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Borel Tukey morphisms and combinatorial cardinal invariants of the continuum

by

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Abstract. We discuss the Borel Tukey ordering on cardinal invariants of the continuum. We observe that this ordering makes sense for a larger class of cardinals than has previously been considered. We then provide a Borel version of a large portion of van Douwen's diagram. For instance, although the usual proof of the inequality $\mathfrak{p} \leq \mathfrak{b}$ does not provide a Borel Tukey map, we show that in fact there is one. Afterwards, we revisit a result of Mildenberger concerning a generalization of the unsplitting and splitting numbers. Lastly, we use our results to give an embedding from the inclusion ordering on $\mathcal{P}(\omega)$ into the Borel Tukey ordering on cardinal invariants.

1. Introduction. Cardinal invariants of the continuum are cardinal numbers which are determined by families of real numbers (or any similar continuum such as $\mathcal{P}(\omega)$, the set of subsets of the natural numbers). For instance, the least size of a Lebesgue nonnull set is a cardinal invariant, one of many derived from properties of measure and category. A second example is the least size of a family of sequences of natural numbers such that any other sequence is eventually dominated by one from the family. This example is one of several which are known as *combinatorial* cardinal invariants.

As with each of these examples, most classical cardinal invariants take on values between $\aleph_1$ and $\mathfrak{c}$. The particular values can vary from one model of set theory to another; for instance, in a model of CH they always have value $\aleph_1 = \mathfrak{c}$. But the pattern of values is not arbitrary: there exist deep connections between them which dictate that certain inequalities must hold in any model of set theory. We refer the reader to Andreas Blass's excellent article [Bla03] in *Handbook of Set Theory* for a survey of this rich area of research.

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In this article we will be interested in a categorical approach to cardinal invariants and their inequalities which is due to Vojtáš. See [Voj93], or [Bla03, Section 4] for a more detailed account. This approach rests on the following definition scheme for cardinal invariants. A *Vojtáš triple* is some $\boldsymbol{A} = (A_-, A_+, \mathsf{A})$, where A is a relation from A_- to A_+ (that is, $\mathsf{A} \subset A_- \times A_+$). The cardinal invariant of the continuum corresponding to such a triple $\boldsymbol{A}$ is defined by

$$\|\boldsymbol{A}\| := \min\{|\mathcal{F}| : \mathcal{F} \text{ is a dominating family with respect to } \boldsymbol{A}\}.$$

Here, a subset $\mathcal{F} \subset A_+$ is said to be a *dominating family* with respect to $\boldsymbol{A}$ iff for all $x \in A_-$ there exists $y \in \mathcal{F}$ such that $x \mathrel{\mathsf{A}} y$. The simplest example is the dominating number $\mathfrak{d}$, which was described in the first paragraph. It is easy to see that $\mathfrak{d}$ is the cardinal invariant corresponding to the triple $(\omega^\omega, \omega^\omega, \leq^*)$, where ω^ω denotes the space of sequences of natural numbers and $\leq^*$ denotes the eventual domination relation.

We will be interested in the Vojtáš triples themselves, and not the corresponding cardinal invariants. This is really a separate pursuit, since it is clear that many different Vojtáš triples may be used to define the same cardinal number. Of course many triples do not define interesting invariants, but our study may be more compelling when they do.

The natural maps between Vojtáš triples are (generalized) Tukey morphisms. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are Vojtáš triples, then a *Tukey morphism* (or just *morphism*) from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a pair of maps

$$\phi\colon B_- \to A_-, \qquad \psi\colon A_+ \to B_+$$

such that for all $b_- \in B_-$ and $a_+ \in A_+$,

$$\phi(b_-) \mathrel{\mathsf{A}} a_+ \;\Rightarrow\; b_- \mathrel{\mathsf{B}} \psi(a_+).$$

In particular, if (ϕ, ψ) is a morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ and $\mathcal{F}$ is a dominating family with respect to $\boldsymbol{A}$, then $\psi(\mathcal{F})$ is a dominating family with respect to $\boldsymbol{B}$. But the existence of a morphism entails more than just this. For instance, the symmetry in the definitions leads to a notion of duality for triples and morphisms. If (ϕ, ψ) is a morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then (ψ, ϕ) is a morphism from $\boldsymbol{B}^\perp$ to $\boldsymbol{A}^\perp$, where $\boldsymbol{A}^\perp$ is the triple defined by $(A_+, A_-, \breve{\mathsf{A}})$ and $a \mathrel{\breve{\mathsf{A}}} a'$ iff $a' \not\mathrel{\mathsf{A}} a$.

As a consequence of the observation that morphisms send dominating families to dominating families, it follows that if there is a morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then $\|\boldsymbol{A}\| \geq \|\boldsymbol{B}\|$. Just as cardinal inequalities can be forced to hold or fail, Tukey morphisms between triples can be forced to exist or not. However, assuming some amount of definability on the triples and morphisms involved, one can use morphisms to establish absolute cardinal inequalities.

1.1. Definition.

- The Vojtáš triple (A_-, A_+, A) is called *Borel* if A_- and A_+ are Borel subsets of Polish spaces, and A is a Borel relation.
- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are Borel, then we say that a morphism (ϕ, ψ) from $\boldsymbol{A}$ to $\boldsymbol{B}$ is *Borel* if both ϕ and ψ are Borel functions.

If there is a Borel morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then we write $\boldsymbol{A} \geq_{\mathrm{BT}} \boldsymbol{B}$. Borel morphisms were initially studied by Blass [Bla96], who first noted that they resolve the absoluteness problem mentioned above. Indeed, if there exists a Borel Tukey morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then the corresponding cardinal inequality $\|\boldsymbol{A}\| \geq \|\boldsymbol{B}\|$ is absolute to forcing extensions. Additionally, Blass was motivated by some more subtle applications of Borel morphisms. For instance, consider the cardinal equalities $\mathfrak{r}_m = \mathfrak{r}_n$ for all m, n, where $\mathfrak{r}_n$ denotes the *n-unsplitting number*: the least cardinality of a family of reals such that every coloring $c \in m^\omega$ is almost constant on some member of the family. In other words, $\mathfrak{r}_n$ is defined by the triple $(n^\omega, [\omega]^\omega, \mathsf{R}_n)$, where $c \mathrel{\mathsf{R}_n} B$ iff c is almost constant on B. (Thus $\mathfrak{r}_2$ is just the usual unsplitting number $\mathfrak{r}$; see the next section.) The proofs of the inequalities $\mathfrak{r}_m \geq \mathfrak{r}_n$ for $2 \leq m < n$ can be seen as involving an operation on Vojtáš triples called *sequential composition*. Blass conjectured that for $2 \leq m < n$ the inequality $\mathfrak{r}_m \geq \mathfrak{r}_n$ is not witnessed by a Borel morphism, which we would take to mean that sequential composition is necessary to prove the inequality.

Since Blass's initial study, however, there have been just a couple of results on Borel morphisms. Blass's conjecture concerning $\mathfrak{r}_n$ was established by Mildenberger, who showed in [Mil02] that there are no such Borel morphisms. Another step was taken in [PR95], where the authors showed that after suitably coding the null and meager ideals, all of the inequalities in Cichoń's diagram are witnessed by Borel (in fact continuous) morphisms. On the other hand, there are a growing number of applications of Borel morphisms appearing in the literature. See, for instance, the body of recent work on parametrized diamond principles initiated in [MHD04], or the results in Borel equivalence relations found in [CS11].

In this paper we wish to renew an interest in the systematic study of the relationships between cardinal invariants with respect to Borel morphisms. We would also like to propose a mild generalization of this study to certain cardinal invariants which are not definable from Vojtáš triples alone. To see what we mean, consider the almost disjointness number $\mathfrak{a}$. This cardinal is the least size of a family which is not only dominating with respect to $\not\perp$, but which is *also* almost disjoint. Presently, we show how to handle cardinals which are definable in this more general sense. (This is motivated in part by Zapletal's [Zap04], where such cardinals are discussed and handled collectively.)

