Ordinal remainders of classical $\psi$-spaces

by

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Abstract. Let $\omega$ denote the set of natural numbers. We prove: for every mod-finite ascending chain $\{T_\alpha : \alpha < \lambda\}$ of infinite subsets of $\omega$, there exists $M \subset [\omega]^\omega$, an infinite maximal almost disjoint family (MADF) of infinite subsets of the natural numbers, such that the Stone–Čech remainder $\beta \psi \setminus \psi$ of the associated $\psi$-space, $\psi = \psi(\omega, M)$, is homeomorphic to $\lambda + 1$ with the order topology. We also prove that for every $\lambda < t^+$, where $t$ is the tower number, there exists a mod-finite ascending chain $\{T_\alpha : \alpha < \lambda\}$, hence a $\psi$-space with Stone–Čech remainder homeomorphic to $\lambda + 1$. This generalizes a result credited to S. Mrówka by J. Terasawa which states that there is a MADF $M$ such that $\beta \psi \setminus \psi$ is homeomorphic to $\omega_1 + 1$.

1. Introduction. Let $\omega$ denote the set of natural numbers. Let $[\omega]^\omega$ denote the set of all countably infinite subsets of $\omega$. Sets $A, B \in [\omega]^\omega$ are said to be almost disjoint provided $A \cap B$ is finite. An infinite family $A \subset [\omega]^\omega$ is called an almost disjoint family (ADF) if any two elements of $A$ are almost disjoint. An ADF $M$ is called a maximal almost disjoint family (MADF) if it is not properly contained in another ADF.

We have considered almost disjoint families of countable subsets of an arbitrary cardinal $\kappa$ in [4], [5], but in this paper we only consider the classical case $\kappa = \omega$.

Almost disjoint families, especially MADF’s, are of interest in set theory (e.g., [8], [14]), topology (e.g., [6], [13]), Boolean algebra (e.g., [1], [3]) and Banach spaces (e.g., [10], [11]).

An important interest in topology of MADF’s concerns the well-known class of topological spaces called $\psi$-spaces (e.g., see [2, §11]; for some historical notes, see [4, §2]). For any ADF $A \subset [\omega]^\omega$, let $\psi(\omega, A)$ denote the space with underlying set $\omega \cup A$ and with the topology having as a base all singletons $\{\alpha\}$ for $\alpha < \omega$ and all sets of the form $\{A\} \cup (A \setminus F)$ where...
A ∈ ℳ and ℱ is finite. We call \( ψ(ω, A) \) the \( ψ \)-space associated with \( A \), and the class of spaces of the form \( ψ(ω, A) \) the class of \( ψ \)-spaces on \( ω \) (or on \( [ω]ω \)). It is well-known that the topology of \( ψ(ω, A) \) is closely related to the ADF \( A \), however all \( ψ \)-spaces on \( ω \), regardless of which ADF \( A \) is used, are first countable, zero-dimensional, Hausdorff, locally compact non-compact spaces in which \( ω \) is an open dense set of isolated points (hence the \( ψ \)-space is separable), and \( ψ \setminus ω = A \) is a closed discrete set. Maximal ADF’s are of additional interest because \( M \) is maximal if and only if the \( ψ \)-space associated with \( M \) is pseudocompact (i.e., every real-valued continuous function defined on \( ψ \) is bounded \([12]\)).

For \( A, B \in [ω]ω \), \( A ↑ B \) means that \( A \setminus B \) is finite, and \( A \subset^* B \) means that \( A \subset B \) and \( B \setminus A \) is infinite. Let \( X \in [ω]ω \). A mod-finite ascending chain of order type \( λ \) in \( X \) (chain for short) is a family \( \{T_α : α < λ\} \subset [ω]^ω \), where \( λ \) is an ordinal, such that \( T_α ↑ X \) for all \( α < λ \), and \( β < α < λ \) imply \( T_β ↑ T_α \). We also say “chain indexed by \( λ \)” when the particular ordinal \( λ \) is needed. Let \( c \) denote the cardinality of the continuum, and let \( t \) denote the tower number, i.e., the smallest cardinality of a tower in \([ω]^ω \) (e.g., see \([2]\) or \([16]\)).

We use the approach to Stone–Cech compactifications in \([7]\). For \( E \subset ψ \), we let \( E^\bullet \) denote the closure of \( E \) in \( βψ \). The closure of \( E \) in \( ψ \) will be denoted by \( cl_ψ(E) \).

J. Terasawa \([15]\) proved that every compact metric space is the Stone–Čech remainder of a \( ψ \)-space on \( ω \) for a suitably chosen MADF \( M \), and he stated that the ordinal \( ω_1 + 1 \) with the order topology is also the remainder of a \( ψ \)-space. He attributed that result to S. Mrówka. Our main theorem generalizes this result as follows.

**Theorem 1.1.** If there exists a chain \( \{T_α : α < λ\} \) in \([ω]^ω \) indexed by the ordinal \( λ \), then there exists a MADF \( M \subset [ω]^ω \) such that \( βψ \setminus ψ \) is homeomorphic to \( λ + 1 \) with the order topology.

Concerning the existence of ascending chains, we prove

**Theorem 1.2.** If \( λ < t^+ \), then there exists an ascending ordered chain in \( ω \) of order type \( λ \). Thus there exists a MADF \( M \subset [ω]^ω \) such that \( βψ \setminus ψ \cong \lambda + 1 \).

