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Categories of directed spaces

by

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Abstract. The main goal of the present paper is to unify two commonly used models of directed spaces: d-spaces and streams. To achieve this, we provide certain "goodness" conditions for d-spaces and streams. Then we prove that the categories of good d-spaces and good streams are isomorphic. Next, we prove that the category of good d-spaces is complete, cocomplete, and cartesian closed (assuming we restrict to compactly generated weak Hausdorff spaces). The category of good d-spaces is large enough to contain many interesting examples of directed spaces, including probably all which are interesting from the point of view of concurrency theory. However it fails to contain some spaces having applications to non-commutative geometry. Next, we define the class of locally d-path-connected spaces (ldpc-spaces); the additional condition allows us to eliminate some exotic examples of directed spaces. Again, we prove that ldpc-spaces and good ldpc-spaces form a category which is complete, cocomplete and cartesian closed.

1. Introduction. Directed Algebraic Topology is a new area of research with its motivations coming mainly from Computer Science. The main objects of interest are directed spaces—topological spaces (whose points represent possible states of some process, for example a computer program) equipped with an additional structure which distinguishes locally some directions; it determines how these states can evolve in time. Many models of directed spaces are in use, including d-spaces, streams, local po-spaces, flows, etc., but the relationships between them are not yet thoroughly studied. The main goal of this paper is to show that the first two concepts are nearly equivalent.

The first are *d-spaces* introduced by Grandis [1]; they are topological spaces with the additional directed structure stating which paths are possible paths of execution of a computer program. In another approach, of streams, introduced by Krishnan [4], the additional information says whether or not a state x can be achieved from a state y while moving inside a given open set. These two approaches are not equivalent, but they are close in the following

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sense: there exist subcategories of these two categories which are isomorphic to each other and contain many interesting examples (like realizations of cubical sets). Therefore they seem to be large enough for applications to concurrency theory. Moreover, these common subcategories have good properties, i.e. they are complete, cocomplete and cartesian closed.

Some results presented in Sections 2 and 3 were obtained independently by Haucourt [3].

The organization of the paper. Section 2 contains the definitions of d-spaces and streams. Next, we recall the construction of adjoint functors between the categories of d-spaces and streams and define good d-spaces and good streams. Then we show that the categories of good d-spaces and of good streams are isomorphic. Next, we provide some criteria allowing us to determine if a d-space (or stream) is good and prove that the category of good d-spaces is complete and cocomplete. In Section 3 we consider the category of good compactly generated weakly Hausdorff d-spaces and prove that it is cartesian closed. Section 4 contains the definition of ldpc-spaces—d-spaces satisfying a certain condition which enforces compatibility between their topologies and d-structures. Furthermore, we prove the completeness and cocompleteness of the categories of lpdc-spaces and good ldpc-spaces. In Section 5 we construct mapping spaces of CGWH ldpc-spaces and prove that they are exponential objects in the categorical sense. Finally, Section 6 considers localizations and colocalizations.

Notation. Throughout, $I = [0, 1]$ is the unit interval and $P(X)$ stands for the space of all paths on a topological space X (with the compact-open topology). If $x, y \in X$, then

$$P(X)_x^y := \{\alpha \in P(X) : \alpha(0) = x \wedge \alpha(1) = y\}.$$

If X, Y are topological spaces, then $\mathrm{map}(X, Y)$ is the mapping space with the compact-open topology, and

$$\mathfrak{F}(A, B) := \{f \in \mathrm{map}(X, Y) : f(A) \subseteq B\} \quad \text{for } A \subseteq X,\ B \subseteq Y.$$

2. d-spaces and streams

d-spaces

Definition 2.1 ([1]). Let X be a topological space. A *d-structure* on X is a collection of paths $\vec{P}(X) \subseteq P(X)$, called *directed paths* or *d-paths*, such that:

- every constant path is directed,
- $\vec{P}(X)$ is closed under reparametrization, that is, $\alpha \circ f \in \vec{P}(X)$ for any $\alpha \in \vec{P}(X)$ and every continuous non-decreasing function $f : I \to I$,

- $\vec{P}(X)$ is closed under concatenation, that is, if $\alpha, \beta \in \vec{P}(X)$, $\alpha(1) = \beta(0)$, then $\alpha * \beta \in \vec{P}(X)$, where
$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{for } t \leq 1/2, \\ \beta(2t-1) & \text{for } t \geq 1/2. \end{cases}$$

A *d-space* is a topological space equipped with a d-structure.

Remark. Equivalently, a d-space X is a triple $(\mathfrak{S}(X), \mathfrak{T}(X), \vec{P}(X))$, where $\mathfrak{S}(X)$ is a set, $\mathfrak{T}(X)$ a topology on $\mathfrak{S}(X)$, and $\vec{P}(X)$ a d-structure on the topological space $(\mathfrak{S}(X), \mathfrak{T}(X))$. We will usually write X instead of $\mathfrak{S}(X)$, unless it leads to confusion.

Definition 2.2. A *d-map* $f : X \to Y$, where X and Y are d-spaces, is a continuous map which preserves the d-structure (i.e. such that $f \circ \alpha \in \vec{P}(Y)$ for $\alpha \in \vec{P}(X)$).

Obviously compositions of d-maps are d-maps. Hence d-spaces and d-maps form a category denoted by **dTop**. As proven in [1], the category **dTop** is complete and cocomplete.

Definition 2.3. If X is a d-space, then a *d-subspace* $A \subseteq X$ is equipped with the d-structure $\vec{P}(A) := P(A) \cap \vec{P}(X)$.

Streams. To provide the definition of streams, we need to recall some notions from [4]:

Definition 2.4. A relation $\leq$ on a set X is a *preorder* if it is:

- reflexive ($x \leq x$ for every $x \in X$),
- transitive (if $x \leq y$ and $y \leq z$, then $x \leq z$).

A set equipped with a preorder will be called a *preordered set*. If R is an arbitrary relation, then its transitive-reflexive closure is a preorder. For a family $\{(X_i, \leq_i)\}_{i\in I}$ of preordered sets, let $\bigvee_{i\in I} \leq_i$ denote the preorder on $\bigcup_{i\in I} X_i$ which is the transitive-reflexive closure of $\bigcup_{i\in I} \leq_i$ (considered as a subset of $\bigcup_{i\in I} X_i \times \bigcup_{i\in I} X_i$). Equivalently, $x(\bigvee_{i\in I} \leq_i)y$ if and only if $x = y$ or there exists a sequence
$$x = x_0 \leq_{i_1} x_1 \leq_{i_2} \cdots \leq_{i_n} x_n = y,$$
where $i_k \in I$, $x_k \in X_{i_k} \cap X_{i_{k+1}}$.