1.2. Definition. If $\boldsymbol{A}$ is a Vojtáš triple and P is an arbitrary property of subsets of A_+, then the cardinal invariant of the continuum corresponding to $\boldsymbol{A}$ and P is

$$\|A\|_P := \min\{|\mathcal{F}| : \mathcal{F} \text{ satisfies property } P \text{ and is a dominating family with respect to } \boldsymbol{A}\}.$$

In the case that P is trivial, that is, the cardinal is definable from a Vojtáš triple alone, we say that the cardinal is *simple*. Again, it makes sense to define Tukey morphisms between cardinals which are not simple.

1.3. Definition. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are Vojtáš triples, and P and Q are properties, then a morphism from $\boldsymbol{A}, P$ to $\boldsymbol{B}, Q$ is a pair of maps $\phi\colon B_- \to A_-$ and $\psi\colon A_+ \to B_+$ satisfying:

(a) if $\mathcal{F}$ satisfies property P, then $\psi(\mathcal{F})$ satisfies property Q, and
(b) $\phi(b_-)\ \mathsf{A}\ a_+ \Rightarrow b_-\ \mathsf{B}\ \psi(a_+)$.

We are proposing to study Borel Tukey morphisms between a number of cardinal invariants definable from some $\boldsymbol{A}$ and P, a more ambitious plan than that of [Bla03]. This extension was proposed by Coskey and Schneider, who encountered the problem in a slightly different context [CS11]. By allowing cardinal definitions where P is nontrivial, we open the door for many important new cardinals to be compared with respect to Borel morphisms. For instance, we can now incorporate into the Borel Tukey order several new entries from the van Douwen diagram of combinatorial cardinal invariants.

We should address the common objection to this programme that the existence of a Borel morphism is much stronger than is needed to prove the corresponding cardinal inequality. To answer this, simply note that the above-mentioned applications of Borel morphisms to parametrized diamond principles and Borel equivalence relations cannot be established on the basis of cardinal inequalities alone. Thus in these areas the Borel Tukey order serves as a dictionary of positive results. Moreover, results concerning Borel morphisms can have combinatorial value. For instance, Mildenberger's discovery that there is no Borel morphism from $\mathfrak{r}_m$ to $\mathfrak{r}_n$ for $m < n$ can be viewed as new information concerning measurable colorings of ω and homogeneity.

This paper is organized as follows. In the next section, we will establish a Borel version of van Douwen's diagram. The third section is devoted to the proof of just one of the edges in this diagram: the construction of a morphism from $\mathfrak{p}$ to $\mathfrak{b}$. In the fourth section, we do for splitting numbers what Blass and Mildenberger did for unsplitting numbers: we define *n-splitting numbers* $\mathfrak{s}_n$ and prove an analog of Mildenberger's theorem. We also consider the infinite versions $\mathfrak{r}_\sigma$ and $\mathfrak{s}_\sigma$ of the unsplitting and splitting numbers. In the final section we give a method for constructing arbitrary patterns in the Borel Tukey order.

2. A Borel van Douwen diagram. In this section, we consider the cardinal invariants in van Douwen's diagram which can be naturally defined using Vojtáš triples. Specifically, we consider the cardinal invariants shown in Figure 1. The aim is to produce a "Borel version" of van Douwen's diagram, with arrows only in the case that the cardinal inequalities are witnessed by Borel morphisms.

$$\begin{tikzcd}
 & \mathfrak{i} \arrow[dl] \arrow[d] & \mathfrak{u} \arrow[dl] \\
\mathfrak{d} \arrow[d] \arrow[dr] & \mathfrak{r} \arrow[d] & \mathfrak{a} \arrow[dl] \\
\mathfrak{s} \arrow[dr] & \mathfrak{b} \arrow[d] & \\
 & \mathfrak{p} &
\end{tikzcd}$$

Fig. 1. Provable size relationships among some combinatorial cardinal invariants. (Here, $\to$ means $\geq$.)

Since each cardinal invariant can be defined by several different Vojtáš triples, the answer to the question of whether a cardinal inequality is witnessed by a Borel morphism will vary depending on the choice of triples. We shall take the approach of choosing at our discretion a particularly natural triple defining each of the cardinal invariants in Figure 1. Afterwords, we can conflate without confusion the cardinal invariants with their chosen defining triples. Thus, the meaning of a shorthand such as $\mathfrak{d} \geq_{\mathrm{BT}} \mathfrak{b}$ may be resolved by examining the definitions in Table 1 below.

Table 1. Natural definitions for the cardinal invariants shown in Figure 1

cardinal	A_-	A_+	A	$P =$ "$\mathcal{F}$ is ..."
$\mathfrak{p}$	$[\omega]^\omega$	$[\omega]^\omega$	$\not\subseteq^*$	centered
$\mathfrak{s}$	$[\omega]^\omega$	2^ω	is split by	–
$\mathfrak{r}$	2^ω	$[\omega]^\omega$	does not split	–
$\mathfrak{b}$	ω^ω	ω^ω	$\not\geq^*$	–
$\mathfrak{d}$	ω^ω	ω^ω	$\leq^*$	–
$\mathfrak{a}$	IC	IC	$\not\perp$	a.d., infinite
$\mathfrak{i}$	IC	IC	does not split	(see discussion)
$\mathfrak{u}$	$[\omega]^\omega$	$[\omega]^\omega$	does not split	centered

Because the table is displayed compactly, it is necessary to explain some of the terminology. First, recall that a function $c \in 2^\omega$ is said to *split* the infinite set A if $c{\restriction}A$ takes both values infinitely often. Next, IC denotes the family of infinite/co-infinite subsets of ω, and $\perp$ denotes the relation "is almost disjoint from". Finally, in row $\mathfrak{i}$, the property $P(\mathcal{F})$ should actually say that "$\mathcal{F}$ is derived from an independent family by taking all intersections of finitely many sets or their complements".

We now consider in turn each edge of the diagram. First, there are a number of easy answers to be reaped.

$\boxed{\mathfrak{d} \to \mathfrak{b}}$ Since dominating families are unbounded, this is just a trivial morphism.

$\boxed{\mathfrak{d} \to \mathfrak{s}}$ The classical proof can be seen as a morphism proof. See [Bla03], Theorem 3.3 and the corresponding discussion in Section 4 of that article.

$\boxed{\mathfrak{r} \to \mathfrak{b}}$ This is dual to $\mathfrak{d} \geq \mathfrak{s}$.

$\boxed{\mathfrak{u} \to \mathfrak{r}}$ The identity maps clearly work.

$\boxed{\mathfrak{i} \to \mathfrak{r}}$ The identity maps clearly work, except that we technically must define the behavior of ϕ on the finite and cofinite sets. In fact, if B is cofinite, then we can let $\phi(B) \in [\omega]^\omega$ be arbitrary.

$\boxed{\mathfrak{i} \to \mathfrak{d}}$ There do not exist Borel such maps. Indeed, suppose that (ϕ, ψ) were such a morphism. Then, in particular, ϕ and ψ satisfy

$$\phi(f) \text{ does not split } A \;\Rightarrow\; f \leq \psi(A).$$

This implies that (ϕ, ψ) is a Borel morphism from $\mathfrak{r}$ to $\mathfrak{d}$. Now, it is well-known that the inequality $\mathfrak{r} \geq \mathfrak{d}$ can be violated in a forcing extension (for instance in the Miller model), and hence in this extension we have $\mathfrak{r} \not\geq_{\mathrm{BT}} \mathfrak{d}$. But since both $\mathfrak{r}$ and $\mathfrak{d}$ are simple, the fact that (ϕ, ψ) is a morphism from $\mathfrak{r}$ to $\mathfrak{d}$ would be preserved to the forcing extension, a contradiction.

