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When $C_p(X)$ is domain representable

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Abstract. Let M be a metrizable group. Let G be a dense subgroup of M^X. We prove that if G is domain representable, then $G = M^X$. The following corollaries answer open questions. If X is completely regular and $C_p(X)$ is domain representable, then X is discrete. If X is zero-dimensional, T_2, and $C_p(X, \mathbb{D})$ is subcompact, then X is discrete.

1. Introduction. Let X be a completely regular space and $\mathbb{R}$ the topological group of real numbers. Let $C_p(X)$ denote the group of continuous functions from X to $\mathbb{R}$ equipped with the topology of pointwise convergence. The space $C_p(X)$ is usually not complete. One can make "usually" precise when one makes the notion "complete" precise. For example, Lutzer and McCoy showed [10, Theorem 8.6] that the following are equivalent: (a) $C_p(X)$ is Čech-complete, (b) X is countable and discrete, and (c) $C_p(X)$ is completely metrizable. They also showed [10, Theorem 8.4 and Remark 8.5] that the following are equivalent when X is a normal space: (a) $C_p(X)$ is pseudo-complete, (b) $C_p(X)$ is weakly α-favorable, and (c) every countable subset of X is closed and discrete. Almost thirty years later, Tkachuk [11] showed that X is discrete iff $C_p(X)$ is subcompact. Inspired by Tkachuk's results and methods, Bennett and Lutzer [2, Main Theorem] showed that the following are equivalent for normal spaces X: (a) $C_p(X)$ is Scott-domain representable, (b) $C_p(X)$ is domain representable, and (c) X is discrete.

For any space M and set X, M^X denotes the space of all functions from X to M with the usual product topology; further notation and terminology is established in Section 2. In Section 3, we briefly discuss completeness properties in general, and then focus on subcompactness and domain representability. In Section 4, we prove our main theorem: If M is a metrizable group and G is a dense, domain representable subgroup of M^X, then

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$G = M^X$. Corollaries to our main theorem continue the line of research of the previous paragraph. In particular, a space X is discrete iff $C_p(X)$ is domain representable, which answers a question of Bennett and Lutzer ([2, Question 5.1] and [6, Question 6.2]); and a zero-dimensional, T_2 space X is discrete iff $C_p(X, \mathbb{D})$ is subcompact, which answers a question of Lutzer, van Mill, and Tkachuk ([11, Question 5.6]). Here $\mathbb{D}$ is the doubleton $\{0, 1\}$ with the discrete topology. In Section 5, we show how to adapt our methods to the case where the range M is the unit interval $\mathbb{I}$. Section 6 contains a remark about measurable cardinals.

2. Notation and terminology. All topological spaces are assumed to be completely regular. When X is a topological space, we let $\tau(X)$ denote the topology on the set X, and we let $\tau^*(X)$ denote the family of nonempty elements of $\tau(X)$. When discussing open filter bases and completeness properties, we often say "$\mathcal{B} \subseteq \tau^*(X)$" is a base for X instead of "$\mathcal{B} \cup \{\emptyset\}$ is a base for X".

When $(M, +)$ is a group, possibly not Abelian, and X is a set, then the product M^X is a group with pointwise operations, that is, $(g + h)(x) := g(x) + h(x)$. When X and M are topological spaces, we will denote the set of continuous functions from X to M by $C(X, M)$. If M is a group, then $C(X, M)$ is a group, too. We write $C_p(X, M)$ when we consider $C(X, M)$ as a subspace of the usual, finite support, product topology on M^X. This is the topology of pointwise convergence on $C(X, M)$. If M is a topological group, then $C_p(X, M)$ is a topological group, too. In particular, $(C(X, \mathbb{R}), +)$ is a subgroup of $(\mathbb{R}^X, +)$. We write $C(X)$ for $C(X, \mathbb{R})$ and $C_p(X)$ for $C_p(X, \mathbb{R})$.

Our main result was proved originally for $C_p(X)$, but it holds whenever the range of the continuous functions is a metrizable group. We use $(M, +)$ to denote the range. *We do not assume that* $(M, +)$ *is Abelian nor that the metric is translation invariant.* Some results hold when M is a metrizable median algebra—for example, a metrizable linearly ordered space. See Section 5 for definitions.

If κ is an infinite cardinal, we let $[X]^{<\kappa}$ denote $\{Y \subseteq X : |Y| < \kappa\}$, the family of subsets of X of cardinality less than κ. Analogously, we set $[X]^\kappa = \{Y \subseteq X : |Y| = \kappa\}$.

DEFINITION 2.1. Let κ be an infinite cardinal and let $G \subseteq M^X$. We say G *covers all* $<\kappa$*-faces of* M^X if for every $Y \in [X]^{<\kappa}$, every function from Y to M extends to an element of G. When $\kappa = \omega$, we say that a subset G of a product M^X *covers all finite faces of* M^X. Similarly, we say G *covers all countable faces of* M^X when $\kappa = \omega_1$.

For any topology on M, if a subset $G \subseteq M^X$ covers all finite faces of M^X then G is dense in M^X, and if M carries the discrete topology then

G covers all finite faces of M^X if and only if G is dense in M^X. Since all spaces considered here are completely regular, we have

LEMMA 2.2. *$C_p(X)$ covers all finite faces of $\mathbb{R}^X$. If X is zero-dimensional and T_2, then $C_p(X, \mathbb{D})$ covers all finite faces of $\mathbb{D}^X$.*

We say that a subset Y is *C-embedded* in a space X if every element of $C(Y)$ extends to an element of $C(X)$.

LEMMA 2.3. *Let M be a space with more than one point. If $C_p(X, M)$ covers all $<\kappa$-faces of M^X, then every $Y \in [X]^{<\kappa}$ is closed and discrete in X. If $C_p(X, M)$ covers all $<|X|^+$-faces of M^X, then X is discrete. $C_p(X)$ covers all $<\kappa$-faces of $\mathbb{R}^X$ iff every $Y \in [X]^{<\kappa}$ is closed, discrete, and C-embedded in X.*

Proof. Choose two points $a, b \in M$. If $Y \subseteq X$ contains a limit point p of itself and $|Y| < \kappa$, then the function $f : Y \to M$ given by $f(y) = a$ if $y \in Y \setminus \{p\}$ and $f(p) = b$ cannot be extended to an element of $C_p(X, M)$. ■

The hypothesis that every small subset is closed discrete does not imply that every small subset is C-embedded. Tkachuk informed us that a slight modification of a construction of Reznichenko [13] provides, for every infinite cardinal κ, a space X_κ with the following properties: (a) $X_\kappa \subset \mathbb{D}^{2^\kappa}$ is pseudocompact and $|X_\kappa| = 2^\kappa$, (b) every $Y \in [X_\kappa]^\kappa$ is closed discrete in X_κ, and (c) X_κ covers all κ-faces of $\mathbb{D}^{2^\kappa}$. Because X_κ is pseudocompact, no infinite subset of X_κ is C-embedded.

