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Ordinal remainders of classical ψ-spaces

by

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Abstract. Let ω denote the set of natural numbers. We prove: for every mod-finite ascending chain $\{T_\alpha : \alpha < \lambda\}$ of infinite subsets of ω, there exists $\mathcal{M} \subset [\omega]^\omega$, an infinite maximal almost disjoint family (MADF) of infinite subsets of the natural numbers, such that the Stone–Čech remainder $\beta\psi \setminus \psi$ of the associated ψ-space, $\psi = \psi(\omega, \mathcal{M})$, is homeomorphic to $\lambda + 1$ with the order topology. We also prove that for every $\lambda < \mathfrak{t}^+$, where $\mathfrak{t}$ is the tower number, there exists a mod-finite ascending chain $\{T_\alpha : \alpha < \lambda\}$, hence a ψ-space with Stone–Čech remainder homeomorphic to $\lambda + 1$. This generalizes a result credited to S. Mrówka by J. Terasawa which states that there is a MADF $\mathcal{M}$ such that $\beta\psi \setminus \psi$ is homeomorphic to $\omega_1 + 1$.

1. Introduction. Let ω denote the set of natural numbers. Let $[\omega]^\omega$ denote the set of all countably infinite subsets of ω. Sets $A, B \in [\omega]^\omega$ are said to be *almost disjoint* provided $A \cap B$ is finite. An infinite family $\mathcal{A} \subset [\omega]^\omega$ is called an *almost disjoint family* (ADF) if any two elements of $\mathcal{A}$ are almost disjoint. An ADF $\mathcal{M}$ is called a *maximal almost disjoint family* (MADF) if it is not properly contained in another ADF.

We have considered almost disjoint families of countable subsets of an arbitrary cardinal κ in [4], [5], but in this paper we only consider the classical case $\kappa = \omega$.

Almost disjoint families, especially MADF's, are of interest in set theory (e.g., [8], [14]), topology (e.g., [6], [13]), Boolean algebra (e.g., [1], [3]) and Banach spaces (e.g., [10], [11]).

An important interest in topology of MADF's concerns the well-known class of topological spaces called ψ-spaces (e.g., see [2, §11]; for some historical notes, see [4, §2]). For any ADF $\mathcal{A} \subset [\omega]^\omega$, let $\psi(\omega, \mathcal{A})$ denote the space with underlying set $\omega \cup \mathcal{A}$ and with the topology having as a base all singletons $\{\alpha\}$ for $\alpha < \omega$ and all sets of the form $\{A\} \cup (A \setminus F)$ where

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$A \in \mathcal{A}$ and F is finite. We call $\psi(\omega, \mathcal{A})$ the ψ-space associated with $\mathcal{A}$, and the class of spaces of the form $\psi(\omega, \mathcal{A})$ the class of *ψ-spaces on ω* (or *on* $[\omega]^\omega$). It is well-known that the topology of $\psi(\omega, \mathcal{A})$ is closely related to the ADF $\mathcal{A}$, however all ψ-spaces on ω, regardless of which ADF $\mathcal{A}$ is used, are first countable, zero-dimensional, Hausdorff, locally compact non-compact spaces in which ω is an open dense set of isolated points (hence the ψ-space is separable), and $\psi \setminus \omega = \mathcal{A}$ is a closed discrete set. Maximal ADF's are of additional interest because $\mathcal{M}$ is maximal if and only if the ψ-space associated with $\mathcal{M}$ is pseudocompact (i.e., every real-valued continuous function defined on ψ is bounded [12]).

For $A, B \in [\omega]^\omega$, $A \subset^* B$ means that $A \setminus B$ is finite, and $A <^* B$ means that $A \subset^* B$ and $B \setminus A$ is infinite. Let $X \in [\omega]^\omega$. A *mod-finite ascending chain of order type λ in X* (*chain* for short) is a family $\{T_\alpha : \alpha < \lambda\} \subset [\omega]^\omega$, where λ is an ordinal, such that $T_\alpha <^* X$ for all $\alpha < \lambda$, and $\beta < \alpha < \lambda$ imply $T_\beta <^* T_\alpha$. We also say "chain indexed by λ" when the particular ordinal λ is needed. Let $\mathfrak{c}$ denote the *cardinality of the continuum*, and let $\mathfrak{t}$ denote the *tower number*, i.e., the smallest cardinality of a tower in $[\omega]^\omega$ (e.g., see [2] or [16]).

We use the approach to Stone–Čech compactifications in [7]. For $E \subset \psi$, we let $\overline{E}$ denote the closure of E in $\beta\psi$. The closure of E in ψ will be denoted by $\mathrm{cl}_\psi(E)$.

J. Terasawa [15] proved that every compact metric space is the Stone–Čech remainder of a ψ-space on ω for a suitably chosen MADF $\mathcal{M}$, and he stated that the ordinal $\omega_1 + 1$ with the order topology is also the remainder of a ψ-space. He attributed that result to S. Mrówka. Our main theorem generalizes this result as follows.