2.1. Question. Is it possible to tinker with the definition of $\mathfrak{i}$ in such a way that this question becomes nontrivial?

$\boxed{\mathfrak{a} \to \mathfrak{b}}$ This question is somewhat trivial, since it is easy to see that there does not exist any morphism (let alone a Borel one) from $\mathfrak{a}$ to $\mathfrak{b}$.

2.2. Proposition. *There do not exist maps $\phi\colon \omega^\omega \to \mathrm{IC}$ and $\psi\colon \mathrm{IC} \to \omega^\omega$ satisfying*

$$\phi(f) \not\perp A \;\Rightarrow\; f \not\geq^* \psi(A).$$

Proof. Suppose that the ϕ and ψ were such maps. Consider the sets O = the odd numbers and E = the even numbers in place of A, and let

$f(n) = \max\{\psi(O)(n), \psi(E)(n)\}$. Now either $\phi(f) \not\leq O$ or else $\phi(f) \not\leq E$, a contradiction in either case. ∎

We find this triviality-of-a-proof unsatisfying, particularly because it exploits a pair of complementary sets—something never present in an infinite a.d. family. This would be resolved by a negative answer to the following, subtler question.

2.3. Question. Can there be a pair of maps satisfying the condition of Proposition 2.2 just for sets A ranging in some mad family of minimal cardinality?

This discussion admits a generalization to cardinals with definitions similar to that of $\mathfrak{a}$. Specifically, for $\mathcal{C}$ a collection of filters on ω let $\mathfrak{p}_{\mathcal{C}}$ bet the cardinal defined by the triple $([\omega]^\omega, [\omega]^\omega, \not\subset^*)$ together with the property $P(\mathcal{F}) =$ "$\mathcal{F}$ generates a filter which is in $\mathcal{C}$". Then $\mathfrak{p}$ is $\mathfrak{p}_{\mathcal{C}}$ where $\mathcal{C}$ consists of all filters. Moreover, it is easy to see that $\mathfrak{a}$ is $\mathfrak{p}_{\mathcal{C}}$ where $\mathcal{C}$ consists of those filters whose dual ideal is generated by an infinite mad family.

2.4. Proposition. *For any class of filters $\mathcal{C}$, we have $\mathfrak{p}_{\mathcal{C}} \not\geq_{\mathrm{BT}} \mathfrak{b}$ (whether the inequality $\mathfrak{p}_{\mathcal{C}} \geq \mathfrak{b}$ is true or false).*

The proof is identical to that of Proposition 2.2 (and we can ask a version of Question 2.3 in this case). Concerning morphisms going the other way, the following observation shows that the problem is closely connected with that of diagonalizing filters.

2.5. Proposition. *Suppose that it is possible to diagonalize any filter in $\mathcal{C}$ without adding dominating reals. Then $\mathfrak{b} \not\geq_{\mathrm{BT}} \mathfrak{p}_{\mathcal{C}}$.*

Proof. Suppose that there is such a morphism, that is, there exist Borel maps $\phi\colon [\omega]^\omega \to \omega^\omega$ and $\psi\colon \omega^\omega \to [\omega]^\omega$ satisfying:

(a) $\psi(\omega^\omega)$ generates a filter in $\mathcal{C}$, and
(b) $A \subset^* \psi(f) \Rightarrow f \leq^* \phi(A)$.

Then by our assumption, it is possible to force to add a pseudo-intersection $\dot{A}$ of $\psi(\omega^\omega)$ without adding dominating reals. Thus there exists $f \in \omega^\omega \cap V$ such that $f \not\leq^* \phi(\dot{A})$. Since ϕ is Borel, in the extension we have, for all $x \in [\omega]^\omega$,

$$x \subset^* \psi(f) \Rightarrow f \leq^* \phi(x).$$

Plugging in $x = \dot{A}$ yields an immediate contradiction. ∎

Of course, in the case of $\mathfrak{p}_{\mathcal{C}} = \mathfrak{a}$, we already know that there is no Borel morphism from $\mathfrak{b}$ to $\mathfrak{a}$ (since the inequality $\mathfrak{b} \geq \mathfrak{a}$ can be forced to fail). It would be interesting to give a proof of this using a diagonalization argument. Notice also that the proof of Proposition 2.5 shows outright that ϕ cannot be Borel. It would be nice to find a condition which implies ψ cannot be Borel.

$\boxed{\mathfrak{a} \to \mathfrak{p}}$ Since we established that $\mathfrak{a} \not\geq_{\mathrm{BT}} \mathfrak{b}$, it is natural to ask whether we even have $\mathfrak{a} \geq_{\mathrm{BT}} \mathfrak{p}$. Indeed, this is the case, since the maps $\psi(A) = \omega \smallsetminus A$ and $\phi = \mathrm{id}$ satisfy the requirements:

(a) if $\mathcal{F}$ is a.d. and infinite then $\psi(\mathcal{F})$ is centered, and
(b) $\phi(A) \not\perp B \Rightarrow A \not\subset^* \psi(B)$.

Thus $\mathfrak{a}$ has not fallen off of the diagram!

$\boxed{\mathfrak{s} \to \mathfrak{p}}$ The following result shows that in fact $\mathfrak{s} \not\geq_{\mathrm{BT}} \mathfrak{p}$.

2.6. Theorem. *Suppose that $\phi, \psi \colon [\omega]^\omega \to [\omega]^\omega$ are maps satisfying:*

(a) *$\psi([\omega]^\omega)$ is centered, and*
(b) *$A \subset^* \psi(B) \Rightarrow B$ does not split $\phi(A)$.*

Then ϕ and ψ cannot both be Borel.

Coskey and Schneider have previously established this result under the additional assumption that ϕ is E_0-invariant (i.e., $A =^* A'$ iff $\phi(A) =^* \phi(A')$). However, that fact is now superseded by the following shorter and stronger argument, which was pointed out to us by Dilip Raghavan.

Proof of Theorem 2.6. Suppose that (ϕ, ψ) are Borel functions satisfying (a) and (b). Letting $\mathcal{F}$ denote the filter generated by $\psi([\omega]^\omega)$, we use the (relativized) Mathias forcing to add a pseudo-intersection $\dot{A}$ for $\mathcal{F}$. This forcing is always ccc, and since $\mathcal{F}$ is analytic, the forcing is Suslin as well (see [BJ95, Definition 3.6.1]). It follows from [BJ95, Lemma 3.6.24] that the ground model is a splitting family in the forcing extension. In particular, there exists $B \in [\omega]^\omega \cap V$ such that B splits $\phi(\dot{A})$. In the ground model, we apply (b) to obtain

$$(\forall x \in [\omega]^\omega)\ x \subset^* \psi(B) \Rightarrow B \text{ does not split } \phi(x).$$

Since ϕ is Borel, the same sentence holds in the extension. It follows that B does not split $\phi(\dot{A})$, which is a contradiction. ∎

We remark that the argument of Theorem 2.6 also shows that there is no Borel morphism from $\mathfrak{s}_\sigma$ to $\mathfrak{p}$ (for the definition of $\mathfrak{s}_\sigma$, see the later section on splitting). We leave open the following question:

2.7. Question. Does there exist a morphism (ϕ, ψ) from $\mathfrak{s}$ to $\mathfrak{p}$ such that just one of the maps is Borel?

$\boxed{\mathfrak{b} \to \mathfrak{p}}$ It is the case that $\mathfrak{b} \geq_{\mathrm{BT}} \mathfrak{p}$. Since the construction is fairly involved, we shall give the proof its own section, below.

