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A class of spaces that admit no sensitive commutative group actions

by

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Abstract. We show that a metric space X admits no sensitive commutative group action if it satisfies the following two conditions: (1) X has property S, that is, for each $\varepsilon > 0$ there exists a cover of X which consists of finitely many connected sets with diameter less than ε; (2) X contains a free n-network, that is, there exists a nonempty open set W in X having no isolated point and $n \in \mathbb{N}$ such that, for any nonempty open set $U \subset W$, there is a nonempty connected open set $V \subset U$ such that the boundary $\partial_X(V)$ contains at most n points. As a corollary, we show that no Peano continuum containing a free dendrite admits a sensitive commutative group action. This generalizes some previous results in the literature.

1. Introduction. Let (X, d) be a metric space. Denote by $\mathrm{Homeo}(X)$ the homeomorphism group of X. A subgroup H of $\mathrm{Homeo}(X)$ is said to be *sensitive* (resp. *expansive*) if there is a constant $c > 0$ such that, for any nonempty open subset U of X, there exists an $h \in H$ such that $\mathrm{diam}(h(U)) > c$ (resp. for any two different points x and y in X, there exists an $h \in H$ such that $d(h(x), h(y)) > c$); the constant c is called a *sensitivity constant* (resp. *expansivity constant*) of H. Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and positive integers respectively. For any $m \in \mathbb{N}$, write $\mathbb{N}_m = \{1, \ldots, m\}$. A homeomorphism $h : X \to X$ is said to be *sensitive* (resp. *expansive*) if the cyclic subgroup $\langle h \rangle \equiv \{h^n : n \in \mathbb{Z}\}$ of $\mathrm{Homeo}(X)$ is sensitive (resp. expansive). In general, expansivity is stronger than sensitivity: if X contains no isolated points, then every expansive subgroup H of $\mathrm{Homeo}(X)$ is sensitive.

Any group homomorphism φ from a group G to $\mathrm{Homeo}(X)$ is called a *group action* on X. A group action $\varphi : G \to \mathrm{Homeo}(X)$ is said to be

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commutative if the group G is commutative, and $\varphi : G \to \mathrm{Homeo}(X)$ is said to be *sensitive* (resp. *expansive*) if the subgroup $\varphi(G)$ of $\mathrm{Homeo}(X)$ is sensitive (resp. expansive).

In the study of dynamical systems, sensitivity is an interesting subject. It gives a description of the phenomenon in chaotic systems that minute errors in experimental readings eventually lead to large scale divergence. In the definition of chaos given by Devaney [6], sensitivity is a key condition (though it can be derived from transitivity and density of periodic points, see [2]). Cairns et al. [4, 5] studied chaotic group actions on compact manifolds and showed that: (1) the circle admits no chaotic group action; (2) every compact surface admits a chaotic $\mathbb{Z}$-action; and (3) every compact triangulable manifold of dimension greater than one admits a faithful chaotic action of every countably generated free group. In [1, 7, 8, 17], the relations between sensitivity and some definitions of chaos were discussed.

An interesting question is: what metric space can admit an expansive or a sensitive commutative group action? It is well known that there are many metric spaces admitting expansive homeomorphisms, such as the Cantor set, the 2-adic solenoid, and the orientable closed surfaces of positive genus (see [25]). Since expansivity implies sensitivity, these spaces also admit sensitive homeomorphisms.

On the other hand, there are also many metric spaces which admit no expansive homeomorphisms, even admit no expansive (or sensitive) commutative group actions. For example, Kato proved that dendroids, Peano continua containing 1-dimensional AR neighborhoods, Peano continua in the plane, and chainable continua admit no expansive homeomorphisms (see [9, 10, 11, 13]). Kato and Mouron proved that hereditarily indecomposable continua do not admit expansive homeomorphisms [14]. Recently, Mouron proved that tree-like continua admit no expansive homeomorphisms [21] and the solenoids are the only circle-like continua that admit expansive homeomorphisms [22]. Mai and Shi showed that graphs admit no sensitive commutative group actions [18], and Peano continua containing free dendrites admit no expansive commutative group actions [19]. In [18], the authors also construct a sensitive homeomorphism on a Suslinian continuum, which answers a question posed by Kato [12].