THEOREM 1.1. *If there exists a chain $\{T_\alpha : \alpha < \lambda\}$ in $[\omega]^\omega$ indexed by the ordinal λ, then there exists a MADF $\mathcal{M} \subset [\omega]^\omega$ such that $\beta\psi \setminus \psi$ is homeomorphic to $\lambda + 1$ with the order topology.*

Concerning the existence of ascending chains, we prove

THEOREM 1.2. *If $\lambda < \mathfrak{t}^+$, then there exists an ascending ordered chain in ω of order type λ. Thus there exists a MADF $\mathcal{M} \subset [\omega]^\omega$ such that $\beta\psi \setminus \psi \cong \lambda + 1$.*

Combining these two theorems we get

THEOREM 1.3. *For every successor ordinal $\lambda + 1 < \mathfrak{t}^+$ there exists a MADF $\mathcal{M} \subset [\omega]^\omega$ such that the Stone–Čech remainder of $\psi(\omega, \mathcal{M})$ is homeomorphic to $\lambda + 1$ with the order topology.*

Theorem 1.3 implies the result attributed to Mrówka because, as is well known, $\mathfrak{t}$ is an uncountable cardinal, hence $\omega_1 + 1 < \mathfrak{t}^+$.

Theorem 1.3 is the best possible in ZFC in the sense that it is consistent that the ordinal $\mathfrak{t}^+ + 1$ is not the Stone–Čech remainder of any ψ-space on ω. This follows because a compact separable space has weight at most $\mathfrak{c}$ [9, 2.3(i)]. Hence a compact separable space cannot contain a subspace with $\mathfrak{c}^+$ isolated points; in particular cannot contain a copy of the ordinal $\mathfrak{c}^+ + 1$. Hence in any model where $\mathfrak{t} = \mathfrak{c}$, also $\mathfrak{t}^+ = \mathfrak{c}^+$, and this implies that $\mathfrak{t}^+ + 1$ is not the Stone–Čech remainder of any ψ-space on ω.

We establish the homeomorphism between the Stone–Čech remainder, $\beta\psi \setminus \psi$, and $\lambda + 1$ using the following simple result.

Lemma 1.4. *Let X be a Hausdorff space and $\{O_\alpha : \alpha < \lambda\}$ a cover of X by compact, clopen sets such that, for all $\beta < \alpha < \lambda$,*

(1) *$O_\beta \subset O_\alpha$, and*
(2) *$|O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta| = 1$ for $\alpha < \lambda$.*

Then X is homeomorphic to λ with the order topology.

Proof. For $\alpha < \lambda$, let $x_\alpha \in X$ be the point such that $\{x_\alpha\} = O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta$, and define $\varphi : \lambda \to X$ by $\varphi(\alpha) = x_\alpha$. Clearly φ is one-to-one and onto X. By the compactness of O_α, the decreasing family $\{O_\alpha \setminus O_\beta : \beta < \alpha\}$ is a local base for x_α in X. We show that φ is a homeomorphism by showing that $\varphi((\gamma, \alpha]) = O_\alpha \setminus O_\gamma$ for all $\gamma < \alpha < \lambda$. If $x \in \varphi((\gamma, \alpha])$, there exists $\gamma < \beta \le \alpha$ such that $x = \varphi(\beta) = x_\beta$. Since $x_\beta \in O_\beta$ and, by (1), $O_\beta \subset O_\alpha$, we see that $\varphi(\beta) \in O_\alpha$. For $\gamma < \beta$, $x_\beta \notin O_\gamma$. Thus $\varphi(\beta) = x_\beta \in O_\alpha \setminus O_\gamma$. Conversely, let $x \in O_\alpha \setminus O_\gamma$, and let $\beta \le \alpha$ be the first ordinal such that $x \in O_\beta$. Then $x \in O_\beta \setminus \bigcup_{\tau<\beta} O_\tau$; so $x = x_\beta$. We have $\gamma < \beta$, so $\gamma < \beta \le \alpha$ and $x = \varphi(\beta)$. Thus $x \in \varphi((\gamma, \alpha])$. This proves the equality $\varphi((\gamma, \alpha]) = O_\alpha \setminus O_\gamma$, and shows that φ is a homeomorphism.

Lemma 1.5. *If Z is a zero set in $\beta\psi$, then $\overline{Z \cap \mathcal{M}} \supset Z \cap (\beta\psi \setminus \psi)$, and if $Z \cap (\beta\psi \setminus \psi) \neq \emptyset$, then $Z \cap \mathcal{M}$ is infinite.*

Proof. Let $p \in Z \cap (\beta\psi \setminus \psi)$. Then p is a free z-ultrafilter on ψ. Let Z' be any zero-set neighborhood of p in $\beta\psi$. Since ψ is pseudocompact, the z-ultrafilter p is countably complete, i.e., if $Z_n \in p$ for $n \in \omega$, then $\bigcap_{n\in\omega} Z_n \in p$ ([7, 8.6]). For $n \in \omega$, $Z_n = \psi \setminus \{n\}$ is a (clopen) zero set in ψ, and since p is free, $\psi \setminus \{n\} \in p$. Now we have $Z'' = Z \cap Z' \cap \bigcap_{n\in\omega} Z_i \in p$, and $Z'' \subset \mathcal{M}$. This shows that the zero-set neighborhood Z' of p has a non-empty intersection with $Z \cap \mathcal{M}$. Since Z' was arbitrary, it follows that $p \in \overline{Z \cap \mathcal{M}}$. The second statement in the lemma follows from the first.