To complete our discussion of van Douwen's diagram, we finally verify that whenever an edge does *not* appear in Figure 1, then there is not a Borel morphism either. Most of this verification is routine, because it is already known that any cardinal inequality not shown in Figure 1 can be violated by forcing. Hence, if there is no edge between *simple* invariants $\|\boldsymbol{A}\|$ and $\|\boldsymbol{B}\|$ in Figure 1, then we automatically obtain $\boldsymbol{A} \not\geq_{\mathrm{BT}} \boldsymbol{B}$.

Even when just one of the invariants involved is simple, a forcing argument will work. Indeed, if there is a Borel morphism from $\boldsymbol{A}, P$ to $\boldsymbol{B}$, then the condition in Definition 1.3(a) is trivial, and so it is preserved to forcing extensions. On the other hand, if there is a Borel morphism from $\boldsymbol{A}$ to $\boldsymbol{B}, Q$, and property Q is downward closed, then the condition in Definition 1.3(a) amounts to saying that all of $\mathrm{im}(\psi)$ has property Q. Since all of the cardinals we are considering are defined by a property Q which is downward closed and very low in complexity, this will again be preserved to forcing extensions.

Hence, we need only handle the inequalities between cardinals which are both not simple. This is done in the next result.

2.8. Proposition. *The invariants* $\mathfrak{i}$, $\mathfrak{u}$ *and* $\mathfrak{a}$ *are incomparable with respect to* $\geq_{\mathrm{BT}}$.

Proof. Referring to the definitions of $\mathfrak{i}$ and $\mathfrak{u}$, it is clear that if we had either $\mathfrak{a} \geq_{\mathrm{BT}} \mathfrak{i}$ or $\mathfrak{a} \geq_{\mathrm{BT}} \mathfrak{u}$, then we would have $\mathfrak{a} \geq_{\mathrm{BT}} \mathfrak{r}$. But now the inequality $\mathfrak{a} \geq \mathfrak{r}$ can be violated by forcing and $\mathfrak{r}$ is simple, so we can use the argument above.

The rest of the cases are similar. If we had $\mathfrak{u} \geq_{\mathrm{BT}} \mathfrak{i}$ then we would also have $\mathfrak{r} \geq_{\mathrm{BT}} \mathfrak{i}$; if we had $\mathfrak{u} \geq_{\mathrm{BT}} \mathfrak{a}$ then we would also have $\mathfrak{r} \geq_{\mathrm{BT}} \mathfrak{a}$; if we had $\mathfrak{i} \geq_{\mathrm{BT}} \mathfrak{u}$ then we would have $\mathfrak{r} \geq_{\mathrm{BT}} \mathfrak{u}$; if we had $\mathfrak{i} \geq_{\mathrm{BT}} \mathfrak{a}$ then we would have $\mathfrak{r} \geq_{\mathrm{BT}} \mathfrak{a}$. In all four of these cases, the argument above applies. ■

The results of this section are summarized in Figure 2.

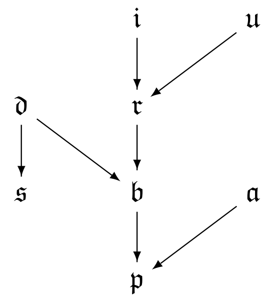

Fig. 2. Borel Tukey morphisms among some combinatorial cardinal invariants. (Here, $\to$ means $\geq_{\mathrm{BT}}$.)

2.9. Question. Is there an interesting alternative set of definitions of these invariants for which the Borel morphisms faithfully reflect all of the inequalities in van Douwen's diagram?

For instance, we know that there is no Borel morphism from $\mathfrak{i}$ to $\mathfrak{d}$ as we have defined them. But it is worth mentioning that if ϕ and ψ are the maps constructed in Theorem 3.1 below, then property 3.1(b) comes very close to giving the condition needed for a morphism from $\mathfrak{i}$ to $\mathfrak{d}$ (with the roles of ϕ and ψ interchanged). Hence it may be possible to give a new proof that $\mathfrak{i} \geq \mathfrak{d}$ by slightly modifying the triple for $\mathfrak{i}$ and the construction in Theorem 3.1.

$\boxed{\mathfrak{t}}$ No discussion involving $\mathfrak{p}$ would be complete without mentioning the tower number, $\mathfrak{t}$. This cardinal is defined by the same triple as $\mathfrak{p}$, together with the property $P(\mathcal{F}) =$ "$\mathcal{F}$ is linearly ordered". Clearly $\mathfrak{t} \geq_{\mathrm{BT}} \mathfrak{p}$, but it has only recently been shown by Malliaris and Shelah [MS12] that $\mathfrak{p} \geq \mathfrak{t}$. Thus, it is desirable to verify that the latter inequality does not have a Borel proof.

2.10. Proposition. *We have* $\mathfrak{p} \not\geq_{\mathrm{BT}} \mathfrak{t}$.

Proof. Suppose towards a contradiction that (ϕ, ψ) satisfy:

(a) if $\mathcal{F}$ is centered then $\psi(\mathcal{F})$ is linearly ordered, and
(b) $\phi(x) \not\subset^* y \Rightarrow x \not\subset^* \psi(y)$.

Let A, B, C be any three infinite sets with empty intersection but such that any two have infinite intersection. Then applying (a) to each pair $\{A, B\}, \{A, C\}, \{B, C\}$ we conclude that $\psi(\{A, B, C\})$ is linearly ordered. Thus $\psi(\{A, B, C\})$ has infinite intersection, and using (b), it follows that $\{A, B, C\}$ does too. This contradicts the choice of A, B, C. ∎

Once again this result is rather trivial, so it would be interesting to rework the question to yield a more poignant theorem. Moreover, we were unable to include $\mathfrak{t}$ in Figure 2 since we do not know its Borel relationship to the other invariants. Thus we are left with the following question:

2.11. Question. Can the result of the next section be improved to show that $\mathfrak{b} \geq_{\mathrm{BT}} \mathfrak{t}$?

3. A Borel morphism from $\mathfrak{b}$ to $\mathfrak{p}$. Although the simplest proof that $\mathfrak{b} \geq \mathfrak{p}$ does not give a Borel morphism, the following result establishes that it is indeed the case that $\mathfrak{b} \geq_{\mathrm{BT}} \mathfrak{p}$.

3.1. Theorem. *There exists a continuous map* $\psi\colon \omega^\omega \to [\omega]^\omega$ *and a Borel map* $\phi\colon [\omega]^\omega \to \omega^\omega$ *satisfying:*

(a) $\psi(\omega^\omega)$ *is centered, and*
(b) $A \subset^* \psi(f) \Rightarrow f \leq^* \phi(A)$.

Proof. The outline of the proof is as follows. We will construct a continuous map ψ which satisfies (a), and has the additional property that, for every $\leq^*$-unbounded subset $S \subset \omega^\omega$, the image $\psi(S)$ does not have a pseudo-intersection. In particular, we will have:

($\star$) for each $A \in [\omega]^\omega$, the set $C_A := \{f \in \omega^\omega : A \subset^* \psi(f)\}$ is $\leq^*$-bounded.

Letting $\phi(A)$ be such a bound, it is easy to see that (ϕ, ψ) satisfy property (b). We will show moreover that bounds $\phi(A)$ can be chosen in a Borel fashion.

We now begin the construction of ψ. Let $n \in \mathbb{N}$, and let T_n denote the tree which is ω-branching for the first n levels, and binary branching afterward.

