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Bootstrapping Kirszbraun's extension theorem

by

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Abstract. We show how Kirszbraun's theorem on extending Lipschitz mappings in Hilbert space implies its own generalization. There is a *continuous extension operator* preserving the Lipschitz constant of every mapping.

1. Introduction. Let A be subset of a Hilbert space X and $f : A \to X$ a Lipschitz mapping. According to Kirszbraun's theorem [Ki] (see also [BL] and [WW]), there is a Lipschitz extension of f defined on X. The extension has the same Lipschitz constant, but in general it is not unique. Can it nevertheless be assigned in a continuous way? In this paper we show that the answer is yes. We combine Kirszbraun's theorem with a simple homotopy argument to show that for bounded mappings the multivalued extension operator is lower semicontinuous. The Michael selection theorem then provides a singlevalued continuous extension operator.

The motivation for asking this question comes from approximating contractions in the topology of uniform convergence by Lipschitz isometries. See [KSS] and [K] for a detailed explanation.

The underlying Banach space for our considerations will be $C(A,X)$, the bounded continuous mappings from a nonempty subset A of a Hilbert space X into X. We use the following notation. If $x \in X$, then $|x|$ stands for the inner product norm of x. If $f : A \to X$ is a mapping, then $|f|_A = \sup_{x\in A} |f(x)|$; if $f \in C(A,X)$ this is the norm of f. In the case when $A = X$ we will simply write $|f|$.

We will be concerned with two families of Lipschitz mappings: $\mathscr{C}(A)$ and $\mathscr{L}(A)$ in $C(A,X)$. The subspace $\mathscr{L}(A)$ consists of all bounded Lipschitz mappings from A to X; in general it is not closed in $C(A,X)$. If $A \subset X$ is compact, then by the Stone–Weierstrass theorem $\mathscr{L}(A)$ is dense in $C(A,X)$.

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A subset of $\mathscr{L}(A)$, the contractions $\mathscr{C}(A)$, contains all bounded mappings from A to X with Lipschitz constant at most one. It is a closed subset of $C(A,X)$, in particular, the supremum norm $|\cdot|$ equips it with a complete metric. If X is finite-dimensional and A is a compact set, then by the Arzelà-Ascoli theorem, any closed and bounded subset of $\mathscr{C}(A)$ is compact in $C(A,X)$. In particular, the set of all contractions vanishing at a given point $a \in A$ is compact.

We consider two multivalued extension mappings. The mapping Φ extends contractions to contractions, possibly increasing the Lipschitz constant. The mapping Ψ provides every Lipschitz mapping with extensions having the same Lipschitz constant. More precisely, $\Phi : \mathscr{C}(A) \to \mathcal{P}(\mathscr{C}(X))$ assigns to each $f \in \mathscr{C}(A)$ *all* of its extensions in $\mathscr{C}(X)$. The extension mapping $\Psi : \mathscr{L}(A) \to \mathcal{P}(\mathscr{L}(X))$ assigns to each $f \in \mathscr{L}(A)$ all of its extensions in $\mathscr{L}(X)$ which *preserve* the Lipschitz constant of f. Here we denote by $\mathcal{P}(M)$ the family of all subsets of a set M.

In Theorem 2.3 we show that Φ is lower semicontinuous. This is equivalent to the restriction mapping $R : \mathscr{C}(X) \to \mathscr{C}(A)$, defined by $R(f) = f \restriction A$, being open. In Theorem 2.5 we show that Ψ is lower semicontinuous as well. Hence (Corollary 2.6), for every closed subset D of $\mathscr{L}(A)$ and every continuous selection $F : D \to \mathscr{L}(X)$ of the extension mapping Ψ, there is a continuous selection $\tilde{F} : \mathscr{L}(A) \to \mathscr{L}(X)$ which extends F. For the extension operator Φ there are such continuous selections as well.

This paper generalizes the results of [K]. There continuity of the extension operator was proved for contractions defined on a bounded subset of a Euclidean space using the ball intersection property. Notice that [KR], assuming the results of this paper, proves continuity also when each of the extensions is moreover required to have the same closed convex hull of the range as the original Lipschitz mapping.

2. Continuity of extension mappings. In this section we will explain how to get a continuous extension operator for Lipschitz mappings on a Hilbert space X.