Let $F \in [\omega]^\omega$ and $\{T_\alpha : \alpha < \lambda\}$ be a chain in X. We say that the set F *diagonalizes the chain in X* (or *in* $[X]^\omega$) provided $F \subset^* X$ and for all $\beta < \alpha$, $|F \cap T_\beta| < \omega$. If $\mathcal{F} \subset [\omega]^\omega$ is a family of sets, we say $\mathcal{F}$ *diagonalizes the chain in X* provided each set $F \in \mathcal{F}$ diagonalizes the chain in X. We note that there exists F that diagonalizes a chain $\{T_\beta : \beta < \alpha\}$ in X if and only if the chain is a not a tower in X (consider $X \setminus F$).

Given disjoint sets $A, B \in [\omega]^\omega$ and $\mathcal{A} \subset [A]^\omega$ ADF, $\mathcal{B} \subset [B]^\omega$ ADF with $|A| = |B|$, the *(Mrówka) join* of $\mathcal{A}$ and $\mathcal{B}$ is defined by $\mathcal{A} \oplus \mathcal{B} = \{A \cup \phi(A) : A \in \mathcal{A}\}$, where $\phi : \mathcal{A} \to \mathcal{B}$ is a bijection (see [4, §4]). Clearly $\mathcal{A} \oplus \mathcal{B}$ is an ADF and is maximal if both $\mathcal{A}$ and $\mathcal{B}$ are maximal.

We call a MADF $\mathcal{M}$ a *Mrówka MADF* provided $|\beta\psi \setminus \psi| = 1$. As proved by S. Mrówka in [13], there exists $\mathcal{M} \subset [\omega]^\omega$ such that $|\beta\psi \setminus \psi| = 1$.

LEMMA 1.6. *Let $N, \omega \setminus N \in [\omega]^\omega$, $\mathcal{B} \subset [N]^\omega$ be a Mrówka MADF, and $\mathcal{A} \subset [\omega \setminus N]^\omega$ an ADF such that $|\mathcal{B}| = |\mathcal{A}| \geq \omega$. If $\mathcal{M} \subset [\omega]^\omega$ is a MADF and $\mathcal{A} \oplus \mathcal{B} \subset \mathcal{M}$ then $\mathcal{A} \oplus \mathcal{B}$ has exactly one limit point in $\beta\psi$, where $\psi = \psi(\omega, \mathcal{M})$.*

Proof. The proof is similar to that of [5, Theorem 1.3, Case 3].

2. Proofs of the main results

Proof of Theorem 1.1. Let $\{T_\beta : \beta < \lambda\}$ be a chain in T_λ. For convenience we identify T_λ with ω. We must construct a MADF $\mathcal{M} \subset [\omega]^\omega$ such that $\beta\psi \setminus \psi$ is homeomorphic to $\lambda + 1$. For λ finite, say $\lambda = n \geq 1$, we prove the theorem by taking n disjoint copies of ω each with a Mrówka MADF (i.e., $|\beta\psi \setminus \psi| = 1$). Then taking unions yields the theorem. If we prove the theorem for a limit ordinal λ, then we get it for successor ordinals, $\lambda + 1, \lambda + 2, \ldots$, because all these ordinal spaces are homeomorphic to $\lambda + 1$. Thus we assume that $\{T_\beta : \beta < \lambda\}$ is a chain and λ is a limit ordinal.

We want our chain to have the following property:

(∗) For $\alpha \leq \lambda$, if $\mathrm{cf}(\alpha) > \omega$ and $\{T_\beta : \beta < \alpha\}$ is not a tower in T_α, then it is not a tower in any $H <^* T_\alpha$.

The given chain $\{T_\beta : \beta < \lambda\}$ may not have this property, so we adjust it: For every $\alpha < \lambda$, if there exists $H <^* T_\alpha$ such that $\{T_\beta : \beta < \alpha\}$ is a tower in H (hence $\mathrm{cf}(\alpha) > \omega$), then we change the definition of T_α by replacing T_α with H (then $T_\alpha = H$). After this change, $\{T_\beta : \beta < \alpha\}$ is a tower in T_α. For those $\alpha < \lambda$ for which this change is made, T_α becomes a bit smaller than it was and the ring $T_{\alpha+1} \setminus T_\alpha$ becomes a bit larger. No change is made for any other $\alpha < \lambda$. This adjusted chain clearly has property (∗).

Now we proceed to the construction of the MADF $\mathcal{M}$. For $\alpha \leq \lambda$ define

$$\xi_\alpha = \{\mathcal{A} \subset [T_\alpha]^\omega : \mathcal{A} \text{ is an ADF diagonalizing } \{T_\beta : \beta < \alpha\} \text{ in } T_\alpha\}.$$

Note that

$$\xi_\alpha = \emptyset \iff \{T_\beta : \beta < \alpha\} \text{ is a tower in } [T_\alpha]^\omega.$$

When $\xi_\alpha \neq \emptyset$, we partially order ξ_α by set inclusion. Then by Zorn's Lemma, there is a family $\mathcal{N}_\alpha \in \xi_\alpha$ such that $\mathcal{N}_\alpha$ is a maximal element in the poset ξ_α. Since there exists a MADF of cardinality $\mathfrak{c}$, we may assume that $|\mathcal{N}_\alpha| = \mathfrak{c}$. Pick $N_\alpha \in \mathcal{N}_\alpha$ and let $\mathcal{B}_\alpha \subset [N_\alpha]^\omega$ be a Mrówka MADF with $|\mathcal{B}_\alpha| = \mathfrak{c}$. Since $\mathcal{N}_\alpha$ diagonalizes the chain, $\mathcal{B}_\alpha$ diagonalizes the chain. Then $\mathcal{B}_\alpha \oplus (\mathcal{N}_\alpha \setminus \{N_\alpha\})$ is an ADF, and diagonalizes the chain $\{T_\beta : \beta < \alpha\}$ (obviously if each of two sets diagonalizes a chain, then their union also does).