3.2. Claim. *There exists a continuous function $\psi_n \colon [T_n] \to [\omega]^\omega$ with the properties:*

(i) *if $f_1, \ldots, f_n \in [T_n]$ then $\psi_n(f_1) \cap \cdots \cap \psi_n(f_n)$ is infinite,*
(ii) *if $f_1, \ldots, f_{n+1} \in [T_n]$ are all distinct, then $\psi_n(f_1) \cap \cdots \cap \psi_n(f_{n+1})$ is finite, and*
(iii) *if $f_1, \ldots, f_{n+1} \in [T_n]$ and $f_1{\restriction}_n, \ldots, f_{n+1}{\restriction}_n$ are all distinct, then we even have $\psi_n(f_1) \cap \cdots \cap \psi_n(f_{n+1}) = \emptyset$.*

Proof. It will be more convenient to construct the map ψ_n from $[T_n]$ into the set $[\Omega]^\omega$, where

$$\Omega = \{(t_1, \ldots, t_n) \in (T_n)^n \colon (\exists l > n)\ (\forall i)\ \operatorname{lev}(t_i) = l \ \&\ \ulcorner(t_1{\restriction}_n, \ldots, t_n{\restriction}_n)\urcorner < l\}.$$

Here, $\ulcorner \cdot \urcorner$ denotes any fixed bijection $(\omega^n)^n \to \omega$. Now, we simply define

$$\psi_n(f) = \{(t_1, \ldots, t_n) \in \Omega : (\exists i)\ t_i \subset f\}.$$

To see (i), let $f_1, \ldots, f_n \in [T_n]$ be given. Then for l large enough we will have $(f_1{\restriction}_l, \ldots, f_n{\restriction}_l) \in \Omega$, and moreover these sequences will lie in $\psi_n(f_1) \cap \cdots \cap \psi_n(f_n)$.

Assertion (iii) just follows from the pigeon-hole principle: no sequence of length n will suffice to match $n+1$ initial segments.

For assertion (ii), if $f_1, \ldots, f_{n+1}$ are all distinct, then there exists a level l such that $f_1{\restriction}_l, \ldots, f_{n+1}{\restriction}_l$ are all distinct. Using the same pigeon-hole argument as above, any element $(t_1, \ldots, t_n)$ of $\psi_n(f_1) \cap \cdots \cap \psi_n(f_{n+1})$ must lie at some level $l' < l$. But there are only finitely many such $(t_1, \ldots, t_n)$, since we require $\ulcorner(t_1{\restriction}_n, \ldots, t_n{\restriction}_n)\urcorner < l'$ and T_n is finitely branching after level n. ∎

To define ψ, we simply "glue together" all of the ψ_n. More precisely, for each n, we regard ω^ω as a subset of $[T_n]$ and therefore think of ψ_n as a function from ω^ω into $[\omega]^\omega$. Using this tacitly, we let ψ be the function from $\omega^\omega \to [\omega \times \omega]^\omega$ defined by placing ψ_n on the nth column. (Even more precisely, for each n, we fix an embedding from $\omega^{<\omega}$ into T_n which is equal to the identity on the first n levels. Letting $\iota_n \colon \omega^\omega \to [T_n]$ denote the induced injection, this allows us to replace ψ_n with $\psi_n \circ \iota_n$ without harming appeals to Claim 3.2(iii). We then let

$$\psi(f) = \{(n, m) : m \in \psi_n(\iota_n(f))\}.$$

Of course, we may also use a pairing function to think of ψ as a function into $[\omega]^\omega$. In our arguments, we will freely elide the use of ι_n and this pairing function.)

With this definition, it is clear that $\psi(\omega^\omega)$ is centered. Indeed, given a sequence $f_1, \ldots, f_n \in \omega^\omega$, we deduce from Claim 3.2(i) that $\psi_n(f_1) \cap \cdots \cap \psi_n(f_n)$ is infinite, and hence so is $\psi(f_1) \cap \cdots \cap \psi(f_n)$. To get property $(\star)$, we use the following auxiliary claim.

3.3. Claim. *If $S \subset \omega^\omega$ is $\leq$-unbounded then $\bigcap_{f \in S} \psi(f)$ is finite.*

Proof. First note that if S is infinite, then for each n we know that $\bigcap_{f \in S} \psi_n(f)$ is finite by Claim 3.2(ii). Hence $\bigcap_{f \in S} \psi(f)$ meets each column of $\omega \times \omega$ in a finite set. Now, if additionally $S \subset \omega^\omega$ is $\leq$-unbounded, then it is not hard to see that there exists a level l and elements $f_1, f_2, \ldots \in S$ such that $f_1{\restriction_l}, f_2{\restriction_l}, \ldots$ are all distinct. It follows from Claim 3.2(iii) that for every $n > l$, we have $\bigcap_i \psi_n(f_i) = \emptyset$. Hence $\bigcap_{f \in S} \psi(f)$ only meets finitely many columns of $\omega \times \omega$. Putting these together, we can conclude that $\bigcap_{f \in S} \psi(f)$ is finite. ∎

We can now conclude that the function ψ satisfies property $(\star)$. Indeed, for all $A \in [\omega]^\omega$, Claim 3.3 implies that for each n the set

$$C_{A,n} := \{f \in \omega^\omega : A \setminus n \subset \psi(f)\}$$

is $\leq$-bounded, and hence C_A is $\leq^*$-bounded. However, to show that a bound $\phi(A)$ can be obtained from A in a Borel fashion, we need one more claim. In the following result, we will let $\mathcal{K}(\omega^\omega)$ denote the space of compact subsets of ω^ω endowed with its usual hyperspace topology (called the Vietoris topology). Note that this space includes all of the $C_{A,n}$ because they are closed and $\leq$-bounded.

3.4. Claim. *For all n, the function $[\omega]^\omega \to \mathcal{K}(\omega^\omega)$ defined by $A \mapsto C_{A,n}$ is Borel.*

Proof. Since the map $A \mapsto A \setminus n$ is continuous, it is enough to treat the case $n = 0$, that is, to show that the map

$$A \mapsto \{f \in \omega^\omega : A \subset \psi(f)\}$$

is Borel. By [Kec95, Theorem 28.8], if X and Y are Polish then $\alpha\colon X \to \mathcal{K}(Y)$ is Borel iff the relation $\{(x, y) : y \in \alpha(x)\}$ is Borel. Thus, to establish the claim, we need only verify that

$$\{(A, f) : A \subset \psi(f)\}$$

is a Borel subset of $[\omega]^\omega \times \omega^\omega$. But this follows easily from Suslin's theorem, because $A \subset \psi(f)$ if and only if there exists $B \in [\omega]^\omega$ such that $B = \psi(f)$ and $A \subset B$, and also $A \subset \psi(f)$ if and only if for all $B \in [\omega]^\omega$ if $B = \psi(f)$ then $A \subset B$. ∎

With this in hand, we can define $\phi(A)$ as follows. For all n, since $C_{A,n}$ is closed and $\leq$-bounded, we can find its least upper bound b_n. (It is an easy exercise to check that the map $\mathcal{K}(\omega^\omega) \to \omega^\omega$ which sends a bounded set to its least upper bound is continuous.) Then, simply diagonalize to find $\phi(A)$ such that for all n, $b_n \leq^* \phi(A)$. This concludes the proof of Theorem 3.1 ∎

We remark that the result cannot be improved to get ϕ continuous too. Indeed, if ψ is even Borel then $\psi(\omega^\omega)$ generates a Baire measurable filter $\mathcal{F}$. A well-known result of Talagrand [Tal80] and Jalali-Naini [JN76] implies that there exists a partition of ω into finite intervals (J_n) such that for all $F \in \mathcal{F}$ and almost all n, $F \cap J_n \neq \emptyset$. Now, let $\mathcal{A}$ be the collection of almost transversals for (J_n), i.e., sets A such that $|A \cap J_n| \leq 1$ for all n and $|A \cap J_n| = 1$ for all but finitely many n. Then $\mathcal{A}$ is σ-compact, and if ϕ is continuous then $\phi(\mathcal{A})$ is σ-compact as well. It follows that $\phi(\mathcal{A})$ is $\leq^*$-bounded, say by f. Now, $\psi(f) \in \mathcal{F}$ clearly contains some subset A such that $A \in \mathcal{A}$. But then property (b) implies that $f \leq^* \phi(A)$, and this is a contradiction.