Definition 2.5. A *circulation* on a topological space X is a family $\{\leq_U\}_{U\in\mathfrak{T}(X)}$ of preorders such that
$$\leq_{\bigcup_{U\in\mathcal{O}} U} = \bigvee_{U\in\mathcal{O}} \leq_U$$
for every $\mathcal{O} \subseteq \mathfrak{T}(X)$ (we do not require any continuity condition on these preorders). A *stream* is a topological space equipped with a circulation.

Remark. A stream X is a triple $(\mathfrak{S}(X), \mathfrak{T}(X), \{\leq_U^X\})$, where $\mathfrak{S}(X)$ is a set, $\mathfrak{T}(X)$ a topology on $\mathfrak{S}(X)$, and $\{\leq_U^X\}$ a circulation on $(\mathfrak{S}(X), \mathfrak{T}(X))$.

Definition 2.6. A continuous map $f : X \to Y$ is a *stream map* if for every open subset $U \subseteq Y$ and every pair $x, y \in p^{-1}(U)$ the condition $x \leq_{p^{-1}(U)} y$ implies that $f(x) \leq_U f(y)$.

Compositions of stream maps are again stream maps; the category of streams and stream maps will be denoted by **Str**.

Proposition 2.7 ([4]). *The category* **Str** *is complete and cocomplete.*

The adjoint functors. For a d-space X and $x, y \in X$ we define

$$\vec{P}(X)_x^y := \{\alpha \in \vec{P}(X) : \alpha(0) = x \wedge \alpha(1) = y\} = \vec{P}(X) \cap P(X)_x^y. \tag{2.8}$$

For every open subset $U \subseteq X$ define a relation on U by

$$x \leq_U^{F(X)} y \ \Leftrightarrow \ \vec{P}(U)_x^y \neq \emptyset, \tag{2.9}$$

i.e. there exists a directed path in U from x to y. Let

$$F(X) := (\mathfrak{S}(X), \mathfrak{T}(X), \{\leq_U^{F(X)}\}). \tag{2.10}$$

Proposition 2.11. *$F(X)$ is a stream.*

Proof. Since constant paths are directed, and concatenations of d-paths are directed, we see that the relations $\leq_U^{F(X)}$ are preorders. For simplicity, we skip the upper index $^{F(X)}$ in the remaining part of the proof. Let $\{U_i\}_{i\in J}$ be a family of open sets in X and let $V := \bigcup_{i\in J} U_i$. If $x(\bigvee_{i\in I} \leq_{U_i})y$, then there is a sequence

$$x = x_0 \leq_{U_{i_1}} x_1 \leq_{U_{i_2}} \cdots \leq_{U_{i_n}} x_n = y,$$

where $x_k \in U_{i_k} \cap U_{i_{k+1}}$. Therefore there are paths $\alpha_k \in \vec{P}(U_{i_k})$ such that $\alpha_k(0) = x_{k-1}$, $\alpha_k(1) = x_k$. The existence of the concatenation $\alpha_1 * \cdots * \alpha_k$ implies that $x = x_0 \leq_V x_n = y$. On the other hand, if $x \leq_V y$, then there exists $\alpha \in \vec{P}(V)$ such that $\alpha(0) = x$, $\alpha(1) = y$. By the compactness of the interval, there exist sequences $0 = t_0 < t_1 < \cdots < t_n = 1$ and $i_1, \ldots, i_n$ such that $\alpha([t_{k-1}, t_k]) \subseteq U_{i_k}$. Then

$$x = \alpha(t_0) \leq_{U_{i_1}} \alpha(t_1) \leq_{U_{i_2}} \cdots \leq_{U_{i_n}} \alpha(t_n) = y.$$

Therefore $\leq_V = \bigvee \leq_{U_i}$ and hence $\{\leq_U^{F(X)}\}$ is a circulation. ∎

Let $\vec{I}$ be the directed interval, i.e. the stream with the circulation

$$x \leq_U y \ \Leftrightarrow \ x \leq y \wedge [x, y] \subseteq U.$$

For every stream Y define the d-space $G(Y)$ by

$$G(Y) := (\mathfrak{S}(Y), \mathfrak{T}(Y), \{f : \vec{I} \to Y : f \text{ is a stream map}\}). \tag{2.12}$$

PROPOSITION 2.13. *The assignments*

$$F : \mathbf{dTop} \ni X \mapsto F(X) \in \mathbf{Str}, \qquad G : \mathbf{Str} \ni Y \mapsto G(Y) \in \mathbf{dTop}$$

are adjoint functors (F *is left adjoint and* G *is right adjoint*).

Proof. The functoriality is obvious. Fix a stream map $f : F(X) \to Y$. If $\alpha \in \vec{P}(X)$, then $f \circ \alpha : \vec{I} \to Y$ is a stream map, hence $f \circ \alpha \in \vec{P}(G(Y))$ and therefore $f : X \to G(Y)$ is a d-map. Now fix a d-map $X \to G(Y)$, $U \in \mathfrak{T}(Y)$ and assume that $x \leq^{F(X)}_{f^{-1}(U)} y$. There exists $\alpha \in \vec{P}(f^{-1}(U))$ such that $\alpha(0) = x$ and $\alpha(1) = y$. Then $f \circ \alpha \in \vec{P}(G(U))$, which implies $x = f(\alpha(0)) \leq^Y_U f(\alpha(1)) = y$. Hence $f : F(X) \to Y$ is a stream map. ∎

Good d-spaces and good streams. For every topological space X, the set of all d-structures on X is equipped with a natural partial order: a d-structure $\vec{P}(X)$ is finer than $\vec{P}'(X)$ iff $\vec{P}(X) \subseteq \vec{P}'(X)$. Similarly, there is a partial order on the set of all circulations on X: a circulation $\{\leq_U\}$ is finer than $\{\leq'_U\}$ iff $x \leq_U y$ implies $x \leq'_U y$. Obviously both F and G preserve this partial order. We will see that when composing the functors F and G, we obtain less fine d-structures and finer circulations.

PROPOSITION 2.14. *For every d-space* X *the d-structure* $\vec{P}(X)$ *is finer than* $\vec{P}(GFX)$. *For every stream* Y *the circulation* $\{\leq^Y\}$ *is less fine than* $\{\leq^{FGY}\}$.

Proof. By the adjointness of F and G, we have the d-map $X \to GFX$ and the stream map $FGY \to Y$. They are both identities (in **Top**). The conclusion follows by the definitions of d-maps and stream maps. ∎

PROPOSITION 2.15. *For every d-space* X *the streams* FX *and* $FGFX$ *are isomorphic. For every stream* Y *the d-spaces* GY *and* $GFGY$ *are isomorphic.*

Proof. By 2.14, $x \leq^{FGFX}_U y$ implies that $x \leq^{FX}_U y$. On the other hand, if $x \leq^{FX}_U y$ then there exists a path $\alpha \in \vec{P}(U)$ such that $\alpha(0) = x$ and $\alpha(1) = y$. But $\alpha \in \vec{P}(GFX)$, hence $x \leq^{FGFX}_U y$.