As an intermediate step before we get to $\mathcal{M}$ we define families C_α.

(1_α) If $\xi_\alpha \neq \emptyset$ we define $\mathcal{C}_\alpha = \mathcal{B}_\alpha \oplus (\mathcal{N}_\alpha \setminus \{N_\alpha\})$ as described above. This case includes all successor ordinals, all ordinals of countable cofinality, and all ordinals of uncountable cofinality for which $\{T_\beta : \beta < \alpha\}$ is not a tower in T_α, hence not a tower in any $H <^* T_\alpha$ (by the adjustment we made to the tower at the beginning of the proof).

(2_α) If $\xi_\alpha = \emptyset$, then we define $\mathcal{C}_\alpha = \emptyset$.

We note that if $\mathcal{C}_\alpha \neq \emptyset$ then $|\mathcal{C}_\alpha| = \mathfrak{c}$.

Claim 1. *For all $\alpha \leq \lambda$, $\mathcal{C}_\alpha \subset [T_\alpha]^\omega$ and $\mathcal{C}_\alpha$ diagonalizes the chain $\{T_\beta : \beta < \alpha\}$ in T_α.*

Proof. If $\mathcal{C}_\alpha = \emptyset$, there is nothing to prove. If $\mathcal{C}_\alpha \neq \emptyset$, then the inclusion $\mathcal{C}_\alpha \subset [T_\alpha]^\omega$ follows because $\mathcal{N}_\alpha \subset [T_\alpha]^\omega$, $\mathcal{B}_\alpha \subset [N_\alpha]^\omega$ and $N_\alpha \in \mathcal{N}_\alpha$. Since $\mathcal{C}_\alpha = \mathcal{B}_\alpha \oplus (\mathcal{N}_\alpha \setminus \{N_\alpha\})$ is a join of two ADF's each of which diagonalizes the chain $\{T_\beta : \beta < \alpha\}$ in T_α, we see that $\mathcal{C}_\alpha$ diagonalizes the chain.

Claim 2. *For all $\alpha \leq \lambda$, if $H \in [T_\alpha]^\omega$ diagonalizes $\{T_\beta : \beta < \alpha\}$ in T_α then there exists $C \in \mathcal{C}_\alpha$ such that $|C \cap H| = \omega$.*

Proof. Assume H diagonalizes $\{T_\beta : \beta < \alpha\}$ in T_α. The existence of H implies that $\xi_\alpha \neq \emptyset$. Hence $\mathcal{C}_\alpha$ was defined by (1_α). Therefore the maximal element $\mathcal{N}_\alpha$ of ξ_α was used in the definition of $\mathcal{C}_\alpha$. We cannot have both $H \notin \mathcal{N}_\alpha$ and $\mathcal{N}_\alpha \cup \{H\}$ an ADF because that would contradict the maximality of $\mathcal{N}_\alpha$ in ξ_α. Either way, there is $N \in \mathcal{N}_\alpha$ such that $|H \cap N| = \omega$, hence there exists $C \in \mathcal{C}_\alpha$ such that $|C \cap H| = \omega$.

Claim 3. *The family $\{C \cap T_\alpha : C \in \bigcup_{\beta \leq \alpha} \mathcal{C}_\beta\}$ is a MADF on T_α.*

Proof. If $H \in [T_\alpha]^\omega$, then there exists a first $\beta \leq \alpha$ such that $|H \cap T_\beta| = \omega$. Then $H \cap T_\beta$ diagonalizes $\{T_\xi : \xi < \beta\}$ in T_β; so by Claim 2, there exists $C \in \mathcal{C}_\beta$ such that $|C \cap H \cap T_\beta| = \omega$. Hence $|C \cap H| = \omega$.

We define
$$\mathcal{M} = \bigcup_{\alpha \leq \lambda} C_\alpha.$$

Claim 4. *$\mathcal{M} \subset [\omega]^\omega$ is a MADF.*

Proof. This follows from Claim 3 and the fact that $T_\lambda = \omega$.

We now consider the topology of $\psi = \psi(\omega, \mathcal{M})$, the ψ-space associated with the MADF $\mathcal{M}$ constructed above. We want to show that $\beta\psi \setminus \psi$ is homeomorphic to $\lambda + 1$ with the order topology. We first establish some useful facts concerning the topology on ψ and $\beta\psi$.

Claim 5. *For $\alpha \le \lambda$, $\overline{T_\alpha} \cap \mathcal{M} \subset \bigcup_{\beta\le\alpha} \mathcal{C}_\alpha$, $\mathrm{cl}_\psi(T_\alpha)$ is clopen in ψ, hence $\overline{T_\alpha}$ is clopen in $\beta\psi$.*