4. Splitting and unsplitting.

In this section, we consider the so-called σ-splitting number $\mathfrak{s}_\sigma$ and the σ-unsplitting number $\mathfrak{r}_\sigma$. These cardinals are closely related to $\mathfrak{s}$ and $\mathfrak{r}$: we have both $\mathfrak{s}_\sigma \geq \mathfrak{s}$ and $\mathfrak{r}_\sigma \geq \mathfrak{r}$, and we do not know whether either of the reverse inequalities are theorems of ZFC. We will show in each case that the unknown inequalities do not hold in the Borel Tukey order. For $\mathfrak{r}_\sigma$, this result follows trivially from Mildenberger's result concerning the cardinals $\mathfrak{r}_n$ which was mentioned in the introduction. For $\mathfrak{s}_\sigma$, we will follow a similar strategy and define a family of cardinals $\mathfrak{s}_n$ which in some sense approximate $\mathfrak{s}_\sigma$.

$\boxed{\mathfrak{r}_\sigma}$ The *σ-unsplitting* number, $\mathfrak{r}_\sigma$, is defined to be the least cardinality of a family of reals such that no countable subset of 2^ω suffices to split them all. In other words, it is defined by the triple $((2^\omega)^\omega, [\omega]^\omega, \mathsf{R}_\sigma)$ where $\langle c_n \rangle \ \mathsf{R}_\sigma \ B$ iff for all n, c_n is almost constant on B. It is clear that $\mathfrak{r}_\sigma \geq \mathfrak{r}$, in fact the trivial maps $\phi(c) = \langle c \rangle$ (the constant sequence) and $\psi = \mathrm{id}$ give a morphism. On the other hand, it is an important open question whether the reverse inequality $\mathfrak{r} \geq \mathfrak{r}_\sigma$ holds.

4.1. Corollary (essentially due to Mildenberger). $\mathfrak{r} \not\geq_{\mathrm{BT}} \mathfrak{r}_\sigma$.

Proof. We have just seen that $\mathfrak{r}_\sigma \geq_{\mathrm{BT}} \mathfrak{r}$, and we can similarly show that $\mathfrak{r}_\sigma \geq_{\mathrm{BT}} \mathfrak{r}_n$ for $n > 2$. Indeed, we require maps $\phi\colon n^\omega \to (2^\omega)^\omega$ and

$\psi\colon [\omega]^\omega \to [\omega]^\omega$ such that:

$$\phi(c)(i) \text{ almost constant on } B\ (\forall i) \Rightarrow c \text{ almost constant on } \psi(B).$$

For this, we simply take $\phi(c)(i)$ to be the 2-coloring which assigns to k the ith bit of $c(k)$, and $\psi = \mathrm{id}$ as before. But now if we had $\mathfrak{r} \geq_{\mathrm{BT}} \mathfrak{r}_\sigma$, then since $\mathfrak{r}_\sigma \geq_{\mathrm{BT}} \mathfrak{r}_n$ we would have $\mathfrak{r} \geq_{\mathrm{BT}} \mathfrak{r}_n$, contradicting the result of Mildenberger that there are no Borel morphisms from $\mathfrak{r}_m$ to $\mathfrak{r}_n$ for $m < n$. ∎

It should be noted that Spinas [Spi04] has strengthened Mildenberger's result, showing that if (ϕ, ψ) is a morphism from $\mathfrak{r}_m$ to $\mathfrak{r}_n$ $(m < n)$ then ϕ is not Borel. This gives us the analogous strengthening in the case of morphisms from $\mathfrak{r}$ to $\mathfrak{r}_\sigma$.

$\boxed{\mathfrak{s}_\sigma}$ The *σ-splitting number* $\mathfrak{s}_\sigma$ is the least cardinality of a family $\mathcal{F}$ such that for any $A_1, A_2, \ldots \in [\omega]^\omega$ there exists $F \in \mathcal{F}$ which splits them all. In other words, it is defined by the triple $((2^\omega)^\omega, [\omega]^\omega, \mathsf{S}_\sigma)$ where $\langle A_n \rangle\ \mathsf{S}_\sigma\ B$ iff for all n, A_n is split by B. It is easy to see that $\mathfrak{s} \leq \mathfrak{s}_\sigma \leq \mathfrak{d}$, and both of these inequalities are witnessed by Borel morphisms. (For the first inequality a trivial morphism works, and for the second inequality the usual proof gives a morphism.) It is not known whether $\mathfrak{s} \geq \mathfrak{s}_\sigma$ is a true inequality, and so it is natural to ask for verification that there is no Borel proof of it.

Emulating the example of $\mathfrak{r}$, $\mathfrak{r}_n$ and $\mathfrak{r}_\sigma$, we can similarly define the cardinal $\mathfrak{s}_n$ to be the least cardinality of an *n-splitting family*, that is, a family $\mathcal{F}$ such that given any $A_1, \ldots, A_n$ there exists $B \in \mathcal{F}$ which splits them all. In other words, $\mathfrak{s}_n$ is defined by the triple $(([\omega]^\omega)^n, 2^\omega, \mathsf{S}_n)$, where $\langle A_1, \ldots, A_n \rangle\ \mathsf{S}_n\ B$ iff for all i, A_i is split by B. Thus $\mathfrak{s}$ and $\mathfrak{s}_1$ are exactly the same by definition. It is clear that $\mathfrak{s}_n = \mathfrak{s}_m$ and $\mathfrak{s}_n \geq_{\mathrm{BT}} \mathfrak{s}_m$ for $m < n$. We have the following analog of Mildenberger's result.

4.2. Theorem. *$\mathfrak{s}_n \not\geq_{\mathrm{BT}} \mathfrak{s}_{n+1}$ for all n.*

Proof. We prove the stronger fact that there is no Baire measurable function ψ which carries n-splitting families to $n+1$-splitting families. For this, we will first focus on the proof that there is no Baire measurable function ψ which carries (1-)splitting families to 2-splitting families. Afterwards, we will show how to modify the argument in the case when $n > 1$.

Suppose, towards a contradiction, that ψ carries splitting families to 2-splitting families and that ψ is continuous on a comeager set G. Let O_n be a decreasing family of dense open sets such that $\bigcap O_n \subset G$. We will construct a partition (I_k) of ω into finite intervals, a sequence of distinct integers a_k, and a family of sequences $\{\theta(s) \in 2^{I_k} : s \in 2^{I_{<k}}\}$ (where $I_{<k}$ denotes $\bigcup_{j<k} I_j$). In our construction, we shall ensure that for each $s \in 2^{I_{<k}}$ the following are satisfied:

(a) $N_{s \cup \theta(s)} \subset O_k$, and
(b) for all $c \in N_{s\cup\theta(s)} \cap G$ we have $\psi(c)(a_k) = 1$.

Here as usual the notation N_s means the basic open neighborhood of 2^ω corresponding to the sequence s. Borrowing some terminology from [Bla94], let us say that θ *predicts* $c \in 2^\omega$ at level k if $\theta(c{\restriction}_{I_{<k}}) = c{\restriction}_{I_k}$. Thus, condition (a) implies that if θ predicts c at infinitely many levels, then c lies in G.