We establish notation for a base of the product space M^X.

DEFINITION 2.4. When (M, d) is a metric space and X is an index set, we will denote the basic open subsets of the product space M^X as

$$O(g, S, \epsilon) = \{f \in M^X : d(f(x), g(x)) < \epsilon \text{ for all } x \in S\}$$

where $g \in M^X$, $S \in [X]^{<\omega}$ and $\epsilon > 0$. If u is a function from a subset $Y \subset X$ to M, then for $S \in [Y]^{<\omega}$ and $\epsilon > 0$, we write $O(u, S, \epsilon)$ for the set $O(g, S, \epsilon)$ where $g \in M^X$ is any function with $g|_S = u|_S$.

3. Some completeness properties. The study of completeness properties strives to generalize completeness from the class of metrizable spaces or from the class of locally compact spaces to more general topological spaces. One strand of properties starts with complete metrizability and proceeds through pseudocompleteness and α-favorability towards the Baire Category Theorem. These properties assert that certain countable filter bases of open sets have nonempty intersection. Another strand starts with compactness and leads to subcompactness and domain representability. These properties assert that certain filter bases, without cardinality restriction, have nonempty intersection. We can define new properties by adding cardinality restrictions—for example, countable compactness and countable subcompactness. In this section we define the notion of subcompactness and introduce a simplified definition of domain representability.

See [6] for definitions of the other properties, history of completeness properties, open questions, and much more.

Definition 3.1. An *upward directed set* is a nonempty set P together with a reflexive and transitive binary relation $\ll$ or $\prec$ with the additional property that every pair of elements has an upper bound. *Downward directed sets* are defined analogously. Let us define $\prec_{\mathrm{cl}}$ on $\tau^*(X)$ via $V \prec_{\mathrm{cl}} U$ iff $\mathrm{cl}\, V \subseteq U$. An *open filter base* on a space X is a nonempty subset $\mathcal{F}$ of $\tau^*(X)$ such that $(\mathcal{F}, \subseteq)$ is downward directed. A *regular open filter base* on a space X is a nonempty subset $\mathcal{F}$ of $\tau^*(X)$ such that $(\mathcal{F}, \prec_{\mathrm{cl}})$ is downward directed. In this example, $U \prec_{\mathrm{cl}} U$ iff U is clopen.

Definition 3.2. A space X is called *subcompact* if it has a base $\mathcal{B} \subseteq \tau^*(X)$ with the property that $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F} \subseteq \mathcal{B}$ is a regular open filter base. We say that a space X is *κ-subcompact* if it has a base $\mathcal{B} \subseteq \tau^*(X)$ with the property that $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F} \subseteq \mathcal{B}$ is a regular open filter base and $|\mathcal{F}| < \kappa$. In this context we say that $\mathcal{B}$ is a *κ-subcompact* base for X.

Observe that if M is a complete metric space and G covers all $<\kappa$-faces of M^X, then $\{O(g, S, \epsilon) \cap G : g \in M^X,\ S \in [X]^{<\omega},\ \epsilon > 0\}$ is a κ-subcompact base for G. In Proposition 4.6 we will show a converse. If G is a dense subgroup of M^X and is κ-subcompact, then G covers all $<\kappa$-faces of M^X.

Another notion of completeness begins with a dcpo, i.e., a directed-complete poset $(P, \sqsubseteq)$, and uses $\sqsubseteq$ to define a new relation $\ll$ on P. One writes that $a \ll b$ (often spoken, "a approximates b") if for each directed set $D \subseteq P$ having $b \sqsubseteq \sup(D)$, some $d \in D$ has $a \sqsubseteq d$. Note that $\ll$ is transitive and antisymmetric. For each $a \in P$ define $\twoheaddownarrow(a) = \{b \in P : b \ll a\}$. The poset P is said to be *continuous* if $\twoheaddownarrow(a)$ is directed and has $a = \sup(\twoheaddownarrow(a))$ for each $a \in P$. Given that $(P, \sqsubseteq)$ is a continuous dcpo, we let $\twoheaduparrow(a) = \{c \in P : a \ll c\}$ for each $a \in P$. Then the collection $\{\twoheaduparrow(a) : a \in P\}$ is a base for what is called the *Scott topology* on P, and the collection $\{\twoheaduparrow(a) \cap \max(P) : a \in P\}$ is a base for the subspace topology on the set $\max(P)$ consisting of all maximal elements of P. When a space X is homeomorphic to the space $\max(P)$ for a continuous dcpo, Martin [12] writes that X has a model, while Bennett and Lutzer [6] write that X is domain representable.

We are able to prove our theorems with what seems, at first, to be a weaker topological property, namely:

DEFINITION 3.3. We say that a triple $(Q, \ll, B)$ *represents* X provided

(1) $B : Q \to \tau^*(X)$ and $\{B(q) : q \in Q\}$ is a base for X,
(2) $\ll$ is a transitive, antisymmetric relation on Q,
(3) for all p, q in Q, $p \ll q$ implies $B(q) \subseteq B(p)$,
(4) for all $x \in X$, $\{q \in Q : x \in B(q)\}$ is upward directed,
(5) if $D \subseteq Q$ and $(D, \ll)$ is upward directed, then $\bigcap\{B(p) : p \in D\} \neq \emptyset$.

We can add a cardinal parameter. For κ an uncountable cardinal, we say that $(Q, \ll, B)$ *κ-represents* X if (1)–(4) and $(5)_\kappa$ hold, where

$(5)_\kappa$ if $D \in [Q]^{<\kappa}$ and $(D, \ll)$ is upward directed, then $\bigcap\{B(p) : p \in D\} \neq \emptyset$.

Next, we discuss the implications among subcompactness, domain representabilty, and the property of Definition 3.3.

LEMMA 3.4. *If X is subcompact, then there is a triple $(Q, \ll, B)$ which represents X.*

Proof. Let $\mathcal{B}$ be a subcompact base for X. Define $Q = \mathcal{B}$, $\ll = \prec_{\mathrm{cl}}$, and $B = \mathrm{id}$, where $\mathrm{id}(B) = B$ for all $B \in \mathcal{B}$. ∎

If the converse of Lemma 3.4 were true, then that converse, together with Lemma 3.5 and Tkachuk's theorem, would give a proof of Theorem 4.1. However, the converse of Lemma 3.4 is false [9].

LEMMA 3.5. *If X is domain representable, then there is a triple $(Q, \ll, B)$ which represents X.*

Proof. Let X be homeomorphic to the subspace $\max(P)$ for a continuous dcpo $(P, \sqsubseteq)$ with defined relation $\ll^P$. Define $Q = \{p \in P : \Uparrow(p) \cap \max(P) \neq \emptyset\}$, $\ll = \ll^P|_Q$, and $B(q) = \Uparrow(q) \cap \max(P)$ for all $q \in Q$. ∎

The converse of Lemma 3.5 is true. Suppose that $(Q, \ll, B)$ represents X. Then the ideal completion of $(Q, \ll)$, denoted $\mathsf{Idl}(Q)$, is a continuous dcpo (see [1, Proposition 2.2.22]), and X is homeomorphic to $\max(\mathsf{Idl}(Q))$ (see [9]). This method is used in [4] to show that subcompactness implies domain representability.