Combining these two theorems we get

**Theorem 1.3.** For every successor ordinal \( λ + 1 < t^+ \) there exists a MADF \( M \subset [ω]^ω \) such that the Stone–Čech remainder of \( ψ(ω, M) \) is homeomorphic to \( λ + 1 \) with the order topology.

Theorem \([13]\) implies the result attributed to Mrówka because, as is well known, \( t \) is an uncountable cardinal, hence \( ω_1 + 1 < t^+ \).
Theorem 1.3 is the best possible in ZFC in the sense that it is consistent that the ordinal \( t^+ + 1 \) is not the Stone–Čech remainder of any \( \psi \)-space on \( \omega \). This follows because a compact separable space has weight at most \( \mathfrak{c} \) [9 2.3(i)]. Hence a compact separable space cannot contain a subspace with \( \mathfrak{c}^+ \) isolated points; in particular cannot contain a copy of the ordinal \( \mathfrak{c}^+ + 1 \).

Hence in any model where \( t = \mathfrak{c} \), also \( t^+ = \mathfrak{c}^+ \), and this implies that \( t^+ + 1 \) is not the Stone–Čech remainder of any \( \psi \)-space on \( \omega \).

We establish the homeomorphism between the Stone–Čech remainder, \( \beta \psi \setminus \psi \), and \( \lambda + 1 \) using the following simple result.

**Lemma 1.4.** Let \( X \) be a Hausdorff space and \( \{ O_\alpha : \alpha < \lambda \} \) a cover of \( X \) by compact, clopen sets such that, for all \( \beta < \alpha < \lambda \),

1. \( O_\beta \subset O_\alpha \), and
2. \( |O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta| = 1 \) for \( \alpha < \lambda \).

Then \( X \) is homeomorphic to \( \lambda \) with the order topology.

**Proof.** For \( \alpha < \lambda \), let \( x_\alpha \in X \) be the point such that \( \{ x_\alpha \} = O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta \), and define \( \varphi : \lambda \to X \) by \( \varphi(\alpha) = x_\alpha \). Clearly \( \varphi \) is one-to-one and onto \( X \). By the compactness of \( O_\alpha \), the decreasing family \( \{ O_\alpha \setminus O_\beta : \beta < \alpha \} \) is a local base for \( x_\alpha \) in \( X \). We show that \( \varphi \) is a homeomorphism by showing that \( \varphi((\gamma, \alpha]) = O_\alpha \setminus O_\gamma \) for all \( \gamma < \alpha < \lambda \). If \( x \in \varphi((\gamma, \alpha]) \), there exists \( \gamma < \beta \leq \alpha \) such that \( x = \varphi(\beta) = x_\beta \). Since \( x_\beta \in O_\beta \) and, by (1), \( O_\beta \subset O_\alpha \), we see that \( \varphi(\beta) \in O_\alpha \). For \( \gamma < \beta \), \( x_\beta \not\in O_\gamma \). Thus \( \varphi(\beta) = x_\beta \in O_\alpha \setminus O_\gamma \).

Conversely, let \( x \in O_\alpha \setminus O_\gamma \), and let \( \beta \leq \alpha \) be the first ordinal such that \( x \in O_\beta \). Then \( x \in O_\beta \setminus \bigcup_{\tau \leq \beta} O_\tau \); so \( x = x_\beta \). We have \( \gamma < \beta \), so \( \gamma < \beta \leq \alpha \) and \( x = \varphi(\beta) \). Thus \( x \in \varphi((\gamma, \alpha]) \). This proves the equality \( \varphi((\gamma, \alpha]) = O_\alpha \setminus O_\gamma \), and shows that \( \varphi \) is a homeomorphism.

**Lemma 1.5.** If \( Z \) is a zero set in \( \beta \psi \), then \( \overline{Z} \cap \mathcal{M} \supset Z \cap (\beta \psi \setminus \psi) \), and if \( Z \cap (\beta \psi \setminus \psi) \neq \emptyset \), then \( Z \cap \mathcal{M} \) is infinite.

**Proof.** Let \( p \in Z \cap (\beta \psi \setminus \psi) \). Then \( p \) is a free z-ultrafilter on \( \psi \). Let \( Z' \) be any zero-set neighborhood of \( p \) in \( \beta \psi \). Since \( \psi \) is pseudocompact, the z-ultrafilter \( p \) is countably complete, i.e., if \( Z_n \in p \) for \( n \in \omega \), then \( \bigcap_{n \in \omega} Z_n \in p \) ([7 8.6]). For \( n \in \omega \), \( Z_n = \psi \setminus \{ n \} \) is a (clopen) zero set in \( \psi \), and since \( p \) is free, \( \psi \setminus \{ n \} \in p \). Now we have \( Z'' = Z \cap Z' \cap \bigcap_{n \in \omega} Z_i \in p \), and \( Z'' \subset \mathcal{M} \). This shows that the zero-set neighborhood \( Z' \) of \( p \) has a non-empty intersection with \( Z \cap \mathcal{M} \). Since \( Z' \) was arbitrary, it follows that \( p \in Z \cap \overline{\mathcal{M}} \).