Proof. Let $M \in \overline{T_\alpha} \cap \mathcal{M}$; then $M \cap T_\alpha$ is infinite by the definition of the topology on ψ. By Claim 3, there exists $C \in \bigcup_{\beta\le\alpha} \mathcal{C}_\alpha$ such that $|C \cap (M \cap T_\alpha)| = \omega$. Since $C, M \in \mathcal{M}$, an almost disjoint family, we have $C = M$; so $M \in \bigcup_{\beta\le\alpha} \mathcal{C}_\alpha$. To see that $\mathrm{cl}_\psi(T_\alpha)$ is clopen in ψ, it suffices to to show this set is open since it is closed by definition. We may assume $\alpha < \lambda$ since $\mathrm{cl}_\psi(T_\lambda) = \mathrm{cl}_\psi(\omega) = \psi$ is clopen in ψ. Let $M \in \mathrm{cl}_\psi(T_\alpha) \cap \mathcal{M}$, hence $M \in \overline{T_\alpha} \cap \mathcal{M}$, so by the first part of Claim 5, $M \in \bigcup_{\beta\le\alpha} \mathcal{C}_\alpha$. Thus by Claim 1, for some $\beta \le \alpha$, $M \subset T_\beta \subset^* T_\alpha$, hence for some finite set F we find that $\{M\} \cup (M \setminus F)$ is a neighborhood of M contained in $\mathrm{cl}_\psi(T_\alpha)$. Thus $\mathrm{cl}_\psi(T_\alpha)$ is clopen in ψ, and therefore $\overline{T_\alpha} = \overline{\mathrm{cl}_\psi(T_\alpha)}$ is clopen in $\beta\psi$ [7, p. 90]. This completes the proof of Claim 5.

In preparation for using Lemma 1.4, we define

$$O_\alpha = \overline{T_\alpha} \cap (\beta\psi \setminus \psi) = \overline{\mathrm{cl}_\psi(T_\alpha)} \cap (\beta\psi \setminus \psi) \quad \text{for } \alpha \le \lambda.$$

Then $\{O_\alpha : \alpha \le \lambda\}$ is an increasing family of compact clopen sets in $\beta\psi \setminus \psi$, $O_\lambda = \beta\psi \setminus \psi$, and satisfies condition (1) in Lemma 1.4. To complete the proof of the theorem, we show that condition (2) also holds for this family, i.e., we show that

$$\Big| O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta \Big| = 1 \quad \text{for } \alpha \le \lambda.$$

First we show that $O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta$ contains at least one point.

Claim 6. *If $\beta < \alpha \le \lambda$ then $O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta \ne \emptyset$.*

Proof. Since $\{O_\alpha \setminus O_\beta : \beta < \alpha\}$ is a decreasing family of compact sets, it suffices to show that $O_\alpha \setminus O_\beta \ne \emptyset$ for all $\beta < \alpha$. If $\mathcal{C}_\alpha \ne \emptyset$ then $\mathcal{C}_\alpha$ is an infinite (in fact uncountable) subset of $\mathrm{cl}_\psi(T_\alpha) \setminus \mathrm{cl}_\psi(T_\beta)$; hence $\overline{T_\alpha} \setminus \overline{T_\beta} \ne \emptyset$. If $\mathcal{C}_\alpha = \emptyset$, then α is a limit ordinal (in fact $\mathrm{cf}(\alpha) > \omega$), hence $\beta + 1 < \alpha$. Since $\mathcal{C}_{\beta+1} \ne \emptyset$, we have $\emptyset \ne O_{\beta+1} \setminus O_\beta \subset O_\alpha \setminus O_\beta$.

The remainder of the proof is devoted to showing that $O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta$ contains at most one point.

Claim 7. For all $\alpha \le \lambda$, $|O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta| \le 1$.

Proof. Assume that for all $\beta < \alpha$ we have proved $|O_\beta \setminus \bigcup_{\tau<\beta} O_\tau| = 1$. We show $|O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta| \le 1$. We have two cases:

Case 1: $\xi_\alpha = \emptyset$. Then $\mathrm{cf}(\alpha) > \omega$, $\{T_\beta : \beta < \alpha\}$ is a tower in T_α and $\mathcal{C}_\alpha = \emptyset$.

Let $X = \bigcup_{\beta<\alpha} O_\beta \subset \beta\psi \setminus \psi$. By the induction hypothesis, $\{O_\beta : \beta < \alpha\}$ satisfies the hypothesis of Lemma 1.4, hence $X \cong \alpha$. Then the one-point compactification of X and the one-point compactification of α are homeomorphic, and since in this case $\mathrm{cf}(\alpha) > \omega$, the only compactfication of α is its one-point compactification, the ordinal $\alpha+1$. Thus the only compactification of X is its one-point compactification, and $\overline{X} \cong \alpha+1$. Moreover $\overline{X} \subset \overline{O_\alpha} = O_\alpha$. To prove the claim it suffices to show that $\overline{X} = O_\alpha$. If $\overline{X} \neq O_\alpha$, then there exists a point $p \in O_\alpha \setminus \overline{X} \subset \overline{T_\alpha}$. Since $\overline{X}$ is compact, there exists a continuous function $f : \beta\psi \to [0,1]$ such that $f(p) = 0$, $f^{-1}(0) \subset \overline{T_\alpha}$ and $f(\overline{X}) = 1$.

Since $\mathcal{C}_\alpha = \emptyset$, by Claim 5, we have

$$f^{-1}(0) \cap \mathcal{M} \subset \overline{T_\alpha} \cap \mathcal{M} \subset \bigcup_{\beta\leq\alpha} \mathcal{C}_\beta = \bigcup_{\beta<\alpha} \mathcal{C}_\beta.$$

By Lemma 1.5, $f^{-1}(0) \cap \mathcal{M}$ is infinite. Pick distinct $C_i \in f^{-1}(0) \cap \mathcal{M}$; say that $C_i \in \mathcal{C}_{\beta_i}$ (for $i \in \omega$). Let $\beta = \sup\{\beta_i : i \in \omega\}$. Then $\beta < \alpha$ because $\mathrm{cf}(\alpha) > \omega$. Let x be a limit point of $\{C_i : i \in \omega\}$ in $\beta\psi \setminus \psi$. Then $f(x) = 0$. Since each $C_i \subset T_{\beta_i} \subset^* T_\beta$, we have $x \in \overline{T_\beta} \subset X$; hence $f(x) = 1$, which is a contradiction. This proves Case 1.