4. Main theorem.

This section is devoted to an inductive proof of a theorem that extends results of Bennett, Lutzer, van Mill, and Tkachuk, and answers questions posed in [2], [6], and [11].

THEOREM 4.1. *Let M be a metrizable group, let G be a dense subgroup of M^X, and let κ be an uncountable cardinal. If there is a triple $(Q, \ll, B)$ which κ-represents X, then G covers all $<\kappa$-faces of M^X. If there is a triple $(Q, \ll, B)$ which represents X, then $G = M^X$. In particular, if G is domain representable, then $G = M^X$.*

Proof. We proceed by induction on κ. For the initial stage, $\kappa = \omega_1$, applying Lemma 4.4 below takes us from our hypothesis that G is dense in M^X to the conclusion that G covers all finite faces of M^X. Then Lemma 4.5 finishes the initial step by showing G in fact covers all countable faces of M^X. The successor stage, from μ to $\mu^+ = \kappa$, is Proposition 4.6. Finally, if κ is a limit cardinal, the stage is trivial because a set of cardinality less than κ has cardinality less than μ for some $\mu < \kappa$. ∎

COROLLARY 4.2. *If X is completely regular and $C_p(X)$ is domain representable, then $C_p(X) = \mathbb{R}^X$. Hence X is discrete. If X is zero-dimensional, T_2, and $C_p(X, \mathbb{D})$ is domain representable, then $C_p(X, \mathbb{D}) = \mathbb{D}^X$. Hence X is discrete.*

Proof. Use Lemma 2.3. ∎

The next theorem and its proof are well known and due to Banach (also when $\mathbb{R}$ is replaced by a completely metrizable topological group). The reader may choose to use the alternative proof which follows as a warm-up since it presents the proof of our main theorem without filter bases, product neighborhoods, and new completeness properties.

THEOREM 4.3. *Let G be a dense subgroup of $\mathbb{R}$. If G has a complete metric, then $G = \mathbb{R}$.*

Proof. It is well known that G dense and completely metrizable implies that G is a dense G_δ. Let $f \in \mathbb{R}$ be arbitrary. Then $G' = \{f - h : h \in G\}$ and $G \cap G'$ are also dense G_δ's. By the Baire Category Theorem, there is an element $f - h = g$ in $G' \cap G$. Then $f = g + h \in G$ because G is a subgroup of $\mathbb{R}$. ∎

Let X be a (possibly uncountable) space. We want to show that the only complete (in some suitable sense) dense subgroup of $\mathbb{R}^X$ is in fact $\mathbb{R}^X$ itself. We cannot consider complete metrizabilty and hope to use Banach's proof because $\mathbb{R}^X$ is (completely) metrizable only when X is countable. Even Čech-completeness is too restrictive for our purposes in light of the theorem of Lutzer and McCoy [10] that $C_p(X)$ is a Čech-complete space if and only if X is countable and discrete. To consider uncountable X, we need a more general completeness property, like subcompactness or domain representability. Then, however, the quick proof above cannot be used, because a space can have disjoint dense subcompact subspaces. In particular, the top arrow and the bottom arrow are disjoint subcompact dense subspaces of the double arrow space. (We thank Tkachuk and Lutzer for independently showing us this example.)

The following proof of Theorem 4.3 is messy, but we can apply this method to $\mathbb{R}^X$ with the hypothesis there is a triple $(Q, \ll, B)$ which represents G.

Alternative proof of Theorem 4.3. Let d be the usual metric on $\mathbb{R}$ and let ρ be a complete metric on G. Let $f \in \mathbb{R}$ be arbitrary. For $n \in \omega$, let W_n be the d-ball of radius 2^{-n} centered at f. By induction on $n \in \omega$, we will construct $\langle g_n + h_n : n \in \omega \rangle$, a sequence of points in G converging to f.

Here is the first step of our induction. Let $g_0 \in G$ and U_0 open in $\mathbb{R}$ satisfy ρ-diam$(U_0 \cap G) \le 1$ and $g_0 \in U_0$. Then

$$-g_0 + f \subseteq (-U_0 + f) \cap (-g_0 + W_0).$$

Because $(-U_0 + f) \cap (-g_0 + W_0)$ is open and G is dense, we may choose $h_0 \in G$ and V_0 open in $\mathbb{R}$ satisfying ρ- diam$(V_0 \cap G) \le 1$, and

$$h_0 \in V_0 \subseteq (-U_0 + f) \cap (-g_0 + W_0).$$

Because $g_0 + h_0$ is in W_0, we have $d(g_0 + h_0, f) \le 1$. Also we observe that $f - V_0 \subseteq f - (-U_0 + f) = U_0$.

Suppose the $(n-1)$th step of the induction is complete. Because $f - V_{n-1}$ is open and G is dense, we may choose $g_n \in G$ and U_n open in $\mathbb{R}$ satisfying ρ-diam$(U_n \cap G) \le 2^{-n}$, $\mathrm{cl}_G(U_n \cap G) \subseteq U_{n-1}$ and $g_n \in U_n \subseteq f - V_{n-1}$. Hence

$$-g_n + f \subseteq (-U_n + f) \cap (-g_n + W_n).$$

Let $h_n \in G$ and V_n open in $\mathbb{R}$ satisfy ρ-diam$(V_n \cap G) \le 2^{-n}$, $\mathrm{cl}_G(V_n \cap G) \subseteq V_{n-1}$, and

$$h_n \in V_n \subseteq (-U_n + f) \cap (-g_n + W_n).$$

Because $g_n + h_n$ and f are in W_n, we have $d(g_n + h_n, f) \le 2^{-n}$. Also we observe that $f - V_n \subseteq f - (-U_n + f) = U_n$.

After ω steps, because ρ is complete, we know that there is a unique point g in the intersection $\bigcap\{\mathrm{cl}_G(U_n \cap G) : n \in \omega\}$ and that the sequence $\langle g_n : n \in \omega \rangle$ converges to g. Similarly, the sequence $\langle h_n : n \in \omega \rangle$ converges to h, the unique point in $\bigcap\{\mathrm{cl}_G(V_n \cap G) : n \in \omega\}$. Because the group operation is continuous, $\langle g_n + h_n : n \in \omega \rangle$ converges to $g + h$.