Admitting the construction, we consider the families

$$\mathcal{S}_{\text{even}} = \{c \in 2^\omega : \theta \text{ predicts } c \text{ at all even levels}\},$$
$$\mathcal{S}_{\text{odd}} = \{c \in 2^\omega : \theta \text{ predicts } c \text{ at all odd levels}\}.$$

Then $\mathcal{S}_{\text{even}}$ can split any subset of $\bigcup_j I_{2j+1}$, and $\mathcal{S}_{\text{odd}}$ can split any subset of $\bigcup_j I_{2j}$. It follows easily that $\mathcal{S}_{\text{even}} \cup \mathcal{S}_{\text{odd}}$ is a splitting family. On the other hand, using (b), we deduce that $\psi(\mathcal{S}_{\text{even}})$ cannot split the set $\{a_{2j} : j \in \omega\}$, and $\psi(\mathcal{S}_{\text{odd}})$ cannot split the set $\{a_{2j+1} : j \in \omega\}$. We therefore conclude that while $\mathcal{S}_{\text{even}} \cup \mathcal{S}_{\text{odd}}$ is a splitting family, $\psi(\mathcal{S}_{\text{even}} \cup \mathcal{S}_{\text{odd}})$ is not a 2-splitting family, a contradiction.

We now turn to the construction. Suppose that I_j, a_j and $\theta(s)$ have been defined for $j < k$ and $s \in 2^{I_{<j}}$. For each $s \in 2^{I_{<k}}$, we first use the fact that O_k is dense open to find a $t(s)$ such that $N_{s\cup t(s)} \subset O_k$. This will imply that after the construction, (a) will be satisfied. Next, roughly speaking, we will use the continuity of ψ on G to find $\theta(s) \supset t(s)$ which decides certain values of $\psi(c)(m)$ for $c \in N_{s\cup\theta(s)} \cap G$. To satisfy (b), we just need to ensure that we can find some m where this value is always decided to be 1.

4.3. Claim. *For each $s \in 2^{I_{<k}}$, there are only finitely many m such that for all $c \in N_{s\cup t(s)} \cap G$ we have $\psi(c)(m) = 0$.*

Proof. Otherwise there would be an infinite subset $Z \subset \omega$ such that for all $c \in N_{s\cup t(s)} \cap G$ we have $\psi(c){\restriction}Z = 0$. But this implies that $\psi(N_{s\cup t(s)} \cap G)$ is not a splitting family, which is a contradiction because $N_{s\cup t(s)} \cap G$ is nonmeager and hence splitting, and ψ takes splitting families to splitting families. ∎

Now, we can choose $a_k > a_{k-1}$ so large that for all $s \in 2^{I_{<k}}$ there exists $c_s \in N_{s\cup t(s)} \cap G$ such that $\psi(c_s)(a_k) = 1$. Using the continuity of ψ, we can choose $\theta(s) \subset c_s$ extending $t(s)$ so that for all $c \in N_{s\cup\theta(s)} \cap G$ we have $\psi(c)(a_k) = 1$. Lengthening $\theta(s)$ if necessary, we can suppose that they all have the same domain, which we take for I_k. This completes the construction, and the proof in the case when $n = 1$.

Finally, we briefly show how to change the argument when $n > 1$. Rather than defining $\mathcal{S}_{\text{even}}$ and $\mathcal{S}_{\text{odd}}$, we simply define $\mathcal{S}_r$ to be the set of c such that θ predicts c at all levels which are congruent to r modulo n. Then it is not hard to verify that $\mathcal{S}_0 \cup \cdots \cup \mathcal{S}_{n-1}$ is an $n-1$-splitting family. But no element of $\psi(\mathcal{S}_0 \cup \cdots \cup \mathcal{S}_{n-1})$ can simultaneously split all of the sets $A_r = \{a_{nj+r} : j \in \omega\}$ for $r < n$. ∎

4.4. Corollary. $\mathfrak{s} \not\geq_{\mathrm{BT}} \mathfrak{s}_\sigma$.

Proof. This is just the same simple argument of Corollary 4.1. Suppose there were a Borel morphism from $\mathfrak{s}$ to $\mathfrak{s}_\sigma$. Then composing it with a (trivial) morphism from $\mathfrak{s}_\sigma$ to $\mathfrak{s}_2$ we would obtain one from $\mathfrak{s}$ to $\mathfrak{s}_2$, contradicting Theorem 4.2. ∎

The argument of Theorem 4.2 can also be used to separate (arbitrary) finite splitting from infinite splitting. That is, if we define the cardinal $\mathfrak{s}_{<\omega}$ by the relation $\mathsf{S}_{<\omega} = \bigcup_{n\in\omega} \mathsf{S}_n$, then we have the following corollary to the proof of Theorem 4.2.

4.5. Corollary. $\mathfrak{s}_{<\omega} \not\geq_{\mathrm{BT}} \mathfrak{s}_\sigma$.

Proof. Suppose towards a contradiction that ψ is a Borel map which carries finitely splitting families to infinitely splitting families, and construct I_k, θ, and a_k as before. Now, we simply put together all of the partial splitting families used in the proof of Theorem 4.2. Namely, let $\mathcal{S}_{n,r}$ denote the set of all c such that θ predicts c at all levels congruent to r modulo n. Then $\bigcup \mathcal{S}_{n,r}$ is clearly n-splitting for all n, but no element of $\psi(\bigcup \mathcal{S}_{n,r})$ can ever simultaneously split all of the sets $A_r = \{a_{nj+r} : j \in \omega\}$ for $n \in \omega$ and $r < n$. ∎

5. A sea of splitting numbers. In this last section we describe a family of Vojtáš triples of size continuum, each of which describes the usual splitting number as a cardinal invariant, but which are Borel Tukey inequivalent. Similar results have appeared before; it is known that there is a continuum of triples which are incomparable even up to ordinary Tukey equivalence. Our result gives a method of producing essentially arbitrary patterns in the $\leq_{\mathrm{BT}}$ ordering.

To begin, we will need to extend the methods of the previous section to produce antichains as well as chains. Building on our earlier notation, for $m \leq n$ we let $\mathfrak{s}_{n,m}$ denote the least cardinality of an *n, m-splitting family*: that is, an $\mathcal{F} \subset 2^\omega$ such that for any sequence $A_1, \ldots, A_n$ of infinite subsets of ω there exists $B \in \mathcal{F}$ which splits at least m of them. Thus $\mathfrak{s}_{n,m}$ is defined by a triple $(([\omega]^\omega)^n, 2^\omega, \mathsf{S}_{n,m})$ where $\mathsf{S}_{n,m}$ denotes the relation "at least m of which are split by". Again, all of the cardinals $\mathfrak{s}_{n,m}$ are equal to $\mathfrak{s}$, but it is not immediately clear which pairs are related by a Borel Tukey map. The following result computes precisely when this is the case.

5.1. Proposition. *Let $m \leq n$ and $m' \leq n'$.*

- *If $m < m'$ then there is no Borel Tukey morphism from $\mathfrak{s}_{n,m}$ to $\mathfrak{s}_{n',m'}$.*
- *If $m \geq m'$ then there is a Borel Tukey morphism from $\mathfrak{s}_{n,m}$ to $\mathfrak{s}_{n',m'}$ if and only if*

$$\lfloor n/n' \rfloor (m'-1) + \min(r, m'-1) < m, \tag{5.2}$$

where r denotes the remainder upon dividing n by n'.

The combinatorial condition in (5.2) means: if you spread n balls evenly over n' ordered buckets (with the remainder spread over the left-most buckets), then among the first $m'-1$ buckets there are fewer than m balls. For a diagram depicting this scenario see Figure 3.

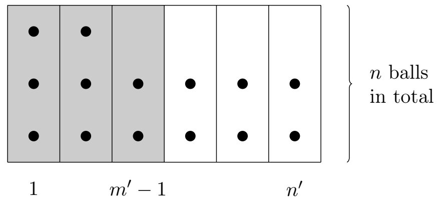

Fig. 3. Deciding when (5.2) holds. In this particular example, $n=14$, $n'=6$, and $m'-1=3$. The number of balls lying in the shaded region corresponds to the left-hand side of (5.2).