In this paper we continue the study of the existence of sensitive commutative group actions. In Section 2 we introduce the notions of n-network and free n-network, and exhibit some examples of n-networks, containing graphs, dendrites and regular curves of finite order. In Section 3 we recall the notion of property S, and introduce some properties of metric spaces having property S. Our main result is the following theorem, which is a generalization of the main results in [9, 10, 18, 19].

THEOREM 4.1. *Let X be a metric space which has property S and contains a free n-network for some $n \in \mathbb{N}$. Then X admits no sensitive commutative group action.*

2. n-networks and free n-networks. In this section we will consider a type of metric spaces, called an n-network, which contains all graphs, all dendrites and all regular curves of order $\leq n$.

For any metric space (X, d) and any nonempty subset Y of X, denote by $\operatorname{diam}(Y) \equiv \sup\{d(x, y) : x, y \in Y\}$ the diameter of Y, by $\operatorname{Int}_X(Y)$ the interior of Y in X, and by $\partial_X(Y) \equiv \overline{Y} - \operatorname{Int}_X(Y)$ the boundary of Y in X. For any $x \in X$ and any $r > 0$, write

$$B(x, r) = \{w \in X : d(w, x) < r\} \quad \text{and} \quad B(Y, r) = \{w \in X : d(w, Y) < r\}.$$

DEFINITION 2.1. Let $n \in \mathbb{N}$. A metric space X having no isolated point is called an *n-network* if for any nonempty open subset V of X there is a nonempty connected open set $U \subset V$ such that the boundary $\partial_X(U)$ contains at most n points.

There are many examples of n-networks.

EXAMPLE 2.2. (1) A space homeomorphic to the interval $[0, 1]$ is called an *arc*. We denote by $\operatorname{End}(A)$ the set of the two endpoints of an arc A. A compact connected metric space G is called a *graph* if there exist finitely many arcs $A_1, \ldots, A_n$ such that $G = \bigcup_{i=1}^n A_i$ and $A_i \cap A_j = \operatorname{End}(A_i) \cap \operatorname{End}(A_j)$ for $1 \leq i < j \leq n$. Evidently, every graph G is a 2-network.

(2) A compact connected metric space is called a *continuum*. A locally connected continuum is called a *Peano continuum*. A simple closed curve is called a *circle*. A Peano continuum containing no circle is called a *dendrite*. From the definition it is easy to show that every dendrite is also a 2-network.

EXAMPLE 2.3. A metric space homeomorphic to the square $[0, 1]^2$ is called a *disk*. For any disk D, let ∂D denote the boundary circle of D and write $\mathring{D} = D - \partial D$. If E is the union of finitely many pairwise disjoint disks $D_1, \ldots, D_n$ in the plane $\mathbb{R}^2$, then we write $\partial E = \bigcup_{i=1}^n \partial D_i$. Let $E_0 = [0, 1]^2$ and $G_0 = \partial E_0$. For $n = 1, 2, \ldots$, let

$$\begin{aligned} E_n = \bigcup \{ [i/3^n, (i+1)/3^n] \times [j/3^n, (j+1)/3^n] : i \text{ and } j \text{ are even integers in} \\ [0, 3^n], \text{ and } [i/3^n, (i+1)/3^n] \times [j/3^n, (j+1)/3^n] \cap G_{n-1} \neq \emptyset \}, \end{aligned}$$

and let

$$G_n = G_{n-1} \cup \partial E_n \ (= \partial E_0 \cup \partial E_1 \cup \cdots \cup \partial E_n).$$

Then E_n is the union of finitely many pairwise disjoint squares in $\mathbb{R}^2$ with side length $1/3^n$ each, and G_n is a connected graph. Let

$$G = \bigcup_{n=0}^{\infty} G_n \quad \text{and} \quad K = \overline{G}$$

(see Fig. 1). Then K is a compact subspace of $\mathbb{R}^2$. Since each G_n is arcwise connected, so is G, which implies that $K = \overline{G}$ is connected. For any $x \in K$ and any $\varepsilon > 0$, there exists a square $Q = Q_{x\varepsilon}$ in $\mathbb{R}^2$ with side length $< \varepsilon$ such that $x \in \mathring{Q}$ and $Q \cap G$ is arcwise connected. This means that $Q \cap K$ is a connected neighborhood of x in the space K. Thus K is locally connected, and hence a Peano continuum. In addition, for any nonempty open subset U of K, it is easy to see that there exists a square $Q = Q_U$ in $\mathbb{R}^2$ such that $\mathring{Q} \cap K$ is a connected open subset of K contained in U, and

$$(K - (\mathring{Q} \cap K)) \cap (Q \cap K) = (K - \mathring{Q}) \cap Q$$

contains exactly two points. Thus K is a 2-network.