Case 2: $\xi_\alpha \neq \emptyset$. We break this case into three subcases depending on properties of α.

Subcase 1: α is a successor ordinal, say $\alpha = \tau + 1$. In this case

$$O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta = O_{\tau+1} \setminus O_\tau.$$

Since $\xi_\alpha \neq \emptyset$, $|\mathcal{C}_\alpha| = \mathfrak{c} \geq \omega$. By Lemma 1.6, $\mathcal{C}_\alpha$ has exactly one limit point in $\beta\psi$, call it x_α. Since $\mathcal{C}_\alpha$ diagonalizes $\{T_\beta : \beta < \alpha\}$ in T_α, we deduce for every $C \in \mathcal{C}_\alpha$ that $C \subset^* T_\alpha \setminus T_\tau$, hence the unique limit point x_α of $\mathcal{C}_\alpha$ is in $\overline{T_\alpha \setminus T_\tau} = \overline{T_\alpha} \setminus \overline{T_\tau}$; so $x_\alpha \in O_\alpha \setminus O_\tau$. If this is the only point in $O_\alpha \setminus O_\tau$, then we are done. So assume there is a point $p \in O_\alpha \setminus O_\tau$ and $p \neq x_\alpha$. Therefore there exists an open neighborhood U of p in $\beta\psi$ such that $U \subset \overline{T_\alpha} \setminus \overline{T_\tau}$ and $U \cap \mathcal{C}_\alpha = \emptyset$. Let $f : \beta\psi \to [0,1]$ be continuous such that $p \in f^{-1}(0) \subset U$. Since $f^{-1}(0) \cap \mathcal{C}_\alpha = \emptyset$, it follows from Claim 5 that

$$f^{-1}(0) \cap \mathcal{M} \subset \overline{T_\alpha} \cap \mathcal{M} \subset \bigcup_{\beta\leq\tau} \mathcal{C}_\beta,$$

and from Lemma 1.5 that $f^{-1}(0) \cap (\beta\psi \setminus \psi) \subset \overline{f^{-1}(0) \cap \mathcal{M}}$. Therefore

$$p \in f^{-1}(0) \cap (\beta\psi \setminus \psi) \subset \overline{f^{-1}(0) \cap \mathcal{M}} \subset \overline{\bigcup_{\alpha\leq\tau} \mathcal{C}_\alpha} \subset \overline{T_\tau};$$

so $p \in O_\tau$. This is a contradiction.

Subcase 2: $\mathrm{cf}(\alpha) = \omega$. By Lemma 1.6, $\mathcal{C}_\alpha$ has exactly one limit point in $\beta\psi$, which we denote by x_α, and $x_\alpha \in O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta$. If this is the only point in $O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta$, then we are done. So assume there is a point $p \in O_\alpha \setminus \bigcup_{\beta<\alpha} O_\beta$ and $p \neq x_\alpha$. Then $p \notin \overline{\mathcal{C}_\alpha}$. Let $f : \beta\psi \to [0,1]$ be a continuous function such that $p \in f^{-1}(0) \subset \overline{T_\alpha}$, and $f^{-1}(0) \cap \mathcal{C}_\alpha = \emptyset$. By Claim 5 and Lemma 1.5, $f^{-1}(0) \cap \bigcup_{\beta<\alpha} C_\beta$ is infinite. If $f^{-1}(0) \cap \mathcal{C}_\beta \neq \emptyset$ for cofinally many $\beta < \alpha$, then we may pick, for each $n \in \omega$, ordinals $\beta_n < \alpha$ and $M_n \in f^{-1}(0) \cap \mathcal{C}_{\beta_n}$ in such a way that $\{\beta_n : n \in \omega\}$ is an increasing and cofinal sequence in α. Since $f(M_n) = 0$ for $n \in \omega$, we may pick by recursion distinct integers a_n with

$$a_n \in M_n \cap \Big(T_\alpha \setminus \bigcup_{i<n} T_{\beta_i}\Big)$$

such that $f(a_n) < 1/(n+1)$ for $n \in \omega$. Then $A = \{a_n : n \in \omega\}$ diagonalizes $\{T_\beta : \beta < \alpha\}$ in T_α. By Claim 2 there exists $C \in C_\alpha$ such that $|C \cap A| = \omega$, but then it follows that $f(C) = 0$; so $C \in f^{-1}(0) \cap \mathcal{C}_\alpha$, which is a contradiction. Therefore it must be the case that there exists $\gamma < \alpha$ such that $f^{-1}(0) \cap \mathcal{M} \subset \bigcup_{\beta\leq\gamma} \mathcal{C}_\beta$. We show this possibility does not occur. If it did, for each $C \in f^{-1}(0) \cap \mathcal{M}$ we have $C \subset^* T_\gamma$, hence $C \in \overline{T_\gamma} \cap \mathcal{M}$. Therefore $\overline{f^{-1}(0) \cap \mathcal{M}} \subset \overline{T_\gamma}$. But $p \in f^{-1}(0) \cap (\beta\psi \setminus \psi) \subset \overline{f^{-1}(0) \cap \mathcal{M}}$ by Lemma 1.5. Thus we have $p \in \overline{T_\gamma}$, which implies $p \in O_\gamma$ where $\gamma < \alpha$, and that is a contradiction.