Similarly, by 2.14, $\vec{P}(GY) \subseteq \vec{P}(GFGY)$. If $\alpha \in \vec{P}(GFGY)$, then $\alpha : \vec{I} \to FGY$ is a stream map. But $\alpha : \vec{I} \to Y$ is also a stream map (by 2.14 again), hence $\alpha \in \vec{P}(GY)$. ∎

As a corollary we have

THEOREM 2.16. *The restricted functors*

$$F : G(\mathbf{Str}) \leftrightarrow F(\mathbf{dTop}) : G$$

are isomorphisms of categories. ∎

Define $\mathcal{G} := G(\mathbf{Str}) = F(\mathbf{dTop})$. We call a d-space X *good* if $X = G(Y)$ for some stream Y. Similarly, a stream Y is *good* if $Y = F(X)$ for some d-space X. Note that a d-space X is good iff $X = GFX$, and a stream Y is good iff $Y = FGY$.

PROPOSITION 2.17. *The functor $GF : \mathbf{dTop} \to \mathbf{dTop}$ is a localization. The functor $FG : \mathbf{Str} \to \mathbf{Str}$ is a colocalization* (cf. Section 6).

Proof. The coaugmentation $\eta_X : X \to GFX$ is adjoint to 1_{FX} and it is idempotent by 2.15. ∎

Properties of good d-spaces and streams

DEFINITION 2.18. Let X be a d-space. A path $\alpha \in P(X)$ is *almost directed* if for any $U \in \mathfrak{T}(X)$ and $a < b$ such that $\alpha([a,b]) \subseteq U$ there exists a directed path $\beta \in \vec{P}(U)$ connecting $\alpha(a)$ and $\alpha(b)$ (i.e. such that $\alpha(a) = \beta(0)$ and $\alpha(b) = \beta(1)$). In other words, $\vec{P}(U)_{\alpha(a)}^{\alpha(b)} \neq \emptyset$.

The following statements are straightforward consequences of the definitions:

PROPOSITION 2.19. *A d-space X is good iff every almost directed path in X is directed. A stream Y is good iff for every $U \in \mathfrak{T}(X)$ and every pair $x, y \in U$ such that $x \leq_U y$ there exists a stream map $f : \vec{I} \to X$ such that $f(0) = x$, $f(1) = y$.*

PROPOSITION 2.20. *If X is a good d-space and $A \subseteq X$ is a d-subspace, then A is good.*

PROPOSITION 2.21. *Let X, Y be d-spaces. Suppose that $\mathfrak{S}(X) = \mathfrak{S}(Y)$, $\vec{P}(X) = \vec{P}(Y)$, $\mathfrak{T}(X) \subseteq \mathfrak{T}(Y)$. If X is good, then so is Y.*

PROPOSITION 2.22. *Let $f : X \to Y$ be a d-map and let $\alpha \in P(X)$ be an almost directed path. Then $f \circ \alpha$ is almost directed.*

Proof. Fix any $U \in \mathfrak{T}(Y)$ and $a < b$ such that $f(\alpha([a,b])) \subseteq U$. Since α is almost directed, there exists $\beta \in \vec{P}(f^{-1}(U))_{\alpha(a)}^{\alpha(b)}$. So $f \circ \beta \in \vec{P}(U)_{f(\alpha(a))}^{f(\alpha(b))}$. ∎

PROPOSITION 2.23. *For every space X the collection $\{\mathfrak{F}([s,t], U)\}$, where $0 \leq s \leq t \leq 1$, $U \in \mathfrak{T}(X)$, is a semibasis of $P(X)$.*

Proof. Fix a compact subset $C \subseteq I$, an open subset $U \subseteq X$ and $f \in \mathfrak{F}(C, U)$. The family of connected components of $f^{-1}(U)$ covers C, hence we can pick a finite covering $\{(s_j, t_j)\}_{j=1}^n$. Now we have

$$f \in \bigcap_{j=1}^{n} \mathfrak{F}(C_j, U) \subseteq \mathfrak{F}(C, U),$$

where $C_j = [\inf(C \cap (s_j, t_j)), \sup(C \cap (s_j, t_j))]$. ∎

Proposition 2.24. *Let X be a d-space. If $\vec{P}(X)$ is a closed subset of $P(X)$, then X is good.*

Proof. Let $\alpha \in P(X)$ be a path which is almost directed but not directed. By the assumption, there is an open neighbourhood $\mathfrak{U}$ of α contained in $P(X) \setminus \vec{P}(X)$. Moreover, by 2.23 we can assume that

$$\mathfrak{U} = \bigcap_{j=1}^{n} \mathfrak{F}(C_j, U_j),$$

where $C_j \subseteq I$ are closed intervals and $U_j \subseteq X$ are open sets. Let $0 = p_0 < p_1 < \cdots < p_m = 1$ be a sequence which contains all the endpoints of the intervals C_j and for $i \in \{1, \ldots, m\}$ define

$$A_i := \{j \in \{1, \ldots, n\} : [p_{i-1}, p_i] \subseteq C_j\}.$$

Obviously $\alpha([p_{i-1}, p_i]) \subseteq U_j$ for all $j \in A_i$. Since α is almost directed, there exist directed paths $\beta_i \in \vec{P}(\bigcap_{j \in A_i} U_j)_{\alpha(p_{i-1})}^{\alpha(p_i)}$ for every $i = 1, \ldots, m$ (if A_i is empty we take $\beta_i \in \vec{P}(X)_{\alpha(p_{i-1})}^{\alpha(p_i)}$). Let β be the path such that

$$\beta(r) = \beta_i\left(\frac{r - p_{i-1}}{p_i - p_{i-1}}\right)$$

for $r \in [p_{i-1}, p_i]$. It is well-defined (since $\beta(p_i) = \beta_i(1) = \beta_{i+1}(0) = \alpha(p_i)$) and directed (since it is a reparametrized concatenation of directed paths). For every $j = 1, \ldots, n$ and every $t \in C_j$ there is an interval $[p_{i-1}, p_i]$ such that $t \in [p_{i-1}, p_i] \subseteq C_j$. Then

$$\beta(t) \subseteq \beta_i(I) \subseteq \bigcap_{l \in A_i} U_l \subseteq U_j.$$

Therefore $\beta(C_j) \subseteq U_j$ and hence $\beta \in \mathfrak{U}$; but this contradicts the assumption that $\mathfrak{U}$ contains no directed paths. ∎

Remark. The inverse statement is not true. The unit interval I with a d-structure given by

$$\alpha \in \vec{P}(I) \ \Leftrightarrow\ \alpha^{-1}(\{0\}) \in \{\emptyset, I\}$$

is good but the space of directed paths is not closed in $P(I)$.

