposition 3.10. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. Every family $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ for $\xi < \omega_1$ is thin.*

Proof. This follows from the fact that the families $\mathcal{A}_\xi$ are thin (cf. [F3]) (which means that if $s, t \in \mathcal{A}_\xi$ and $s \propto t$, then $s = t$). ∎

Proposition 3.11. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and $\xi \geq 1$ be a countable ordinal. Then*

(i) *every infinite orderly sequence $\vec{s} = (s_n)_{n \in \mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ has canonical representation with respect to $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$, which means that there exists a unique strictly increasing sequence $(m_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$ so that $(s_1, \ldots, s_{m_1}) \in \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ and $(s_{m_{n-1}+1}, \ldots, s_{m_n}) \in \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ for every $n > 1$;*
(ii) *every non-empty finite orderly sequence $\mathbf{s} = (s_1, \ldots, s_k) \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ has canonical representation with respect to $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$, so either $\mathbf{s} \in (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ or there exist unique $n \in \mathbb{N}$ and $m_1, \ldots, m_n \in \mathbb{N}$ with $m_1 < \cdots < m_n \leq k$ such that either*

$$(s_1, \ldots, s_{m_1}), \ldots, (s_{m_{n-1}+1}, \ldots, s_{m_n}) \in \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \quad \textit{and} \quad m_n = k,$$

or

$$(s_1, \ldots, s_{m_1}), \ldots, (s_{m_{n-1}+1}, \ldots, s_{m_n}) \in \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon),$$
$$(s_{m_n+1}, \ldots, s_k) \in (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon).$$

Proof. This follows from the fact that every non-empty increasing sequence (finite or infinite) in $\mathbb{N}$ has canonical representation with respect to $\mathcal{A}_\xi$ (cf. [F3]) and that the family $\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$ is thin (Proposition 3.10). ∎

Now, using Proposition 3.11, we can give an alternative description of the second horn of the dichotomy described in Theorem 3.5 in case the partition family is a tree.

Proposition 3.12. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\xi \geq 1$ a countable ordinal and $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ be a tree. Then*

$$\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) \setminus \mathcal{F}$$
$$\Leftrightarrow\ \mathcal{F} \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon))^* \setminus \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon).$$

Proof. Suppose $\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) \setminus \mathcal{F}$ and $\mathbf{s} = (s_1,\ldots,s_k) \in \mathcal{F} \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u})$. Then $\mathbf{s}$ has canonical representation with respect to $\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$ (Proposition 3.11), hence either $\mathbf{s} \in (\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon))^* \setminus \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$, as required, or there exists $\mathbf{s}_1 \in \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$ such that $\mathbf{s}_1 \propto \mathbf{s}$. The second case is impossible. Indeed, since $\mathcal{F}$ is a tree and $\mathbf{s} \in \mathcal{F} \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u})$, we have $\mathbf{s}_1 \in \mathcal{F} \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \cap \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$, a contradiction to our assumption. Hence $\mathcal{F} \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon))^* \setminus \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$. ∎

Definition 3.13. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. Identifying every $\mathbf{s} \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ and every $\vec{s} \in \widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ with their characteristic functions $x_{r(\mathbf{s})} \in \{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$ and $x_{r(\vec{s})} \in \{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$ respectively (where $r(\mathbf{s}) = \{s_1,\ldots,s_k\}$ is the range of $\mathbf{s} = (s_1,\ldots,s_k) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$, $r(\vec{s}) = \{s_n : n \in \mathbb{N}\}$ is the range of $\vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ and $r(\emptyset) = \emptyset$), we say that a family $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ is *pointwise closed* if the family $\{x_{r(\mathbf{s})} : \mathbf{s} \in \mathcal{F}\}$ is closed in the product topology (equivalently, the pointwise convergence topology) of $\{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$, and by analogy a family $\mathcal{U} \subseteq \widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ is *pointwise closed* if $\{x_{r(\vec{s})} : \vec{s} \in \mathcal{U}\}$ is closed in $\{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$ with the product topology.

Proposition 3.14. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions.*

(i) *If $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ is a tree, then $\mathcal{F}$ is pointwise closed if and only if there does not exist an infinite sequence $(\mathbf{s}_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\mathbf{s}_n \propto \mathbf{s}_{n+1}$ and $\mathbf{s}_n \neq \mathbf{s}_{n+1}$ for all $n \in \mathbb{N}$.*

(ii) *If $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ is hereditary, then $\mathcal{F}$ is pointwise closed if and only if there does not exist $\vec{s} \in \widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ such that $\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}) \subseteq \mathcal{F}$.*

(iii) *The hereditary family $(\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}))_*$ is pointwise closed for every countable ordinal ξ and $\vec{u} \in \widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$.*

Proof. (i) follows from the fact that $\widetilde{\mathrm{EV}}^{<\infty}(\mathbf{s})$ is finite for every $\mathbf{s} \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$, and (ii) follows from (i). The statement (iii) can be proved by induction on ξ, using (ii). The main idea of the proof is that given $\vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ with $\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}) \subseteq (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}))_*$, then, according to the pigeonhole principle, there exists $k \leq \min \mathrm{dom}^+(s_1)$ such that $(s_2, \ldots, s_n) \in (\widetilde{L}^{\xi_k}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}))_*$ for every $n \in \mathbb{N}$ (using Proposition 3.4). ■

Let $\vec{s} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$. For a hereditary and pointwise closed family $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ we will define the strong Cantor–Bendixson index $sO_{\vec{s}}(\mathcal{F})$ of $\mathcal{F}$ with respect to $\vec{s}$.