Let \( F \in [\omega]^{\omega} \) and \( \{ T_\alpha : \alpha < \lambda \} \) be a chain in \( X \). We say that the set \( F \) diagonalizes the chain in \( X \) (or in \( [X]^{\omega} \)) provided \( F \subset X \) and for all \( \beta < \alpha \), \( |F \cap T_\beta| < \omega \). If \( \mathcal{F} \subset [\omega]^{\omega} \) is a family of sets, we say \( \mathcal{F} \) diagonalizes the chain in \( X \) provided each set \( F \in \mathcal{F} \) diagonalizes the chain in \( X \). We
note that there exists $F$ that diagonalizes a chain $\{T_\beta : \beta < \lambda\}$ in $X$ if and only if the chain is a not a tower in $X$ (consider $X \setminus F$).

Given disjoint sets $A, B \in [\omega]^\omega$ and $A \subset [A]^\omega$ ADF, $B \subset [B]^\omega$ ADF with $|A| = |B|$, the (Mrówka) join of $A$ and $B$ is defined by $A \oplus B = \{A \cup \phi(A) : A \in A\}$, where $\phi : A \to B$ is a bijection (see [4, §4]). Clearly $A \oplus B$ is an ADF and is maximal if both $A$ and $B$ are maximal.

We call a MADF $M$ a Mrówka MADF provided $|\beta \psi \setminus \psi| = 1$. As proved by S. Mrówka in [13], there exists $M \subset [\omega]^\omega$ such that $|\beta \psi \setminus \psi| = 1$.

**Lemma 1.6.** Let $N, \omega \setminus N \in [\omega]^\omega$, $B \subset [N]^\omega$ be a Mrówka MADF, and $A \subset [\omega \setminus N]^\omega$ an ADF such that $|B| = |A| \geq \omega$. If $M \subset [\omega]^\omega$ is a MADF and $A \oplus B \subset M$ then $A \oplus B$ has exactly one limit point in $\beta \psi$, where $\psi = \psi(\omega, M)$.

**Proof.** The proof is similar to that of [5, Theorem 1.3, Case 3].

2. Proofs of the main results

**Proof of Theorem 1.4.** Let $\{T_\beta : \beta < \lambda\}$ be a chain in $T_\lambda$. For convenience we identify $T_\lambda$ with $\omega$. We must construct a MADF $M \subset [\omega]^\omega$ such that $\beta \psi \setminus \psi$ is homeomorphic to $\lambda + 1$. For $\lambda$ finite, say $\lambda = n \geq 1$, we prove the theorem by taking $n$ disjoint copies of $\omega$ each with a Mrówka MADF (i.e., $|\beta \psi \setminus \psi| = 1$). Then taking unions yields the theorem. If we prove the theorem for a limit ordinal $\lambda$, then we get it for successor ordinals, $\lambda + 1, \lambda + 2, \ldots$, because all these ordinal spaces are homeomorphic to $\lambda + 1$. Thus we assume that $\{T_\beta : \beta < \lambda\}$ is a chain and $\lambda$ is a limit ordinal.

We want our chain to have the following property:

(*) For $\alpha \leq \lambda$, if $\text{cf}(\alpha) > \omega$ and $\{T_\beta : \beta < \alpha\}$ is not a tower in $T_\alpha$, then it is not a tower in any $H <^* T_\alpha$.

The given chain $\{T_\beta : \beta < \lambda\}$ may not have this property, so we adjust it: For every $\alpha < \lambda$, if there exists $H <^* T_\alpha$ such that $\{T_\beta : \beta < \alpha\}$ is a tower in $H$ (hence $\text{cf}(\alpha) > \omega$), then we change the definition of $T_\alpha$ by replacing $T_\alpha$ with $H$ (then $T_\alpha = H$). After this change, $\{T_\beta : \beta < \alpha\}$ is a tower in $T_\alpha$. For those $\alpha < \lambda$ for which this change is made, $T_\alpha$ becomes a bit smaller than it was and the ring $T_{\alpha+1} \setminus T_\alpha$ becomes a bit larger. No change is made for any other $\alpha < \lambda$. This adjusted chain clearly has property (*).

Now we proceed to the construction of the MADF $M$. For $\alpha \leq \lambda$ define

$\xi_\alpha = \{A \subset [T_\alpha]^\omega : A$ is an ADF diagonalizing $\{T_\beta : \beta < \alpha\}$ in $T_\alpha\}$.

Note that

$\xi_\alpha = \emptyset \iff \{T_\beta : \beta < \alpha\}$ is a tower in $[T_\alpha]^\omega$.

When $\xi_\alpha \neq \emptyset$, we partially order $\xi_\alpha$ by set inclusion. Then by Zorn's Lemma, there is a family $N_\alpha \in \xi_\alpha$ such that $N_\alpha$ is a maximal element in
the poset \( \xi_\alpha \). Since there exists a MADF of cardinality \( c \), we may assume that \(|N_\alpha| = c\). Pick \( N_\alpha \in N_\alpha \) and let \( B_\alpha \subset [N_\alpha]^{\omega} \) be a Mrówka MADF with \(|B_\alpha| = c\). Since \( N_\alpha \) diagonalizes the chain, \( B_\alpha \) diagonalizes the chain. Then \( B_\alpha \oplus (N_\alpha \setminus \{N_\alpha\}) \) is an ADF, and diagonalizes the chain \( \{T_\beta : \beta < \alpha\} \) (obviously if each of two sets diagonalizes a chain, then their union also does).

As an intermediate step before we get to \( M \) we define families \( C_\alpha \).