Limits and colimits

Proposition 2.25. *Let $\{X_i\}_{i \in I}$ be a collection of d-spaces. Then*

(1) *$(\coprod X_i, \coprod \vec{P}(X_i))$ is a direct sum of $\{X_i\}$ (in* **dTop***)* (see [1]),
(2) *if all X_i's are good, then $\coprod X_i$ is good,*
(3) *$(\prod X_i, \prod dX_i)$ is a direct product of $\{X_i\}$ (in* **dTop***)* (see [1]),
(4) *if all X_i's are good, then $\prod X_i$ is good.*

Proof. Only the last statement is not obvious. Let $\alpha = (\alpha_i)_{i\in I}$ be an almost directed path in $\prod_{i\in I} X_i$. Fix $j \in I$, $U \in \mathfrak{T}(X_j)$ and $0 \le a < b \le 1$ such that $\alpha_j([a,b]) \subseteq U$. Since α is almost directed, there exists a directed path $\beta = (\beta_i) \in \vec{P}(U \times \prod_{j\neq i\in I} X_i)$ with $\beta(0) = \alpha(a)$ and $\beta(1) = \alpha(b)$. In particular, $\beta_j \in \vec{P}(U)$ and $\beta_j(0) = \alpha_j(a)$ and $\beta_j(1) = \alpha_j(b)$. Thus α_j is almost directed and by the goodness of X_j it is directed. Therefore for every $j \in I$ the path α_j is directed and so α is directed. ∎

PROPOSITION 2.26. *Let X, Y be d-spaces and let $f, g : X \to Y$ be d-maps. We have*

$$\lim_{\mathbf{dTop}}(X \overset{f}{\underset{g}{\rightrightarrows}} Y) = E,$$

where $E = \{x \in X : f(x) = g(x)\}$ and $\vec{P}(E) = \vec{P}(X) \cap P(E)$. Moreover, if X is good, then E is good.

Proof. It is obvious that E is a limit in **dTop**, and it is good by 2.20. ∎

COROLLARY 2.27. *The category $\mathcal{G}$ is complete.*

Coequalizers of diagrams of good d-spaces are not necessarily good.

EXAMPLE 2.28. Let $r : \vec{S}^1 \to \vec{S}^1$ be the rotation by an angle α such that $\alpha/2\pi$ is not rational, and let

$$C = \operatorname{colim}_{\mathbf{dTop}}(\vec{S}^1 \overset{\mathrm{id}}{\underset{r}{\rightrightarrows}} \vec{S}^1). \tag{2.29}$$

Then C is an uncountable antidiscrete space. All paths in C are almost directed, but not all are directed.

PROPOSITION 2.30. *The category $\mathcal{G}$ is cocomplete.*

Proof. The functor $Q := GF$ with an augmentation $X \to GFX$ is a localization on the category **dTop** (cf. 2.17); condition (6.1) below is satisfied because of 2.15. Therefore Q preserves colimits. As a consequence, for every functor $A : \mathcal{C} \to \mathcal{G}$ we have

$$\operatorname{colim}_{\mathcal{G}} A = Q(\operatorname{colim}_{\mathbf{dTop}} A). \ ∎$$

REMARK. Example 2.28 is closely related to the examples given by Grandis in [2, 2.5]. It also shows that the category $\mathcal{G}$ is not suitable for applications in non-commutative geometry.

3. Mapping spaces. The main goal of this section is to prove that the category of good d-spaces with compactly generated weak Hausdorff underlying spaces is cartesian closed.

Compactly generated weak Hausdorff d-spaces

DEFINITION 3.1. A subset A of a topological space X is *k-open* if $f^{-1}(A)$ is open in K for every compact Hausdorff space K and every continuous map $f : K \to X$. A space X is *compactly generated* if every k-open subset of X is open.

DEFINITION 3.2. A topological space X is *weakly Hausdorff* if for every compact Hausdorff space K and every continuous map $f : K \to X$ the image $f(K)$ is closed in X.

The classical result of Steenrod [7] states that the category of compactly generated weakly Hausdorff topological spaces (CGWH-spaces for short) is complete, cocomplete and cartesian closed.

The functor

$$k : \mathbf{Top} \ni (X, \mathfrak{T}(X)) \mapsto (X, \{U \subseteq X : U \text{ is k-open})\} \in \mathbf{Top} \tag{3.3}$$

with the identity augmentation is a colocalization. The class of its colocal objects consists of all compactly generated spaces. The functor

$$w : \mathbf{Top} \ni X \mapsto X/E \in \mathbf{Top}, \tag{3.4}$$

where E is the smallest closed equivalence relation on X, is a localization. A space is w-local iff it is weakly Hausdorff.

Let $\mathcal{G}_k$ be the category of good compactly generated weakly Hausdorff d-spaces.

PROPOSITION 3.5. *The category $\mathcal{G}_k$ is complete and cocomplete.*

Proof. Both functors F and G (cf. (2.10), (2.12)) preserve underlying topological spaces. As a consequence, the composition $Q_k := GF : \mathbf{dTop}_k \to \mathbf{dTop}_k$ is well-defined and is a localization. Then the cocompleteness of $\mathcal{G}_k$ is a consequence of the cocompleteness of $\mathbf{dTop}_k$. Let $A : \mathcal{C} \to \mathcal{G}_\mathbf{c}$ be a functor from a small category. Following [8, 2.30], the limit of A in $\mathbf{Top}_\mathbf{c}$ is $k(L)$, where

$$L := \Big\{(x_A) \in \prod_{A \in \mathcal{A}} F(A) : \forall_{a:A \to A' \in \mathcal{A}}\, F(a)(x_A) = x_{A'}\Big\}.$$

Goodness is preserved by products (cf. 2.25), by passing to subspaces (cf. 2.20) and by passing to a richer topology (cf. 2.21). Hence $k(L)$ is good. ∎

Mapping spaces

DEFINITION 3.6. Let X and Y be d-spaces. Let $\vec{\mathrm{map}}(X, Y)$ be the space of all d-maps from X to Y with the compact-open topology. A path $\alpha \in P(\vec{\mathrm{map}}(X, Y))$ is *directed* if its adjoint map $\vec{I} \times X \to Y$ is a d-map.

Definition 3.7. For arbitrary d-spaces X and Y let $X \tilde{\times} Y$ be the d-space with the underlying space $X \times Y$ and $\vec{P}(X \tilde{\times} Y)$ generated by the paths $(\alpha, \mathrm{const}_y)$ and $(\mathrm{const}_x, \beta)$ for $x \in X$, $y \in Y$, $\alpha \in \vec{P}(X)$, $\beta \in \vec{P}(Y)$.