Subcase 3: $\mathrm{cf}(\alpha) > \omega$. This subcase has some similarities with Case 1 since in both cases $\mathrm{cf}(\alpha) > \omega$. However, in Case 1, we have $\mathcal{C}_\alpha = \emptyset$, while in this subcase we have $\mathcal{C}_\alpha \neq \emptyset$. Put $X = \bigcup_{\beta<\alpha} O_\beta$. As in Case 1, $\overline{X}$ is the one-point compactification of $\bigcup_{\beta<\alpha} O_\beta$, and $\overline{X} \subset O_\alpha$. We need to show that $O_\alpha = \overline{X}$. If not, there exists a point $p \in O_\alpha \setminus \overline{X}$. Let $f : \beta\psi \to [0,1]$ be a continuous function such that $p \in f^{-1}(0)$, $\overline{X} \subset f^{-1}(1)$, and $f^{-1}(0) \subset \overline{T_\alpha}$. We will derive a contradiction.

By Lemma 1.5 and Claim 5,

$$p \in \overline{\bigcup_{\beta\leq\alpha} \mathcal{C}_\beta} = \overline{\bigcup_{\beta<\alpha} \mathcal{C}_\beta} \cup \overline{\mathcal{C}_\alpha}.$$

As in Case 1, $p \notin \overline{\bigcup_{\beta<\alpha} \mathcal{C}_\beta}$ because $f^{-1}([0,1/2))$ is neighborhood of p and $f^{-1}([0,1/2)) \cap \bigcup_{\beta<\alpha} \mathcal{C}_\beta$ is finite (since $\mathrm{cf}(\alpha) > \omega$ and $f^{-1}([0,1/2)) \cap X = \emptyset$). Thus $p \in \overline{\mathcal{C}_\alpha}$. By Lemma 1.6, p is the only limit point of $\mathcal{C}_\alpha$ in $\beta\psi$. Since $f^{-1}([0,1/2))$ is a neighborhood of p in $\beta\psi$, we have $\mathcal{C}_\alpha \subset^* f^{-1}([0,1/2))$. Let $F_0 = \mathcal{C}_\alpha \setminus f^{-1}([0,1/2))$, a finite set. By Claim 1, $\mathcal{C}_\alpha \subset \overline{T_\alpha}$, thus $F_0 \subset \overline{T_\alpha}$, and moreover, for each $M \in F_0$, $M \subset T_\alpha$. Define $K_0 = F_0 \cup \bigcup F_0$. Then K_0 is clopen and compact in ψ, hence clopen and compact in $\beta\psi$. Define f_0 to be equal to f on $\beta\psi \setminus K_0$ and $f_0 = 0$ on K_0. Then $f_0 : \beta\psi \to [0,1]$ is continuous and has the property that $\mathcal{C}_\alpha \subset f_0^{-1}([0,1/2))$ (true subset). In addition f_0 retains three relevant properties of f: $p \in f_0^{-1}(0)$, $\overline{X} \subset f_0^{-1}(1)$ (because f and f_0 agree on $\beta\psi \setminus \psi$), and $f_0^{-1}(0) \subset \overline{T_\alpha}$ (because $K_0 \subset \overline{T_\alpha}$).

We see that $f_0^{-1}((1/2,1])$ is an open set containing $\overline{X}$, and $\bigcup_{\beta<\alpha} \mathcal{C}_\beta \subset^* f_0^{-1}((1/2,1])$ (since $\mathrm{cf}(\alpha) > \omega$ and $f_0^{-1}([0,1/2)) \cap X = \emptyset$). Let $F_1 = \bigcup_{\beta<\alpha} \mathcal{C}_\beta \setminus f_0^{-1}((1/2,1])$, a finite set. Define $K_1 = F_1 \cup \bigcup F_1$. Then K_1 is clopen and compact in ψ, hence clopen and compact in $\beta\psi$. Define f_1 to be equal to f_0 on $\beta\psi \setminus K_1$ and $f_1 = 1$ on K_1. Then $f_1 : \beta\psi \to [0,1]$ is continuous and has the property that $\bigcup_{\beta<\alpha} \mathcal{C}_\beta \subset f_1^{-1}((1/2,1])$ (true subset). In addition f_1 retains four relevant properties of f_0: $\mathcal{C}_\alpha \subset f_1^{-1}([0,1/2))$ (since f_0 and f_1 agree on $\mathcal{C}_\alpha$), $p \in f_1^{-1}(0)$, $\overline{X} \subset f_1^{-1}(1)$ (because f_0 and f_1 agree on $\beta\psi \setminus \psi$), and $f_1^{-1}(0) \subset \overline{T_\alpha}$ (because $f_1^{-1}(0) \subset f_0^{-1}(0)$).