DEFINITION 3.15. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\vec{s} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ and let $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ be a hereditary and pointwise closed family. For every $\xi < \omega_1$ we define the families $(\mathcal{F})^{\xi}_{\vec{s}}$ inductively as follows: We define $(\mathcal{F})^{0}_{\vec{s}} = \mathcal{F}$. For every $\mathbf{w} = (w_1, \ldots, w_l) \in \mathcal{F} \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s})$ we set

$$A_{\mathbf{w}} = \{t \in \widetilde{\mathrm{EV}}(\vec{s}) : (w_1, \ldots, w_l, t) \notin \mathcal{F}\}, \qquad A_{\emptyset} = \{t \in \widetilde{\mathrm{EV}}(\vec{s}) : (t) \notin \mathcal{F}\}.$$

We define

$$(\mathcal{F})^{1}_{\vec{s}} = \{\mathbf{w} \in \mathcal{F} \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}) \cup \{\emptyset\} : A_{\mathbf{w}} \text{ does not contain an infinite orderly sequence}\}.$$

It is easy to verify that $(\mathcal{F})^{1}_{\vec{s}}$ is hereditary, hence it is pointwise closed since $\mathcal{F}$ is pointwise closed (Proposition 3.14). So, we can define for every $\xi > 1$ the ξ-derivatives of $\mathcal{F}$ recursively as follows:

$$\begin{aligned} (\mathcal{F})^{\zeta+1}_{\vec{s}} &= ((\mathcal{F})^{\zeta}_{\vec{s}})^{1}_{\vec{s}} && \text{for all } \zeta < \omega_1, \\ (\mathcal{F})^{\xi}_{\vec{s}} &= \bigcap_{\beta<\xi} (\mathcal{F})^{\beta}_{\vec{s}} && \text{for } \xi \text{ a limit ordinal.} \end{aligned}$$

The *strong Cantor–Bendixson index* $sO_{\vec{s}}(\mathcal{F})$ of $\mathcal{F}$ on $\vec{s}$ is the smallest countable ordinal ξ such that $(\mathcal{F})^{\xi}_{\vec{s}} = \emptyset$.

REMARK 3.16. Let $\vec{s} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ and let $\mathcal{F}, \mathcal{R} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ be non-empty, hereditary and pointwise closed families.

(i) $sO_{\vec{s}}(\mathcal{F})$ is a countable successor ordinal less than or equal to the "usual" Cantor–Bendixson index $O(\mathcal{F})$ of $\mathcal{F}$ into $\{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$ (cf. [KM]).
(ii) $sO_{\vec{s}}(\mathcal{F} \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s})) = sO_{\vec{s}}(\mathcal{F})$.
(iii) $sO_{\vec{s}}(\mathcal{F}) \leq sO_{\vec{s}}(\mathcal{R})$ if $\mathcal{F} \subseteq \mathcal{R}$.

(iv) If $\vec{t} = (t_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ with $(t_{k+n})_{n\in\mathbb{N}} \prec \vec{s}$ for some $k \in \mathbb{N}\cup\{0\}$ and $\mathbf{w} \in (\mathcal{F})^\xi_{\vec{s}}$, then for every $\mathbf{w}_1 \in \widetilde{\mathrm{EV}}^{<\infty}(\vec{t})$ such that $\mathbf{w}_1 \in \widetilde{\mathrm{EV}}^{<\infty}(\mathbf{w})$ we have $\mathbf{w}_1 \in (\mathcal{F})^\xi_{\vec{t}}$.

(v) If $\vec{s}_1 \prec \vec{s}$, then $sO_{\vec{s}_1}(\mathcal{F}) \geq sO_{\vec{s}}(\mathcal{F})$, according to (iv).

(vi) If $r(\vec{s}_1) \setminus r(\vec{s})$ is a finite set, then $sO_{\vec{s}_1}(\mathcal{F}) \geq sO_{\vec{s}}(\mathcal{F})$.

In Proposition 3.18 below, we will prove that the corresponding strong Cantor–Bendixson index for Schreier families of order ξ is equal to $\xi+1$. For the proof we will need the following lemma, which is analogous to Lemma 2.8 in [F5] and has an analogous proof.

LEMMA 3.17. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\xi \geq 1$ a countable ordinal, $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$, $\vec{s}_1 \prec \vec{s}$, and let $\mathcal{F} \subseteq \widetilde{\mathrm{EV}}^{<\infty}(\vec{s})$ be such that $\mathcal{F}_*$ and $(\mathcal{F}(t))_*$ are pointwise closed for every $t \in \widetilde{\mathrm{EV}}(\vec{s})$.*

(i) $((\mathcal{F}(t))_*)^\xi_{\vec{s}_1} \subseteq (\mathcal{F}_*)^\xi_{\vec{s}_1}(t)$ *for every* $t \in \widetilde{\mathrm{EV}}(\vec{s})$.

(ii) *If* $\mathbf{w} = (w_1, \ldots, w_l) \neq \emptyset$ *and* $\mathbf{w} \in (\mathcal{F}_*)^\xi_{\vec{s}_1}$, *then there exist* $\vec{s}_2 \prec \vec{s}_1$ *and* $t \in \widetilde{\mathrm{EV}}(\vec{s})$ *with* $t <_{R_1} w_1$ *or* $\mathrm{dom}(t) \subseteq \mathrm{dom}(w_1)$ *such that* $\mathbf{w} - t \in ((\mathcal{F}(t))_*)^\xi_{\vec{s}_2}$.

Proof. (i) This can be proved by induction on ξ, using Definition 3.15 and the inclusion $(\mathcal{F}(t))_* \subseteq \mathcal{F}_*(t)$.