Notice that a path $\omega \in P(X \times Y)$ is almost directed in $X \tilde{\times} Y$ iff it is directed in $X \times Y$. Equivalently, $Q(X \tilde{\times} Y) = X \times Y$.

Proposition 3.8. *Let X, Y be d-spaces. Assume that Y is good. Then $\alpha \in P(\vec{\mathrm{map}}(X, Y))$ is directed if and only if for every x the evaluation α_x is a d-path in Y.*

Proof. Fix $\alpha \in P(\vec{\mathrm{map}}(X, Y))$ and assume that $\alpha_x \in \vec{P}(Y)$ for every $x \in X$. Then the map $f : \vec{I} \tilde{\times} X \to Y$ adjoint to α is a d-map. By the goodness of Y, $f : \vec{I} \times X \to Y$ is a d-map. The converse is clear. ∎

Proposition 3.9. *Let X and Y be d-spaces. If Y is good, then* $\vec{\mathrm{map}}(X, Y)$ *is good.*

Proof. Let α be an almost directed path in $\vec{\mathrm{map}}(X, Y)$. Choose $x \in X$, $a < b \in [0, 1]$ and $U \in \mathfrak{T}(Y)$ such that $\alpha_x([a, b]) \subseteq U$. Since $\mathfrak{F}(\{x\}, U)$ is open in $\vec{\mathrm{map}}(X, Y)$ and α is almost directed, we see that there is a d-path $\beta \in \vec{P}(\mathfrak{F}(\{x\}, U))_{\alpha(i)}^{\alpha(j)}$. Hence $\beta_x \in \vec{P}(U)_{\alpha_x(i)}^{\alpha_x(j)}$. This proves that α_x is directed for every $x \in X$, and then by, 3.8, α is directed. ∎

Proposition 3.10. *The category $\mathcal{G}_k$ is cartesian closed.*

Proof. We have to prove that the functor $\vec{\mathrm{map}}(X, -)$ is adjoint to $X \times -$. For any good d-spaces X, Y, Z consider the pair of maps

$$\vec{\mathrm{map}}(X, \vec{\mathrm{map}}(Y, Z)) \underset{A}{\overset{B}{\leftrightarrows}} \vec{\mathrm{map}}(X \times Y, Z)$$

given by the obvious formulas $A(f)(x, y) = f(x)(y)$, $B(g)(x)(y) = g(x, y)$.

A is well-defined. Let $f \in \vec{\mathrm{map}}(X, \vec{\mathrm{map}}(Y, Z))$. We have $A(f)(\mathrm{const}_x, \beta) = f(x)(\beta) \in \vec{P}(Z)$ for $x \in X$ and $\beta \in \vec{P}(Y)$ (since $f(x) \in \vec{\mathrm{map}}(Y, Z)$) and $A(f)(\alpha, \mathrm{const}_y) = f(\alpha)(y) \in \vec{P}(Z)$ for $\alpha \in \vec{P}(X)$, $y \in Y$ (since it is the evaluation of the path $f(\alpha)$ at y). Therefore the map

$$\vec{I} \tilde{\times} \vec{I} \ni (s, t) \mapsto A(f)(\alpha(s), \beta(t)) \in \vec{\mathrm{map}}(X \times Y, Z)$$

is a d-map. The diagonal path $\vec{I} \ni t \mapsto (t, t) \in \vec{I} \tilde{\times} \vec{I}$ is almost directed; thus the path

$$\vec{I} \ni t \mapsto A(f)(\alpha(t), \beta(t)) \in \vec{\mathrm{map}}(X \times Y, Z)$$

is also almost directed (and hence directed by 3.9) for every $\alpha \in \vec{P}(X)$, $\beta \in \vec{P}(Y)$. Thus $\mathrm{Im}(A) \subseteq \vec{\mathrm{map}}(X \times Y, Z)$.

B is well-defined. Fix $g \in \vec{\mathrm{map}}(X \times Y, Z)$, $x \in X$ and $\beta \in \vec{P}(Y)$. We have $B(g)(x)(\beta) = g(\mathrm{const}_x, \beta) \in \vec{P}(Z)$; thus $B(g)(x) \in \vec{\mathrm{map}}(Y, Z)$. If $\alpha \in$

$\vec{P}(X)$ and $y \in Y$, then $B(g)(\alpha)(y) = g(\alpha, \mathrm{const}_y) \in \vec{P}(Z)$. Thus $B(g)(\alpha) \in \vec{P}(\mathrm{m\vec{a}p}(Y, Z))$ by 3.8.

Both A and B preserve paths. The map $\vec{P}(A)$ is the composition

$$\vec{P}(\mathrm{map}(X \times Y, Z)) = \mathrm{m\vec{a}p}(\vec{I} \times X \times Y, Z) \xrightarrow{A} \mathrm{map}(\vec{I} \times X, \mathrm{m\vec{a}p}(Y, Z)) = \vec{P}(X, \mathrm{m\vec{a}p}(Y, Z)))$$

and is well-defined (since so are A and B for all d-spaces). A similar argument shows that also $\vec{P}(B)$ is well-defined. ∎

4. Locally d-path-connected spaces. The motivation for introducing path-weak spaces is the following

EXAMPLE 4.1. Consider the set $I \times I$ with the d-structure

$$d := \{(\alpha, \beta) \in P(I \times I) : \alpha, \beta \in \vec{P}(\vec{I}) \wedge (\alpha, \beta)^{-1}(1/2, 1/2) \in \{\emptyset, I\}\}.$$

Let $\mathfrak{T}$ be the standard product topology on $I \times I$ and let $\mathfrak{T}'$ be the topology generated by $\mathfrak{T}$ and $\{(1/2, 1/2)\}$. Finally, let X and X' be d-spaces with the underlying set $I \times I$, the d-structure d and the topologies $\mathfrak{T}$ and $\mathfrak{T}'$ respectively. Both these d-spaces model the same computational problem (the middle point cannot be reached by a directed path from any other point) but are not equivalent. It seems that X' is a better space than X: in X the middle point is computationally distant but not topologically distant; in X it is just another component.

In this section we construct a colocalization which modifies topologies of d-spaces to ensure that "computationally distant" points are also "topologically distant".

DEFINITION 4.2. Let X be a topological space and let $\vec{P}(X)$ be a d-structure on X. The *inverse d-structure* $\vec{P}^{\mathrm{op}}(X)$ is defined by

$$\vec{P}^{\mathrm{op}}(X) := \{\alpha \in P(X) : \alpha^{\mathrm{op}} := (t \mapsto \alpha(1-t)) \in \vec{P}(X)\}.$$

The symmetric closure of $\vec{P}(X)$, denoted by $\vec{P}^s(X)$, is the smallest d-structure on X which contains both $\vec{P}(X)$ and $\vec{P}^{\mathrm{op}}(X)$.