In the special case when $A \subset X$ is closed and convex, there is a 1-Lipschitz retraction $r : X \to A$, namely the nearest point projection. An extension defined by $\tilde{f} = f \circ r$ preserves the Lipschitz constant, and clearly has the property that $|\tilde{f} - \tilde{g}|_X = |f - g|_A$ for any two Lipschitz mappings f and g on A. However, any Lipschitz retract of X has to be connected, so this simple approach does not work for a general set A.

Recall that a multivalued mapping is lower semicontinuous if the "large pre-images" of open sets are open. For example, Ψ is lower semicontinuous if for any open $V \subset \mathscr{L}(X)$, the set

$$\Psi^{-1}(V) = \{g \in \mathscr{L}(A) : \Psi(g) \cap V \neq \emptyset\}$$

is open in $\mathscr{L}(A)$. Loosely speaking, given an open neighborhood V of a mapping f defined on X, every mapping g which is close enough to f on A admits an extension contained in V.

Here is a simple example when this is not possible, motivating why in this paper we deal only with *bounded* Lipschitz mappings.

Assume $A = [0,1]$ and $f(x) = x$ is the identity on $X = \mathbb{R}$. Then given any $0 < \delta < 1$, if $g(x) = (1-\delta)x$ on A, then $|f-g|_A \le \delta$; nevertheless for any Lipschitz constant preserving extension of g to $\mathbb{R}$ we have $|f-g|_{\mathbb{R}} = \infty$.

LEMMA 2.1. *Let X be a Hilbert space, $\emptyset \neq A \subset X$, and let $f \in \mathscr{C}(X)$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $g \in \mathscr{C}(A)$ is such that $|f-g|_A < \delta$, then g admits a contractive extension to X so that $|f-g|_X < \varepsilon$.*

Proof. Assume $0 < \varepsilon < 1$, and define $\delta = \varepsilon^2/(8\max\{1,|f|\})$.

If $g \in \mathscr{C}(A)$ is so that $|f-g|_A < \delta$, we define a mapping h on a subset of the ℓ_2-sum $X \times \mathbb{R}$ as follows:

$$h : X \times \{0\} \cup A \times \{\varepsilon\} \to X,$$
$$h(x,0) = f(x) \quad \text{for } x \in X, \qquad h(y,\varepsilon) = g(y) \quad \text{for } y \in A;$$

see also Fig. 1. We show that the Lipschitz constant of h does not exceed

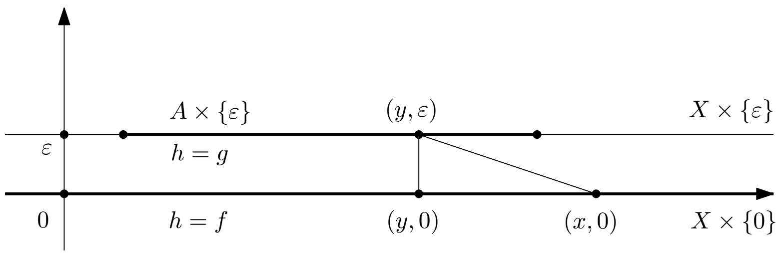

Fig. 1. Contraction h as "parallel copies" of f and g

one, hence by Kirszbraun's theorem, h can be extended to a contraction on $X \times \mathbb{R}$. It is enough to estimate for $x \in X$ and $y \in A$:

$$\begin{aligned}|h(x,0) - h(y,\varepsilon)|^2 &= |f(x) - g(y)|^2 \le (|f(x) - f(y)| + |f(y) - g(y)|)^2 \\ &\le (|f(x) - f(y)| + \delta)^2 \le |f(x) - f(y)|^2 + 4\delta|f| + \delta^2 \\ &\le |x-y|^2 + 4\delta|f| + \delta^2 \le |x-y|^2 + \varepsilon^2.\end{aligned}$$

For $x \in X$, we define the extension of g by $g(x) = h(x,\varepsilon)$. Then

$$|g(x) - f(x)| = |h(x,\varepsilon) - h(x,0)| \le \varepsilon. \ \blacksquare$$

The method of extending mappings simultaneously from closely placed parallel copies of "A" and "X" in $X\times\mathbb{R}$ to keep the mappings close together on X if they were close together on A already appears in a different setting in Theorem 5 of [Bo]. We will exploit the method one more time in a slightly more complicated way in the proof of the next lemma.

Let $\mathrm{Lip}(f,A)$ denote the Lipschitz constant of a mapping f restricted to a set A. When $A=X$ we will sometimes write just $\mathrm{Lip}(f)$.