(ii) The proof is by induction on ξ. The main argument is contained in the proof of the case $\xi = 1$. Let $\mathbf{w} = (w_1, \ldots, w_l) \in (\mathcal{F}_*)^1_{\vec{s}_1}$. For every $u \in \widetilde{\mathrm{EV}}(\vec{s}_1) \setminus A_{\mathbf{w}}$ there exists $v_u = (v^u_1, \ldots, v^u_{l_u}) \in \mathcal{F}$ such that $(w_1, \ldots, w_l, u) \in \widetilde{\mathrm{EV}}^{<\infty}(v_u)$. Then $v^u_1 \leq w_1$ (which means that $v^u_1 <_{R_1} w_1$ or $\mathrm{dom}(v^u_1) \subseteq \mathrm{dom}(w_1)$) for every $u \in \widetilde{\mathrm{EV}}(\vec{s}_1) \setminus A_{\mathbf{w}}$ and the set $\{v \in \widetilde{\mathrm{EV}}(\vec{s}) : v \leq w_1\}$ is finite. Since $\mathbf{w} \in (\mathcal{F}_*)^1_{\vec{s}_1}$, by Theorem 2.6 (case $m = 1$), there exist $\vec{s}_2 \prec \vec{s}_1$ and $t \in \widetilde{\mathrm{EV}}(\vec{s})$ with $t \leq w_1$ such that $\widetilde{\mathrm{EV}}(\vec{s}_2) \subseteq \widetilde{\mathrm{EV}}(\vec{s}_1) \setminus A_{\mathbf{w}}$ and $v^u_1 = t$ for every $u \in \widetilde{\mathrm{EV}}(\vec{s}_2)$. Then $\mathbf{w} - t \in ((\mathcal{F}(t))_*)^1_{\vec{s}_2}$. ∎

PROPOSITION 3.18. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\xi < \omega_1$ be an ordinal and $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. Then*

$$sO_{\vec{s}_1}\big((\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big) = \xi + 1 \quad \textit{for every } \vec{s}_1 \prec \vec{s}.$$

Proof. We have $(\widetilde{L}^0(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_* = \{\emptyset\}$ for every $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ and $sO_{\vec{s}_1}(\{\emptyset\}) = 1$ for $\vec{s}_1 \prec \vec{s}$, since $(\{\emptyset\})^1_{\vec{s}} = \emptyset$, so the conclusion holds for $\xi = 0$.

The families $(\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*$, $(\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{s})(t))_*$ are hereditary and pointwise closed for every $1 \leq \xi \leq \omega_1$, $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ and $t \in \widetilde{\mathrm{EV}}(\vec{s})$, according to Proposition 3.14, since in case $\min \mathrm{dom}^+(t) = n$, Proposition 3.4 implies that

$(\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))(t)=\widetilde{L}^{\xi_n}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}-t)$ for some $\xi_n<\xi$.

In order to prove the proposition, it is enough to prove by induction on ξ that $\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{\xi}_{\vec{s}_1}=\{\emptyset\}$ for every $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$, $\vec{s}_1\prec\vec{s}$ and $1\le\xi<\omega_1$. Since $(\widetilde{L}^{1}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*=\{(t):t\in\widetilde{\mathrm{EV}}(\vec{s})\}\cup\{\emptyset\}$, we have $\big((\widetilde{L}^{1}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{1}_{\vec{s}_1}=\{\emptyset\}$ for every $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ and $\vec{s}_1\prec\vec{s}$.

Let $\xi>1$ and assume that $\big((\widetilde{L}^{\zeta}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{\zeta}_{\vec{s}_1}=\{\emptyset\}$ for every $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$, $\vec{s}_1\prec\vec{s}$ and $1\le\zeta<\xi$. Let $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ and $\vec{s}_1\prec\vec{s}$. For every $t\in\widetilde{\mathrm{EV}}(\vec{s})$ with $\min\mathrm{dom}^+(t)=n$ we have, by Remark 3.16(vi),

$$\begin{aligned}\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))(t)_*\big)^{\xi_n}_{\vec{s}_1}&=\big((\widetilde{L}^{\xi_n}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}-t))_*\big)^{\xi_n}_{\vec{s}_1}\\&=\big((\widetilde{L}^{\xi_n}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}-t))_*\big)^{\xi_n}_{\vec{s}_1-t}\\&=\{\emptyset\}.\end{aligned}$$

This gives $\emptyset\in\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))(t)_*\big)^{\xi_n}_{\vec{s}_1}$. By Lemma 3.17(i) we have $(t)\in\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{\xi_n}_{\vec{s}_1}$. Hence, $\emptyset\in\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{\xi}_{\vec{s}_1}$ for every $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ and $\vec{s}_1\prec\vec{s}$. Indeed, if $\xi=\zeta+1$, then $(t)\in\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{\zeta}_{\vec{s}_1}$ for every $t\in\widetilde{\mathrm{EV}}(\vec{s}_1)$, and if ξ is a limit ordinal, then $\sup\xi_n=\xi$ and $\emptyset\in\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{\xi_n}_{\vec{s}_1}$ for every $n\in\mathbb{N}$.

We will prove that $\{\emptyset\}=\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{\xi}_{\vec{s}_1}$ for every $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ and $\vec{s}_1\prec\vec{s}$. Indeed, let

$$\mathbf{w}=(w_1,\dots,w_l)\in\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{\xi}_{\vec{s}_1}$$

for some $l\in\mathbb{N}$, $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ and $\vec{s}_1\prec\vec{s}$. By Lemma 3.17(ii) there exist $\vec{s}_2\prec\vec{s}_1$ and $t\in\widetilde{\mathrm{EV}}(\vec{s})$ such that $\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))(t)_*\big)^{\xi}_{\vec{s}_2}\neq\emptyset$. If $\min\mathrm{dom}^+(t)=n$, analogously to the previous paragraph we have

$$\begin{aligned}\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))(t)_*\big)^{\xi}_{\vec{s}_2}&=\big((\widetilde{L}^{\xi_n}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}-t))_*\big)^{\xi}_{\vec{s}_2}\\&=\big((\widetilde{L}^{\xi_n}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}-t))_*\big)^{\xi}_{\vec{s}_2-t}\\&\neq\emptyset.\end{aligned}$$

A contradiction, since $\xi_n<\xi$ and $\big((\widetilde{L}^{\xi_n}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}-t))_*\big)^{\xi}_{\vec{s}_2-t}=\emptyset$, by the induction hypothesis.