For each n, we noted that $d(g_n + h_n, f) \le 2^{-n}$; hence $\langle g_n + h_n : n \in \omega \rangle$ also converges to f. We conclude that $f = g + h$, as desired. ∎

The next lemma follows the pattern of the alternative proof of Theorem 4.3. Rather than specifically the real line, it applies to any metrizable topological group $(M, +)$, whose group operation is not necessarily Abelian and whose metric is not necessarily translation invariant. The ambient space is M^X, so we will use the basic open sets $O(g, S, \epsilon)$ of Definition 2.4. Moreover, instead of assuming that G is completely metrizable, we assume that there is a triple $(Q, \ll, B)$ which represents G.

LEMMA 4.4. *Let G be a dense subgroup of M^X. If there is a triple $(Q, \ll, B)$ which ω_1-represents G, then G covers all finite faces of M^X.*

Proof. Let $(Q, \ll, B)$ ω_1-represent G. Let $Y \in [X]^{<\omega}$ and $w : Y \to M$ be arbitrary. Let $f \in M^X$ extend w. For $n \in \omega$, set $W_n = O(f, Y, 2^{-n})$. By induction on $n \in \omega$, we construct $\langle g_n + h_n : n \in \omega\rangle$, a sequence of points in G such that $\langle (g_n + h_n)(y) : n \in \omega\rangle$ converges to $f(y)$ for all $y \in Y$.

Here is the first step of our induction. Let $g_0 \in G$ be arbitrary. Choose $p_0 \in Q$ and a basic open set $U_0 = O(g_0, S_0, \epsilon_0)$, where $Y \subseteq S_0 \in [X]^{<\omega}$ and $\epsilon_0 < 1$, satisfying

$$g_0 \in U_0 \cap G \subseteq B(p_0).$$

Because $(-U_0 + f) \cap (-g_0 + W_0)$ is open and G is dense, we may choose $h_0 \in G$, $q_0 \in Q$, and a basic open set $V_0 = O(h_0, T_0, \eta_0)$, where $S_0 \subseteq T_0$ and $\eta_0 < 1$, satisfying

$$h_0 \in V_0 \cap G \subseteq B(q_0) \subseteq (-U_0 + f) \cap (-g_0 + W_0).$$

Because $g_0 + h_0$ is in W_0, we have $d((g_0 + h_0)(y), f(y)) < 1$ for all $y \in Y$. Also we observe that $(f - V_0) \subseteq f - (-U_0 + f) = U_0$.

Suppose the $(n-1)$th step of the induction is complete. Because $f - V_{n-1}$ is open and G is dense, we may choose $g_n \in G$, $p_n \in Q$, and a basic open set $U_n = O(g_n, S_n, \epsilon_n)$, where $T_{n-1} \subseteq S_n$ and $\epsilon_n < 2^{-n}$, satisfying

$$g_n \in U_n \cap G \subseteq B(p_n) \subseteq f - V_{n-1} \subseteq U_{n-1}.$$

Since $g_n \in U_{n-1} \cap G \subseteq B(p_{n-1})$, we see that $g_n \in B(p_{n-1}) \cap B(p_n)$. Replacing p_n with the r guaranteed by Definition 3.3(4), we assume that $p_{n-1} \ll p_n$. Because $(-U_n + f) \cap (-g_n + W_n)$ is open and G is dense, we may choose $h_n \in G$, $q_n \in Q$, and a basic open set $V_n = O(h_n, T_n, \eta_n)$, where $S_{n-1} \subseteq T_n$ and $\eta_n < 2^{-n}$, satisfying

$$h_n \in V_n \cap G \subseteq B(q_n) \subseteq (-U_n + f) \cap (-g_n + W_n).$$

Because $g_n + h_n$ and f are in W_n, we have $d((g_n + h_n)(y), f(y)) \leq 2^{-n}$ for all $y \in Y$. Also we observe that $f - V_n \subseteq f - (-U_n + f) = U_n$. By the same reasoning used with the g_n, we may assume that $q_{n-1} \ll q_n$.

Suppose that the induction is complete. Set $S = \bigcup\{S_n : n \in \omega\}$. Note that $Y \subseteq S = \bigcup\{T_n : n \in \omega\}$. Because $\{p_n : n \in \omega\}$ is $\ll$-directed, by Definition 3.3(5), there is $g \in \bigcap\{B(p_n) : n \in \omega\}$. Observe that for all n and all $m > n$,

$$g, g_m \in U_n = O(g_n, S_n, \epsilon_n).$$

Hence $\langle g_n(x) : n \in \omega\rangle$ converges to $g(x)$ for all $x \in S$. Similarly, there is $h \in \bigcap\{B(q_n) : n \in \omega\}$ and $\langle h_n(x) : n \in \omega\rangle$ converges to $h(x)$ for all $x \in S$. Because $+$ is continuous, $\langle g_n(x) + h_n(x) : n \in \omega\rangle$ converges to $(g + h)(x)$ for all $x \in S$.

From $d((g_n + h_n)(y), f(y)) < 2^{-n}$ for all $n \in \omega$ and for all $y \in Y$, we may conclude that $(g + h)(y) = f(y)$ for all $y \in Y$. We have found $g + h \in G$ extending w as desired. ■

The next proof follows the same pattern with a few differences. Because $Y = \{y_n : n \in \omega\} \in [X]^\omega$ is infinite, we cannot require $Y \subseteq S_0$. Instead, in the induction we require $y_n \in S_n$. For each $n \in \omega$, either we define $h_\ell(y_n) = -g_\ell(y_n) + f(y_n)$ for some $\ell \leq n$, or we define $g_{\ell+1}(y_n) = f(y_n) - h_\ell(y_n)$ for some $\ell \leq n$. As a result, the sequences converge by being eventually constant.

LEMMA 4.5. *Let G be a subgroup of M^X which covers all finite faces of M^X. If there is a triple $(Q, \ll, B)$ which ω_1-represents G, then G covers all countable faces of M^X.*

Proof. Let $(Q, \ll, B)$ ω_1-represent G. Let $Y = \{y_n : n \in \omega\} \in [X]^\omega$ and $w : Y \to M$ be arbitrary. Let $f \in M^X$ extend w. By induction on $n \in \omega$, we construct $\langle g_n : n \in \omega\rangle$ and $\langle h_n : n \in \omega\rangle$, sequences of points in G such that $\langle g_n(y) + h_n(y) : n \in \omega\rangle$ converges to $f(y)$ for all $y \in Y$.