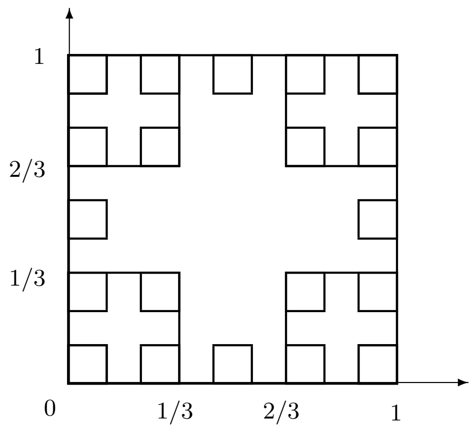

Fig. 1. The 2-network $K = \overline{\bigcup_{n=0}^{\infty} G_n}$ (note that $G_2 = \bigcup_{i=0}^{2} \partial E_i$ is drawn)

Example 2.4. A continuum X is called a *regular curve* if for any $x \in X$ and any open neighborhood U of x in X there is an open neighborhood V of x contained in U such that the cardinality of $\partial_X(V)$ is finite; if this cardinality is not greater than some fixed $n \in \mathbb{N}$, then X is called a *regular curve of order* $\leq n$ (see [16, p. 274]). From the definition it is easy to show that each regular curve is a Peano continuum.

An example of a regular curve is the triangular Sierpiński curve, which is of order ≤ 4 (see [16, p. 276]). It is easy to check that the triangular Sierpiński curve is a 3-network. The 2-network K of Example 2.3 is a regular curve of order ≤ 3. In addition, all dendrites are regular curves, but not all are of finite order.

From the definitions we see that every regular curve of order $\leq n$ is an n-network. But an n-network (even if it is a Peano continuum) may not be a regular curve. For example, let $Y = \{(x, y, 0) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ be a square in $\mathbb{R}^3$, let $\{(x_n, y_n, 0) : n = 1, 2, \ldots\}$ be a countable dense subset of Y and let $Z = \{(x_n, y_n, t) : 0 \leq t \leq 1/n,\ n = 1, 2, \ldots\}$. Then $Y \cup Z$ is a 2-network and a Peano continuum, but not a regular curve.

Recall that a subspace W of a metric space X is called a *free arc* (resp. a *free dendrite*) in X if W itself is an arc (resp. a dendrite) and there is a connected open subset U of X such that $\overline{U} = W$. Kawamura [15] proved that no Peano continuum containing a free arc admits an expansive homeomorphism. Mai and Shi [19] further showed that no Peano continuum containing a free dendrite admits an expansive commutative group action. In this paper we will extend this study to free n-networks.

Definition 2.5. Let X be a metric space. A nonempty subspace W of X is called a *free n-network* in X if W itself is an n-network, and there exists an open subset U of X such that $U \subset W \subset \overline{U}$.

It follows from Example 2.2 that every free dendrite in a metric space is also a free 2-network. But the converse is not true. For instance, the space K in Example 2.3 is a 2-network, and the triangular Sierpiński curve is a 3-network, but they contain no free dendrite.

The following lemma is trivial, but it is useful in Section 4.

Lemma 2.6. *Let X be a metric space, V, W and Z be nonempty subsets of X with $V \subset W \subset \overline{V}$, $n \in \mathbb{N}$, $\varepsilon_0 > 0$, and $h \in \mathrm{Homeo}(X)$.*

(1) *If $\partial_X(V)$ contains at most n points, then so does $\partial_X(h(V))$.*
(2) *If V is an n-network, then so is W.*
(3) *If V is a free n-network in X, then so are W, $h(W)$ and any nonempty open subset U of X contained in W.*
(4) *If Z is connected, $\mathrm{diam}(Z) > \varepsilon_0$, $\partial_X(Z) \neq \emptyset$, and $\partial_X(Z)$ contains at most n points, then $Z - B(\partial_X(Z), \varepsilon_0/(2n)) \neq \emptyset$.* ∎

3. Metric spaces having property S. The following property was introduced by R. L. Moore [20] in 1922 (see also [24, p. 120]).