Hence, $\big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*\big)^{\xi}_{\vec{s}_1}=\{\emptyset\}$ and $sO_{\vec{s}_1}((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{\mathrm{EV}}^{<\infty}(\vec{s}))_*)=\xi+1$ for every $\xi<\omega_1$. ∎

Corollary 3.19. *Let ξ_1, ξ_2 be countable ordinals with $\xi_1 < \xi_2$ and $\vec{w} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. Then there exists $\vec{u} \prec \vec{w}$ such that*

$$(\widetilde{L}^{\xi_1}(\Sigma, \vec{k}; \upsilon))_* \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^{\xi_2}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi_2}(\Sigma, \vec{k}; \upsilon).$$

Proof. The family $(\widetilde{L}^{\xi_1}(\Sigma, \vec{k}; \upsilon))_* \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ is a tree. According to Theorem 3.5 and Proposition 3.12 there exists $\vec{u} \prec \vec{w}$ such that either

- $\widetilde{L}^{\xi_2}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^{\xi_1}(\Sigma, \vec{k}; \upsilon))_*$, or
- $(\widetilde{L}^{\xi_1}(\Sigma, \vec{k}; \upsilon))_* \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^{\xi_2}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi_2}(\Sigma, \vec{k}; \upsilon)$.

The first alternative is impossible, since by Proposition 3.18,

$$\xi_2 + 1 = sO_{\vec{u}}((\widetilde{L}^{\xi_2}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}))_*) \le sO_{\vec{u}}((\widetilde{L}^{\xi_1}(\Sigma, \vec{k}; \upsilon))_*) = \xi_1 + 1. \blacksquare$$

Theorem 3.21 below, the main result in this section, refines Theorem 3.5 in case the partition family is a tree. We denote by $\sup X$, where X is a set of ordinals, the least ordinal α such that $\beta \le \alpha$ for every $\beta \in X$.

Definition 3.20. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$. We set $\mathcal{F}_h = \{\mathbf{w} \in \mathcal{F} : \widetilde{\mathrm{EV}}^{<\infty}(\mathbf{w}) \subseteq \mathcal{F}\} \cup \{\emptyset\}$.

Of course, $\mathcal{F}_h$ is the largest subfamily of $\mathcal{F} \cup \{\emptyset\}$ which is hereditary.

Theorem 3.21. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, let $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ be a tree and $\vec{w} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. Then we have the following cases:*

[Case 1] *The family $\mathcal{F}_h \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{w})$ is not pointwise closed.*

Then there exists $\vec{u} \prec \vec{w}$ such that $\widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$.

[Case 2] *The family $\mathcal{F}_h \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{w})$ is pointwise closed.*

Then, setting

$$\zeta_{\vec{w}}^{\mathcal{F}} = \sup\{sO_{\vec{u}}(\mathcal{F}_h) : \vec{u} \prec \vec{w}\},$$

which is a countable ordinal, the following subcases obtain:

2(i) *If $\xi + 1 < \zeta_{\vec{w}}^{\mathcal{F}}$, then there exists $\vec{u} \prec \vec{w}$ such that*

$$\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F};$$

2(ii) *if $\xi + 1 > \xi > \zeta_{\vec{w}}^{\mathcal{F}}$, then for every $\vec{w}_1 \prec \vec{w}$ there exists $\vec{u} \prec \vec{w}_1$ such that $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$ (equivalently $\mathcal{F} \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$);*

2(iii) *if $\xi + 1 = \zeta_{\vec{w}}^{\mathcal{F}}$ or $\xi = \zeta_{\vec{w}}^{\mathcal{F}}$, then there exists $\vec{u} \prec \vec{w}$ such that either $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, or $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.*

Proof. [CASE 1] If the hereditary family $\mathcal{F}_h \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{w})$ is not pointwise closed, then, according to Proposition 3.14, there exists $\vec{u} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ such that $\widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}_h \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{w}) \subseteq \mathcal{F}$. Of course, $\vec{u} \prec \vec{w}$.

[CASE 2] If the hereditary family $\mathcal{F}_h \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{w})$ is pointwise closed, then $\zeta^{\mathcal{F}}_{\vec{w}}$ is a countable ordinal, since the "usual" Cantor–Bendixson index $O(\mathcal{F}_h)$ of $\mathcal{F}_h$ into $\{0,1\}^{\widetilde{L}(\Sigma,\vec{k},\upsilon}$ is countable (Remark 3.16(i)) and also $sO_{\vec{u}}(\mathcal{F}_h) \leq O(\mathcal{F}_h)$ for every $\vec{u} \prec \vec{w}$.

2(i) Let $\xi+1 < \zeta^{\mathcal{F}}_{\vec{w}}$. Then there exists $\vec{u}_1 \prec \vec{w}$ such that $\xi+1 < sO_{\vec{u}_1}(\mathcal{F}_h)$. By Theorem 3.5 and Proposition 3.12, there exists $\vec{u} \prec \vec{u}_1$ such that either

- $\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}_h \subseteq \mathcal{F}$, or
- $\mathcal{F}_h \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \subseteq (\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon))^*$ $\subseteq (\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon))_*$.

The second alternative is impossible, for if $\mathcal{F}_h \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon))_*$, then, according to Remark 3.16 and Proposition 3.18,

$$sO_{\vec{u}_1}(\mathcal{F}_h) \leq sO_{\vec{u}}(\mathcal{F}_h) = sO_{\vec{u}}(\mathcal{F}_h \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u})) \leq sO_{\vec{u}}((\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon))_*) = \xi+1,$$

a contradiction. Hence, $\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$.

2(ii) Let $\xi + 1 > \xi > \zeta^{\mathcal{F}}_{\vec{w}}$ and $\vec{w}_1 \prec \vec{w}$. According to Theorem 3.5, there exists $\vec{u_1} \prec \vec{w}_1$ such that either

- $\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}_1) \subseteq \mathcal{F}_h$, or
- $\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u_1}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}_h$.

Proposition 3.18 gives $\zeta^{\mathcal{F}}_{\vec{w}} + 1 = sO_{\vec{u}_1}((\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}_1))_*)$. Since $sO_{\vec{u}_1}(\mathcal{F}_h) \leq \zeta^{\mathcal{F}}_{\vec{w}}$, the first alternative is impossible, by Remark 3.16(iii). So,

$$\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}_1) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}_h. \tag{1}$$

According to Theorem 3.5, there exists $\vec{u} \prec \vec{u}_1$ such that either

- $\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, or
- $\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.