REMARK. Let J_n be the interval $[0,1]$ with the d-structure generated by the paths $\beta_i : t \mapsto (2i+t)/n$ $(0 \le 2i < n)$ and $\gamma_j : t \mapsto (2j+1-t)/n$ $(0 \le 2j{+}1 \le n)$. Then $\alpha \in \vec{P}^s(X)$ iff it can be represented as a reparametrization of a d-map $J_n \to X$ for some n.

DEFINITION 4.3. A d-space X is *locally d-path-connected* (or is an *ldpc-space* for short) if for every $x \in X$ and every $U \in \mathfrak{T}(X)$ the path component of x in U,

$$C_x^U := \{y \in U : \exists_{\alpha \in \vec{P}^s(U)}\ \alpha(0) = x \wedge \alpha(1) = y\},$$

is open in X. Let $\mathbf{dTop}_p$ be the category of ldpc-spaces and d-maps.

We provide another definition of ldpc-spaces. We shall say that a subset A of a d-space X is *d-path-connected* if every pair of points of A can be connected by a symmetric d-path contained in A.

Proposition 4.4. *A d-space X is ldpc iff it has a basis consisting of d-path-connected sets.*

Proof. If X is ldpc, then $\{C_x^U : U \in \mathfrak{T}(X),\ x \in U\}$ is a d-path-connected basis. Conversely, if a d-space X has a d-path-connected basis, then for every $U \in \mathfrak{T}(X)$ and every $x \in U$ there exists a d-path-connected open set V such that $x \in V \subseteq U$. Obviously $V \subseteq C_x^U$ and then C_x^U is open. ■

For a d-space X let $\mathfrak{T}_{\mathfrak{W}}(X)$ be the smallest topology which contains the sets $\{C_x^U\}$ for every $U \in \mathfrak{T}(X)$ and $x \in U$. Let $W(X)$ denote the d-space with the same set of points as X, the same d-structure and the topology $\mathfrak{T}_{\mathfrak{W}}(X)$. It is easy to observe that all d-paths of X remain continuous when we pass to the topology $\mathfrak{T}_{\mathfrak{W}}$.

Proposition 4.5. *For every basis $B \subseteq \mathfrak{T}(X)$ the set $B_{\mathfrak{W}} := \{C_x^U\}_{U\in B}^{x\in U}$ is a basis of $\mathfrak{T}_{\mathfrak{W}}(X)$.*

Proof. Fix $U \in \mathfrak{T}_{\mathfrak{W}}$ and $x \in A$. There exists a finite collection $\{U_i\} \subseteq \mathfrak{T}(X)$, where $i = 1, \ldots, n$, such that $x \in \bigcap_{i=1}^n C_{x_i}^{U_i} \subseteq A$. Choose $V \in \mathfrak{B}$ such that $x \in V \subseteq \bigcap U_i$. We have

$$x \in C_x^V \subseteq C_x^{\bigcap U_i} \subseteq \bigcap C_x^{U_i} \subseteq U,$$

and obviously $C_x^V \in B_{\mathfrak{W}}$. ■

Corollary 4.6. *For every space X,*

$$(\mathfrak{T}_{\mathfrak{W}})_{\mathfrak{W}} = \mathfrak{T}_{\mathfrak{W}}.$$

Proof. Since $C_x^{C_y^U} = C_x^{C_x^U} = C_x^U$, the topologies $(\mathfrak{T}_{\mathfrak{W}})_{\mathfrak{W}}$ and $\mathfrak{T}_{\mathfrak{W}}$ have a common basis. ■

Proposition 4.7. *If $f : X \to Y$ is a d-map, then so is $f : W(X) \to W(Y)$.*

Proof. Fix $V \in \mathfrak{T}(Y)$ and $x \in f^{-1}(V)$. For every $x' \in C_x^{f^{-1}(V)}$ there exists a path $\alpha \in \vec{P}^s(f^{-1}(V))_x^{x'}$. Hence $f(\alpha) \in \vec{P}^s(V)$ connects $f(x)$ and $f(x')$. Therefore $C_x^{f^{-1}(V)} \subseteq f^{-1}(C_V^{f(x)})$. ■

As a consequence, $W : \mathbf{dTop} \to \mathbf{dTop}$ is a functor.

Proposition 4.8. *The functor W is a colocalization. A space X is W-colocal iff it is an ldpc-space.*

Proof. The augmentation $\eta_X : W(X) \to X$ is the identity map (as a map between sets). The idempotency is a consequence of 4.6, and the last statement follows immediately from the definition. ■

Examples

EXAMPLE 4.9. Let X be a topological space with trivial (i.e. minimal) d-structure. Then $W(X)$ is a discrete d-space.

EXAMPLE 4.10. Let X and X' be the d-spaces defined in Example 4.1. Then $W(X) = X'$.

EXAMPLE 4.11. Let $X = I \times I$ with the d-structure generated by the paths $t \mapsto (x,t)$ for $0 \le x \le 1$ and $t \mapsto (t,y)$ for $y \in \{0,1\}$. Then the topology on $W(X)$ is the richest topology such that these paths are continuous.

Limits and colimits

PROPOSITION 4.12. *A colimit* (*in* **dTop**) *of ldpc-spaces is an ldpc-space. In particular,* $\mathbf{dTop}_p$ *is cocomplete.*

Proof. For every family $\{X_i\}_{i\in I}$ of ldpc-spaces the union $\coprod X_i$ is obviously ldpc. Consider a pair of maps $f, g : X \to Y$ between ldpc-spaces. Let E be its coequalizer and $p : Y \to E$ be the obvious projection. Fix $z \in E$, $V \in \mathfrak{T}(E)$ and $y \in f^{-1}(C_z^V)$. Since $p(C_y^{p^{-1}(V)}) \subseteq C_{p(y)}^{p(V)} = C_z^{p(V)}$, we see that $C_y^{p^{-1}(V)}$ is an open neighbourhood of y contained in $f^{-1}(C_z^{p(V)})$. Hence $f^{-1}(C_z^{p(V)}) \in \mathfrak{T}(Y)$ and then $C_z^{p(V)} \in \mathfrak{T}(E)$. ∎

PROPOSITION 4.13. *Finite products of lpdc-spaces are ldpc.*

Proof. It is easy to check that $C_{(x_1,\ldots,x_n)}^{U_1\times\cdots\times U_n} = C_{x_1}^{U_1} \times \cdots \times C_{x_n}^{U_n}$ for every collection of d-spaces $X_1, \ldots, X_n$ with $x_i \in U_i \in \mathfrak{T}(X_i)$. ∎

In general, limits of lpdc-spaces are not necessarily ldpc. For example, the Cantor set $\{0,1\}^\omega$ (with its only, trivial d-structure) is not ldpc though its factors are. However, since W is a colocalization it preserves limits, therefore

PROPOSITION 4.14. *The category* $\mathbf{dTop}_p$ *is complete.* ∎

Let $\mathcal{G}_p$ be the category of good ldpc-spaces.