Now we define

$$H = f_1^{-1}((1/2,1]) \cap T_\alpha.$$

Then $H \subset \omega$ is infinite, and has the following three properties:

- (i) $T_\beta \subset^* H$ for all $\beta < \alpha$, since if for some $\beta < \alpha$ we have $|T_\beta \setminus H| = \omega$, then by Claim 3, $\bigcup_{\tau\le\beta} \mathcal{C}_\alpha$ is a MADF on T_β, hence there exists $C \in \bigcup_{\tau\le\beta} \mathcal{C}_\alpha$ such that $|C \cap (T_\beta \setminus H)| = \omega$, but this implies $f_1(C) \le 1/2$, which contradicts the definition of f_1.
- (ii) $H <^* T_\alpha$, because by Case 2, $C_\alpha \neq \emptyset$, and for any $C \in \mathcal{C}_\alpha$, $C \subset^* T_\alpha \setminus H$ (since $f_1(C) < 1/2$).
- (iii) $\{T_\beta : \beta < \alpha\}$ is a tower in H. We need only check the maximality condition of a tower; so suppose $K < H$ and $T_\beta \subset^* K$ for all $\beta < \alpha$. Then $H \setminus K$ is an infinite subset of T_α and diagonalizes $\{T_\beta : \beta < \alpha\}$ in T_α. Hence by Claim 2 there exists $C \in \mathcal{C}_\alpha$ such that $|C \cap (H \setminus K)| = \omega$. This implies that $f_1(C) \ge 1/2$, but this is impossible because $f_1(C) < 1/2$. This proves that $\{T_\beta : \beta < \alpha\}$ is a tower in $H <^* T_\alpha$.

But by hypothesis of Case 2, $\xi_\alpha \neq \emptyset$; so $\{T_\beta : \beta < \alpha\}$ is not a tower in T_α (as noted following the definition of ξ_α), hence not a tower in H by property $(*)$ of our adjusted chain. That contradicts (iii) and completes the proof of Subcase 3 of Claim 7, and therefore Claim 7 is proved.

By Claims 6 and 7, the family $\{O_\alpha : \alpha < \lambda + 1\}$ satisfies the hypothesis of Lemma 1.4, hence $\beta\psi \setminus \psi$ is homeomorphic to $\lambda + 1$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. If we have a chain indexed by an ordinal λ, then clearly we have chains indexed by all ordinals $\beta < \lambda$. There exists a chain (in fact a tower) $\{S_\alpha : \alpha < \mathfrak{t}\} \subset [\omega]^\omega$ indexed by $\mathfrak{t}$. It suffices to prove that for every ordinal $\lambda < \mathfrak{t}^+$ with $\mathrm{cf}(\lambda) = \mathfrak{t}$, there is a chain indexed by λ. The proof is by induction. Assume we have (ascending) ordered chains in ω of order type β for every $\beta < \lambda$ where $\lambda < \mathfrak{t}^+$, and $\mathrm{cf}(\lambda) = \mathfrak{t}$. We construct a chain indexed by λ. Let $\varphi : \mathfrak{t} \to \lambda$ be a strictly increasing function onto a set of ordinals cofinal in λ. Let $\{E_\alpha : \alpha < \mathfrak{t}\}$ be a pairwise disjoint family of copies of ω, and let $f_\alpha : E_\alpha \to S_{\alpha+1} \setminus S_\alpha$ be a bijection. For each $\alpha < \kappa$ let

$$\{U^\alpha_\tau : \varphi(\alpha) \leq \tau < \varphi(\alpha+1)\}$$

be an ascending mod-finite chain in $[E_\alpha]^\omega$ indexed by the interval of ordinals $[\varphi(\alpha), \varphi(\alpha+1))$ (considered as a subset of a chain indexed by the ordinal $\varphi(\alpha+1)$) with the one extra requirement that $U^\alpha_{\varphi(\alpha)} = \emptyset$. By "ascending" we mean that if $\varphi(\alpha) < \tau < \mu < \varphi(\alpha+1)$ then $U^\alpha_\tau <^* U^\alpha_\mu <^* E_\alpha$. Now we define a chain indexed by λ as follows: For $\tau < \lambda$, let $\alpha < \mathfrak{t}$ be the unique ordinal such that $\varphi(\alpha) \leq \tau < \varphi(\alpha+1)$, and define

$$T_\tau = S_\alpha \cup f_\alpha(U^\alpha_\tau) \quad \text{ for } \varphi(\alpha) \leq \tau < \varphi(\alpha+1).$$

By our definitions, $T_{\varphi(\alpha)} = S_\alpha$. It remains to show "mod-finite ascending". Suppose $\tau < \mu < \lambda$. Let $\alpha < \mathfrak{t}$ be such that $\varphi(\alpha) \leq \tau < \varphi(\alpha+1)$. If $\mu < \varphi(\alpha+1)$, then on E_α we have $U^\alpha_\tau <^* U^\alpha_\mu$, hence $f_\alpha(U^\alpha_\tau) <^* f_\alpha(U^\alpha_\mu)$, hence

$$T_\tau = S_\alpha \cup f_\alpha(U^\alpha_\tau) <^* S_\alpha \cup f_\alpha(U^\alpha_\mu) = T_\mu.$$

If $\varphi(\alpha+1) \leq \mu$, let $\beta < \mathfrak{t}$ be such that $\varphi(\beta) \leq \mu < \varphi(\beta+1)$. Then $\alpha+1 \leq \beta$ and we have

$$T_\tau = S_\alpha \cup f(U^\alpha_\tau) <^* S_{\alpha+1} \subset^* S_\beta \subset S_\beta \cup f_\beta(U^\beta_\mu) = T_\mu.$$

Thus $\{T_\alpha : \alpha < \lambda\}$ is a chain, and this completes the proof of Theorem 1.2.

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