Here is the first step of our induction. Let $g_0 \in G$ be arbitrary. Choose $p_0 \in P$ and a basic open set $U_0 = O(g_0, S_0, \epsilon_0)$, where $y_0 \in S_0 \in [X]^{<\omega}$ and $\epsilon_0 < 1$, satisfying

$$g_0 \in U_0 \cap G \subseteq B(p_0).$$

Because G covers all finite faces of M^X, we may choose $h_0 \in G$, $q_0 \in P$, and a basic open set $V_0 = O(h_0, T_0, \eta_0)$, where $S_0 \subseteq T_0$ and $\eta_0 < 1$, satisfying $h_0(x) = (-g_0 + f)(x)$ for all $x \in S_0$ and

$$h_0 \in V_0 \cap G \subseteq B(q_0) \subseteq -U_0 + f.$$

Suppose the $(n-1)$th step of the induction is complete. Because G covers all finite faces of M^X, we may choose $g_n \in G$, $p_n \in P$, and a basic open set $U_n = O(g_n, S_n, \epsilon_n)$, where $\{y_n\} \cup T_{n-1} \subseteq S_n$ and $\epsilon_n < 2^{-n}$, satisfying $g_n(x) = (f - h_{n-1})(x)$ for all $x \in T_{n-1}$ and

$$g_n \in U_n \cap G \subseteq B(p_n) \subseteq f - V_{n-1} \subseteq f - (-U_{n-1} + f) = U_{n-1}.$$

Observe that $S_{n-1} \subseteq T_{n-1} \subseteq S_n$. Hence for all $x \in S_n$, we have

$$g_n(x) = f(x) - h_n(x) = f(x) - (-g_{n-1}(x) + f(x)) = g_{n-1}(x).$$

Since $g_n \in U_{n-1} \cap G \subseteq B(p_{n-1})$, we see that $g_n \in B(p_{n-1}) \cap B(p_n)$. Replacing p_n with the r guaranteed by Definition 3.3(4), we assume that $p_{n-1} \ll p_n$.

Because G covers all finite faces of M^X, we may choose $h_n \in G$, $q_n \in P$, and a basic open set $V_n = O(h_n, T_n, \eta_n)$, where $S_{n-1} \subseteq T_n$ and $\eta_n < 2^{-n}$, satisfying $h_n(x) = (-g_n + f)(x)$ for all $x \in S_n$ and

$$h_n \in V_n \cap G \subseteq B(q_n) \subseteq -U_n + f \subseteq -(f - V_{n-1}) + f = V_{n-1}.$$

By the same reasoning used with g_n, we have $h_n(x) = h_{n-1}(x)$ for all $x \in T_{n-1}$ and we assume that $q_{n-1} \ll q_n$.

Suppose that the induction is complete. Set $S = \bigcup\{S_n : n \in \omega\}$. Note that $Y \subseteq S = \bigcup\{T_n : n \in \omega\}$. Observe that for all n, all $x \in S_n$, and all $m > n$,

$$g_n(y_n) = g_m(y_n).$$

Therefore there is a function $\tilde{g} : S \to M$ such that $\langle g_n(x) : n \in \omega\rangle$ converges to $\tilde{g}(x)$ for all $x \in S$. Because $\{p_n : n \in \omega\}$ is $\ll$-directed, by Definition 3.3(5)$_\kappa$, there is g satisfying

$$(*) \qquad g \in \bigcap\{B(p_n) : n \in \omega\} = \bigcap\{O(g_n, S_n, \epsilon_n) : n \in \omega\} \cap G.$$

From $\epsilon_n \to 0$, we see that $g|_S = \tilde{g}$. Hence $\langle g_n(x) : n \in \omega\rangle$ converges to $g(x)$ for all $x \in S$. There are functions $\tilde{h} : S \to M$ and $h \in G$ with analogous properties.

If $n \le m$, then $h_m(y_n) = (-g_m + f)(y_n)$ and $g_{m+1}(y_n) = (f - h_m)(y_n)$. Hence

$$w = f|_Y = (\tilde{g} + \tilde{h})|_Y = (g + h)|_Y,$$

and $g + h \in G$ is the desired function. ■

The next proposition is the successor step in the inductive proof of Theorem 4.1.

Proposition 4.6. *Let μ be an uncountable cardinal and let $\kappa = \mu^+$ be its cardinal successor. Let $G \subseteq M^X$ cover all $<\mu$-faces of M^X. If there is a triple $(Q, \ll, B)$ which κ-represents G, then G covers all $<\kappa$-faces of M^X.*

Compared to the proof of Lemma 4.5, the proof of Proposition 4.6 (to be found at the end of this section) is longer with auxiliary notions. However, the key ideas are the same. Definition 4.7 and Lemma 4.8 establish the analogue of equation $(*)$ above.

Definition 4.7. Suppose X, M and $(Q, \ll, B)$ are as in Proposition 4.6. We say that (Y, D, u) is a *neat triple* if

(1) Y is a subset of X,
(2) D is a directed subset of $(Q, \ll)$,
(3) u is a function from Y to M,
(4) for every $p \in D$, there are $S \in [Y]^{<\omega}$ and $m \in \omega$ such that $O(u, S, 2^{-m}) \cap G \subset B(p)$,
(5) for every $S \in [Y]^{<\omega}$ and $m \in \omega$ there is $p \in D$ such that $B(p) \subset O(u, S, 2^{-m})$.

For example, $(S, \{p_n : n \in \omega\}, g|_S)$ and $(S, \{q_n : n \in \omega\}, h|_S)$ from the proof of Lemma 4.5 are neat triples.

We make a few observations about neat triples.

LEMMA 4.8. *Assume the hypotheses of Proposition* 4.6.

(1) *Let (Y, D, u) be a neat triple with $|D| < \kappa$. Then there exists*
$$g \in \bigcap\{B(p) : p \in D\} = \bigcap\{O(u, S, 2^{-m}) : S \in [Y]^{<\omega},\ m \in \omega\} \cap G$$
and hence $g|_Y = u$.

(2) *Let (Y_i, D_i, u_i), $i < \delta$, be an increasing chain of neat triples. Then the triple of unions is a neat triple.*

(3) *Suppose that (Y, D, u) is a neat triple, that u' is a function with $\operatorname{dom} u \cap \operatorname{dom} u' = \emptyset$, and that $|Y \cup D \cup u'| + \omega = \nu < \mu$. Then there is a neat triple (Z, E, v) satisfying $Y \subseteq Z$, $D \subseteq E$, $u \cup u' \subseteq v$, and $|Z \cup E| \leq \nu$.*

Proof. (1) We note that D is directed and $|D| < \kappa$. By Definition $3.3(5)_\kappa$, there is g in the first intersection. The two intersections are equal because of items (4) and (5) of Definition 4.7.

(2) The union of an increasing chain of sets is a set; the union of an increasing chain of directed sets is a directed set; and the union of an increasing chain of functions is a function.

(3) Because G covers all $<\mu$-faces of M^X, there is $g \in G$ such that $u \cup u' \subset g$. For each $p \in P$ such that $g \in B(p)$, choose $S(p) \in [X]^{<\omega}$ and $m(p) \in \omega$ satisfying $O(g, S(p), 2^{-m(p)}) \cap G \subset B(p)$. For each $p, q \in P$ such that $g \in B(p) \cap B(q)$, choose $r(p, q) \in P$ satisfying $g \in B(r(p, q)) \subseteq B(p) \cap B(q)$. Also, for each $m \in \omega$ and $S \in [X]^{<\omega}$, choose $q(S, m) \in P$ satisfying $g \in B(q(S, m)) \subset O(g, S, 2^{-m})$.