(1\( \alpha \)) If \( \xi_\alpha \neq \emptyset \) we define \( C_\alpha = B_\alpha \oplus (N_\alpha \setminus \{N_\alpha\}) \) as described above. This case includes all successor ordinals, all ordinals of countable cofinality, and all ordinals of uncountable cofinality for which \( \{T_\beta : \beta < \alpha\} \) is not a tower in \( T_\alpha \), hence not a tower in any \( H <^* T_\alpha \) (by the adjustment we made to the tower at the beginning of the proof).

(2\( \alpha \)) If \( \xi_\alpha = \emptyset \), then we define \( C_\alpha = \emptyset \).

We note that if \( C_\alpha \neq \emptyset \) then \(|C_\alpha| = c\).

**Claim 1.** For all \( \alpha \leq \lambda \), \( C_\alpha \subset [T_\alpha]^{\omega} \) and \( C_\alpha \) diagonalizes the chain \( \{T_\beta : \beta < \alpha\} \) in \( T_\alpha \).

*Proof.* If \( C_\alpha = \emptyset \), there is nothing to prove. If \( C_\alpha \neq \emptyset \), then the inclusion \( C_\alpha \subset [T_\alpha]^{\omega} \) follows because \( N_\alpha \subset [T_\alpha]^{\omega} \), \( B_\alpha \subset [N_\alpha]^{\omega} \) and \( N_\alpha \in N_\alpha \). Since \( C_\alpha = B_\alpha \oplus (N_\alpha \setminus \{N_\alpha\}) \) is a join of two ADF’s each of which diagonalizes the chain \( \{T_\beta : \beta < \alpha\} \) in \( T_\alpha \), we see that \( C_\alpha \) diagonalizes the chain.

**Claim 2.** For all \( \alpha \leq \lambda \), if \( H \in [T_\alpha]^{\omega} \) diagonalizes \( \{T_\beta : \beta < \alpha\} \) in \( T_\alpha \) then there exists \( C \in C_\alpha \) such that \(|C \cap H| = \omega\).

*Proof.* Assume \( H \) diagonalizes \( \{T_\beta : \beta < \alpha\} \) in \( T_\alpha \). The existence of \( H \) implies that \( \xi_\alpha \neq \emptyset \). Hence \( C_\alpha \) was defined by (1\( \alpha \)). Therefore the maximal element \( N_\alpha \) of \( \xi_\alpha \) was used in the definition of \( C_\alpha \). We cannot have both \( H \notin N_\alpha \) and \( N_\alpha \cup \{H\} \) an ADF because that would contradict the maximality of \( N_\alpha \) in \( \xi_\alpha \). Either way, there is \( N \in N_\alpha \) such that \(|H \cap N| = \omega\), hence there exists \( C \in C_\alpha \) such that \(|C \cap H| = \omega\).

**Claim 3.** The family \( \{C \cap T_\alpha : C \in \bigcup_{\beta \leq \alpha} C_\beta\} \) is a MADF on \( T_\alpha \).

*Proof.* If \( H \in [T_\alpha]^{\omega} \), then there exists a first \( \beta \leq \alpha \) such that \(|H \cap T_\beta| = \omega\). Then \( H \cap T_\beta \) diagonalizes \( \{T_\xi : \xi < \beta\} \) in \( T_\beta \); so by Claim 2, there exists \( C \in C_\beta \) such that \(|C \cap H \cap T_\beta| = \omega\). Hence \(|C \cap H| = \omega\).

We define

\[
M = \bigcup_{\alpha \leq \lambda} C_\alpha.
\]

**Claim 4.** \( M \subset [\omega]^{\omega} \) is a MADF.

*Proof.* This follows from Claim 3 and the fact that \( T_\lambda = \omega \).
We now consider the topology of $\psi = \psi(\omega, M)$, the $\psi$-space associated with the MADF $M$ constructed above. We want to show that $\beta\psi \setminus \psi$ is homeomorphic to $\lambda + 1$ with the order topology. We first establish some useful facts concerning the topology on $\psi$ and $\beta\psi$.

**Claim 5.** For $\alpha \leq \lambda$, $T_\alpha \cap M \subset \bigcup_{\beta \leq \alpha} C_\alpha$, $\text{cl}_\psi(T_\alpha)$ is clopen in $\psi$, hence $\overline{T_\alpha}$ is clopen in $\beta\psi$.

**Proof.** Let $M \in T_\alpha \cap M$; then $M \cap T_\alpha$ is infinite by the definition of the topology on $\psi$. By Claim 3, there exists $C \in \bigcup_{\beta \leq \alpha} C_\alpha$ such that $|C \cap (M \cap T_\alpha)| = \omega$. Since $C, M \in M$, an almost disjoint family, we have $C = M$; so $M \in \bigcup_{\beta \leq \alpha} C_\alpha$.