Proof of Proposition 5.1. We begin with the first claim. As in the proof of Theorem 4.2, we assume towards a contradiction that there is a Borel map ψ carrying (n,m)-splitting families to (n',m')-splitting families. We then carry out the construction of θ, I_k, and a_k satisfying (a) and (b) from the proof of Theorem 4.2. This done, we again let $\mathcal{S}_{n',r}$ denote the family consisting of those $c \in 2^\omega$ such that θ predicts c at all levels which are congruent to r modulo n'. We then let

$$\mathcal{S} = \bigcup_{r_1<\cdots<r_{n'-m}} \bigcap_i \mathcal{S}_{n',r_i},$$

that is, the set of all c such that θ predicts c on at least $n'-m$ congruence classes of levels. Then it is not hard to verify that $\mathcal{S}$ is an (n,m)-splitting family—in fact, it is an m-splitting family. But by the construction, no element of $\psi(\mathcal{S})$ can split $m+1$ of the sets $A_r = \{a_{n'j+r} : j \in \omega\}$ for $r < n'$.

For the second claim, first suppose that (5.2) holds. We shall argue that every (n,m)-splitting family is in fact (n',m')-splitting, and hence the identity morphism will suffice. Indeed, suppose that a family $\mathcal{S}$ is (n,m)-splitting and let $B_1,\dots,B_{n'}$ be infinite subsets of ω. Partition each B_i into either $\lfloor n/n' \rfloor$ or $\lfloor n/n' \rfloor + 1$ infinite subsets C_i^j in such a way that there are n many C_i^j in total. Since $\mathcal{S}$ is (n,m)-splitting, there exists $c \in \mathcal{S}$ which splits at least m of these subsets. It now follows from (5.2) that c splits at least m' of the original n' sets. (To visualize this, refer to Figure 4.) Thus $\mathcal{S}$ is (n',m')-splitting.

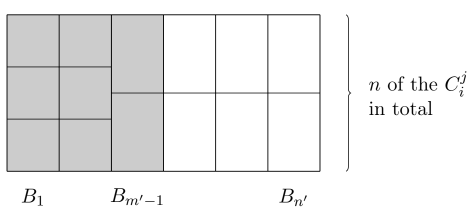

Fig. 4. If the shaded region contains fewer than m regions, then any set which splits at least m regions must also split at least m' of the columns B_i.

Now suppose that (5.2) fails. Note that in this case we have $n' < n$. Once more we take the contradiction approach. Suppose there is a Borel map ψ which carries (n, m)-splitting families to (n', m')-splitting families, and construct θ, I_k, and a_k. We again consider the families $\mathcal{S}_{n,r}$ as above, and for $x \subset n$ we let

$$\mathcal{S}_x = \bigcap_{r \notin x} \mathcal{S}_{n,r} \quad \text{and} \quad \mathcal{S} = \bigcup \{\mathcal{S}_x : |x \cap n'| \le m' - 1\}.$$

Then $\mathcal{S}$ is (n, m)-splitting. Indeed, given $B_1, \dots, B_n$, for each i there exists $r_i < n$ such that B_i has infinite intersection with $\bigcup_j I_{nj+r_i}$. Since (5.2) fails, some m many $r_{i_1}, \dots, r_{i_m}$ must lie in some x with $|x \cap n'| \le m' - 1$ (see Figure 3). Then some $c \in \mathcal{S}_x$ splits $B_{i_1}, \dots, B_{i_m}$.

On the other hand, $\psi(\mathcal{S})$ is not (n', m')-splitting, since by the construction no element of $\psi(\mathcal{S})$ can split m' of the sets $A_r = \{a_{nj+r} : j \in \omega\}$ for $r < n'$. This completes the proof of the second claim of Proposition 5.1. ∎

It is not hard to see from equation (5.2) that if n/m is much larger than n'/m' then there will not be a morphism from $\mathfrak{s}_{n,m}$ to $\mathfrak{s}_{n',m'}$. This will allow us to show that there are infinite antichains among the $\mathfrak{s}_{n,m}$ in the Borel Tukey order.

5.3. Theorem. *The triples which define $\mathfrak{s}_{2^m,m}$, where m varies over the natural numbers ≥ 3, form an antichain in the $\le_{\mathrm{BT}}$ ordering.*

Proof. If $m < m'$ then there is no Borel Tukey map from $\mathfrak{s}_{2^m,m}$ to $\mathfrak{s}_{2^{m'},m'}$ by the first claim in Proposition 5.1. If $m > m'$, we must evaluate whether (5.2) holds with $n = 2^m$ and $n' = 2^{m'}$. It is not difficult to see that this equation fails, since in this case $r = 0$ and

$$\begin{aligned} \frac{2^m}{2^{m'}}(m'-1) = 2^{m-m'}(m'-1) &\ge ((m-m')+1)(m'-1) \\ &\ge ((m-m')+1) + (m'-1) = m. \end{aligned}$$

(For the second inequality we note that $AB \ge A + B$ for $A, B \ge 2$.) Thus it follows from the second claim in Proposition 5.1 that there is again no Borel Tukey map from $\mathfrak{s}_{2^m,m}$ to $\mathfrak{s}_{2^{m'},m'}$. ∎

Finally, we can combine members of this countable antichain to produce more complex patterns.

5.4. Corollary. *The superset ordering on $\mathcal{P}(\omega)$ embeds into the $\le_{\mathrm{BT}}$ ordering on triples. In fact, this ordering embeds into the $\le_{\mathrm{BT}}$ ordering on triples that define $\mathfrak{s}$.*

Proof. Let us work with $\mathcal{P}(\omega \setminus 3)$ in place of $\mathcal{P}(\omega)$. For $X \subset \omega \setminus 3$, we say that a family $\mathcal{S}$ is *X-splitting* if it is $(2^m, m)$-splitting for all $m \in X$. Clearly, the "cardinals" $\mathfrak{s}_X$ defined by the corresponding relation are all equal to $\mathfrak{s}$.

Moreover, if $X \supset Y$ then X-splitting implies Y-splitting and so there is a trivial morphism from $\mathfrak{s}_X$ to $\mathfrak{s}_Y$.

Conversely, assume that there exists $m_0 \in Y \setminus X$, and suppose towards a contradiction that there exists a Borel map ψ which carries X-splitting families to Y-splitting families. By Theorem 5.3 and Proposition 5.1, for each $m \in X$ there exists a $(2^m, m)$-splitting family $\mathcal{S}^{(m)}$ and a sequence of sets $A_1^{(m)}, \ldots, A_{2^{m_0}}^{(m)}$ such that no element of $\psi(\mathcal{S}^{(m)})$ splits m_0 of the $A_i^{(m)}$.

Now, if $\mathcal{S} = \bigcup_{m\in X} S^{(m)}$, then clearly $\mathcal{S}$ is X-splitting. We claim that $\psi(\mathcal{S})$ is not Y-splitting, in fact that it is not even $(2^{m_0}, m_0)$-splitting. To see this, note that the proof of Proposition 5.1 implies that for each i, $A_i^{(m)}$ can be taken to be of the form $\{a_{nj+i} : j \in \omega\}$ for some n. That is, the indices are taken from the ith congruence class modulo n. It follows easily that there exists a single set A_i such that $A_i \subset^* A_i^{(m)}$ for every $m \in X$. Now, no element of $\psi(\mathcal{S}^{(m)})$ can split m_0 of the A_i, since that would imply that it splits m_0 of the $A_i^{(m)}$. Hence $\psi(\mathcal{S})$ is not $(2^{m_0}, m_0)$-splitting, as desired. ∎

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