Since the family $\mathcal{F}$ is a tree, the first alternative does not hold. Indeed, if $\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, then $(\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}))^* \subseteq \mathcal{F}^* = \mathcal{F}$. Consequently, from Proposition 3.11 it follows that $(\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon))^* \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) = (\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}))^* \subseteq \mathcal{F}$. Since $\xi > \zeta^{\mathcal{F}}_{\vec{w}}$, by Corollary 3.19 there exists $\vec{t} \prec \vec{u}$ such that

$$(\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon))_* \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{t}) \subseteq (\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon))^* \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{u}) \subseteq \mathcal{F}.$$

Hence, $(\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon))_* \cap \widetilde{\mathrm{EV}}^{<\infty}(\vec{t}) \subseteq \mathcal{F}_h$, contrary to (1).

2(iii) If $\zeta^{\mathcal{F}}_{\vec{w}} = \xi + 1$ or $\zeta^{\mathcal{F}}_{\vec{w}} = \xi$, we use Theorem 3.5. ∎

The following immediate corollary to Theorem 3.21 is more useful for applications. A quite simplified consequence of Theorem 3.21, one not involving Schreier-type families of variable ω-$\mathbb{Z}^*$-located words, is equivalent to the particular case of Theorem 2.5 in which the partition families of $\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$ are clopen sets in the product topology (Theorem 3.23 below).

COROLLARY 3.22. *Let $\mathcal{F}\subseteq\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ be a tree and let $\vec{w}\in\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$. Then either*

(i) *there exists $\vec{u}\prec\vec{w}$ such that $\widetilde{\mathrm{EV}}^{<\infty}(\vec{u})\subseteq\mathcal{F}$, or*
(ii) *for every countable ordinal $\xi>\zeta^{\mathcal{F}}_{\vec{w}}$ there exists $\vec{u}\prec\vec{w}$ such that for every $\vec{u}_1\prec\vec{u}$ the unique initial segment of $\vec{u}_1$ which is an element of $\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$ belongs to $\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)\setminus\mathcal{F}$.*

Corollary 3.22 can be considered as a strengthening of Theorem 3.23 below; we will prove this by giving a reformulation of Theorem 3.23 in Theorem 3.24.

THEOREM 3.23. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. If $\mathcal{U}\subseteq\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$ is a pointwise closed family and $\vec{w}\in\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$, then there exists $\vec{u}\prec\vec{w}$ such that*

$$\text{either}\quad \widetilde{\mathrm{EV}}^{\infty}(\vec{u})\subseteq\mathcal{U},\quad\text{or}\quad \widetilde{\mathrm{EV}}^{\infty}(\vec{u})\subseteq\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)\setminus\mathcal{U}.$$

Theorem 3.23 has the following reformulation.

THEOREM 3.24. *Let $\mathcal{F}\subseteq\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ be a tree and let $\vec{w}\in\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$. Then either*

(i) *there exists $\vec{u}\prec\vec{w}$ such that $\widetilde{\mathrm{EV}}^{<\infty}(\vec{u})\subseteq\mathcal{F}$, or*
(ii) *there exists $\vec{u}\prec\vec{w}$ such that for every $\vec{u}_1\prec\vec{u}$ there exists an initial segment of $\vec{u}_1$ which belongs to $\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)\setminus\mathcal{F}$.*

REMARK 3.25. (1) Theorem 3.24 implies Theorem 3.23. Indeed, let $\mathcal{U}\subseteq\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$ be a closed family in the product topology and $\vec{w}\in\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$. Set

$$\mathcal{F}_{\mathcal{U}}=\{\mathbf{w}\in\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon):\text{there exists }\vec{s}\in\mathcal{U}\text{ such that }\mathbf{w}\propto\vec{s}\}.$$

Since the family $\mathcal{F}_{\mathcal{U}}$ is a tree, we use Corollary 3.22. So, we have the following two cases:

[CASE 1] There exists $\vec{u}\prec\vec{w}$ such that $\widetilde{\mathrm{EV}}^{<\infty}(\vec{u})\subseteq\mathcal{F}_{\mathcal{U}}$. Then $\widetilde{\mathrm{EV}}^{\infty}(\vec{u})\subseteq\mathcal{U}$. Indeed, if $\vec{z}=(z_n)_{n\in\mathbb{N}}\in\widetilde{\mathrm{EV}}^{\infty}(\vec{u})$, then $(z_1,\ldots,z_n)\in\mathcal{F}_{\mathcal{U}}$ for every $n\in\mathbb{N}$. Hence, for each $n\in\mathbb{N}$ there exists $\vec{s}_n\in\mathcal{U}$ such that $(z_1,\ldots,z_n)\propto\vec{s}_n$. Since $\mathcal{U}$ is pointwise closed, we have $\vec{z}\in\mathcal{U}$ and so $\widetilde{\mathrm{EV}}^{\infty}(\vec{u})\subseteq\mathcal{U}$.

[CASE 2] There exists $\vec{u}\prec\vec{w}$ such that for every $\vec{u}_1\prec\vec{u}$ there exists an initial segment of $\vec{u}_1$ which belongs to $\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)\setminus\mathcal{F}_{\mathcal{U}}$. Hence, $\widetilde{\mathrm{EV}}^{\infty}(\vec{u})\subseteq\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)\setminus\mathcal{U}$.

(2) Theorem 3.23 implies Theorem 3.24. Indeed, let $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ be a tree and let $\vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$. Set

$$\mathcal{U}_{\mathcal{F}} = \{\vec{t} = (t_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon) : \text{there exists } k \in \mathbb{N} \text{ such that } (t_1, \ldots, t_k) \in \mathcal{F}\}.$$

Since $\widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{U}_{\mathcal{F}}$ is a closed family in the product topology, using Theorem 3.23, we obtain the conclusion of Theorem 3.24.

Using Corollary 3.22, we can get the corresponding result for variable ω-located words, which extends Corollary 3.8 and implies the particular case of Theorem 2.3 in case the partition family $\mathcal{F}$ is clopen.