To see that $\text{cl}_\psi(T_\alpha)$ is clopen in $\psi$, it suffices to to show this set is open since it is closed by definition. We may assume $\alpha < \lambda$ since $\text{cl}_\psi(T_\lambda) = \text{cl}_\psi(\omega) = \psi$ is clopen in $\psi$. Let $M \in \text{cl}_\psi(T_\alpha) \cap M$, hence $M \in \overline{T_\alpha} \cap M$, so by the first part of Claim 5, $M \in \bigcup_{\beta \leq \alpha} C_\alpha$. Thus by Claim 1, for some $\beta \leq \alpha$, $M \subset T_\beta \subset T_\alpha$, hence for some finite set $F$ we find that $\{M\} \cup (M \setminus F)$ is a neighborhood of $M$ contained in $\text{cl}_\psi(T_\alpha)$. Thus $\text{cl}_\psi(T_\alpha)$ is clopen in $\psi$, and therefore $\overline{T_\alpha} = \text{cl}_\psi(T_\alpha)$ is clopen in $\beta\psi$ [7, p. 90]. This completes the proof of Claim 5.

In preparation for using Lemma 1.4, we define

$$O_\alpha = \overline{T_\alpha} \cap (\beta\psi \setminus \psi) = \text{cl}_\psi(T_\alpha) \cap (\beta\psi \setminus \psi) \quad \text{for } \alpha \leq \lambda.$$ 

Then $\{O_\alpha : \alpha \leq \lambda\}$ is an increasing family of compact clopen sets in $\beta\psi \setminus \psi$, $O_\lambda = \beta\psi \setminus \psi$, and satisfies condition (1) in Lemma 1.4. To complete the proof of the theorem, we show that condition (2) also holds for this family, i.e., we show that

$$|O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta| = 1 \quad \text{for } \alpha \leq \lambda.$$

First we show that $O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta$ contains at least one point.

**Claim 6.** If $\beta < \alpha \leq \lambda$ then $O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta \neq \emptyset$.

**Proof.** Since $\{O_\alpha \setminus O_\beta : \beta < \alpha\}$ is a decreasing family of compact sets, it suffices to show that $O_\alpha \setminus O_\beta \neq \emptyset$ for all $\beta < \alpha$. If $C_\alpha \neq \emptyset$ then $C_\alpha$ is an infinite (in fact uncountable) subset of $\text{cl}_\psi(T_\alpha) \setminus \text{cl}_\psi(T_\beta)$; hence $T_\alpha \setminus T_\beta \neq \emptyset$. If $C_\alpha = \emptyset$, then $\alpha$ is a limit ordinal (in fact $\text{cf}(\alpha) > \omega$), hence $\beta + 1 < \alpha$. Since $C_{\beta + 1} \neq \emptyset$, we have $\emptyset \neq O_{\beta + 1} \setminus O_\beta \subset O_\alpha \setminus O_\beta$.

The remainder of the proof is devoted to showing that $O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta$ contains at most one point.

**Claim 7.** For all $\alpha \leq \lambda$, $|O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta| \leq 1$.

**Proof.** Assume that for all $\beta < \alpha$ we have proved $|O_\beta \setminus \bigcup_{\tau < \beta} O_\tau| = 1$. We show $|O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta| \leq 1$. We have two cases:

**Case 1:** $\xi_\alpha = \emptyset$. Then $\text{cf}(\alpha) > \omega$, so $\{T_\beta : \beta < \alpha\}$ is a tower in $T_\alpha$ and $C_\alpha = \emptyset$. 

Let $X = \bigcup_{\beta < \alpha} O_{\beta} \subset \beta\psi \setminus \psi$. By the induction hypothesis, \( \{O_{\beta} : \beta < \alpha\} \) satisfies the hypothesis of Lemma 1.4 hence \( X \cong \alpha \). Then the one-point compactification of \( X \) and the one-point compactification of \( \alpha \) are homeomorphic, and since in this case \( \text{cf}(\alpha) > \omega \), the only compactification of \( \alpha \) is its one-point compactification, the ordinal \( \alpha + 1 \). Thus the only compactification of \( X \) is its one-point compactification, and \( X \cong \alpha + 1 \). Moreover \( X \subset \overline{O_{\alpha}} = \alpha \). To prove the claim it suffices to show that \( \overline{X} = \alpha \). If \( \overline{X} \neq \alpha \), then there exists a point \( p \in O_{\alpha} \setminus X \subset T_{\alpha} \). Since \( \overline{X} \) is compact, there exists a continuous function \( f : \beta\psi \to [0, 1] \) such that \( f(p) = 0, f^{-1}(0) \subset T_{\alpha} \) and \( f(\overline{X}) = 1 \).

Since \( C_{\alpha} = \emptyset \), by Claim 5, we have

\[
f^{-1}(0) \cap \mathcal{M} \subset T_{\alpha} \cap \mathcal{M} \subset \bigcup_{\beta \leq \alpha} C_{\beta} = \bigcup_{\beta < \alpha} C_{\beta}.
\]

By Lemma 1.5, \( f^{-1}(0) \cap \mathcal{M} \) is infinite. Pick distinct \( C_{i} \in f^{-1}(0) \cap \mathcal{M} \); say that \( C_{i} \in \mathcal{C}_{\beta_{i}} \) (for \( i \in \omega \)). Let \( \beta = \sup\{\beta_{i} : i \in \omega\} \). Then \( \beta < \alpha \) because \( \text{cf}(\alpha) > \omega \). Let \( x \) be a limit point of \( \{C_{i} : i \in \omega\} \) in \( \beta\psi \setminus \psi \). Then \( f(x) = 0 \). Since each \( C_{i} \subset T_{\beta_{i}} \supset \ast T_{\beta} \), we have \( x \in T_{\beta} \subset X \); hence \( f(x) = 1 \), which is a contradiction. This proves Case 1.