Set $Y(0) = Y \cup \operatorname{dom} u'$ and $D(0) = D$. Suppose that $Y(n)$ and $D(n)$ are defined and $|Y(n)|+|D(n)| \leq \nu$. Set $Y(n+1) = Y(n) \cup \bigcup\{S(p) : p \in D(n)\}$; observe that $|Y(n+1)| \leq |Y(n)| + |D(n)| \leq \nu$. Set
$$\begin{aligned} D(n+1) = D(n) &\cup \{r(p, q) : p, q \in D(n)\} \\ &\cup \{q(T, m) : T \in [D(n)]^{<\omega} \text{ and } m \in \omega\}. \end{aligned}$$
Observe that $|D(n+1)| \leq |D(n)|+|D(n)|+|D(n)|\cdot\omega \leq \nu$. Set $Z = \bigcup\{D(n) : n \in \omega\}$, $E = \bigcup\{D(n) : n \in \omega\}$, and $v = g|_Z$. Then (Z, E, v) is a neat triple, and $|Z \cup E| \leq \nu \cdot \omega < \mu$. ∎

Definition 4.9 and Lemma 4.10 establish the analogue of "for each $n \in \omega$, either we define $h_\ell(y_n) = -g_\ell(y_n) + f(y_n)$ for some $\ell \leq n$, or we define $g_{\ell+1}(y_n) = f(y_n) - h_\ell(y_n)$ for some $\ell \leq n$" before Lemma 4.5. The notion of aiming quintuple is in the spirit of acceptable quadruple of [2].

DEFINITION 4.9. Suppose X, M and $(Q, \ll, B)$ are as in Proposition 4.6. We say that a quintuple (Z, D, u, E, v) *aims at* a function w from a subset Y of X to M if

(1) (Z, D, u) and (Z, E, v) are neat triples,
(2) $u(x) + v(x) = w(x)$ for all $x \in Y \cap Z$.

For example, in the proof of Lemma 4.5, the quintuple $(S, \{p_n : n \in \omega\}, g|_S, \{q_n : n \in \omega\}, h|_S)$ aims at $w : Y \to M$.

LEMMA 4.10. *Assume the hypotheses of Proposition* 4.6. *Let* $Y \in [X]^\mu$ *and* $w : Y \to M$ *be arbitrary. Suppose that* (Z, D, u, E, v) *is a quintuple which aims at* w, *that* $y \in Y$, *and that* $|Z \cup D \cup E| + \omega = \nu < \mu$. *Then there is a quintuple* (Z', D', u', E', v') *which aims at* w *such that* $|Z' \cup D' \cup E'| = \nu$ *and* $Z \cup \{y\} \subset Z'$.

Proof. Let $f \in M^X$ extend w. Set $(S_0, D_0, u_0) = (Z, D, u)$ and $(T_0, E_0, v_0) = (Z, E, v)$. Set $a_0 = \{(y, w(y))\}$. Apply Lemma 4.8(3) to obtain (S_1, D_1, u_1) such that $u_0 \cup a_0 \subset u_1$ and $|S_1 \cup D_1| = \nu$. If $(S_{n+1}, D_{n+1}, u_{n+1})$ has been defined, set $b_n = \{(x, -u_{n+1}(x) + f(x)) : x \in S_{n+1} \setminus T_n\}$. Apply Lemma 4.8(3) to obtain $(T_{n+1}, E_{n+1}, v_{n+1})$ such that $v_n \cup b_n \subset v_{n+1}$ and $|T_{n+1} \cup E_{n+1}| = \nu$. If (T_n, E_n, v_n), $n > 0$, has been defined, set $a_{n+1} = \{(x, f(x) - v_n(x)) : x \in T_n \setminus S_n\}$. Apply Lemma 4.8(3) to obtain $(S_{n+1}, D_{n+1}, u_{n+1})$ such that $u_n \cup a_{n+1} \subset u_{n+1}$ and $|S_{n+1} \cup D_{n+1}| = \nu$.

After ω steps, set $Z' = \bigcup\{S(n) : n \in \omega\} = \bigcup\{T(n) : n \in \omega\}$, $D' = \bigcup\{D(n) : n \in \omega\}$, $u' = \bigcup\{u_n : n \in \omega\}$, $E' = \bigcup\{E(n) : n \in \omega\}$, and $v' = \bigcup\{v_n : n \in \omega\}$. Note that all of these sets have cardinality $\nu \cdot \omega = \nu$. Note also that D' and E' are directed sets and that u' and v' are functions.

Observe that $\mathcal{Z} = \{Z\} \cup \{\operatorname{dom} a_n : n \in \omega\} \cup \{\operatorname{dom} b_n : n \in \omega\}$ is pairwise disjoint. Let $x \in Y \cap Z'$ where $Z' = \bigcup \mathcal{Z}$. If $x \in Z$, then $u'(x) + v'(x) = u(x) + v(x) = w(x)$ because (Z, D, u, E, v) aims at w. If $x \in \operatorname{dom} a_n$, then $u'(x) + v'(x) = (f(x) - v'(x)) + v(x) = w(x)$ by definition of $u'(x)$. If $x \in \operatorname{dom} b_n$, then $u'(x) + v'(x) = u'(x) + (-u'(x) + f(x)) = w(x)$ by definition of $v'(x)$. ∎

In the proof of Lemma 4.5, we constructed $S = \bigcup\{S_n : n \in \omega\}$ where $y_n \in S_n$ and each S_n was finite. Below we will construct $Z = \bigcup\{Z_\alpha : \alpha \in \mu\}$ where $y_\alpha \in Z_{\alpha+1}$ and each Z_α satisfies $|Z_\alpha| < \mu$.

Proof of Proposition 4.6. Let $Y = \{y_\alpha : \alpha < \mu\}$ and $w : Y \to M$ be arbitrary. By induction on $\alpha \le \mu$, we define Z_α, D_α, u_α, E_α, and v_α satisfying:

(1) If $\beta < \alpha$, then $Z_\beta \subset Z_\alpha$, $D_\beta \subset D_\alpha$, $u_\beta \subset u_\alpha$, $E_\beta \subset E_\alpha$, and $v_\beta \subset v_\alpha$.
(2) $\{y_\beta : \beta < \alpha\} \subset Z_\alpha$.
(3) $(Z_\alpha, D_\alpha, u_\alpha, E_\alpha, v_\alpha)$ aims at w.

Set $Z_0 = D_0 = u_0 = E_0 = v_0 = \emptyset$. If δ is a limit ordinal, set $Z_\delta = \bigcup\{Z_\alpha : \alpha < \delta\}$, $D_\delta = \bigcup\{D_\alpha : \alpha < \delta\}$, $u_\delta = \bigcup\{u_\alpha : \alpha < \delta\}$, $E_\delta = \bigcup\{E_\alpha : \alpha < \delta\}$, and $v_\delta = \bigcup\{v_\alpha : \alpha < \delta\}$.