Definition 3.1. Let X be a metric space. A cover $\mathcal{Y}$ of X is said to be *connected* if each $Y \in \mathcal{Y}$ is connected. The space X is said to have *property S* if for each $\varepsilon > 0$ there is a finite connected cover $\mathcal{Y}$ of X such that $\mathrm{diam}(Y) < \varepsilon$ for each $Y \in \mathcal{Y}$.

It is easy to check that every Peano continuum has property S. Besides Peano continua, many metric spaces have this property. For instance, the space $X = \{(r, s) \in [0, 1]^2 : r \text{ or } s \text{ is a rational number}\}$ and every bounded convex set in $\mathbb{R}^n$ have this property. In addition, the following evident proposition yields more spaces with this property.

PROPOSITION 3.2. *Let X be a metric space, $X_0, X_1, \ldots, X_n$ be subspaces of X which have property S, $n \in \mathbb{N}$, and $X_0 \subset Y \subset \overline{X}_0$. Then Y and $\bigcup_{i=0}^n X_i$ also have property S.*

Recall that a metric space is said to be *totally bounded* if for each $\varepsilon > 0$, the open cover $\{B(x, \varepsilon) : x \in X\}$ has a finite subcover. From the definitions we get at once

PROPOSITION 3.3. *Every metric space having property S is totally bounded, and has only finitely many connected components.* ∎

The following proposition is well known (for example, see [24, Theorems 8.3 and 8.4]).

PROPOSITION 3.4. (1) *Any metric space X having property S is locally connected.*

(2) *A compact metric space X is locally connected if and only if X has property S.*

The following proposition may be folklore, but we still present a simple proof.

PROPOSITION 3.5. *Let X be a metric space which has property S. Then the completion $\widetilde{X}$ of X is the union of finitely many Peano continua.*

Proof. We can regard X as a dense subspace of $\widetilde{X}$. By Proposition 3.2, $\widetilde{X}$ also has property S. Hence $\widetilde{X}$ is locally connected. By Proposition 3.3, it is totally bounded, and has only finitely many connected components $\widetilde{X}_1, \ldots, \widetilde{X}_m$. Since every totally bounded complete metric space is compact (see [23, p. 275]), $\widetilde{X}_1, \ldots, \widetilde{X}_m$ are all Peano continua. ∎

From Proposition 3.5 we get

COROLLARY 3.6. *Let X be a connected metric space which has property S. Then the completion $\widetilde{X}$ of X is a Peano continuum.* ∎

EXAMPLE 3.7. There exists a locally connected metric space X which has only finitely many connected components $X_1, \ldots, X_m$, and for each $i \in \{1, \ldots, m\}$ the completion $\widetilde{X}_i$ of X_i is a Peano continuum, but X fails property S. For instance, $X = [0,1]^2 - ((\{0\} \cup \{1/k : k \in \mathbb{N}\}) \times (0,1])$ is such a space. From this example we see that the converse of Proposition 3.5 is not true.

For more properties of metric spaces having property S, see [24, Chap. 8, Sec. 1].

4. Main result and its proof. We now state the main result of this paper.

THEOREM 4.1. *Let X be a metric space which has property S and contains a free n-network for some $n \in \mathbb{N}$. Then X admits no sensitive commutative group action.*

Proof. Suppose Homeo(X) has a sensitive commutative subgroup H. Let W be a free n-network in X. Since H is sensitive, there exist an interior point x_1 of W and $f_1 \in H$ such that $f_1(x_1) \neq x_1$, and by Lemma 2.6(3) we have

CLAIM 1. *There exists a nonempty connected open set U_1 such that $x_1 \in U_1 \subset W$, $f_1(U_1) \cap U_1 = \emptyset$, and U_1 is a free n-network in X.*