Case 2: \( \xi_{\alpha} \neq \emptyset \). We break this case into three subcases depending on properties of \( \alpha \).

Subcase 1: \( \alpha \) is a successor ordinal, say \( \alpha = \tau + 1 \). In this case

\[
O_{\alpha} \setminus \bigcup_{\beta < \alpha} O_{\beta} = O_{\tau + 1} \setminus O_{\tau}.
\]

Since \( \xi_{\alpha} \neq \emptyset \), \( |C_{\alpha}| = \mathfrak{c} \geq \omega \). By Lemma 1.6, \( C_{\alpha} \) has exactly one limit point in \( \beta\psi \), call it \( x_{\alpha} \). Since \( C_{\alpha} \) diagonalizes \( \{T_{\beta} : \beta < \alpha\} \) in \( T_{\alpha} \), we deduce for every \( C \in C_{\alpha} \) that \( C \subset \ast T_{\alpha} \setminus T_{\tau} \), hence the unique limit point \( x_{\alpha} \) of \( C_{\alpha} \) is in \( T_{\alpha} \setminus T_{\tau} = T_{\alpha} \setminus T_{\tau} \); so \( x_{\alpha} \in O_{\alpha} \setminus O_{\tau} \). If this is the only point in \( O_{\alpha} \setminus O_{\tau} \), then we are done. So assume there is a point \( p \in O_{\alpha} \setminus O_{\tau} \) and \( p \neq x_{\alpha} \). Therefore there exists an open neighborhood \( U \) of \( p \) in \( \beta\psi \) such that \( U \subset T_{\alpha} \setminus T_{\tau} \) and \( U \cap C_{\alpha} = \emptyset \). Let \( f : \beta\psi \to [0, 1] \) be continuous such that \( p \in f^{-1}(0) \subset U \). Since \( f^{-1}(0) \cap C_{\alpha} = \emptyset \), it follows from Claim 5 that

\[
f^{-1}(0) \cap \mathcal{M} \subset T_{\alpha} \cap \mathcal{M} \subset \bigcup_{\beta \leq \tau} C_{\beta},
\]

and from Lemma 1.5 that \( f^{-1}(0) \cap (\beta\psi \setminus \psi) \subset f^{-1}(0) \cap \mathcal{M} \). Therefore

\[
p \in f^{-1}(0) \cap (\beta\psi \setminus \psi) \subset f^{-1}(0) \cap \mathcal{M} \subset \bigcup_{\alpha \leq \tau} C_{\alpha} \subset T_{\tau}.
\]

so \( p \in O_{\tau} \). This is a contradiction.

Subcase 2: \( \text{cf}(\alpha) = \omega \). By Lemma 1.6, \( C_{\alpha} \) has exactly one limit point in \( \beta\psi \), which we denote by \( x_{\alpha} \), and \( x_{\alpha} \in O_{\alpha} \setminus \bigcup_{\beta < \alpha} O_{\beta} \). If this is the
only point in \( O_\alpha \setminus \bigcup_{\beta < \omega} O_\beta \), then we are done. So assume there is a point \( p \in O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta \) and \( p \neq x_\alpha \). Then \( p \notin C_\alpha \). Let \( f : \beta \psi \to [0,1] \) be a continuous function such that \( p \in f^{-1}(0) \subset T_\alpha \), and \( f^{-1}(0) \cap C_\alpha = \emptyset \). By Claim 5 and Lemma 1.5, \( f^{-1}(0) \cap \bigcup_{\beta < \alpha} C_\beta \) is infinite. If \( f^{-1}(0) \cap C_\beta \neq \emptyset \) for cofinally many \( \beta < \alpha \), then we may pick, for each \( n \in \omega \), ordinals \( \beta_n < \alpha \) and \( M_n \in f^{-1}(0) \cap C_{\beta_n} \) in such a way that \( \{ \beta_n : n \in \omega \} \) is an increasing and cofinal sequence in \( \alpha \). Since \( f(M_n) = 0 \) for \( n \in \omega \), we may pick by recursion distinct integers \( a_n \) with
\[
a_n \in M_n \cap \left( T_\alpha \setminus \bigcup_{i < n} T_{\beta_i} \right)
\]
such that \( f(a_n) < 1/(n+1) \) for \( n \in \omega \). Then \( A = \{ a_n : n \in \omega \} \) diagonalizes \( \{ T_\beta : \beta < \alpha \} \) in \( T_\alpha \). By Claim 2 there exists \( C \in C_\alpha \) such that \( |C \cap A| = \omega \), but then it follows that \( f(C) = 0 \); so \( C \in f^{-1}(0) \cap C_\alpha \), which is a contradiction. Therefore it must be the case that \( f^{-1}(0) \cap M \subset \bigcup_{\beta \leq \gamma} C_\beta \). We show this possibility does not occur. If it did, for each \( C \in f^{-1}(0) \cap M \) we have \( C \subset^* T_\gamma \), hence \( C \in \overline{T_\gamma} \cap M \). Therefore \( f^{-1}(0) \cap M \subset \overline{T_\gamma} \). But \( p \in f^{-1}(0) \cap (\beta \psi \setminus \psi) \subset f^{-1}(0) \cap M \) by Lemma 1.5. Thus we have \( p \in \overline{T_\gamma} \), which implies \( p \in O_\gamma \) where \( \gamma < \alpha \), and that is a contradiction.