If $(Z_\alpha, D_\alpha, u_\alpha, E_\alpha, v_\alpha)$ has been defined we apply Lemma 4.10 to $(Z_\alpha, D_\alpha, u_\alpha, E_\alpha, v_\alpha)$ and y_a and call the result $(Z_{\alpha+1}, D_{\alpha+1}, u_{\alpha+1}, E_{\alpha+1}, v_{\alpha+1})$.

By (2), $\operatorname{dom} w \subset Z_\mu = \operatorname{dom} u_\mu = \operatorname{dom} v_\mu$. By (2) of Definition 4.9, $u_\mu(x) + v_\mu(x) = w(x)$ for all $x \in Y \cap Z_\mu$. Because (Z_μ, D_μ, u_μ) is a neat triple, Lemma 4.8(2) gives $g \in G$ with $u_\mu \subset g$. Similarly, there is $h \in G$ with $v_\mu \subset h$. Then $g + h \in G$ is the desired extension of w. ∎

5. Medians. To apply the results of the previous section, M must be a topological group. Some important cases, for example $C_p(X, \mathbb{I})$, are excluded. However, the method of proof can be applied when the space M carries another operation called a median operation. For example, if $(M, \le)$ is a linearly ordered space, then $\operatorname{med}(r, s, t)$ defined to be the median of $\{r, s, t\}$ is a median operation. More generally, if M is a distributive lattice, then Birkoff's [7] self-dual ternary median

$$\operatorname{med}(r, s, t) = (r \vee s) \wedge (s \vee t) \wedge (t \vee r)$$

is a median operation. (In fact, for the next theorem, we only need a weaker property, namely that $\operatorname{med}(r, s, t) = x$ whenever two or more coordinates of (r, s, t) equal x.) We can extend a median on M to a median on a product M^X by defining operations pointwise: $\operatorname{med}(g, h, k)(x) = \operatorname{med}(g(x), h(x), k(x))$. Because the operation is defined pointwise, med is continuous on M^X if med is continuous on M. We say that $G \subseteq M^X$ is *closed under* med if $\operatorname{med}(g, h, k)$ is in G whenever g, h, and k are in G. For example, in the case that M is the linearly ordered metric space $\mathbb{I}$, $G = C_p(X, \mathbb{I})$ is closed under Birkhoff's med operation defined above.

An analogue of Theorem 4.1 holds when we replace the group operation with med.

THEOREM 5.1. *Let M be a metrizable space carrying a continuous median operation, let X be an index set, let G be a subset of M^X closed under* med, *let G cover all finite faces of M^X, and let κ be an uncountable cardinal. If there is a triple $(Q, \ll, B)$ which κ-represents G, then G covers all $<\kappa$-faces of M^X. Consequently, if G is domain representable, then $G = M^X$.*

In particular, if $C_p(X, \mathbb{I})$ is domain representable, then X is discrete.

We state and prove the analogue of Lemma 4.5, leaving the other lemmas to interested readers.

LEMMA 5.2. *Let $G \subseteq M^X$ be closed under* med *and cover all finite faces of M^X. If there is a triple $(Q, \ll, B)$ which ω_1-represents G, then G covers all countable faces of M^X.*

Proof. Let $(Q, \ll, B)$ ω_1-represent G. Let $Y = \{y_n : n \in \omega\} \in [X]^\omega$ and $w : Y \to M$ be arbitrary. Let $f \in M^X$ extend w. By induction on $n \in \omega$, we construct sequences $\langle g_n : n \in \omega\rangle$, $\langle h_n : n \in \omega\rangle$, and $\langle k_n : n \in \omega\rangle$ from G such that for all $n \in \omega$ one of the following holds:

(1) for all $m \geq n$, $g_m(y_n) = h_m(y_n) = w(y_n)$,
(2) for all $m \geq n$, $g_m(y_n) = k_m(y_n) = w(y_n)$, or
(3) for all $m \geq n$, $h_m(y_n) = k_m(y_n) = w(y_n)$.

From our construction, we obtain g, h, and k in G such that $\mathrm{med}(g,h,k)|_Y = w$.

Here is the $n = 0$ step of our induction. Because G covers all finite faces of M^X, we may choose $g_0 \in G$ such that $g_0(y_0) = w(y_0)$. Choose $p_0 \in Q$ and a basic open set $U_0 = O(g_0, S_0, \epsilon_0)$, where $y_0 \in S_0 \in [X]^{<\omega}$ and $\epsilon_0 < 1$, satisfying

$$g_0 \in U_0 \cap G \subseteq B(p_0).$$

Because G covers all finite faces of M^X, we may choose $h_0 \in G$ such that $h_0|_{S_0} = f|_{S_0}$. Choose $q_0 \in Q$ and a basic open set $V_0 = O(h_0, T_0, \eta_0)$, where $S_0 \subseteq T_0$ and $\eta_0 < 1$, satisfying

$$h_0 \in V_0 \cap G \subseteq B(q_0).$$

Because G covers all finite faces of M^X, we may choose $k_0 \in G$ such that $k_0|_{T_0} = f|_{T_0}$. Choose $r_0 \in Q$ and a basic open set $W_0 = O(k_0, T_0, \zeta_0)$, where $T_0 \subseteq R_0$ and $\zeta_0 < 1$, satisfying

$$k_0 \in W_0 \cap G \subseteq B(r_0).$$

This completes the $n = 0$ step.

Suppose that the $(n-1)$th step has been completed. Because G covers all finite faces of M^X, we may choose $g_n \in G$ such that $g_n|_{S_{n-1}} = g_{n-1}|_{S_{n-1}}$, $g_{n-1}|_{R_{n-1}\setminus S_{n-1}} = f|_{R_{n-1}\setminus S_{n-1}}$, and, if $y_n \notin R_{n-1}$, then $g_n(y_n) = w(y_n)$. Choose $p_n \in Q$ and a basic open set $U_n = O(g_n, S_n, \epsilon_n)$, where $y_n \in S_n$ and $\epsilon_n < 2^{-n}$, satisfying $p_{n-1} \ll p_n$ and

$$g_n \in U_n \cap G \subseteq B(p_n).$$

Because G covers all finite faces of M^X, we may choose $h_n \in G$ such that $h_n|_{T_{n-1}} = h_{n-1}|_{T_{n-1}}$ and $h_n|_{S_n\setminus T_{n-1}} = f|_{S_n\setminus T_{n-1}}$. Choose $q_n \in Q$ and a basic open set $V_n = O(h_n, T_n, \eta_n)$, where $S_n \subseteq T_n$ and $\eta_n < 2^{-n}$, satisfying $q_{n-1} \ll q_n$ and

$$h_n \in V_n \cap G \subseteq B(q_n).$$

Because G covers all finite faces of M^X, we may choose $k_n \in G$ such that $k_n|_{R_{n-1}} = k_{n-1}|_{R_{n-1}}$ and $k_n|_{R_n\setminus R_{n-1}} = f|_{S_n\setminus T_{n-1}}$. Choose $r_n \in Q$ and a basic open set $W_n = O(k_n, R_n, \zeta_n)$, where $T_n \subseteq R_n$ and $\zeta_n < 2^{-n}$, satisfying $r_{n-1} \ll r_n$ and

$$k_n \in W_n \cap G \subseteq B(r_n).$$

Suppose that the induction is complete. Set $S = \bigcup\{S_n : n \in \omega\}$. Note that $Y \subseteq S = \bigcup\{T_n : n \in \omega\} = \bigcup\{R_n : n \in \omega\}$. Observe that for all n, all