Let ε_0 be a sensitivity constant of H and let $\varepsilon = \varepsilon_0/(8n)$. Since X has arbitrarily small finite connected covers, there is a finite connected cover $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ of X such that $\operatorname{diam}(Y_i) < \varepsilon$ for $1 \leq i \leq m$. Write $c = \max\{d(Y_i, Y_j) : 1 \leq i < j \leq m\} + 2\varepsilon$. Then $\operatorname{diam}(X) < c$. For any $x, y \in X$, let

$$\xi(x,y) = \begin{cases} c & \text{if } x \text{ and } y \text{ are not in the same connected component of } X, \\ \inf\{\operatorname{diam}(Z) : Z \text{ is a connected subset of } X \text{ containing } x \text{ and } y\} \\ \quad \text{if } x \text{ and } y \text{ are in the same connected component of } X. \end{cases} \tag{4.1}$$

For any two nonempty subsets V and V' of X, let

$$\xi(V, V') = \inf\{\xi(x,y) : x \in V \text{ and } y \in V'\}. \tag{4.2}$$

Notice that if $V \subset U$ and $V' \subset U'$ then $\xi(U, U') \leq \xi(V, V')$. For any nonempty finite set $\{g_1, \ldots, g_k\} \subset H$, define

$$S(g_1, \ldots, g_k) = \sum_{j=1}^{m} \xi\Big(Y_j, \bigcup_{i=1}^{k} g_i(Y_j)\Big). \tag{4.3}$$

Then

$$0 \leq S(g_1, \ldots, g_k) < mc. \tag{4.4}$$

Moreover, for any $\{g_1, \ldots, g_k, g_{k+1}\} \subset H$ and any $j \in \{1, \ldots, m\}$, from (4.3) and (4.2) we get

$$S(g_1, \ldots, g_k) - S(g_1, \ldots, g_k, g_{k+1}) \geq \xi\Big(Y_j, \bigcup_{i=1}^{k} g_i(Y_j)\Big) - \xi\Big(Y_j, \bigcup_{i=1}^{k+1} g_i(Y_j)\Big) \geq 0.$$

CLAIM 2. *Let $k \in \mathbb{N}$. If there exist $\{f_1, \ldots, f_k\} \subset H$ and a free n-network U_k in X such that $U_k \cap \bigcup_{i=1}^{k} f_i(U_k) = \emptyset$, then there exist an $f_{k+1} \in H$ and* *a free n-network* U_{k+1} *in* X *such that* $U_{k+1} \cap \bigcup_{i=1}^{k+1} f_i(U_{k+1}) = \emptyset$ *and*

$$S(f_1, \ldots, f_k) - S(f_1, \ldots, f_k, f_{k+1}) > \varepsilon. \tag{4.5}$$

Proof of Claim 2. Since U_k is a free n-network in X, there is a connected open subset W_0 of X contained in U_k such that $\mathrm{diam}(W_0) < \varepsilon$, $\partial_X(W_0) \neq \emptyset$, and $\partial_X(W_0)$ contains at most n points. Note that W_0 is a free n-network in X. Since H is sensitive with the sensitivity constant $\varepsilon_0 = 8n\varepsilon$, it follows from Lemma 2.6 that, for $i = 1, 2, \ldots$, there exist $\{h_i, Z_i, W_i\}$ satisfying the following three conditions :

(i) $h_i \in H$ with $\mathrm{diam}(h_i(W_{i-1})) > 8n\varepsilon$;
(ii) $Z_i = h_i(W_{i-1})$, which is a connected open set and a free n-network in X, $\partial_X(Z_i) = h_i(\partial_X(W_{i-1})) \neq \emptyset$, and $\partial_X(Z_i)$ contains at most n points;
(iii) W_i is a connected open subset of Z_i, $\mathrm{diam}(W_i) < \varepsilon$, $d\big(W_i, \partial_X(Z_i)\big) > 3\varepsilon$, $\partial_X(W_i) \neq \emptyset$, and $\partial_X(W_i)$ contains at most n points.