Corollary 3.26. *Let $\mathcal{F} \subseteq L^{<\infty}(\Sigma, \vec{k}; \upsilon)$ be a tree and let $\vec{w} \in L^{\infty}(\Sigma, \vec{k}; \upsilon)$. Then either*

(i) *there exists $\vec{u} \prec \vec{w}$ such that $\mathrm{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, or*
(ii) *there exists $\xi_0 < \omega_1$ such that for every countable ordinal $\xi > \xi_0$ there exists $\vec{u} \prec \vec{w}$ such that for every $\vec{u}_1 \prec \vec{u}$ the unique initial segment of $\vec{u}_1$ which is an element of $L^{\xi}(\Sigma, \vec{k}; \upsilon)$ belongs to $L^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.*

4. Applications to the Ramsey theory of the rationals with addition. T. Budak, N. Işik and J. Pym [BIP, Theorem 4.2] introduced a representation of rational numbers with specific properties, which implies that a non-zero rational number can be identified with an ω-$\mathbb{Z}^*$-located word over the alphabet $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$, where $\alpha_{-n} = \alpha_n = n$ for $n \in \mathbb{N}$, dominated by $(k_n)_{n\in\mathbb{Z}^*}$, where $k_{-n} = k_n = n$ for $n \in \mathbb{N}$. Hence, all the results concerning ω-$\mathbb{Z}^*$-located words over $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$ dominated by a $(k_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N}$, proved in the previous sections, can be translated into statements concerning rational numbers.

In this section we present a strengthened van der Waerden theorem for the set of rational numbers (Theorem 4.1), using Theorem 2.6 (case $m = 1$), an extended Ramsey-type partition theorem for the set of rational numbers (Theorem 4.2) as a consequence of Theorem 3.5, and a partition theorem for infinite orderly sequences of rational numbers (Theorem 4.3) as a consequence of Theorem 2.5.

Analytically, according to [BIP], every rational number q has a unique expression in the form

$$\sum_{s=1}^{\infty} q_{-s}\frac{(-1)^s}{(s+1)!} + \sum_{r=1}^{\infty} q_r(-1)^{r+1}r!,$$

where $(q_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N} \cup \{0\}$ with $0 \le q_{-s} \le s$ for every $s > 0$, $0 \le q_r \le r$ for every $r > 0$ and $q_{-s} = q_r = 0$ for all but finitely many r, s. So, for a non-zero rational number q, there exist unique $l \in \mathbb{N}$, $\{t_1 < \cdots < t_l\} = \mathrm{dom}(q) \in [\mathbb{Z}^*]^{<\omega}_{>0}$ and $\{q_{t_1}, \dots, q_{t_l}\} \subseteq \mathbb{N}$ with $1 \leq q_{t_i} \leq -t_i$ if $t_i < 0$ and $1 \leq q_{t_i} \leq t_i$ if $t_i > 0$ for every $1 \leq i \leq l$, such that defining $\mathrm{dom}^-(q) = \{t \in \mathrm{dom}(q) : t < 0\}$ and $\mathrm{dom}^+(q) = \{t \in \mathrm{dom}(q) : t > 0\}$ we have

$$q = \sum_{t \in \mathrm{dom}^-(q)} q_t \frac{(-1)^{-t}}{(-t+1)!} + \sum_{t \in \mathrm{dom}^+(q)} q_t (-1)^{t+1} t!$$

(we set $\sum_{t \in \emptyset} = 0$). Observe that

$$e^{-1} - 1 = -\sum_{t=1}^{\infty} \frac{2t-1}{(2t)!} < \sum_{t \in \mathrm{dom}^-(q)} q_t \frac{(-1)^{-t}}{(-t+1)!} < \sum_{t=1}^{\infty} \frac{2t}{(2t+1)!} = e^{-1}$$

and

$$\sum_{t \in \mathrm{dom}^+(q)} q_t (-1)^t (t+1)! \in \mathbb{Z}^* \quad \text{ if } \mathrm{dom}^+(q) \neq \emptyset.$$

Let $\alpha_{-n} = \alpha_n = n$ and $k_{-n} = k_n = n$ for $n \in \mathbb{N}$. We set $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$ and $\vec{k} = (k_n)_{n \in \mathbb{Z}^*}$. For $\upsilon = 0$ we consider the function $g : \widetilde{L}(\Sigma \cup \{0\}, \vec{k}) \to \mathbb{Q}$ which sends a word $w = q_{t_1} \dots q_{t_l} \in \widetilde{L}(\Sigma \cup \{0\}, \vec{k})$ to the rational number

$$g(w) = \sum_{t \in \mathrm{dom}^-(w)} q_t \frac{(-1)^{-t}}{(-t+1)!} + \sum_{t \in \mathrm{dom}^+(w)} q_t (-1)^{t+1} t!.$$

It is easy to see that the restriction of g to the set of constant words $\widetilde{L}(\Sigma, \vec{k})$ is one-to-one and onto $\mathbb{Q} \setminus \{0\}$, and that $g(w_1 \star w_2) = g(w_1) + g(w_2)$ for every $w_1 <_{R_1} w_2 \in \widetilde{L}(\Sigma \cup \{0\}, \vec{k})$. Also, observe that, via the function g, each variable word $w = q_{t_1} \dots q_{t_l} \in \widetilde{L}(\Sigma, \vec{k}; 0)$ corresponds to a function q which sends every $(i,j) \in \mathbb{N} \times \mathbb{N}$ with $1 \leq i \leq -\max \mathrm{dom}^-(w)$ and $1 \leq j \leq \min \mathrm{dom}^+(w)$ to

$$\begin{aligned} q(i,j) &= g(T_{(j,i)}(w)) \\ &= \sum_{t \in C^-} q_t \frac{(-1)^{-t}}{(-t+1)!} + i \sum_{t \in V^-} \frac{(-1)^{-t}}{(-t+1)!} + \sum_{t \in C^+} q_t (-1)^{t+1} t! + j \sum_{t \in V^+} (-1)^{t+1} t!, \end{aligned}$$

where $C^- = \{t \in \mathrm{dom}^-(w) : q_t \in \Sigma\}$, $V^- = \{t \in \mathrm{dom}^-(w) : q_t = 0\}$ and $C^+ = \{t \in \mathrm{dom}^+(w) : q_t \in \Sigma\}$, $V^+ = \{t \in \mathrm{dom}^+(w) : q_t = 0\}$.