PROPOSITION 4.15. *The category* $\mathcal{G}_p$ *is complete and cocomplete.*

Proof. If X is a good d-space, then $W(X)$ is also good. Therefore $W|_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ is a colocalization and thus preserves limits, hence $\mathcal{G}_p$ is complete. On the other hand, the functor Q preserves ldpc-spaces, and consequently $Q|_{\mathbf{dTop}_p}$ is a localization and preserves colimits. Therefore $\mathcal{G}_p$ is complete. ∎

5. Compactly generated ldpc-spaces. Let $\mathbf{dTop}_{pk}$ be the category of compactly generated weakly Hausdorff ldpc-spaces. In this section, we prove that $\mathbf{dTop}_{pk}$ is complete, cocomplete and cartesian closed.

First, we construct a colocalization on **dTop** whose colocal objects are CGWH lpdc-spaces. For every ordinal number α let

$$W^\alpha(X) := \begin{cases} k(w(X)) & \text{if } \alpha = 0, \\ k(W(W^{\alpha-1}(X))) & \text{if } \alpha \text{ is a successor ordinal}, \\ \lim_{\beta<\alpha} W^\beta(X) & \text{if } \alpha \text{ is a limit ordinal}. \end{cases} \tag{5.1}$$

If $\alpha < \beta$, then $\mathfrak{T}(W^\alpha) \subseteq \mathfrak{T}(W^\beta)$. Therefore the sequence W^α must eventually stabilize; denote its limit by $W^\infty(X)$.

Proposition 5.2. *The functor $W^\infty(X)$: **dTop** $\to$ **dTop** is a colocalization. A d-space X is W-colocal iff it is a compactly generated weakly Hausdorff ldpc-space.*

Proof. Since both functors k and W preserve weakly Hausdorff spaces, we find that $W^\infty(X)$ is weakly Hausdorff. It is obviously both compactly generated and ldpc. ∎

Proposition 5.3. *The category $\mathbf{dTop}_{pk}$ is complete and cocomplete.*

Proof. The completeness is a consequence of the completeness of **dTop** and the existence of the colocalization W^∞. By 4.12 colimits of ldpc-spaces are ldpc, and colimits of compactly generated spaces are again compactly generated. Moreover, both these classes are preserved by the functor w (cf. (3.4)). Then $\lim_{\mathbf{dTop}_{pk}} A = w \circ \lim_{\mathbf{dTop}} A$ for every diagram $A : \mathcal{C} \to \mathbf{dTop}_{pk}$. As a consequence, $\mathbf{dTop}_{pk}$ is cocomplete. ∎

Mapping spaces

Proposition 5.4. *Let X, Y be ldpc-spaces. Then the map*

$$W(\vec{\mathrm{map}}(X, W(Y))) \to W(\vec{\mathrm{map}}(X, Y))$$

is a d-homeomorphism.

Proof. By the universal property of W the map

$$\Phi : \vec{\mathrm{map}}(X, W(Y)) \to \vec{\mathrm{map}}(X, Y)$$

is a bijection. Obviously it is continuous, since the topology of $W(Y)$ is richer than the topology of Y. Every directed path in $\vec{P}(\vec{\mathrm{map}}(X, Y))$ is represented by a d-map $\vec{I} \times X \to Y$ which factors uniquely through $W(Y)$ (since $\vec{I} \times X$ is ldpc (cf. 4.13)). This implies that it is a d-map. The only non-trivial part is to prove that Φ is open. Fix a d-map $f : X \to Y$, a compact subset $K \subseteq X$, an open subset $V \subseteq Y$ and $y \in V$. Assume that $f(K) \subseteq V$ and fix $k \in K$. We will prove that

$$C_f^{\mathfrak{F}(K,V)} \subseteq C_f^{\mathfrak{F}(K, C^V_{f(k)})}$$

(the reverse inclusion is obvious). If $g : X \to Y$ belongs to $C_f^{\mathfrak{F}(K,V)}$, then there is a d-map

$$H : J_n \times X \to Y$$

such that J_n is the interval with a d-structure defined in §4, $H(0,x) = g(x)$, $H(1,x) = f(x)$ and $H(t,x) \in V$ for every $x \in K$, $t \in J_n$. In particular, for every $x \in K$ the restriction $H(-,x)$ is a symmetric directed path contained in V and connecting $f(x)$ and $g(x)$. As a consequence $g(x) \in C_{f(k)}^V$ (since $f(x) \in C_{f(k)}^V$)) and therefore $g \in C_f^{\mathfrak{F}(K,C_y^V)}$. ∎

For $X, Y \in \mathbf{dTop}_{pk}$ define

$$\vec{\mathrm{map}}_{pk}(X,Y) := W^\infty(\vec{\mathrm{map}}(X,Y)) \tag{5.5}$$

PROPOSITION 5.6. *If $X, Y \in \mathbf{dTop}_{pk}$, then both maps*

$$\mathrm{inj}_{X,Y} : Y \ni y \mapsto \mathrm{const}_y \in \vec{\mathrm{map}}_{pk}(X, W^\infty(X \times Y)),$$
$$\mathrm{ev}_{X,Y} : X \times \vec{\mathrm{map}}_{pk}(X,Y) \ni (x,f) \mapsto f(x) \in Y$$

are d-maps.

Proof. The map $\mathrm{ev}_{X,Y}$ is the composition

$$X \times \vec{\mathrm{map}}_{pk}(X,Y) = X \times W^\infty(\vec{\mathrm{map}}(X,Y)) \xrightarrow{\mathrm{Id}_X \times \eta} X \times \vec{\mathrm{map}}(X,Y) \xrightarrow{\mathrm{ev}} Y$$

and hence is continuous. The map $\mathrm{inj}_{X,Y}$ is continuous iff the map

$$X \to \vec{\mathrm{map}}(Y, W^\infty(X \times Y))$$

is continuous. We will prove inductively that

$$\mathrm{inj}_{X,Y}^\alpha : X \to \vec{\mathrm{map}}(Y, W^\alpha(X \times Y))$$

is continuous for every ordinal α—this is sufficient since the sequence W^α stabilizes. If $\alpha = 0$, then the claim follows from the cartesian closedness of the category of compactly generated weakly Hausdorff d-spaces (see 3.10). If α is a successor, then there is a d-homeomorphism

$$W(\vec{\mathrm{map}}(X, W(W^{\alpha-1}(X \times Y)))) \to W(\vec{\mathrm{map}}(X, W^{\alpha-1}(X \times Y)));$$

therefore (by 5.4) $\mathrm{inj}_{X,Y}^{\alpha-1}$ factors through $\vec{\mathrm{map}}(X, W(W^{\alpha-1}(X \times Y)))$. Similarly, there is a d-homeomorphism

$$k(\vec{\mathrm{map}}(X, k(W(W^{\alpha-1}(X \times Y))))) \to k(W(\vec{\mathrm{map}}(X, W^{\alpha-1}(X \times Y)))),$$

which proves that it factors through

$$\vec{\mathrm{map}}(X, k(W(W^{\alpha-1}(X \times Y)))) = \vec{\mathrm{map}}(X, W^\alpha(X \times Y)).$$