Since $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ is a cover of X, there exist integers $1 \leq a < b \leq m+1$ and $\mu \in \mathbb{N}_m$ such that $W_a \cap Y_\mu \neq \emptyset$ and $W_b \cap Y_\mu \neq \emptyset$. Pick $y \in W_b \cap Y_\mu$. Since W_a and Y_μ are both connected with diameter $< \varepsilon$, $W_a \cup Y_\mu$ is a connected set and $W_a \cup Y_\mu \subset B(y, 2\varepsilon)$. Since $d(y, \partial_X(Z_b)) \geq d(W_b, \partial_X(Z_b)) > 3\varepsilon$, we have $d(W_a \cup Y_\mu, \partial_X(Z_b)) > \varepsilon$ and so $\overline{W_a \cup Y_\mu} \subset Z_b$. Let $\varphi = h_b h_{b-1} \cdots h_{a+1}$ and let $f_{k+1} = \varphi^{-1}$. Then $\{\varphi, f_{k+1}\} \subset H$, $\varphi(W_a) \supset Z_b$, and $f_{k+1}(\overline{W}_a) \subset f_{k+1}(Z_b) \subset W_a$. Let $V_a = f_{k+1}(W_a)$. Then $\overline{V}_a \subset W_a$. Since $W_a - \overline{V}_a \supset f_{k+1}(Z_b) - f_{k+1}(\overline{W}_a) = f_{k+1}(Z_b - \overline{W}_a) \neq \emptyset$, there exists a free n-network $U_{k+1} \subset W_a - \overline{V}_a$ that satisfies

$$U_{k+1} \cap f_{k+1}(U_{k+1}) \subset U_{k+1} \cap f_{k+1}(W_a) = U_{k+1} \cap V_a = \emptyset. \tag{4.6}$$

Select $u \in U_{k+1}$. Choose $p \in \mathbb{N}_m$ such that $u \in Y_p$. By (4.1) and (4.2) we have

$$\xi(Y_p, f_{k+1}(Y_p)) \leq \xi(u, f_{k+1}(u)) \leq \mathrm{diam}(W_a) < \varepsilon. \tag{4.7}$$

Let $\psi = h_a h_{a-1} \cdots h_1$. Then $\psi \in H$, and

$$u \in U_{k+1} \subset W_a \subset Z_a \subset \psi(W_0) \subset \psi(U_k). \tag{4.8}$$

For any $i \in \mathbb{N}_k$, since H is commutative and $U_k \cap f_i(U_k) = \emptyset$, from (4.8) we get

$$\begin{aligned} Z_a \cap f_i(Z_a) &\subset \psi(U_k) \cap f_i\psi(U_k) = \psi(U_k) \cap \psi f_i(U_k) \\ &= \psi\big(U_k \cap f_i(U_k)\big) = \emptyset, \end{aligned} \tag{4.9}$$

which with (4.8) and (4.6) implies $U_{k+1} \cap \bigcup_{i=1}^{k+1} f_i(U_{k+1}) = \emptyset$. Noting that Y_p is connected, $Y_p \subset B(u, \varepsilon)$, and $d(u, \partial_X(Z_a)) \geq d(W_a, \partial_X(Z_a)) > 3\varepsilon$, we obtain

$$d(Y_p, \partial_X(Z_a)) > 2\varepsilon. \tag{4.10}$$

Thus $Y_p \subset Z_a$, which together with (4.9) implies that

$$f_i(Y_p) \subset f_i(Z_a) \subset X - Z_a \quad \text{for any } i \in \mathbb{N}_k. \tag{4.11}$$

By (4.11), any connected set in X intersecting both Y_p and $\bigcup_{i=1}^{k} f_i(Y_p)$ must intersect $\partial_X(Z_a)$. Hence, from (4.1), (4.2) and (4.10) we get

$$\xi\Big(Y_p, \bigcup_{i=1}^{k} f_i(Y_p)\Big) > 2\varepsilon, \tag{4.12}$$

which together with (4.3), (4.2) and (4.7) implies that

$$\begin{aligned} S(f_1,\dots,f_k) - S(f_1,\dots,f_k,f_{k+1}) &\geq \xi\Big(Y_p, \bigcup_{i=1}^{k} f_i(Y_p)\Big) - \xi\Big(Y_p, \bigcup_{i=1}^{k+1} f_i(Y_p)\Big) \\ &\geq \xi\Big(Y_p, \bigcup_{i=1}^{k} f_i(Y_p)\Big) - \xi(Y_p, f_{k+1}(Y_p)) \\ &> 2\varepsilon - \varepsilon = \varepsilon. \end{aligned}$$

Thus (4.5) holds, and the proof of Claim 2 is complete.