$x \in S_n$, and all $m > n$,

$$g_n(y_n) = g_m(y_n).$$

Therefore there is a function $\tilde{g} : S \to M$ such that $\langle g_n(x) : n \in \omega \rangle$ converges to $\tilde{g}(x)$ for all $x \in S$. Because $\{p_n : n \in \omega\}$ is $\ll$-directed, by Definition 3.3(5)$_\kappa$, there is g satisfying

$$g \in \bigcap \{B(p_n) : n \in \omega\} = \bigcap \{O(g_n, S_n, \epsilon_n) : n \in \omega\} \cap G.$$

From $\epsilon_n \to 0$, we see that $g|_S = \tilde{g}$. Hence $\langle g_n(x) : n \in \omega \rangle$ converges to $g(x)$ for all $x \in S$. There are functions $\tilde{h} : S \to M$, $\tilde{k} : S \to M$, $h \in G$, and $k \in G$ with analogous properties.

Set $Z^0 = S_0 \cup \bigcup \{S_{n+1} \setminus R_n : n \in \omega\}$, $Z^1 = \bigcup \{T_n \setminus S_n : n \in \omega\}$, and $Z^2 = \bigcup \{R_n \setminus T_n : n \in \omega\}$. Then $\{Z^0, Z^1, Z^2\}$ is a partition of S. For $x \in Z^0$, $\tilde{h}(x) = \tilde{k}(x) = f(x)$. For $x \in Z^1$, $\tilde{g}(x) = \tilde{k}(x) = f(x)$. For $x \in Z^2$, $\tilde{g}(x) = \tilde{h}(x) = f(x)$. Hence $\operatorname{med}(g, h, k)$ is an element of G satisfying $\operatorname{med}(g, h, k)|_Y = \operatorname{med}(\tilde{g}, \tilde{h}, \tilde{k})|_Y = f|_Y = w$. ∎

6. Measurable cardinals. An early version of this paper contained an interesting result worth mentioning. Instead of Theorem 4.1, we had: if X is completely regular and $C_p(X)$ is domain representable, then every subset of X is C-embedded in X. We then asked whether the conclusion implies that X is discrete.

THEOREM 6.1. *The following are equivalent:*

(1) *If every subset of X is C-embedded in X, then X is discrete.*
(2) *There are no measurable cardinals.*

Proof. $\neg(2) \Rightarrow \neg(1)$. Let κ be a measurable cardinal and fix a countably complete ultrafilter p on κ. Let X be a set of cardinality κ and identify the points of X with the set $\kappa + 1$. Define a topology on X in which every $\alpha \in \kappa$ is isolated and the neighborhoods of κ are of the form $A \cup \{\kappa\}$ where $A \in p$. Let Y be a subset of X and let $f \in C(Y)$. If $Y \setminus \{\kappa\} \notin p$, or if $\kappa \in Y$ and $Y \setminus \{\kappa\} \in p$, then f can easily be extended to a continuous function on X. Suppose, on the other hand, that $Y \setminus \{\kappa\} \in p$ and $\kappa \notin Y$. It suffices to extend f continuously to $Y \cup \{\kappa\}$. For each $n \in \omega$, let $P_n = \{P_n^m : m \in \omega\}$ be any partition of $\mathbb{R}$ into sets of diameter less than $1/n$. Since p is countably complete, for each $n \in w$ there is exactly one $m(n) \in \omega$ such that $f^{\leftarrow}[P_n^{m(n)}] \in p$. Furthermore, $A = \bigcap \{f^{\leftarrow}[P_n^{m(n)}] : n \in \omega\} \in p$. Since $\operatorname{diam} P_n^{m(n)} < 1/n$, f must be constant on A. Therefore, we can extend f continuously to κ.

$(2) \Rightarrow (1)$. Suppose X is not discrete. Then there is some $x \in X$ with the property that $x \in \operatorname{cl}(X \setminus \{x\})$. Let $\mathcal{U}$ be a maximal pairwise disjoint collection of nonempty open subsets of X that satisfies $x \notin \operatorname{cl} U$ for all $U \in \mathcal{U}$. For each open neighborhood N of x, define $\mathcal{U}(N) = \{U \in \mathcal{U} : N \cap U \neq \emptyset\}$. Set

$p = \{\mathcal{U}(N) : N \in \mathcal{N}(x)\}$. Since $\mathcal{U}$ is maximal, we have $\emptyset \notin p$, and p has the finite intersection property. Extend p to an ultrafilter q. Because $x \notin \operatorname{cl} U$ for each $U \in \mathcal{U}$, q is free. By the no measurable cardinals hypothesis, q is not countably complete. That is, there exists $\{\mathcal{V}_n : n \in \omega\} \subset q$ with $\mathcal{V}_{n+1} \subseteq \mathcal{V}_n$ for all $n \in \omega$ and $\bigcap\{\mathcal{V}_n : n \in \omega\} = \emptyset$. Set $Y = \bigcup\{\mathcal{V}_n : n \in \omega\}$ and define $f : Y \to \mathbb{R}$ by $f(x) = n$ iff $x \in \bigcup \mathcal{V}_n \setminus \bigcup \mathcal{V}_{n+1}$. Since Y is C-embedded in X, there is a continuous extension of f to $\hat{f} \in C(X)$. This is a contradiction since $x \in \operatorname{cl} \bigcup\{\mathcal{V}_i : i \geq n\}$ for all $n \in \omega$. ∎

A search of the literature showed that this result had been obtained by Terada [14] in 1975.

7. Questions. As discussed in the introduction of [6], the class of subcompact spaces and the class of domain representable spaces are closed under the formation of arbitrary products. We wonder if the converse is known. In particular, we ask:

QUESTION 7.1. *If M is a metrizable space, and M^X is subcompact for some index set X with $|X| \geq 2$, must M be completely metrizable? More generally, if S is a topological space such that for some cardinal $\kappa \geq 2$ the product space S^κ is subcompact, must S be subcompact?*

QUESTION 7.2. *Is it true that every domain representable topological group is subcompact?*

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