Finally, if α is a limit ordinal, then every set U which is open in $W^\alpha(\vec{\mathrm{map}}(X,Y))$ is also open in $W^\beta(\vec{\mathrm{map}}(X,Y))$ for some $\beta < \alpha$. By the induction hypothesis, $\mathrm{inj}_{X,Y}^{-1}(U)$ is open in X. The induction step is complete. ∎

Proposition 5.7. *The maps*

$$A : \vec{\mathrm{map}}_{pk}(X \times_{pk} Y, Z) \leftrightarrow \vec{\mathrm{map}}_{pk}(X, \vec{\mathrm{map}}_{pk}(Y, Z)) : B,$$

where $A(f)(x)(y) = f(x, y)$, $B(g)(x, y) = g(x)(y)$, *are d-homeomorphisms.*

Proof. Let $X, Y, Z \in \mathrm{Ob}(\mathbf{dTop}_{pk})$. First, we will prove A and B are well-defined. If $f \in \vec{\mathrm{map}}_{pk}(X \times_{pk} Y, Z)$, then $A(f)$ is the composition

$$X \xrightarrow{\mathrm{inj}_{X,Y}} \vec{\mathrm{map}}_{pk}(Y, X \times_{pk} Y) \xrightarrow{\vec{\mathrm{map}}(Y,f)} \vec{\mathrm{map}}_{pk}(Y, Z)$$

and is a d-map by 5.6. Similarly, if $g \in \vec{\mathrm{map}}_{pk}(X, \vec{\mathrm{map}}_{pk}(Y, Z))$, then $B(g)$ is the composition

$$X \times Y \xrightarrow{g \times \mathrm{Id}_Y} \vec{\mathrm{map}}_{pk}(Y, Z) \times Y \xrightarrow{\mathrm{ev}_{Y,Z}} Z$$

and again it is a d-map for the same reason. We have proven that A and B are mutually inverse bijections. For any objects W, X, Y and Z there are natural bijections

$$\begin{aligned} \vec{\mathrm{map}}_{pk}(W, \vec{\mathrm{map}}_{pk}(X, \vec{\mathrm{map}}_{pk}(Y, Z))) &\simeq \vec{\mathrm{map}}_{pk}(W \times_{pk} X, \vec{\mathrm{map}}_{pk}(Y, Z)) \\ &\simeq \vec{\mathrm{map}}_{pk}(W \times_{pk} X \times_{pk} Y, Z) \simeq \vec{\mathrm{map}}_{pk}(W, \vec{\mathrm{map}}_{pk}(X \times_{pk} Y, Z)). \end{aligned}$$

Therefore $\vec{\mathrm{map}}_{pk}(X, \vec{\mathrm{map}}_{pk}(Y, Z))$ and $\vec{\mathrm{map}}_{pk}(X \times_{pk} Y, Z)$ represent the same contravariant functor $\mathbf{dTop}_{pk} \to \mathbf{Set}$. By Yoneda's lemma they are isomorphic. ∎

Corollary 5.8. *The category* $\mathbf{dTop}_{pk}$ *is cartesian closed; the exponential object of* X *and* Y *is* $\vec{\mathrm{map}}_{pk}(X, Y)$.

Good ldpc-spaces. Let $\mathcal{G}_{pk}$ be the category of good compactly generated, weakly Hausdorff ldpc-spaces.

Proposition 5.9. *The category* $\mathcal{G}_{pk}$ *is complete, cocomplete and cartesian closed.*

Proof. Since mapping spaces of good d-spaces are good (cf. 3.9), and goodness is preserved by the functors k and W, all the statements above concerning the category $\mathbf{dTop}_{pk}$ remain true for $\mathcal{G}_{pk}$. ∎

6. Localizations and colocalizations

Localizations. Let $\mathcal{C}$ be a category. A *localization* of $\mathcal{C}$ is a pair (L, η), where $L : \mathcal{C} \to \mathcal{C}$ is a functor and $\eta : 1_{\mathcal{C}} \to L$ a natural transformation (called *coaugmentation*) such that for every $X \in \mathcal{C}$ the maps

(6.1) $$L(\eta_X), \eta_{L(X)} : L(X) \to L(L(X))$$

are equal isomorphisms. We say that $X \in \mathcal{C}$ is L-*local* if $\eta_X : X \to L(X)$ is an isomorphism. Let $L(\mathcal{C})$ be a (full) subcategory of L-local objects of $\mathcal{C}$.

Universal property. If $X, Y \in \mathcal{C}$ and Y is L-local, then the map

$$\mathrm{Mor}(LX, Y) \xrightarrow{\eta_X^*} \mathrm{Mor}(X, Y)$$

is a bijection. Equivalently, every map from X into an L-local object factors uniquely through $L(X)$. The inverse assigns to each $f : X \to Y$ a morphism $\eta_Y^{-1} \circ L(F)$.

Adjointness. Every localization functor $L : \mathcal{C} \to \mathcal{C}$ is left adjoint to the inclusion $I : L(\mathcal{C}) \to C$. On the other hand, if $I : \mathcal{D} \to \mathcal{C}$ is a full subcategory, then its left adjoint functor L (if it exists) is a localization (the coaugmentation η_Y is adjoint to 1_{LX}) and $\mathcal{D}$ is a class of L-local objects.

Colimits. Since left adjoint functors preserve colimits, for each small category $\mathcal{A}$ we have $\mathrm{colim}_{\mathcal{A}}^{L(\mathcal{C})} = L \circ \mathrm{colim}_{\mathcal{A}}^{\mathcal{C}}$. Therefore if $\mathcal{C}$ is cocomplete, then so is $L(\mathcal{C})$.

Colocalizations. Colocalizations are dual to localizations, i.e. a *colocalization* is a functor $L : \mathcal{C} \to \mathcal{C}$ with an augmentation $\eta : L \to 1_{\mathcal{C}}$. A colocalization is right adjoint to the embedding of the category of L-colocal objects; hence they preserve limits.

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