For two non-zero rational numbers $q_1, q_2 \in g(\widetilde{L}(\Sigma, \vec{k}))$ we set

$$q_1 \prec q_2 \;\Leftrightarrow\; g^{-1}(q_1) <_{R_1} g^{-1}(q_2).$$

Notation. Let $(X, +)$ be an arbitrary semigroup. For $(x_n)_{n \in \mathbb{N}} \subseteq X$ we set

$$\mathrm{FS}[(x_n)_{n \in \mathbb{N}}] = \{x_{n_1} + \cdots + x_{n_l} : n_1 < \cdots < n_l \in \mathbb{N}\}.$$

THEOREM 4.1. *Let* $\mathbb{Q} = Q_1 \cup \cdots \cup Q_r$ *for* $r \in \mathbb{N}$. *Then there exist* $1 \le i_0 \le r$ *and, for every* $n \in \mathbb{N}$, *a function*

$$q_n : \{1,\ldots,n\} \times \{1,\ldots,n\} \cup \{(0,0)\} \to \mathbb{Q}$$

with

$$\begin{aligned} q_n(i,j) = &\sum_{t\in C_n^-} q_t^n \frac{(-1)^{-t}}{(-t+1)!} + i \sum_{t\in V_n^-} \frac{(-1)^{-t}}{(-t+1)!} \\ &+ \sum_{t\in C_n^+} q_t^n (-1)^{t+1} t! + j \sum_{t\in V_n^+} (-1)^{t+1} t!, \end{aligned}$$

where $C_n^-, V_n^- \in [\mathbb{Z}^-]^{<\omega}_{>0}$, $C_n^+, V_n^+ \in [\mathbb{N}]^{<\omega}_{>0}$ *with* $C_n^- \cap V_n^- = \emptyset = C_n^+ \cap V_n^+$, $q_t^n \in \mathbb{N}$ *with* $1 \le q_t^n \le -t$ *for* $t \in C_n^-$, $1 \le q_t^n \le t$ *for* $t \in C_n^+$, *which satisfy* $q_n(i_n, j_n) \prec q_{n+1}(i_{n+1}, j_{n+1})$ *for every* $n \in \mathbb{N}$, *and*

$$\mathrm{FS}\big[\big(q_n(i_n,j_n)\big)_{n\in\mathbb{N}}\big] \subseteq Q_{i_0}$$

for all $((i_n,j_n))_{n\in\mathbb{N}} \subseteq \mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ *with* $0 \le i_n, j_n \le n$ *for every* $n \in \mathbb{N}$.

Proof. Let $g : \widetilde{L}(\Sigma,\vec{k};0) \to \mathbb{Q}$ be as defined above. By Theorem 2.6 there exist $(w_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma,\vec{k};0)$ and $1 \le i_0 \le r$ such that $T_{(i_1,j_1)}(w_{n_1}) \star \cdots \star T_{(i_\lambda,j_\lambda)}(w_{n_\lambda}) \in g^{-1}(Q_{i_0})$ for every $\lambda \in \mathbb{N}$, $n_1 < \cdots < n_\lambda \in \mathbb{N}$, $(i_l,j_l) \in \mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ such that $0 \le i_l, j_l \le n_l$ for every $1 \le l \le \lambda$ and $(0,0) \in \{(i_1,j_1),\ldots,(i_\lambda,j_\lambda)\}$. Let $w_n = w^n_{m_1^n} \ldots w^n_{m^n_{l_n}}$ for every $n \in \mathbb{N}$. Set $q_n(i,j) = g(w_{3n-2} \star T_{(j,i)}(w_{3n-1}) \star T_{(1,1)}(w_{3n}))$ for every $n \in \mathbb{N}$ and $(i,j) \in \mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ with $0 \le i,j \le n$. The functions q_n have the required properties. ∎

NOTATION. For an arbitrary semigroup $(X,+)$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$, for $y_1 = x_{n_1} + \cdots + x_{n_l}$, $y_2 = x_{m_1} + \cdots + x_{m_\nu} \in \mathrm{FS}[(x_n)_{n\in\mathbb{N}}]$ we write $y_1 < y_2$ if $n_l < m_1$, and

$$\big[\mathrm{FS}[(x_n)_{n\in\mathbb{N}}]\big]^{<\infty}_{>0} = \{(y_1,\ldots,y_m) : m \in \mathbb{N}, y_1 < \cdots < y_m \in \mathrm{FS}[(x_n)_{n\in\mathbb{N}}]\}.$$

For every countable ordinal $\xi \ge 1$ and every $n \in \mathbb{N}$ we set

$$\begin{aligned} \mathbb{Q}^{<\infty} &= \{(q_1,\ldots,q_l) : l \in \mathbb{N},\, q_1 \prec \cdots \prec q_l \in \mathbb{Q}\setminus\{0\}\} \cup \{\emptyset\}, \\ \mathbb{Q}^{\xi} &= \{(q_1,\ldots,q_l) \in \mathbb{Q}^{<\infty} : \{\min \mathrm{dom}^+(q_1),\ldots,\min\mathrm{dom}^+(q_l)\} \in \mathcal{A}_\xi\}. \end{aligned}$$

Combining Theorem 3.5 with the representation of rational numbers via the function g, analogously to Theorem 4.1, we get the following Ramsey type partition theorem for every countable order ξ for the set of rational numbers. The case $\xi = 1$ corresponds to Theorem 4.1.