LEMMA 2.2. *Let X be a Hilbert space and $\emptyset\neq A\subset X$. Let $f\in\mathscr{L}(X)$ be such that $\mathrm{Lip}(f,A)=\mathrm{Lip}(f,X)$. Then for every $\varepsilon>0$ there exists $\delta>0$ with the following property. If $g\in\mathscr{L}(A)$ is such that $|f-g|<\delta$ on A, then g admits an extension to X so that $|f-g|<\varepsilon$ on X and $\mathrm{Lip}(g,A)=\mathrm{Lip}(g,X)$.*

Proof. If f is a constant, then $\delta=\varepsilon$ and any Lipschitz constant preserving extension of g composed with the retraction on the ε-ball around the unique value of f works.

Assume f is not a constant and $\varepsilon>0$. Let $s<1$ be very close to one; how close exactly will be specified later. Choose $\delta>0$ so that if $|f-g|_A<\delta$, then $\mathrm{Lip}(g,A)\geq s\,\mathrm{Lip}(f,A)$. One more condition on the smallness of δ, depending only on f and ε, will be given later. Let $\eta=\varepsilon/\max\{1,\mathrm{Lip}(f)\}$.

Now let $g\in\mathscr{L}(A)$ be so that $|f-g|_A<\delta$. We distinguish two cases:

(i) $s\,\mathrm{Lip}(f)\leq\mathrm{Lip}(g,A)\leq 2\,\mathrm{Lip}(f)$,
(ii) $2\,\mathrm{Lip}(f)<\mathrm{Lip}(g,A)$.

In case (i), we define a mapping h on a subset of the ℓ_2-sum $X\times\mathbb{R}$ as follows:

$$h: X\times\{0\}\cup A\times\{\eta\}\to X,$$
$$h(x,0)=sf(x)\quad\text{for } x\in X,\qquad h(y,\eta)=g(y)\quad\text{for } y\in A.$$

We show the Lipschitz constant of h does not exceed that of g, hence by Kirszbraun's theorem, h can be extended to $X\times\mathbb{R}$ with Lipschitz constant $\mathrm{Lip}(g,A)$. As $\mathrm{Lip}(sf,X)\leq\mathrm{Lip}(g,A)$, it is enough to estimate for $x\in X$ and $y\in A$:

$$
\begin{aligned}
&|h(x,0)-h(y,\eta)|^2=|sf(x)-g(y)|^2\\
&\leq[s|f(x)-f(y)|+|f(y)-g(y)|+(1-s)|f(y)|]^2\\
&\leq[s|f(x)-f(y)|+\delta+(1-s)|f|]^2\\
&\leq s^2|f(x)-f(y)|^2+4s|f|(\delta+(1-s)|f|)+(\delta+(1-s)|f|)^2\\
&\leq s^2\,\mathrm{Lip}^2(f)\big[|x-y|^2+[4s|f|(\delta+(1-s)|f|)+(\delta+(1-s)|f|)^2]/s^2\,\mathrm{Lip}^2(f)\big]\\
&\leq\mathrm{Lip}^2(g)(|x-y|^2+\eta^2),
\end{aligned}
$$

if $\delta>0$ is small enough and $s<1$ is close enough to one. We also make sure that $(1-s)|f|<\varepsilon$. For $x\in X$, we define the extension of g by $g(x)=h(x,\eta)$.

Then

$$|g(x) - f(x)| \le |g(x) - sf(x)| + (1-s)|f| \le |h(x,\eta) - h(x,0)| + \varepsilon$$
$$\le \eta \operatorname{Lip}(g) + \varepsilon \le 2\eta \operatorname{Lip}(f) + \varepsilon \le 3\varepsilon.$$

In case (ii), we first enlarge the set A by defining g to be equal to f at all points which are far away from A. More precisely, let $\tilde{A} = \{x \in X : \operatorname{dist}(A, x) \ge 2\delta/\operatorname{Lip}(g, A)\}$, and

$$\tilde{g} = \begin{cases} g & \text{on } A, \\ f & \text{on } \tilde{A}; \end{cases}$$

see also Fig. 2. To see that $\operatorname{Lip}(g, A) = \operatorname{Lip}(\tilde{g}, A \cup \tilde{A})$ let $x \in \tilde{A}$ and $y \in A$.