**SUBCASE 3:** \( \text{cf}(\alpha) > \omega \). This subcase has some similarities with Case 1 since in both cases \( \text{cf}(\alpha) > \omega \). However, in Case 1, we have \( C_\alpha = \emptyset \), while in this subcase, we have \( C_\alpha \neq \emptyset \). Put \( X = \bigcup_{\beta < \alpha} O_\beta \). As in Case 1, \( X \) is the one-point compactification of \( \bigcup_{\beta < \alpha} O_\beta \), and \( X \subset O_\alpha \). We need to show that \( O_\alpha = X \). If not, there exists a point \( p \in O_\alpha \setminus X \). Let \( f : \beta \psi \to [0,1] \) be a continuous function such that \( p \in f^{-1}(0) \), \( X \subset f^{-1}(1) \), and \( f^{-1}(0) \subset T_\alpha \). We will derive a contradiction.

By Lemma 1.5 and Claim 5,
\[
p \in \bigcup_{\beta \leq \alpha} C_\beta = \bigcup_{\beta < \alpha} C_\beta \cup \overline{C_\alpha}.
\]
As in Case 1, \( p \notin \bigcup_{\beta < \alpha} C_\beta \) because \( f^{-1}([0,1/2)) \) is neighborhood of \( p \) and \( f^{-1}([0,1/2)) \cap \bigcup_{\beta < \alpha} C_\beta \) is finite (since \( \text{cf}(\alpha) > \omega \) and \( f^{-1}([0,1/2)) \cap X = \emptyset \)). Thus \( p \in \overline{C_\alpha} \). By Lemma 1.6, \( p \) is the only limit point of \( C_\alpha \) in \( \beta \psi \). Since \( f^{-1}([0,1/2)) \) is a neighborhood of \( p \) in \( \beta \psi \), we have \( C_\alpha \subset^* f^{-1}([0,1/2)) \). Let \( F_0 = C_\alpha \setminus f^{-1}([0,1/2)) \), a finite set. By Claim 1, \( C_\alpha \subset \overline{T_\alpha} \), thus \( F_0 \subset \overline{T_\alpha} \), and moreover, for each \( M \in F_0 \), \( M \subset T_\alpha \). Define \( K_0 = F_0 \cup \bigcup_{\beta \leq \alpha} K_0 \). Then \( K_0 \) is clopen and compact in \( \psi \), hence clopen and compact in \( \beta \psi \). Define \( f_0 \) to be equal to \( f \) on \( \beta \psi \setminus K_0 \) and \( f_0 = 0 \) on \( K_0 \). Then \( f_0 : \beta \psi \to [0,1] \) is continuous and has the property that \( C_\alpha \subset f_0^{-1}([0,1/2)) \) (true subset).
addition \( f_0 \) retains three relevant properties of \( f \): \( p \in f_0^{-1}(0) \), \( \overline{X} \subset f_0^{-1}(1) \) (because \( f \) and \( f_0 \) agree on \( \beta \psi \setminus \psi \)), and \( f_0^{-1}(0) \subset \overline{T_\alpha} \) (because \( K_\alpha \subset \overline{T_\alpha} \)).

We see that \( f_0^{-1}((1/2, 1]) \) is an open set containing \( \overline{X} \), and \( \bigcup_{\beta < \alpha} C_\beta \subset^* f_0^{-1}((1/2, 1]) \) (since \( \text{cf}(\alpha) > \omega \) and \( f_0^{-1}([0, 1/2)) \cap X = \emptyset \)). Let \( F_1 = \bigcup_{\beta < \alpha} C_\beta \setminus f_0^{-1}((1/2, 1]) \), a finite set. Define \( K_1 = F_1 \cup \bigcup F_1 \). Then \( K_1 \) is clopen and compact in \( \psi \), hence clopen and compact in \( \beta \psi \). Define \( f_1 \) to be equal to \( f_0 \) on \( \beta \psi \setminus K_1 \) and \( f_1 = 1 \) on \( K_1 \). Then \( f_1 : \beta \psi \to [0, 1] \) is continuous and has the property that \( \bigcup_{\beta < \alpha} C_\beta \subset f_1^{-1}((1/2, 1]) \) (true subset). In addition \( f_1 \) retains four relevant properties of \( f_0 \): \( C_\alpha \subset f_1^{-1}([0, 1/2)) \) (since \( f_0 \) and \( f_1 \) agree on \( C_\alpha \)), \( p \in f_1^{-1}(0) \), \( \overline{X} \subset f_1^{-1}(1) \) (because \( f_0 \) and \( f_1 \) agree on \( \beta \psi \setminus \psi \)), and \( f_1^{-1}(0) \subset \overline{T_\alpha} \) (because \( f_1^{-1}(0) \subset f_0^{-1}(0) \)).

Now we define \( H = f_1^{-1}((1/2, 1]) \cap T_\alpha \).

Then \( H \subset \omega \) is infinite, and has the following three properties:

(i) \( T_\beta \subset^* H \) for all \( \beta < \alpha \), since if for some \( \beta < \alpha \) we have \( |T_\beta \setminus H| = \omega \), then by Claim 3, \( \bigcup_{\tau \leq \beta} C_\tau \) is a MADF on \( T_\beta \), hence there exists \( C \in \bigcup_{\tau \leq \beta} C_\tau \) such that \( |C \cap (T_\beta \setminus H)| = \omega \), but this implies \( f_1(C) \leq 1/2 \), which contradicts the definition of \( f_1 \).