THEOREM 4.2. *Let* $\xi \ge 1$ *be a countable ordinal and a family* $G \subseteq \mathbb{Q}^{<\infty}$. *Then for each* $n \in \mathbb{N}$ *there exists a function*

$$q_n : \{1,\ldots,n\} \times \{1,\ldots,n\} \cup \{(0,0)\} \to \mathbb{Q}$$

with

$$q_n(i,j) = \sum_{t\in C_n^-} q_t^n \frac{(-1)^{-t}}{(-t+1)!} + i \sum_{t\in V_n^-} \frac{(-1)^{-t}}{(-t+1)!} + \sum_{t\in C_n^+} q_t^n (-1)^{t+1} t! + j \sum_{t\in V_n^+} (-1)^{t+1} t!,$$

where $C_n^-, V_n^- \in [\mathbb{Z}^-]^{<\omega}_{>0}$, $C_n^+, V_n^+ \in [\mathbb{N}]^{<\omega}_{>0}$ *with* $C_n^- \cap V_n^- = \emptyset = C_n^+ \cap V_n^+$, $q_t^n \in \mathbb{N}$ *with* $1 \le q_t^n \le -t$ *for* $t \in C_n^-$, $1 \le q_t^n \le t$ *for* $t \in C_n^+$, *which satisfy* $q_n(i_n, j_n) \prec q_{n+1}(i_{n+1}, j_{n+1})$ *for every* $n \in \mathbb{N}$, *and either*

- $\mathbb{Q}^\xi \cap \big[\mathrm{FS}[(q_n(i_n,j_n))_{n\in\mathbb{N}}]\big]^{<\infty}_{>0} \subseteq G$, *or*
- $\mathbb{Q}^\xi \cap \big[\mathrm{FS}[(q_n(i_n,j_n))_{n\in\mathbb{N}}]\big]^{<\infty}_{>0} \subseteq \mathbb{Q}^{<\infty} \setminus G$

for all $((i_n, j_n))_{n\in\mathbb{N}} \subseteq \mathbb{N}\times\mathbb{N} \cup \{(0,0)\}$ *with* $0 \le i_n, j_n \le n$ *for every* $n \in \mathbb{N}$.

Notation. For a semigroup $(X,+)$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$, we set

$$\big[\mathrm{FS}[(x_n)_{n\in\mathbb{N}}]\big]^{\mathbb{N}} = \{(y_n)_{n\in\mathbb{N}} : y_n \in \mathrm{FS}[(x_n)_{n\in\mathbb{N}}] \text{ and } y_n < y_{n+1} \text{ for all } n \in \mathbb{N}\}.$$

As a corollary of Theorem 2.5 we have the following partition theorem for infinite ordered sequences of rational numbers.

Theorem 4.3. *Let* $\mathcal{U}$ *be a Borel subset of* $\mathbb{Q}^{\mathbb{N}}$ *(in the product topology, considering* $\mathbb{Q}$ *with the discrete topology). Then for each* $n \in \mathbb{N}$ *there exists a function* $q_n : \{1,\dots,n\} \times \{1,\dots,n\} \cup \{(0,0)\} \to \mathbb{Q}$ *with*

$$q_n(i,j) = \sum_{t\in C_n^-} q_t^n \frac{(-1)^{-t}}{(-t+1)!} + i \sum_{t\in V_n^-} \frac{(-1)^{-t}}{(-t+1)!} + \sum_{t\in C_n^+} q_t^n (-1)^{t+1} t! + j \sum_{t\in V_n^+} (-1)^{t+1} t!,$$

where $C_n^-, V_n^- \in [\mathbb{Z}^-]^{<\omega}_{>0}$, $C_n^+, V_n^+ \in [\mathbb{N}]^{<\omega}_{>0}$ *with* $C_n^- \cap V_n^- = \emptyset = C_n^+ \cap V_n^+$, $q_t^n \in \mathbb{N}$, *with* $1 \le q_t^n \le -t$ *for* $t \in C_n^-$, $1 \le q_t^n \le t$ *for* $t \in C_n^+$, *which satisfy* $q_n(i_n, j_n) \prec q_{n+1}(i_{n+1}, j_{n+1})$ *for every* $n \in \mathbb{N}$, *and*

$$\text{either} \quad \big[\mathrm{FS}[(q_n(i_n,j_n))_{n\in\mathbb{N}}]\big]^{\mathbb{N}} \subseteq \mathcal{U}, \quad \text{or} \quad \big[\mathrm{FS}[(q_n(i_n,j_n))_{n\in\mathbb{N}}]\big]^{\mathbb{N}} \subseteq \mathbb{Q}^{\mathbb{N}} \setminus \mathcal{U}$$

for all $(i_n, j_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}\times\mathbb{N} \cup \{(0,0)\}$ *with* $0 \le i_n, j_n \le n$ *for every* $n \in \mathbb{N}$.

Proof. Let $\Sigma = \{\alpha_n : n \in \mathbb{Z}^*\}$, $\vec{k} = (k_n)_{n\in\mathbb{Z}^*}$, where $\alpha_{-n} = \alpha_n = n$ and $k_{-n} = k_n = n$ for every $n \in \mathbb{N}$, and $\upsilon = 0$. We define $\hat{g} : \widetilde{L}^\infty(\Sigma, \vec{k}; 0) \to \mathbb{Q}^{\mathbb{N}}$ by $\hat{g}((w_n)_{n\in\mathbb{N}}) = (g(w_n))_{n\in\mathbb{N}}$. The family $\hat{g}^{-1}(\mathcal{U}) \subseteq \widetilde{L}^\infty(\Sigma, \vec{k}; 0)$ is pointwise closed, since the function $\hat{g}$ is continuous. So, by Theorem 2.5 there exists $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; 0)$ such that

$$\text{either} \quad \widetilde{\mathrm{EV}}^\infty(\vec{w}) \subseteq \hat{g}^{-1}(\mathcal{U}), \quad \text{or} \quad \widetilde{\mathrm{EV}}^\infty(\vec{w}) \subseteq \widetilde{L}^\infty(\Sigma, \vec{k}; 0) \setminus \hat{g}^{-1}(\mathcal{U}).$$

From this point on, the proof is analogous to the one of Theorem 4.1. ∎

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