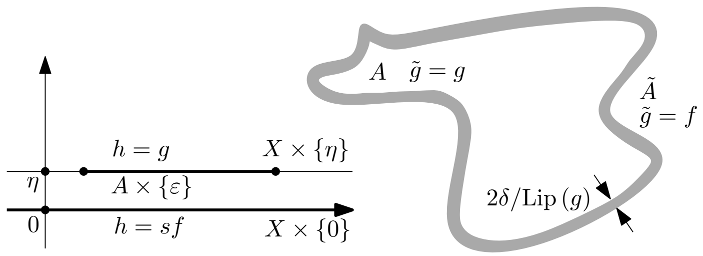

Fig. 2. Extending g when $\operatorname{Lip}(g)$ is small or large

Then

$$|\tilde{g}(x) - \tilde{g}(y)| = |f(x) - g(y)| \le |f(x) - f(y)| + |f(y) - g(y)|$$
$$\le \operatorname{Lip}(f)|x-y| + \delta \le (\operatorname{Lip}(f) + \operatorname{Lip}(g, A)/2)|x-y|$$
$$\le \operatorname{Lip}(g, A)|x-y|.$$

Let g be any extension of $\tilde{g}$ to X so that $\operatorname{Lip}(g, X) = \operatorname{Lip}(\tilde{g}, A \cup \tilde{A}) = \operatorname{Lip}(g, A)$.

If $x \in A \cup \tilde{A}$, then obviously $|g(x) - f(x)| \le \delta$. If $x \in X \setminus (A \cup \tilde{A})$, then there is $y \in A$ so that $|x - y| < 2\delta/\operatorname{Lip}(g)$. Hence

$$|g(x) - f(x)| \le |g(x) - g(y)| + |g(y) - f(y)| + |f(y) - f(x)|$$
$$\le \operatorname{Lip}(g) \cdot 2\delta/\operatorname{Lip}(g) + \delta + \operatorname{Lip}(f) \cdot 2\delta/\operatorname{Lip}(g) \le 4\delta. \ \blacksquare$$

THEOREM 2.3. *Let X be a Hilbert space and $\emptyset \ne A \subset X$. The mapping $\Phi : \mathscr{C}(A) \to \mathcal{P}(\mathscr{C}(X))$ is lower semicontinuous.*

Proof. We have to show that for every open $V \subset \mathscr{C}(X)$ the set

$$\Phi^{-1}(V) = \{g \in \mathscr{C}(A) : \Phi(g) \cap V \ne \emptyset\}$$

is open in $\mathscr{C}(A)$. According to Lemma 2.1, this is indeed the case. ∎

The theorem reformulates without any further effort as follows.

THEOREM 2.4. *Let X be a Hilbert space and $\emptyset \ne A \subset X$. The restriction mapping $R : \mathscr{C}(X) \to \mathscr{C}(A)$ defined by $R(f) = f{\restriction}A$ is open.*

The multivalued extension mapping preserving the Lipschitz constants is lower semicontinuous also on the space of all bounded Lipschitz mappings.

THEOREM 2.5. *Let X be a Hilbert space and $\emptyset \neq A \subset X$. The mapping $\Psi : \mathscr{L}(A) \to \mathcal{P}(\mathscr{L}(X))$ is lower semicontinuous.*

Proof. We have to show that for every open $V \subset \mathscr{L}(X)$ the set

$$\Psi^{-1}(V) = \{g \in \mathscr{L}(A) : \Psi(g) \cap V \neq \emptyset\}$$

is open in $\mathscr{L}(A)$. According to Lemma 2.2, this is indeed the case. ∎

Both Φ and Ψ are lower semicontinuous multivalued mappings defined on the metric spaces $\mathscr{C}(A)$, respectively $\mathscr{L}(A)$. They have convex and closed values in the Banach space $C(X, X)$; hence Michael's selection theorem (see, e.g., [BL] or [Mi]) directly yields continuous extension operators.

COROLLARY 2.6. *Let X be a Hilbert space and $\emptyset \neq A \subset X$. Then for every closed subset D of $\mathscr{C}(A)$ and every continuous selection $F : D \to \mathscr{C}(X)$ of the extension mapping $\Phi : \mathscr{C}(A) \to \mathcal{P}(\mathscr{C}(X))$, there is a continuous selection $\tilde{F} : \mathscr{C}(A) \to \mathscr{C}(X)$ which extends F.*

Similarly, for every closed subset D of $\mathscr{L}(A)$ and every continuous selection $F : D \to \mathscr{L}(X)$ of the extension mapping $\Psi : \mathscr{L}(A) \to \mathcal{P}(\mathscr{L}(X))$, there is a continuous selection $\tilde{F} : \mathscr{L}(A) \to \mathscr{L}(X)$ which extends F.

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