We now continue the proof of Theorem 4.1. Let β be an integer greater than mc/ε. By Claims 1 and 2, there exist $\{f_1,\dots,f_{\beta+1}\} \subset H$ such that (4.5) holds for all $k \in \mathbb{N}_\beta$. Therefore, from (4.4) and (4.5) we obtain

$$S(f_1) \geq S(f_1) - S(f_1,\dots,f_{\beta+1}) > \beta\varepsilon > mc.$$

But this contradicts (4.4). ■

Since every Peano continuum has property S, and every dendrite is a 2-network, from Theorem 4.1 and Example 2.4 we obtain the following corollary at once.

Corollary 4.2. *If X is a metric space which satisfies one of the following three conditions, then X admits no sensitive commutative group action:*

(1) *X is a Peano continuum and contains a free n-network*;
(2) *X is a regular curve of finite order*;
(3) *X is a Peano continuum and contains a free dendrite.* ■

From Corollary 4.2(2) we see that the space K in Example 2.3 and the triangular Sierpiński curve admit no sensitive commutative group action.

In [19, Theorem 3.2] it is proved that *if X is a Peano continuum containing a free dendrite, then X admits no expansive commutative group action.* Since expansivity is stronger than sensitivity, we can also derive this result from Corollary 4.2(3).

Recall that a metric space X is called an *absolute retract* (abbr. AR) if it has the universal extension property, and an *absolute neighborhood retract* (abbr. ANR) if it has the universal neighborhood extension property (see

[23, pp. 221 and 216]). It is known that a continuum X is a 1-dimensional AR if and only if it is a dendrite (see [3, (13.5)]). In [10, Theorem 3.2] Kato proved that *if X is a Peano continuum which has a neighborhood M such that $\overline{M}$ is a* 1*-dimensional* AR*, then X does not admit an expansive homeomorphism; in particular, no* 1*-dimensional compact* ANR *admits an expansive homeomorphism* [23]. For any $h \in \mathrm{Homeo}(X)$, since the cyclic group $\langle h \rangle \equiv \{h^n : n \in \mathbb{Z}\}$ is commutative, the above result of Kato can also be derived from Corollary 4.2(3).

In addition, since every graph is a Peano continuum containing a free arc, from of Corollary 4.2(3) we can derive the result of [18, Theorem 2.1], which says that *if G is a graph, then no commutative subgroup of* Homeo(G) *is sensitive.*

REMARK 4.3. There are examples to show that neither of the two conditions "X has property S" and "X contains a free n-network" in Theorem 4.1 can be removed.

(1) For any $r \in [0, \infty)$, write $z_r = re^{2\pi \mathbf{i} r}/(r+1) \in \mathbb{C}$, where $\mathbf{i} = \sqrt{-1}$. Let $L = \{z_r : r \in [0, \infty)\}$. Then L is an infinite spiral curve contained in the unit disk. Let d_E be the Euclidean metric on $\mathbb{C}$, defined by $d_E(z, z') = |z - z'|$ for any $z, z' \in \mathbb{C}$. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $W = L \cup S^1$. Then W is a connected compact metric space with the metric induced by d_E. Identifying the points $0 \in L$ and $1 \in S^1$, we obtain a space $X = (W - \{0, 1\}) \cup \{\{0, 1\}\}$ with the identification topology. Write $0^* = 1^* = \{0, 1\}$. Define a metric d on X by $d(0^*, 0^*) = 0$, $d(z, 0^*) = \min\{d_E(z, 0), d_E(z, 1)\}$ for all $z \in X - \{0^*\}$, and $d(z, w) = \min\{|z - w|, d(z, 0^*) + d(w, 0^*)\}$ for all $\{z, w\} \subset X - \{0^*\}$. Then X is an arcwise connected continuum which contains free arcs but fails property S. In [18] we showed that such a continuum X admits a sensitive homeomorphism. Thus the condition "X has property S" in Theorem 4.1 cannot be removed.

(2) Let T^2 be a 2-dimensional torus in $\mathbb{R}^3$. Then T^2 has property S but contains no n-network, for any $n \in \mathbb{N}$. It is well known that the torus T^2 admits sensitive or even expansive homeomorphisms. Thus the condition "X contains a free n-network" in Theorem 4.1 cannot be removed either.

Of course, the questions whether the condition "X has property S" or "X contains a free n-network" in Theorem 4.1 can be weakened, or replaced by other conditions, remain to be studied.

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