(ii) \( H \subset^* T_\alpha \), because by Case 2, \( C_\alpha \neq \emptyset \), and for any \( C \in C_\alpha \), \( C \subset^* T_\alpha \setminus H \) (since \( f_1(C) \subset 1/2 \)).

(iii) \( \{T_\beta : \beta < \alpha \} \) is a tower in \( H \). We need only check the maximality condition of a tower; so suppose \( K < H \) and \( T_\beta \subset^* K \) for all \( \beta < \alpha \). Then \( H \setminus K \) is an infinite subset of \( T_\alpha \) and diagonalizes \( \{T_\beta : \beta < \alpha \} \) in \( T_\alpha \). Hence by Claim 2 there exists \( C \in C_\alpha \) such that \( |C \cap (H \setminus K)| = \omega \). This implies that \( f_1(C) \geq 1/2 \), but this is impossible because \( f_1(C) < 1/2 \). This proves that \( \{T_\beta : \beta < \alpha \} \) is a tower in \( H \subset^* T_\alpha \).

But by hypothesis of Case 2, \( \xi_\alpha \neq \emptyset \); so \( \{T_\beta : \beta < \alpha \} \) is not a tower in \( T_\alpha \) (as noted following the definition of \( \xi_\alpha \)), hence not a tower in \( H \) by property (\( * \)) of our adjusted chain. That contradicts (iii) and completes the proof of Subcase 3 of Claim 7, and therefore Claim 7 is proved.

By Claims 6 and 7, the family \( \{O_\alpha : \alpha < \lambda + 1\} \) satisfies the hypothesis of Lemma \( \ref{claim:ordered} \), hence \( \beta \psi \setminus \psi \) is homeomorphic to \( \lambda + 1 \). This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2** If we have a chain indexed by an ordinal \( \lambda \), then clearly we have chains indexed by all ordinals \( \beta < \lambda \). There exists a chain (in fact a tower) \( \{S_\alpha : \alpha < t\} \subset [\omega]^{\omega} \) indexed by \( t \). It suffices to prove that for every ordinal \( \lambda < t^+ \) with \( \text{cf}(\lambda) = t \), there is a chain indexed by \( \lambda \). The proof is by induction. Assume we have (ascending) ordered chains in \( \omega \) of
order type $\beta$ for every $\beta < \lambda$ where $\lambda < t^+$, and $\text{cf}(\lambda) = t$. We construct a chain indexed by $\lambda$. Let $\varphi : t \to \lambda$ be a strictly increasing function onto a set of ordinals cofinal in $\lambda$. Let $\{E_\alpha : \alpha < t\}$ be a pairwise disjoint family of copies of $\omega$, and let $f_\alpha : E_\alpha \to S_{\alpha + 1} \setminus S_\alpha$ be a bijection. For each $\alpha < \kappa$ let
\[
\{U_\tau^\alpha : \varphi(\alpha) \leq \tau < \varphi(\alpha + 1)\}
\]
be an ascending mod-finite chain in $[E_\alpha]^\omega$ indexed by the interval of ordinals $[\varphi(\alpha), \varphi(\alpha + 1))$ (considered as a subset of a chain indexed by the ordinal $\varphi(\alpha + 1)$) with the one extra requirement that $U_\tau^\alpha = \emptyset$. By “ascending” we mean that if $\varphi(\alpha) < \tau < \mu < \varphi(\alpha + 1)$ then $U_\tau^\alpha <* U_\mu^\alpha <* E_\alpha$. Now we define a chain indexed by $\lambda$ as follows: For $\tau < \lambda$, let $\alpha < t$ be the unique ordinal such that $\varphi(\alpha) \leq \tau < \varphi(\alpha + 1)$, and define
\[
T_\tau = S_\alpha \cup f_\alpha(U_\tau^\alpha) \quad \text{for} \quad \varphi(\alpha) \leq \tau < \varphi(\alpha + 1).
\]
By our definitions, $T_{\varphi(\alpha)} = S_\alpha$. It remains to show “mod-finite ascending”. Suppose $\tau < \mu < \lambda$. Let $\alpha < t$ be such that $\varphi(\alpha) \leq \tau < \varphi(\alpha + 1)$. If $\mu < \varphi(\alpha + 1)$, then on $E_\alpha$ we have $U_\tau^\alpha <* U_\mu^\alpha$, hence $f_\alpha(U_\tau^\alpha) <* f_\alpha(U_\mu^\alpha)$, hence
\[
T_\tau = S_\alpha \cup f_\alpha(U_\tau^\alpha) <* S_\alpha \cup f_\alpha(U_\mu^\alpha) = T_\mu.
\]
If $\varphi(\alpha + 1) \leq \mu$, let $\beta < t$ be such that $\varphi(\beta) \leq \mu < \varphi(\beta + 1)$. Then $\alpha + 1 \leq \beta$ and we have
\[
T_\tau = S_\alpha \cup f(U_\tau^\alpha) <* S_{\alpha + 1} \subset* S_\beta \subset S_\beta \cup f_\beta(U_\beta^\alpha) = T_\mu.
\]
Thus $\{T_\alpha : \alpha < \lambda\}$ is a chain, and this completes the proof of Theorem 1.2.

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References

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